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QUANTUM ALGEBRA APPROACH TO q-BESSEL FUNCTIONS

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Abstract

Quantum algebras provide a natural algebraic setting for q-special functions. Taking as example the quantum Euclidean algebra in two dimensions, we shall explain how generating functions, orthogonal relations and addition formulas for q-analogs of the ordinary Bessel functions can be obtained in this framework.

Lie theory gives a natural setting for an algebraic interpretation of the special functions of mathematical physics. Using the exponential mapping from the algebra to the corresponding group, one computes the matrix elements of group operators in specific irreducible representations and finds that these are tipically expressible in terms of hypergeometric series.[1]

A similar connection has now been established[2-5] between quantum algebras and the so called basic or q-special functions.[6] In this case, one considers matrix elements of operators built with q-exponentials of the algebra generators; these elements turn out to be expressible in terms of q-hypergeometric series. Also, q-special functions appear as basis of irreducible representations of quantum algebras. This algebraic setting is extremely usefull in studying properties of these basic special functions. Indeed, various generating relations, orthogonality properties and addition formulas can be obtained within this "group-theoretical" interpretations. We shall illustrate this algebraic approach to q-special functions by considering the two-dimensional quantum Euclidean algebra $\mathcal{E}_q(2)$ and its connection to q-Bessel functions.[2-4]

The algebra $\mathcal{E}_q(2)$ is defined by the following commutation relations of its three generators:

$$[J, P_\pm] = \pm P_\pm \ , \qquad [P_+, P_-] = 0 \ . \tag{1}$$

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It is endowed with a non-trivial Hopf structure. Each irreducible representation $Q(\omega, m_0)$ of $\mathcal{E}_q(2)$ is characterized by two complex numbers ω and m_0, with $\omega \neq 0$ and $0 \leq \Re e\, m_0 < 1$.[1,2] In this representation the spectrum of J is the set $S = \{m_0 + n : \ n \in \mathbf{Z}\}$, and the representation space has basis vectors f_m, $m \in S$, such that

$$P_\pm f_m = \omega f_{m\pm 1} \ , \tag{2a}$$

$$J f_m = m f_m \ , \tag{2b}$$

$$P_+ P_- f_m = \omega^2 f_m \ , \tag{2c}$$

where $C \equiv P_+ P_-$ is the Casimir operator.

An explicit realization of the representation $Q(\omega, m_0)$ can be easily obtained by taking[1,2]

$$J = m_0 + z\frac{d}{dz} \ , \qquad P_+ = \omega z \ , \qquad P_- = \frac{\omega}{z} \ . \tag{3}$$

These operators act on the space of all linear combinations of the functions z^n, $z \in \mathbf{C}$, $n \in \mathbf{Z}$, with basis vectors $f_m(z) = z^n$, where $m = m_0 + n$, n an integer.

In analogy with ordinary Lie theory, it is useful to consider the following elements in the universal enveloping algebra of $\mathcal{E}_q(2)$,[3-5]

$$U(\alpha,\beta) = e_q(\alpha P_+)\, e_q(\beta P_-) \ , \tag{4a}$$

$$\widetilde{U}(\alpha,\beta) = E_q(\alpha P_+)\, E_q(\beta P_-) \ , \tag{4b}$$

with α and β complex parameters. The functions e_q and E_q are q-analogs of the exponential function; they are defined by[6]

$$e_q(z) = \sum_{n=0}^{\infty} \frac{1}{(q;q)_n}\, z^n = \frac{1}{(z;q)_\infty} \ , \qquad |z| < 1 \ , \tag{5a}$$

$$E_q(z) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}}}{(q;q)_n}\, z^n = (-z;q)_\infty \ , \tag{5b}$$

where $(a;q)_n$ denotes the q-shifted factorial, given by $(a;q)_\alpha = (a;q)_\infty/(aq^\alpha;q)_\infty$, where $(a;q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k)$, $|q| < 1$. Note that $e_q(z)\, E_q(-z) = 1$ and that $\lim_{q\to 1^-} e_q\big(z(1-q)\big) = \lim_{q\to 1^-} E_q\big(z(1-q)\big) = e^z$. In the following we shall always assume $|q| < 1$.

The matrix elements of the operator $U(\alpha,\beta)$ are defined through

$$U(\alpha,\beta)\, f_{m_0+n} = \sum_{k=-\infty}^{\infty} U_{kn}(\alpha,\beta)\, f_{m_0+k} \ , \tag{6}$$

and similarly for $\widetilde{U}(\alpha,\beta)$. They can be explicitly computed using the realization (3); one finds[2-4]

$$U_{kn}(\alpha,\beta) = \left(-\frac{\alpha}{\beta}\right)^{(k-n)/2} J^{(1)}_{k-n}\Big(2\omega\,(-\alpha\beta)^{1/2};q\Big) \ , \tag{7a}$$

$$\widetilde{U}_{kn}(\alpha,\beta) = q^{\frac{1}{2}(k-n)^2}\left(-\frac{\alpha}{\beta}\right)^{(k-n)/2} J^{(2)}_{k-n}\left(2\omega\left(-\frac{\alpha\beta}{q}\right)^{1/2};q\right) \ . \tag{7b}$$

The functions $J^{(1)}_\nu(z;q)$ and $J^{(2)}_\nu(z;q)$ are q-generalizations of the Bessel functions of the first kind, and are explicitly given by:[6]

$$J^{(1)}_\nu(z;q) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(q;q)_n\,(q;q)_{n+\nu}} \left(\frac{z}{2}\right)^{2n+\nu} \ , \tag{8a}$$

$$J^{(2)}_\nu(z;q) = \sum_{n=0}^{\infty} q^{n(n+\nu)}\, \frac{(-1)^n}{(q;q)_n\,(q;q)_{n+\nu}} \left(\frac{z}{2}\right)^{2n+\nu} \ . \tag{8b}$$

Generating relations for these q-Bessel function can straightforwardly be obtained. Inserting (7) back into (6), after some simple redefinitions one gets, for the two cases:[2,4]

$$e_q\left(\frac{xz}{2}\right) e_q\left(-\frac{x}{2z}\right) = \sum_{k=-\infty}^{\infty} z^k\, J^{(1)}_k(x;q) \ , \tag{9a}$$

$$E_q\left(\frac{xz}{2}\right) E_q\left(-\frac{qx}{2z}\right) = \sum_{k=-\infty}^{\infty} q^{\frac{1}{2}k(k-1)}\, z^k\, J^{(2)}_k(x;q) \ . \tag{9b}$$

These are q-analogs of the generating relation for the ordinary Bessel functions $J_k(x)$, $e^{x(z-z^{-1})/2} = \sum_{k=-\infty}^{\infty} J_k(x)\, z^k$, to which both formulas (9) reduce in the limit $q \to 1^-$.

A biorthogonality relation involving the two functions $J^{(1)}_k(x;q)$ and $J^{(2)}_k(x;q)$ can be obtained by combining together the two generating formulas (9). First, let $x \to q^{-1/2}x$, $z \to q^{1/2}z$ in (9b), and then multiply it by (9a). Using $e_q(x)\, E_q(-x) = 1$ and equating the coefficients of equal powers of z on both sides , one finally finds, for $|x^2| < 4$,

$$\sum_{k=-\infty}^{\infty} q^{(k+n)^2/2}\, J^{(1)}_{k-m}(x;q)\, J^{(2)}_{n-k}(-q^{-1/2}x;q) = \delta_{m,n} \ . \tag{10}$$

To obtain addition fromulas involving the q-Bessel functions (8), one needs to introduce a realization of the representation $Q(\omega, m_0)$ on the space of functions of two complex variables, x and y. Such a realization is provided by the operators[4]

$$P_+ = y\left(\frac{2}{x}\right)\left(1 - T_y^{-1/2}\, T_x^{1/2}\right) \ , \tag{11a}$$

$$P_- = y^{-1}\left(\frac{2}{x}\right)\left(1 - T_y^{1/2}\, T_x^{1/2}\right) \ , \tag{11b}$$

$$J = y\frac{\partial}{\partial y} \ , \tag{11c}$$

and the basis functions

$$f_m(x,y) = y^m\, J^{(1)}_{-m}(x;q) \ , \qquad m = m_0 + n \ , \ n \in \mathbf{Z} \ , \tag{12}$$

where T_z is the q-dilatation operator, such that: $T_z\varphi(z) = \varphi(qz)$, on any function of the variable z. We have taken $\omega = 1$ for simplicity. Note that the equation (2c) becomes now a second-order difference equation for $J^{(1)}_m(x;q)$:

$$\left[\frac{4}{x^2}\left(1 - q^{m/2}\, T_x^{1/2}\right)\left(1 - q^{-m/2}\, T_x^{1/2}\right) - 1\right] J^{(1)}_m(x;q) = 0 \ . \tag{13}$$

This is a q-analog of the Bessel differential equation, to which it reduces in the limit $q \to 1^-$.

Since we have a realization of the representation $Q(1, m_0)$, one gets from (6), with $m = m_0 + n$ and $U(\alpha, \beta)$ replaced by $\widetilde{U}(\alpha, \beta)$,

$$\widetilde{U}(\alpha,\beta)\, f_m(x,y) = \sum_{k=-\infty}^{\infty} \widetilde{U}_{k\,m-m_0}(\alpha,\beta)\, f_{m_0+k}(x,y)\ , \tag{14}$$

where the model independent matrix elements $\widetilde{U}_{kn}(\alpha,\beta)$ are still given by the formulas (7b), with $\omega = 1$. To get addition formulas for q-Bessel functions, one now needs to evaluate explicitly the l.h.s. of (14), *i.e.* to compute directly the action of $\widetilde{U}(\alpha,\beta)$ on the basis functions (12), when P_+ and P_- are realized as in (11). One finds[3] that this action can be expressed in terms of the following q-hypergeometric series:

$${}_2\phi_1(a,b;c;q,z) = \sum_{n=0}^{\infty} \frac{(a;q)_n\,(b;q)_n}{(q;q)_n\,(c;q)_n}\, z^n\ , \qquad |z| < 1\ . \tag{15}$$

With this result, from (14), (12) and (7b) one finally gets the following relations, taking $\alpha = 0$, $\beta = 0$ and $\alpha\beta \neq 0$, respectively:

$$\left(\frac{x}{2}\right)^m \frac{(-2\beta/x;q)_m}{(q;q)_m}\, {}_2\phi_1\left(0, -\frac{2\beta}{x}q^m; q^{m+1}; q, -\frac{x^2}{4}\right) = \sum_{l=0}^{\infty} q^{l(l-1)/2}\, \frac{\beta^l}{(q;q)_l}\, J^{(1)}_{m-l}(x;q)\ , \tag{16}$$

$$\left(\frac{x}{2}\right)^m \frac{1}{(q;q)_m}\, {}_2\phi_1\left(\frac{2\alpha}{x}, 0; q^{m+1}; q, -\frac{x^2}{4}\right) = \sum_{l=0}^{\infty} q^{l(l-1)/2}\, \frac{\alpha^l}{(q;q)_l}\, J^{(1)}_{m+l}(x;q)\ , \tag{17}$$

$$\begin{aligned}\left(\frac{x}{2}\right)^m \frac{(zq/xy;q)_m}{(q;q)_m}\, {}_2\phi_1\left(\frac{yz}{x}, \frac{z}{xy}q^{m+1}; q^{m+1}; q, -\frac{x^2}{4}\right) \\ = \sum_{l=-\infty}^{\infty} q^{l(l-1)/2}\, y^l\, J^{(2)}_l(z;q)\, J^{(1)}_{m+l}(x;q)\ .\end{aligned} \tag{18}$$

Equations (16) and (17) are q-analogs of the Lommel summation formulas[7] for ordinary Bessel functions, while (18) is the q-analog of the Graf summation theorem.[7]

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