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QUANTUM ALGEBRAS AND QUANTUM GROUPS IN q-SPECIAL FUNCTION THEORY

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Abstract

The quantum algebra and quantum group interpretation of q-special functions is reviewed. Taking the algebra $\mathcal{U}_q(sl(2))$ as example, we shall see how its representation theory can be used to make advances in the study of the q-hypergeometric series ${}_2\phi_1(a,b;c;q,z)$.

The connection between quantum algebras and q-special functions is now well-established. As in standard Lie theory,[1] these functions arise as matrix elements of certain operators in the algebra generators and also as basis vectors of the corresponding representation spaces. This algebraic setting naturally leads to generating relations, orthogonality properties and addition formulas involving the q-special functions.[2-13]

Here, we shall illustrate the power of this "group-theoretic" interpretation by examining the relation between the quantum algebra $\mathcal{U}_q(sl(2))$ and the q-hypergeometric series ${}_2\phi_1(a,b;c;q,z)$.[6,9,11] Though we mainly work within the simpler quantum algebra framework, we shall also make connection with the corresponding "dual" approach, based on the quantum group $SL_q(2)$. Indeed, matrix elements of corepresentations of $SL_q(2)$ are also seen to involve the function ${}_2\phi_1$.[14] We shall show that the algebra and the group settings are completely equivalent, so that results obtained in one approach can be rephrased in the other.[9]

Finally, using the quantum algebraic interpretation, we shall give explicit examples of generating functions and summation formulas involving the q-hypergeometric function

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${}_2\phi_1$.[11] These relations should first be looked at as identities between formal power series; it could happen that they converge only over a finite radius or only when the series terminate.

In the quantum algebra interpretation of the q-hypergeometric function ${}_2\phi_1$, an important role is played by the following q-analogs of the exponential function[15]

$$e_q(z) = \sum_{n=0}^{\infty} \frac{1}{(q;q)_n} z^n = \frac{1}{(z;q)_\infty} , \qquad |z| < 1 , \tag{1a}$$

$$E_q(z) = \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n-1)}}{(q;q)_n} z^n = (-z;q)_\infty , \tag{1b}$$

where $(a;q)_\alpha = (a;q)_\infty/(aq^\alpha;q)_\infty$ is the q-shifted factorial (a and α being arbitrary complex numbers), with $(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)$, $|q| < 1$. Note that $e_q(z)\,E_q(-z) = 1$, and that $\lim_{q\to 1^-} e_q(z(1-q)) = \lim_{q\to 1^-} E_q(z(1-q)) = e^z$. We shall denote by T_z the q-dilatation operator which acts as $T_z\,\varphi(z) = \varphi(qz)$, on functions of the variable z; out of it, the q-difference operators

$$D_z^+ = z^{-1}(1-T_z) , \tag{2a}$$

$$D_z^- = z^{-1}(1-T_z^{-1}) , \tag{2b}$$

are constructed. Observe that $\frac{1}{(1-q)}D_z^+ \to d/dz$ and $\frac{1}{(1-q^{-1})}D_z^- \to d/dz$ as $q \to 1$, and that the q-exponentials obey

$$D_z^+\, e_q(\lambda z) = \lambda e_q(\lambda z) , \tag{3a}$$

$$D_z^-\, E_q(\lambda z) = -q^{-1}\lambda E_q(\lambda z) , \tag{3b}$$

where λ is a complex parameter. The basic hypergeometric series ${}_r\phi_s$ is defined by[15]

$$\begin{aligned} &{}_r\phi_s(a_1,a_2,\ldots,a_r;b_1,\ldots,b_s;q;z) \\ &\quad = \sum_{n=0}^{\infty} \frac{(a_1;q)_n(a_2;q)_n\ldots(a_r;q)_n}{(q;q)_n(b_1;q)_n\ldots(b_s;q)_n} \left[(-1)^n q^{\frac{n(n-1)}{2}}\right]^{1+s-r} z^n , \end{aligned} \tag{4}$$

with $q \neq 0$ when $r > s+1$. Since $(q^{-m};q)_n = 0$, for $n = m+1, m+2, \ldots$, the series ${}_r\phi_s$ terminates if one of the numerator parameters $\{a_i\}$ is of the form q^{-m} with $m = 0,1,2\ldots$, and $q \neq 0$. By the ratio test, when $0 < |q| < 1$, the ${}_r\phi_s$ series converges absolutely for all z if $r \leq s$, and for $|z| < 1$ if $r = s+1$. This series also converges absolutely when $|q| > 1$ and $|z| < |b_1b_2\ldots b_s|/|a_1a_2\ldots a_r|$. It diverges for $z \neq 0$ when $0 < |q| < 1$ and $r > s+1$, and when $|q| > 1$ and $|z| > |b_1b_2\ldots b_s|/|a_1a_2\ldots a_r|$, unless it terminates. In the following, we shall concentrate on the case

$${}_2\phi_1(a,b;c;q,z) = \sum_{n=0}^{\infty} \frac{(a;q)_n(b;q)_n}{(q;q)_n(c;q)_n} z^n , \qquad |z| < 1 , \tag{5}$$

in terms of which various matrix elements will be expressed. Notice that as $c \to q^{1-m}$, with m a positive integer, this function satisfies the following limit relation[6]

$$\frac{1}{(q;q)_{-m}}\, {}_2\phi_1(a,b,q^{1-m};q;z) = z^m\, {}_2\phi_1(aq^m, bq^m, q^{m+1};q,z)\, \frac{(a;q)_m(b;q)_m}{(q;q)_m} . \tag{6}$$

Furthermore, in the following we shall always assume $|q| < 1$.

The quantum universal enveloping algebra $\mathcal{U}_q(sl(2))$ is the Hopf algebra generated by the elements k, k^{-1}, e and f satisfying the relations[16,17]

$$k\,e\,k^{-1} = q^{1/2}e , \qquad k\,f\,k^{-1} = q^{-1/2}f , \qquad [e,f] = \frac{k^2-k^{-2}}{q^{1/2}-q^{-1/2}} , \tag{7}$$

and $k\,k^{-1} = k^{-1}k = 1$. The coproduct $\Delta : \mathcal{U}_q(sl(2)) \to \mathcal{U}_q(sl(2)) \otimes \mathcal{U}_q(sl(2))$, antipode $S : \mathcal{U}_q(sl(2)) \to \mathcal{U}_q(sl(2))$ and counit $\varepsilon : \mathcal{U}_q(sl(2)) \to \mathbf{C}$ are defined by:

$$\begin{aligned} &\Delta(k) = k\otimes k , && \Delta(e) = e\otimes k + k^{-1}\otimes e , && \Delta(f) = f\otimes k + k^{-1}\otimes f , \\ &S(k) = k^{-1} , && S(e) = -q^{1/2}\,e , && S(f) = -q^{-1/2}\,f , \\ &\varepsilon(k) = 1 , && \varepsilon(e) = 0 , && \varepsilon(f) = 0 . \end{aligned} \tag{8}$$

The algebra $\mathcal{U}_q(sl(2))$ has a Poincaré-Birkhoff-Witt basis given by: $e^\mu k^\rho f^\nu$, with $\rho \in \mathbf{Z}$ and $\mu,\nu \in \mathbf{N}$.

We now introduce a left $\mathcal{U}_q(sl(2))$-module $V^{(\lambda,m_0)} = \bigoplus_{j\in I}\mathbf{C}\xi_j$, where $I = \{i|\ i = m_0+n,\ n \in \mathbf{Z}\}$, and λ, m_0 are complex numbers.[9,14] $V^{(\lambda,m_0)}$ is infinite dimensional, unless $\lambda+m_0$ and $\lambda-m_0$ are both positive integers. The corresponding representation is characterized by the following action of the generators on the basis vectors ξ_j, $j \in I$:

$$\begin{aligned} k\,\xi_j &= q^{-j/2}\,\xi_j , \\ e\,\xi_j &= q^{(1-2\lambda)/4}\,\frac{1-q^{\lambda+j}}{1-q}\,\xi_{j-1} , \\ f\,\xi_j &= q^{(1-2\lambda)/4}\,\frac{1-q^{\lambda-j}}{1-q}\,\xi_{j+1} . \end{aligned} \tag{9}$$

Given any $a \in \mathcal{U}_q(sl(2))$, its matrix elements $W_{ij}(a)$ in this representation are defined by

$$a\,\xi_j = \sum_{i\in\mathbf{Z}+m_0} \xi_i\, W_{ij}(a) ; \tag{10}$$

it clearly follows that

$$W_{ij}(ab) = \sum_{k\in\mathbf{Z}+m_0} W_{ik}(a)\, W_{kj}(b) . \tag{11}$$

Analogy with ordinary Lie theory[2] suggests to consider the following element in the completion of $\mathcal{U}_q(sl(2))$:

$$U(\alpha,\beta,\gamma) = E_q(\alpha\,e)\,e_q(\beta\,f)\,k^\gamma , \tag{12}$$

where α, β and γ are complex parameters. Indeed, set $k = q^{-h/2}$ and $q = e^{-\eta}$. In the limit $q \to 1^-$, $\eta \to 0^+$, the defining relations (7) become those of $sl(2)$: $[h,e] = -e$, $[h,f] = f$, $[e,f] = -2h$, and $U\big((1-q)\alpha, (1-q)\beta, 2\gamma/\eta\big)$ goes into the $SL(2)$ group element $e^{\alpha e} e^{\beta f} e^{\gamma h}$.

The matrix elements of $U(\alpha,\beta,\gamma)$ on $V^{(\lambda,m_0)}$ turn out to be expressible in terms of the function ${}_2\phi_1(a,b;c;q,z)$. In fact, with the help of (1), using (9) and identities involving q-shifted factorials, it is straightforward to show that

$$W_{ij}\Big(U(\alpha,\beta,\gamma)\Big) = q^{-\gamma j/2} \left(\beta \frac{q^{(1-2\lambda)/4}}{1-q}\right)^{i-j} \frac{\big(q^{\lambda-i+1};q\big)_{i-j}}{(q;q)_{i-j}} \times {}_2\phi_1\Big(q^{\lambda+i+1}, q^{i-\lambda}; q^{i-j+1}; q, -\alpha\beta \frac{q^{(1-2i)/2}}{(1-q)^2}\Big) , \quad \text{if } i-j \geq 0 , \tag{13a}$$

$$W_{ij}\Big(U(\alpha,\beta,\gamma)\Big) = q^{-\gamma j/2}\, q^{(j-i)(j-i-1)/2} \left(\alpha \frac{q^{(1-2\lambda)/4}}{1-q}\right)^{j-i} \frac{\big(q^{\lambda+i+1};q\big)_{j-i}}{(q;q)_{j-i}} \times {}_2\phi_1\Big(q^{\lambda+j+1}, q^{j-\lambda}; q^{j-i+1}; q, -\alpha\beta \frac{q^{(1-2i)/2}}{(1-q)^2}\Big) , \quad \text{if } i-j \leq 0 , \tag{13b}$$

with $i,j \in \mathbb{Z} + m_0$. This establishes most simply the connection between the basic hypergeometric series ${}_2\phi_1$ and $\mathcal{U}_q(sl(2))$. Notice that by using the limiting formula (6), the above two expressions for $W_{ij}\Big(U(\alpha,\beta,\gamma)\Big)$ are valid irrespective of the sign of $i-j$.

Before using (13) to obtain properties for the function ${}_2\phi_1$, let us make contact with the quantum group approach[18-20,14] and indicate in particular how the results described so far enable one to recover the matrix elements of the $SL_q(2)$ corepresentation given in Ref.[14].

Let $\mathcal{A}$ be the space dual to $\mathcal{U}_q(sl(2))$. To introduce the coordinate ring $A(SL_q(2))$ of the quantum group $SL_q(2)$, consider the fundamental representation $X : \mathcal{U}_q(sl(2)) \to \mathcal{E}nd\,\mathbf{C}^2$, defined by

$$X(k) = \begin{pmatrix} q^{1/4} & 0 \\ 0 & q^{-1/4} \end{pmatrix} , \qquad X(e) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \qquad X(f) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} . \tag{14}$$

Since X is a representation, we have $X(ab) = X(a)X(b)$, for any $a,b \in \mathcal{U}_q(sl(2))$. The matrix elements of X, that we shall denote as

$$X = \begin{pmatrix} x & u \\ v & y \end{pmatrix} , \tag{15}$$

define mappings $\mathcal{U}_q(sl(2)) \to \mathbf{C}$, via the evaluation map $a \to X(a)$; hence x, u, v and y are elements of $\mathcal{A}$. We shall call $A(SL_q(2))$ the subalgebra of $\mathcal{A}$ generated by these coordinate elements. This algebra is seen to inherit from $\mathcal{U}_q(sl(2))$ a Hopf structure. The coproduct $\Delta_{\mathcal{A}} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, antipode $S_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ and counit $\varepsilon_{\mathcal{A}} : \mathcal{A} \to \mathbf{C}$ are defined by

$$\Delta_{\mathcal{A}}(X)\,(a \otimes b) = X(ab) , \qquad a,b \in \mathcal{U}_q(sl(2)) , \tag{16}$$

and

$$S_{\mathcal{A}}(X) = \begin{pmatrix} y & -q^{1/2}u \\ -q^{-1/2}v & x \end{pmatrix} , \qquad \varepsilon_{\mathcal{A}}(X) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{17}$$

In terms of matrix elements, the definition (16) is equivalent to $\Delta_{\mathcal{A}}(X_{rs}) = \sum_{t=1,2} X_{rt}\, X_{ts}$, $r,s = 1,2$, with the understanding that $(\Phi \otimes \Psi)(a \otimes b) = \Phi(a)\Psi(b)$ for $\Phi, \Psi \in \mathcal{A}$, $a,b \in \mathcal{U}_q(sl(2))$. The product $\mu_{\mathcal{A}} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is determined by

$$\mu_{\mathcal{A}}(\Phi \otimes \Psi)(a) \equiv \Phi\Psi(a) = \Phi \otimes \Psi\big(\Delta(a)\big) . \tag{18}$$

With these definitions one finds that

$$\begin{aligned} x^L u^M v^N \big(e^\mu k^\rho f^\nu\big) &= q^{-\frac{1}{4}\rho(\mu+\nu-L) - \frac{1}{4}L(\mu+\nu) - \frac{1}{4}(\mu(\mu-1)+\nu(\nu-1)) + \frac{1}{2}(\mu-M)(\nu-N)} \\ &\qquad \times \frac{(q;q)_L}{(q;q)_{\mu-M}\,(q;q)_{L-\mu+M}} \frac{(q;q)_\mu\,(q;q)_\nu}{(1-q)^{\mu+\nu}} , \\ &\qquad \text{for } \mu - M = \nu - N, \quad M \leq \mu \leq M+L , \\ &= 0 , \qquad \text{otherwise} , \end{aligned} \tag{19a}$$

and

$$\begin{aligned} u^M v^N y^L \big(e^\mu k^\rho f^\nu\big) &= q^{-\frac{1}{4}\rho(\mu+\nu+L) - \frac{1}{4}L(\mu+\nu) - \frac{1}{4}(\mu(\mu-1)+\nu(\nu-1))} \\ &\qquad \times \frac{(q;q)_\mu\,(q;q)_\nu}{(1-q)^{\mu+\nu}}\, \delta_{\mu,M}\, \delta_{\nu,N} , \end{aligned} \tag{19b}$$

with M, N, and L nonnegative integers. Given these evaluations on the Poincaré-Birkhoff-Witt basis elements of $\mathcal{U}_q(sl(2))$, it is immediate to see that the coordinate elements obey the following commutation relations

$$q^{1/2}xu = ux \quad q^{1/2}xv = vx \quad q^{1/2}uy = yu \quad q^{1/2}vy = yv \quad uv = vu , \tag{20}$$

and

$$\det{}_q X = xy - q^{-1/2}uv = yx - q^{1/2}uv = \mathbf{1}_{\mathcal{A}} . \tag{21}$$

The matrix elements W_{ij} defined in (10) can also be viewed as elements of $\mathcal{A}$; they indeed provide linear mappings from $\mathcal{U}_q(sl(2))$ into $\mathbf{C}$, $a \to W_{ij}(a)$, $a \in \mathcal{U}_q(sl(2))$. Introducing an infinite matrix W, with elements W_{ij}, the composition relation (11) can be rewritten in the form

$$\Delta_{\mathcal{A}}\, W = W \otimes W , \tag{22}$$

and we conclude that W defines a corepresentation of $SL_q(2)$.

Starting from the formulas (13), the evaluation of W_{ij} on $U(\alpha,\beta,\gamma)$, it is now possible to derive the analytic expressions for the elements W_{ij} of W in terms of the coordinates x, u, v and y of $\mathcal{A}$. In fact, the element $U(\alpha,\beta,\gamma) = E_q(\alpha e)\, e_q(\beta f)\, k^\gamma$ generates the complete basis of $\mathcal{U}_q(sl(2))$:

$$(-1)^\mu q^{\mu(2-\rho)/2} \Big(\big(D_\alpha^-\big)^\mu \big(D_\beta^+\big)^\nu U(\alpha,\beta,\rho)\Big)\Big|_{\alpha=\beta=0} = e^\mu\, k^\rho\, f^\nu , \tag{23}$$

and therefore an arbitrary element of $\mathcal{A}$ is completely specified once its evaluation on $U(\alpha,\beta,\gamma)$ is given. Since $x^L u^M v^N$ and $u^M v^N y^L$ span $A(SL_q(2))$, with the help of the formulas (19), the expressions for W_{ij} can be easily abstracted from the matrix elements $W_{ij}(U(\alpha,\beta,\gamma))$ given before. The details can be found in Ref.[9]. One obtains

$$W_{ij} = q^{(i-j)(\lambda-j)} \frac{(q^{\lambda+i+1};q)_{j-i}}{(q;q)_{j-i}} \, {}_2\phi_1\Big(q^{\lambda+j+1}, q^{j-\lambda}; q^{j-i+1}; q, -q^{1/2}uv\Big) u^{j-i} y^{i+j} \,, \qquad \text{if } i-j \le 0 \,,\ i+j \ge 0 \,, \tag{24a}$$

$$W_{ij} = q^{(i-j)(\lambda+i)} \frac{(q^{\lambda+i+1};q)_{j-i}}{(q;q)_{j-i}} \, x^{-(i+j)} u^{j-i} \, {}_2\phi_1\Big(q^{\lambda-i+1}, q^{-i-\lambda}; q^{j-i+1}; q, -q^{1/2}uv\Big) \,, \qquad \text{if } i-j \le 0 \,,\ i+j \le 0 \,, \tag{24b}$$

$$W_{ij} = q^{(j-i)(\lambda-i)} \frac{(q^{\lambda-i+1};q)_{i-j}}{(q;q)_{i-j}} \, {}_2\phi_1\Big(q^{\lambda+i+1}, q^{i-\lambda}; q^{i-j+1}; q, -q^{1/2}uv\Big) v^{i-j} y^{i+j} \,, \qquad \text{if } i-j \ge 0 \,,\ i+j \ge 0 \,, \tag{24c}$$

$$W_{ij} = q^{(j-i)(\lambda+j)} \frac{(q^{\lambda-i+1};q)_{i-j}}{(q;q)_{i-j}} \, x^{-(i+j)} v^{i-j} \, {}_2\phi_1\Big(q^{\lambda-j+1}, q^{-j-\lambda}; q^{i-j+1}; q, -q^{1/2}uv\Big) \,, \qquad \text{if } i-j \ge 0 \,,\ i+j \le 0 \,. \tag{24d}$$

These four cases need to be distinguished in order for the elements W_{ij} to be analytic in x, u, v and y. The matrix elements (24) were computed in a different way in Ref.[14]. The quantum algebra derivation, that we have skeched, is straightforward and simpler, and explicitly shows the equivalence between the quantum group and the quantum algebra approach to q-special functions.

For simplicity, up to now all our considerations were based on the specific element (12) of $\mathcal{U}_q(sl(2))$. Off course, other combinations of little and big exponentials can also be used. For instance, take

$$\widetilde{U}(\alpha,\beta,\gamma) = E_q(\beta f)\, e_q(\alpha e)\, k^\gamma \,; \tag{25}$$

its matrix elements in the representation (9) can be easily worked out and one explicitly finds ($i,j \in \mathbb{Z}+m_0$)

$$W_{ij}\Big(\widetilde{U}(\alpha,\beta,\gamma)\Big) = q^{-\gamma j/2}\, q^{(i-j)(i-j-1)/2} \left(\beta \frac{q^{(1-2\lambda)/4}}{1-q}\right)^{i-j} \frac{(q^{\lambda-i+1};q)_{i-j}}{(q;q)_{i-j}} \times {}_2\phi_1\Big(q^{\lambda-j+1}, q^{-j-\lambda}; q^{i-j+1}; q, -\alpha\beta \frac{q^{(1+2i)/2}}{(1-q)^2}\Big) \,, \qquad \text{if } i-j \ge 0 \,, \tag{26a}$$

$$W_{ij}\Big(\widetilde{U}(\alpha,\beta,\gamma)\Big) = q^{-\gamma j/2} \left(\alpha \frac{q^{(1-2\lambda)/4}}{1-q}\right)^{j-i} \frac{(q^{\lambda+i+1};q)_{j-i}}{(q;q)_{j-i}} \times {}_2\phi_1\Big(q^{\lambda-i+1}, q^{-i-\lambda}; q^{j-i+1}; q, -\alpha\beta \frac{q^{(1+2i)/2}}{(1-q)^2}\Big) \,, \qquad \text{if } i-j \le 0 \,. \tag{26b}$$

The case of $e_q(\alpha e)\, E_q(\beta f)\, k^\gamma$ has been considered in Ref.[9], while the matrix elements of operators in $\mathcal{U}_q(sl(2))$ involving two little or two big q-exponentials can be expressed in terms of the q-hypergeometric series ${}_3\phi_1$ and ${}_2\phi_2$, and will not be discussed here. Using the explicit expressions (13) and (26) for the matrix elements of the operators $U(\alpha,\beta,\gamma)$ and $\widetilde{U}(\alpha,\beta,\gamma)$, one can now obtain various identities involving the q-hypergeometric function ${}_2\phi_1$.

As a first example, let us work out an orthogonality relation involving two ${}_2\phi_1$ functions. By recalling that $e_q(z)\, E_q(-z) = 1$ and setting $\gamma = 0$, one sees that

$$U(\alpha,\beta,0)\, \widetilde{U}(-\alpha,-\beta,0) = \mathbf{1} \,; \tag{27}$$

acting on ξ_j, one then finds ($i,j \in \mathbb{Z}+m_0$)

$$\delta_{i-j,0} = \sum_{l\in\mathbb{Z}+m_0} W_{il}\Big(U(\alpha,\beta,0)\Big)\, W_{lj}\Big(\widetilde{U}(-\alpha,-\beta,0)\Big) \,. \tag{28}$$

Insert now for the matrix elements $W_{il}(U)$ and $W_{lj}(\widetilde{U})$ the expression (13b) and the one that it is obtained from (26a) with the use of the following transformation rule for the ${}_2\phi_1$ series,[15]

$${}_2\phi_1(q^a, q^b; q^c; q, z) = (q^{a+b-c}z; q)_{c-a-b}\, {}_2\phi_1(q^{c-a}, q^{c-b}; q^c; q, q^{a+b-c}z) \,. \tag{29}$$

After some simplifications and the redefinition $z = -\alpha\beta\, q^{1/2}/(1-q)^2$, one finally arrives at the following relation[11]

$$\delta_{i-j,0} = \frac{q^{(i^2+j^2)/2}\,(q/z;q)_j}{(q^{\lambda+1};q)_i\,(q^{-\lambda};q)_j} \sum_{l\in\mathbb{Z}+m_0} q^{l(l-1)/2}\big(-zq^{-i-j}\big)^l \frac{(z;q)_l\,(q^{\lambda+1};q)_l\,(q^{-\lambda};q)_l}{(q;q)_{l-j}\,(q;q)_{l-i}} \times {}_2\phi_1\big(q^{\lambda+l+1}, q^{l-\lambda}; q^{l-i+1}; q, zq^{-i}\big)\, {}_2\phi_1\big(q^{\lambda+l+1}, q^{l-\lambda}; q^{l-j+1}; q, zq^{-j}\big) \,. \tag{30}$$

To get generating relations for the ${}_2\phi_1$, one first notices that it is possible to give a one-variable model for the representation (9), where the generators are expressed as q-difference operators in the complex variable z acting on the space of all linear combinations of the functions z^n, $n \in \mathbb{Z}$. Indeed, by taking

$$\begin{aligned} k &= q^{-m_0/2}\, T_z^{-1/2} \,, \\ e &= q^{(1-2\lambda)/4} \left(\frac{1}{1-q} D_z^+ + \frac{1-q^{\lambda+m_0}}{1-q}\, \frac{1}{z}\, T_z\right) \,, \\ f &= q^{(1-2\lambda)/4} \left(\frac{z^2}{1-q} D_z^- + \frac{1-q^{\lambda-m_0}}{1-q}\, z\, T_z^{-1}\right) \,, \end{aligned} \tag{31}$$

and $\xi_j = z^n$, $j = m_0 + n$, $n \in \mathbb{Z}$, for the basis vectors, one can check that the relations (9) are satisfied. Let us now act directly with the operator $U(\alpha,\beta,\gamma)$, with e, f and k expressed as in (31), on z^{j-m_0}. With some manipulations, one finds that this action can be expressed in terms of a ${}_2\phi_0$ q-hypergeometric series,

$$\begin{aligned} U(\alpha,\beta,\gamma)\, z^{j-m_0} &= z^{j-m_0}\, q^{-j\gamma/2} \left(-\frac{\alpha}{z}\,\frac{q^{(1-2\lambda)/4}}{1-q};q\right)_{\lambda+j} \\ &\quad \times {}_2\phi_0\left(-\frac{\alpha}{z}\,\frac{q^{j+(1+2\lambda)/4}}{1-q}, q^{j-\lambda}; q, \beta z\,\frac{q^{-j+(1+2\lambda)/4}}{1-q}\right) . \end{aligned} \tag{32}$$

Since the series ${}_2\phi_0(a,b;q,z)$ does not converge, unless it terminates or $z = 0$, the action of $U(\alpha,\beta,\gamma)$ on the module $V^{(\lambda,m_0)}$ is ill-defined in this model, unless $V^{(\lambda,m_0)}$ is finite-dimensional or $\beta = 0$. Nevertheless, by proceeding formally one can obtain a generating relation for the q-hypergeometric function ${}_2\phi_1$. Recall the definition (10) for the matrix elements of $U(\alpha,\beta,\gamma)$, and insert (32) for the l.h.s., while in the r.h.s. substitute for $W_{ij}\big(U(\alpha,\beta,\gamma)\big)$ the result (13a). After using the transformation formula (29), set $\gamma = 0$, $j = m_0$, $x = -\alpha\beta\, q^{m_0+1/2}/(1-q)^2$ and $y = -(1-q)q^{(2\lambda+3)/4}\, z/\alpha$, to get[11]

$$\begin{aligned} &(zq^{-2m_0};q)_{2m_0}\,(q/y;q)_{m_0+\lambda}\; {}_2\phi_0\big(q^{m_0+\lambda+1}/y, q^{m_0-\lambda}; q, zyq^{-2m_0-1}\big) \\ &\qquad = \sum_{l\in\mathbb{Z}} \frac{(q^{m_0-\lambda};q)_l}{(q^{2m_0+1}/z;q)_l\,(q;q)_l}\, y^l\; {}_2\phi_1\big(q^{\lambda-m_0+1}, q^{-m_0-\lambda}; q^{l+1}; q, z\big) . \end{aligned} \tag{33}$$

The l.h.s. of this relation between formal series yields the ${}_2\phi_1$ as coefficients when expanded in powers of y. If $\lambda + m_0$ and $\lambda - m_0$ are both positive numbers, so that $V^{(\lambda,m_0)}$ is finite-dimensional, the relation (33) becomes an identity between polynomial functions. Note that here again the summand in the r.h.s. is well defined for all integers l, thanks to the limiting relation (6). An equivalent generating formula can be similarly obtained starting with the operator $\widetilde{U}(\alpha,\beta,\gamma)$.

To get further properties of the q-hypergeometric function ${}_2\phi_1$, observe that the matrix elements $W_{ij}\big(U(\alpha,\beta,\gamma)\big)$ and $W_{ij}\big(\widetilde{U}(\alpha,\beta,\gamma)\big)$ themselves define models of the module $V^{(\lambda,m_0)}$. For simplicity, set $\gamma = 0$. Then for each element $a \in \mathcal{U}_q(sl(2))$ define the operator $\pi(a)$ acting on the variables α, β such that $\pi(a)\,U(\alpha,\beta,0) = U(\alpha,\beta,0)\,a$. Upon operating on ξ_j, one obtains

$$\pi(a)\, W_{ij}\big(U(\alpha,\beta,0)\big) = \sum_{l\in\mathbb{Z}+m_0} W_{il}\big(U(\alpha,\beta,0)\big)\, W_{lj}(a) , \tag{34}$$

that is, the functions $W_{ij}\,(U(\alpha,\beta,0))$ transform like the vectors ξ_j. From the properties (3) of the q-exponentials it is easy to construct the operators π for k, e and f, and thus for any $a \in \mathcal{U}_q(sl(2))$ by composition. For instance, one finds that

$$\pi^{(l)}(k) = q^{-l/2}\, T_\alpha^{-1/2}\, T_\beta^{1/2} , \tag{35a}$$

$$\pi^{(l)}(e) = -qD_\alpha^- - \frac{q^{1/2}}{(1-q)^2}\,\beta\left(q^l\, T_\alpha T_\beta^{-1} - q^{-l} T_\alpha^{-1}\right) , \tag{35b}$$

$$\pi^{(l)}(f) = D_\beta^+ , \tag{35c}$$

acting on the basis vectors

$$\xi_j^{(l)}(\alpha,\beta) = W_{li}\big(U(\alpha,\beta,0)\big) , \qquad j,l \in \mathbb{Z} + m_0 , \tag{36}$$

obey the same commutation relations as k, e and f. Similar results are obtained using $W_{ij}\big(\widetilde{U}(\alpha,\beta,0)\big)$.

Since we have a two-variable model of the module $V^{(\lambda,m_0)}$, from the general definition (10) one can write, recalling (36),

$$\widetilde{U}(\alpha,\beta,0)\; \xi_j^{(l)}(x,y) = \sum_{i\in\mathbb{Z}+m_0} W_{li}\big(U(x,y,0)\big)\; W_{ij}\big(\widetilde{U}(\alpha,\beta,0)\big) , \tag{37}$$

where the model independent matrix elements $W_{ij}\big(\widetilde{U}(\alpha,\beta,0)\big)$ are still given by (26). This allows deriving addition formulas for the q-hypergeometric series ${}_2\phi_1$: one just needs to evaluate explicitly the l.h.s. of (37), *i.e.* to compute directly the action of $\widetilde{U}(\alpha,\beta,0)$ on the basis functions $\xi_j^{(l)}(x,y)$, when e and f are realized as in (35), α and β being replaced by x and y.

We shall not give here the details of this evaluation (see Ref.[11]), but only collect the final results. In the case $\alpha = 0$, one finds the following generating relation

$$\begin{aligned} &\frac{(wq^{j-\lambda};q)_{l-j}\,(q^{\lambda-l+1};q)_{l-j}}{(q;q)_{l-j}}\; {}_3\phi_2\big(q^{\lambda+l+1}, q^{l-\lambda}, wq^{l-\lambda}; q^{l-j+1}, 0; q, z\big) \\ &\quad = \sum_{k=0}^{\infty} \frac{(q^{\lambda-l+1};q)_{l-j-k}\,(q^{j-\lambda};q)_k}{(q;q)_{l-j-k}\,(q;q)_k}\; w^k\; {}_2\phi_1\big(q^{\lambda+l+1}, q^{l-\lambda}; q^{l-j-k+1}; q, z\big) , \quad |z| < 1 , \end{aligned} \tag{38}$$

while for $\alpha\beta \neq 0$, one has

$$\begin{aligned} {}_2\phi_1\big(q^{\lambda+l+1}, q^{l-\lambda}; 0; q, z\big)\; {}_2\phi_0\big(q^{\lambda-j+1}, q^{-\lambda-j}; q, w/q\big) &= \sum_{k\in\mathbb{Z}} \frac{(-1)^k q^{k(k+1)/2}}{(q;q)_{l-j-k}\,(q;q)_k} \\ &\quad \times {}_2\phi_1\big(q^{\lambda+l+1}, q^{l-\lambda}; q^{l-j-k+1}; q, z\big)\; {}_2\phi_1\big(q^{\lambda-j+1}, q^{-j-\lambda}; q^{k+1}; q, wq^k\big) . \end{aligned} \tag{39}$$

Concerning the convergence of this last result, the remarks made after Eq.(33) also apply. In particular, when the module $V^{(m_0,\lambda)}$ is finite-dimensional, (39) becomes an identity among polynomials.

Indeed, take $m_0 = 0$ and λ a positive integer. Recalling the definition of the little q-Jacobi polynomials,[15]

$$p_n(z;a,b;q) = {}_2\phi_1(q^{-n}, abq^{n+1}; aq; q, qz) , \tag{40}$$

and setting for simplicity $l = j$, $m = \lambda + j$ and $n = \lambda - j$, from (39) one gets

$$\begin{aligned} &c_n\big(q^{m+1}; -q^n/z; q\big)\; \mathcal{P}_m\big(wq^{-m}; q^{n+1}; q\big) \\ &\quad = \sum_{k=-m}^{n} q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(q^{m+1};q)_k}{(q;q)_k}\, z^k\; p_{n-k}\big(z; q^k, q^{n-m}; q\big)\; p_m\big(wq^k; q^k, q^{n-m-k}; q\big) . \end{aligned} \tag{41}$$

The c_n are q-Charlier polynomials,[15]

$$c_n(z;a;q) = {}_2\phi_1(q^{-n}, z; 0; q, -q^{n+1}/a) \ , \tag{42}$$

while the polynomials $\mathcal{P}_n$ are defined by

$$\mathcal{P}_n(z;a;q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (a;q)_k \, z^k \ , \tag{43}$$

with

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k \, (q;q)_{n-k}} \ . \tag{44}$$

They satisfy the following three-term recursion relation

$$\mathcal{P}_{n+1}(z;a;q) = \left[1 + z(1 - aq^n)\right] \mathcal{P}_n(z;a;q) - z\,(1 - q^n)\,\mathcal{P}_{n-1}(z;a;q) \ , \tag{45}$$

and reduce to the Rogers-Szegö polynomials[15,21-22] in the limit $a \to 0$.

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