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THE QUANTUM ALGEBRA APPROACH TO q-SPECIAL FUNCTIONS

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Abstract

The quantum algebra interpretation of q-special functions is presented. The connection between these functions and representations of q-algebras is explained and it is shown how generating functions and addition formulas are obtained in this approach. The relation between $\mathcal{U}_q(sl(2))$and the ${}_2\phi_1$ series is used for illustration purposes.

Most special functions of mathematical physics have q-analogues, that is generalizations to a base q. The best-known examples are the basic hypergeometric series [1],

$$
\begin{aligned}
&{}_r\phi_s(a_1, a_2, \ldots, a_r; b_1, \ldots, b_s; q; z) \\
&\qquad = \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n \ldots (a_r;q)_n}{(q;q)_n (b_1;q)_n \ldots (b_s;q)_n} \left[(-1)^n q^{\frac{n(n-1)}{2}}\right]^{1+s-r} z^n \,,
\end{aligned}
\tag{1}
$$

which are the q-analogues of the (generalized) hypergeometric series ${}_rF_s$. The q-shifted factorials $(a;q)_\alpha$ that enter in the definition (1) are defined as follows for α complex:

$$
(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty} \,, \qquad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k) \,, \qquad |q| < 1 \,. \tag{2}
$$

The q-deformed functions, like the ordinary ones, satisfy various properties; they have appeared in the evaluation of physical quantities and are certainly worthy of interest.

The standard special functions are also known to have Lie interpretations. They arise for instance as matrix elements, basis vectors, etc. of representations of Lie groups and algebras. We have provided (see for instance [2,3,4]) similar interpretations of the q-special functions using quantum algebras. We summarize here some of our results taking as example the relation between the quantum algebra $\mathcal{U}_q(sl(2))$ and the basic hypergeometric series ${}_2\phi_1$.

The following two q-analogues of the exponential play a central role in our approach:

$$
e_q(z) = \sum_{n=0}^{\infty} \frac{1}{(q;q)_n} z^n \,, \qquad E_q(z) = \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n-1)}}{(q;q)_n} z^n \,. \tag{3}
$$

These functions satisfy $e_q(z)E_q(-z) = 1$ and

$$D_z^+ e_q(\lambda z) = \lambda e_q(\lambda z), \quad D_z^- E_q(\lambda z) = -q^{-1}\lambda E_q(\lambda z), \tag{4}$$

with the q-derivatives $D_z^\pm$ defined by $D_z^\pm \varphi(z) = [\varphi(z) - \varphi(q^{\pm 1} z)]/z$.

The basic idea in establishing the connection between quantum algebras and q-special functions is to replace the exponential map from Lie algebras into Lie groups by q-exponentials of quantum algebra generators. To see how this works, let us consider $\mathcal{U}_q(sl(2))$. This quantum algebra is generated by k, k^{-1}, e and f subjected to

$$kek^{-1} = q^{1/2}e, \quad kfk^{-1} = q^{-1/2}f, \quad kk^{-1} = k^{-1}k = e, \quad [e,f] = \frac{k^2 - k^{-2}}{q^{1/2} - q^{-1/2}}. \tag{5}$$

It can be equipped with a coproduct, counit and antipode to define a Hopf algebra. It also admits a module $V^{(\lambda,m_0)}$ with $\lambda, m_0 \in \mathbf{C}$, characterized by the following action of the generators on the basis vectors ξ_j, $j = m_0 + n$, $n \in \mathbf{Z}$:

$$k\xi_j = q^{-j/2}\xi_j, \quad e\xi_j = q^{(1-2\lambda)/4}\frac{1-q^{\lambda+j}}{1-q}\xi_{j-1}, \quad f\xi_j = q^{(1-2\lambda)/4}\frac{1-q^{\lambda-j}}{1-q}\xi_{j+1}. \tag{6}$$

Given any $a \in \mathcal{U}_q(sl(2))$, its matrix elements $W_{ij}(a)$ in this representation are defined by

$$a\xi_j = \sum_i \xi_j W_{ij}(a). \tag{7}$$

In analogy with Lie theory, one considers the following elements in the completion of $\mathcal{U}_q(sl(2))$:

$$U(\alpha,\beta,\gamma) = E_q(\alpha e)\, e_q(\beta f)\, k^\gamma, \tag{8}$$

with α, β, γ complex parameters. It is then easy to show that

$$\begin{aligned} W_{ij}(U(\alpha,\beta,\gamma)) &= q^{-\gamma j/2}\left(\beta\frac{q^{(1-2\lambda)/4}}{1-q}\right)^{i-j}\frac{(q^{\lambda-i+1};q)_{i-j}}{(q;q)_{i-j}} \\ &\quad \times {}_2\phi_1\left(q^{\lambda+i+1}, q^{i-\lambda}; q^{i-j+1}; q, -\alpha\beta\frac{q^{(1-2i)/2}}{(1-q)^2}\right), \quad \text{if } i-j \geq 0, \\ W_{ij}(U(\alpha,\beta,\gamma)) &= q^{-\gamma j/2} q^{(j-i)(j-i-1)/2}\left(\alpha\frac{q^{(1-2\lambda)/4}}{1-q}\right)^{j-i}\frac{(q^{\lambda+i+1};q)_{j-i}}{(q;q)_{j-i}} \\ &\quad \times {}_2\phi_1\left(q^{\lambda+j+1}, q^{j-\lambda}; q^{j-i+1}; q, -\alpha\beta\frac{q^{(1-2i)/2}}{(1-q)^2}\right), \quad \text{if } i-j \leq 0. \end{aligned} \tag{9}$$

This establishes most straightforwardly the connection between $\mathcal{U}_q(sl(2))$ and Heine's ${}_2\phi_1$. We shall now briefly indicate how this connection can be put to use.

It is of course possible to take instead of (8), different combinations of e_q and E_q. In particular for

$$\widetilde{U}(\alpha,\beta,\gamma) = k^\gamma E_q(\beta f)\, e_q(\alpha e), \tag{10}$$

the matrix elements $W_{ij}[\widetilde{U}(\alpha,\beta,\gamma)]$ are again expressed in terms of ${}_2\phi_1$. (See Ref. [4] for the explicit formulas.) Now note that $U(\alpha,\beta,0)\widetilde{U}(-\alpha,-\beta,0) = 1$ or in terms of matrix elements that $\sum_l W_{il}(U(\alpha,\beta,0))\, W_{lj}\left(\widetilde{U}(-\alpha,-\beta,0)\right) = \delta_{i-j,0}$. Substituting in this last relation the expressions for $W(U)$ and $W(\widetilde{U})$one arrives at the following orthogonality relation:

$$\begin{aligned} \delta_{i-j,0} = &\frac{q^{(i^2+j^2)/2}(q/z;q)_j}{(q^{\lambda+1};q)_i(q^{-\lambda};q)_j}\sum_{l\in\mathbf{Z}+m_0} q^{l(l-1)/2}\left(-zq^{-i-j}\right)^l \frac{(z;q)_l(q^{\lambda+1};q)_l(q^{-\lambda};q)_l}{(q;q)_{l-j}(q;q)_{l-i}} \\ &\times {}_2\phi_1\left(q^{\lambda+l+1}, q^{l-\lambda}; q^{l-i+1}; q, zq^{-i}\right)\, {}_2\phi_1\left(q^{\lambda+l+1}, q^{l-\lambda}; q^{l-j+1}; q, zq^{-j}\right). \end{aligned} \tag{11}$$

A generating relation is obtained through the construction of a one-variable model for $V^{(\lambda,m_0)}$. Let z be a complex variable and take the basis vectors to be

$$\xi_j = z^n, \quad j = m_0 + n \quad n \in \mathbf{Z}. \tag{12}$$

Let the generators be the following q-difference operators

$$\begin{aligned} k &= q^{-m_0/2}\, T_z^{-1/2}, \\ e &= q^{(1-2\lambda)/4}\left(\frac{1}{1-q}D_z^+ + \frac{1-q^{\lambda+m_0}}{1-q}\frac{1}{z}T_z\right), \\ f &= q^{(1-2\lambda)/4}\left(\frac{z^2}{1-q}D_z^- + \frac{1-q^{\lambda-m_0}}{1-q}\, z\, T_z^{-1}\right), \end{aligned} \tag{13}$$

where T_z is defined by $T_z\varphi(z) = \varphi(q\,z)$. It is easy to check that the identifications (12) and (13) provide a realization of the representation (λ, m_0). One now acts with $U(\alpha,\beta,0)$ on the basis vectors $\xi_j = z^{j-m_0}$ using for e and f the expressions given above. By construction $U(\alpha,\beta,0)z^{j-m_0} = \sum_i z^{i-m_0} W_{ij}[U(\alpha,\beta,0)]$. On the r.h.s., the expansion coefficients of the series in z are given in terms of ${}_2\phi_1$. The l.h.s. therefore defines a generating function for these ${}_2\phi_1$. This function can be evaluated with the help of the q-binomial theorem $\sum_{n=0}^\infty \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}$ and after some redefinitions one obtains the following identity between formal power series:

$$\begin{aligned} &(xq^{-2m_0};q)_{2m_0}(q/y;q)_{m_0+\lambda}\, {}_2\phi_0\left(q^{m_0+\lambda+1}/y, q^{m_0-\lambda}; q, xyq^{-2m_0-1}\right) \\ &\quad = \sum_{k\in\mathbf{Z}} \frac{(q^{m_0-\lambda};q)_k}{(q^{2m_0+1}/x;q)_k(q;q)_k} y^k\, {}_2\phi_1\left(q^{\lambda-m_0+1}, q^{-m_0-\lambda}; q^{k+1}; q, x\right). \end{aligned} \tag{14}$$

Addition formulas are arrived at by observing that the matrix elements $W_{ij}[U(\alpha,\beta,0)]$ and $W_{ij}[\widetilde{U}(\alpha,\beta,0)]$ also provide models for the module $V^{(\lambda,m_0)}$. That this is true is shown by presenting operators $\pi^{(i)}(a)$ acting on the variables α and β and such that $\pi^{(i)}(a)W_{ij}[U(\alpha,\beta,0)] = \sum_k W_{ik}[U(\alpha,\beta,0)]W_{kj}(a)$. It suffices of course to give $\pi^i(e)$, $\pi^i(f)$ and $\pi^i(k)$;these are easily obtained by exploiting the properties of the following analogue of the B-C-H formula

$$\begin{aligned} E_q(\zeta X)\, Y\, e_q(-\zeta X) &= \sum_{n=0}^\infty \frac{\zeta^n}{(q;q)_n}[X,Y]_n\,, \\ [X,Y]_0 = Y, \qquad [X,Y]_{n+1} &= q^n X\,[X,Y]_n - [X,Y]_n\, X, \quad \text{for } n = 1,2,\ldots. \end{aligned} \tag{15}$$

One finds

$$\begin{aligned}
\pi^{(i)}(k) &= q^{-i/2}\, T_\alpha^{-1/2}\, T_\beta^{1/2}\ , \\
\pi^{(i)}(e) &= -qD_\alpha^- - \frac{q^{1/2}}{(1-q)^2}\,\beta\left(q^i T_\alpha T_\beta^{-1} - q^{-i} T_\alpha^{-1}\right)\ , \\
\pi^{(i)}(f) &= D_\beta^+\ .
\end{aligned} \tag{16}$$

This defines a 2-variable model of $V^{(\lambda,m_0)}$ with $\xi_j^{(i)} = W_{ij}\,(U(\alpha,\beta,0))$. From here, one can derive the second- order q-difference equation that the ${}_2\phi_1$ satisfy. The Casimir operator C of $\mathcal{U}_q\,(sl(2))$ is given by $C = (q^{1/2}-q^{-1/2})^{-2}(q^{1/2}k^2 + q^{-1/2}k^{-2} - 2) + fe$ and on $V^{(\lambda,m_0)}$ it takes the value $C^{(\lambda)} = (q^{1/2}-q^{-1/2})^2(q^{\lambda+1/2} + q^{-\lambda-1/2} - 2)$. From $\pi^{(i)}(C)\; W_{lj}\,(U(\alpha,\beta,0)) = C^{(\lambda)} W_{lj}\,(U(\alpha,\beta,0))$ one sees that ${}_2\phi_1$ obeys:

$$\begin{aligned}
\Big\{ z(c-abqz)(D_z^+)^2 + \Big[(1-c) + ((1-a)(1-b) - (1-c))z\Big] D_z^+ \\
-(1-a)(1-b)\Big\}\, {}_2\phi_1(a,b;c;q,z) = 0\ .
\end{aligned} \tag{17}$$

Now the addition formula. Since the $W_{ij}\,(U(\alpha,\beta,0))$ form models for $V^{(\lambda,m_0)}$ we must have $\pi^{(i)}[\widetilde{U}(\alpha,\beta,0)]W_{ij}\,(U(\alpha,\beta,0)) = \sum_k W_{ik}\,(U(\alpha,\beta,0))\, W_{kj}\left(\widetilde{U}(\alpha,\beta,0)\right)$. Substituting in the r.h.s. the expressions for the matrix elements in terms of ${}_2\phi_1$ and using the specifics of the two-variable model to evaluate directly the l.h.s.,one can derive after some operations (see Ref.[4]) the following relation between formal series

$$\begin{aligned}
{}_2\phi_1\left(q^{\lambda+l+1}, q^{l-\lambda}; 0; q, z\right)\; {}_2\phi_0\left(q^{\lambda-j+1}, q^{-\lambda-j}; q, w/q\right) = \sum_{k\in\mathbb{Z}} \frac{(-1)^k q^{k(k+1)/2}}{(q;q)_{l-j-k}\,(q;q)_k} \\
\times\, {}_2\phi_1\left(q^{\lambda+l+1}, q^{l-\lambda}; q^{l-j-k+1}; q, z\right)\; {}_2\phi_1\left(q^{\lambda-j+1}, q^{-j-\lambda}; q^{k+1}; q, wq^k\right)\ .
\end{aligned} \tag{18}$$

With this we conclude. We would first like to stress that the examples presented here are far from covering all the applications that the connection between q-special functions and quantum algebras entail. We would also like to point out that our simple quantum algebra approach allows one to recover [3] the results that are obtained in the quantum group picture (see for instance [5]) where one relates the q-special functions to the matrix elements of corepresentations of quantum groups.

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