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ASPECTS OF PERTURBED CONFORMAL FIELD THEORY, AFFINE TODA FIELD THEORY AND EXACT S-MATRICES

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1. INTRODUCTION

Recently, Zamolodchikov has suggested a way of exploring the properties of the Ising model in a background magnetic field, *i.e.* away from criticality[1]. The critical temperature is fixed so that unperturbed the theory is described by a $c = \frac{1}{2}$ conformal field theory. The Hamiltonian is then perturbed by the addition of a non conformally invariant piece depending upon the background magnetic field. An obvious question to ask concerns the nature of this perturbed two-dimensional field theory. Via a sequence of ingenious arguments Zamolodchikov suggests the hypothesis that the perturbed theory is actually an integrable theory of eight massive bosons. Following this development, our aim is to try to discover what this integrable theory is by conjecturing a candidate (inspired by comments of Hollowood and Mansfield[2]), and testing Zamolodchikov's arguments. On the way, we have discovered[3] (and sometimes rediscovered) a number of interesting facts concerning affine Toda field theory and two-dimensional factorisable S-matrices. It seems to us that this by itself warrants further investigation. Much of this material has also been found by Christe and Mussardo[4], whose work has considerable overlap with ours.

We begin by outlining Zamolodchikov's strategy. The unperturbed $c = \frac{1}{2}$ conformal field theory has field content I, σ and ϵ, with conformal weights $(0,0)$, $(\frac{1}{16},\frac{1}{16})$ and $(\frac{1}{2},\frac{1}{2})$, (together with their descendants), with respect to the two sets of mutually commuting sets of Virasoro generators L_n and $\bar{L}_n$. The Virasoro generators correspond to the moments of the holomorphic and anti-holomorphic parts of the energy-momentum tensor

$$T(z) = \sum_{-\infty}^{\infty} L_n z^{-n-2} \qquad \bar{T}(\bar{z}) = \sum_{-\infty}^{\infty} \bar{L}_n \bar{z}^{-n-2},$$

in turn $(2,0)$ and $(0,2)$ descendants of I. There are many 'conserved' quantities $P_s^{(k)}$ (or $P_{-s}^{(k)}$)

$$P_s^{(k)} = \oint dz\, T_{s+1}^{(k)} \tag{1.1}$$

in the sense that $\partial_{\bar{z}} P_s^{(k)} \equiv 0$. Indeed, in this case all the quantities $T_{s+1}^{(k)}$ belong to the holomorphic part of the conformal family of I. The conserved quantities are labelled by their spin s (or $-s$ for the anti-holomorphic part) and k labels their multiplicity. Any quantitiy $T_{s+1}^{(k)}$ which is itself a derivative with respect to z is to be regarded as trivial in this context since it will not lead to an interesting conserved charge.

In the perturbed theory, where the Hamiltonian is given by

$$H = H_{CFT} + \lambda\sigma(z,\bar{z}), \tag{1.2}$$

most of the conserved quantities will cease to be so, but not all. The idea is to seek out those densities $T_{s+1}(z,\bar{z})$ which at $\lambda = 0$ belong to the previous set, and which satisfy

$$\partial_{\bar{z}} T_{s+1} = \lambda R_s = \partial_z S_{s-1} \tag{1.3}$$

for some operator S_{s-1}. From (1.2), λ is assigned conformal dimensions $(\frac{15}{16},\frac{15}{16})$ and thus R_s has dimension $(s+\frac{1}{16},\frac{1}{16})$ and therefore belongs to the conformal family of σ. Comparing dimensions of the spaces of operators of spin $s+1$ in the conformal family of I modulo z-derivatives with the space of operators of spin s in the conformal family of σ modulo z-derivatives, it is found that for $s = 1,7,11,13,17,19$ the dimension of the first space exceeds by one the dimension of the second. In these cases, the mapping defined by $\partial_{\bar{z}}$ (eq(1.3)) followed by projection onto the second space (*i.e.* removing the z-derivative pieces) certainly has non trivial kernel. In other words, there will be operators S_{s-1} for at least these spins, and hence conserved quantities. Zamolodchikov further suggests that a complete set of conserved quantities will correspond to those whose spins are the integers coprime to 30, the first six of which are those given above. A more interesting observation perhaps is that these integers are also the exponents of the Lie algebra E_8 modulo 30 (and 30 is the Coxeter number of E_8)[5]. That there is a coset construction of the $c = \frac{1}{2}$ conformal theory based on E_8 ($E_8^{(1)} \times E_8^{(1)}/E_8^{(2)}$) is well known[6]. The remarks about conserved quantities makes it plausible that the perturbation λ is somehow probing the coset structure.

The next stage in Zamolodchikov's argument is to explore a theory with the above set of conserved charges (is it uniquely determined?) using techniques from S-matrix theory to find a minimal solution to the bootstrap equation consistent with the conjectured conserved quantities. The outcome is a theory of eight self-conjugate massive scalar bosons. In the next section we briefly review the main features of the S-matrix and bootstrap ideas to be employed.

2. FACTORISABLE S-MATRICES[1,7]

We shall be brief in this section and assume in the first instance that the interesting theories contain self-conjugate particles only, with distinct masses. (This assumption is too strong for most of the Toda theories we consider later but is sufficient for the

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present discussion. For E_8, E_7 and D_{even} the particles are self-conjugate, but only in the first two cases are the particles distinguished by mass alone. For all other simply laced cases, some particles appear in conjugate pairs.) In particular, there is no reflection and multiparticle S-matrix elements are products of two particle S-matrices. Thus, the two particle S-matrix is

$$|a(\theta_1), b(\theta_2)\rangle_{in} = S_{ab}(\theta)\,|a(\theta_1), b(\theta_2)\rangle_{out} \tag{2.1}$$

where θ_i is the rapidity, *i.e.*

$$p_i = m_i(\cosh\theta_i, \sinh\theta_i) \qquad \theta = \theta_1 - \theta_2. \tag{2.2}$$

The S-matrix is

(i) unitary, which in this case means that for each element

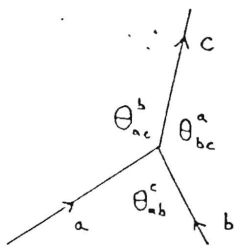
$$S(\theta)S^{\dagger}(\theta) \equiv S(\theta)S(-\theta) = 1; \tag{2.3}$$

(ii) crossing symmetric, which in this case requires

$$S(\theta) = S(i\pi - \theta), \tag{2.4}$$

and is therefore $2i\pi$ periodic taking (2.3) and (2.4) together.

(iii) Additionally, particles may occur as bound state poles in S-matrix elements for $0 \le \theta \le i\pi$, with positive residue. Thus, particle c may occur as a bound state in the process $ab \to ab$ if there is a simple pole at $\theta = \theta_b - \theta_a = i\theta^c_{ab}$ in the S_{ab} S-matrix element (with a suitable residue). In that case (see figure)



we have,

$$\theta_a = \theta_c - i\left(\pi - \theta^b_{ac}\right) \qquad \theta_b = \theta_c + i\left(\pi - \theta^a_{bc}\right), \tag{2.5}$$

and

$$\theta_b - \theta_a = i\left(2\pi - \theta^b_{ac} - \theta^a_{bc}\right) = i\theta^c_{ab}.$$

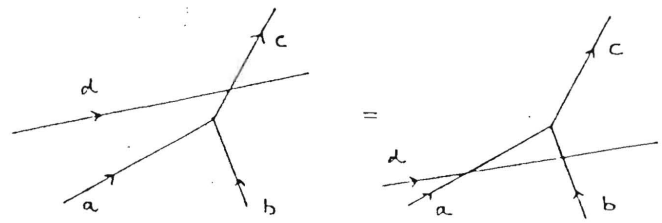
On the other hand,

$$\begin{aligned} m_c^2 = p_c^2 = (p_a + p_b)^2 &= m_a^2 + m_b^2 + 2m_a m_b \cosh(\theta_b - \theta_a) \\ &= m_a^2 + m_b^2 + 2m_a m_b \cos\theta^c_{ab}, \end{aligned} \tag{2.6}$$

and the bound state masses correspond to a particular rapidity value. Indeed, in view of (2.6) the three masses m_a, m_b, m_c form the sides of a triangle Δ_{abc}

with internal angles $\tilde\theta^c_{ab}, \tilde\theta^a_{bc}, \tilde\theta^b_{ca}$ $(\equiv \pi - \theta^c_{ab}, \pi - \theta^a_{bc}, \pi - \theta^a_{bc})$. Equally, particle a or b may occur as a physical bound state in the processes $bc \to bc$ or $ca \to ca$, respectively, although attention must be paid to the sign of the residue.

(iv) The bootstrap property represents a non trivial consistency requirement on the scattering of three particles, implied by the existence of bound states which are themselves possible asymptotic states. It is indicated by the diagram



and expressed by the formula

$$S_{dc}(\theta) = S_{da}(\theta - i\tilde\theta^b_{ac})S_{db}(\theta + i\tilde\theta^a_{bc}), \tag{2.7}$$

where $\theta = \theta_c - \theta_d$.

(v) The conserved quantities mentioned in the previous section must be consistent with the S-matrix. Since they are not in general scalars, this is a non trivial requirement. Thus, when two particles bind to form a third we would expect (using an analytic continuation argument), that if

$$P_s\,|a(\theta)\rangle = q^a_s e^{s\theta}\,|a(\theta)\rangle\,,$$

then

$$q^a_s e^{-is\tilde\theta^b_{ac}} + q^b_s e^{is\tilde\theta^a_{bc}} = q^c_s \tag{2.8}$$

for all possible fusing channels. There is a similar set of equations for the companion quantities bearing the opposite spins P_{-s}, but with no extra content.

Actually, eqs(2.8) are very strong in the sense that given a conjecture for the spins of the conserved quantities the possible fusing angles are highly constrained and hence also the possible particle masses. With a few extra assumptions, properties (i)-(iii) and in particular (iv) and (v) uniquely determine the 'minimal' model consistent with the set of conserved charges. Zamolodchikov's starting point assumes the existence of two particles 1 and 2 say, each containing itself and the other one as a possible 'fusion'. Provided the mass ratio m_2/m_1 is the 'golden ratio' (*i.e.* $2\cos\pi/5 \approx 1.618$) the fusion angles are consistent with the conserved quantities suggested in the previous section. However, the manifestly crossing symmetric unitary S-matrix

$$S_{11}(\theta) = \left(\frac{1}{3}\right)\left(\frac{1}{6}\right)\left(\frac{1}{5}\right)\left(\frac{3}{10}\right), \tag{2.9}$$

where

$$(\alpha) = \frac{\sinh\left(\frac{\theta}{2} + i\pi\alpha\right)}{\sinh\left(\frac{\theta}{2} - i\pi\alpha\right)},$$

does not yet satisfy the first of the bootstrap conditions (eq(2.7) with $a = b = c = d = 1$), to wit

$$S_{11}(\theta) = S_{11}\left(\theta - \frac{i\pi}{3}\right) S_{11}\left(\theta + \frac{i\pi}{3}\right).$$

However, this is remedied by the addition of one more pole at $\theta = i\pi/15$, with its crossed channel partner at $14\pi/15$ and an overall $-$ sign. Thus, instead of (2.9) we are forced to set

$$S_{11}(\theta) = -\left(\frac{1}{3}\right)\left(\frac{1}{6}\right)\left(\frac{1}{5}\right)\left(\frac{3}{10}\right)\left(\frac{1}{30}\right)\left(\frac{14}{30}\right), \tag{2.10}$$

and recognise the need for a third particle 3 for which $m_3/m_1 = 2\cos\pi/30 \approx 1.989$. The next step uses the bootstrap to set up candidates for $S_{12}(\theta)$, $S_{13}(\theta)$ and so on, introducing extra particles as necessary to render the matrix elements consistent. Remarkably, the procedure closes on a 'minimal' solution containing eight particles in the computed mass ratios (to m_1):

1.618 1.989 2.405 2.956 3.218 3.891 4.783.

The question is, what is really going on and can we understand this minimal solution field theoretically? Perhaps there is no field theory for which the minimal solution occurs as a perturbative solution.

In the next section we shall describe a simple field theory which is certainly related to the S-matrix Zamolodchikov describes and which might be used as a framework within which to explore further. In any case, it seems to us that the connection with Toda theory is by itself interesting and revives interest in work begun many years ago.

3. AFFINE TODA FIELD THEORY

Integrable Toda theory associated to each Lie algebra Dynkin diagram has been much discussed in the past[8] and its quantum theory is known to be conformal[9]. It is natural then, when considering a perturbed conformal field theory, to seek a perturbed Toda theory, perturbed in such a way as to preserve integrability. It is also known that the conserved quantities of these theories do indeed occur at spins corresponding to the exponents of the underlying Lie algebra whose simple roots define the theory[10]. These considerations and the remarks in the previous two sections concerning E_8 suggest that the scalar field theory we seek might be affine Toda theory[11] (*i.e.* Toda theory with an additional exponential term depending on the extra root), whose Lagrangian after a field shift could be taken to be

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a - \frac{m^2}{\beta^2}\sum_{i=0}^{r} n_i\, e^{\beta\alpha_i\cdot\phi} \tag{3.1}$$

where $\sum_0^r n_i\alpha_i = 0$ and the r-component simple roots are selected from any one of the affine Dynkin diagrams (of rank $r+1$) in Kac' list[5]. The roots α_i are normalised so that the long roots have length $\sqrt{2}$, and the parameter β keeps track of the overall scale of the roots (classically it is irrelevant). We have organised (3.1) in such a way

that the minimum of the potential (for real β) occurs at $\phi^a = 0$, $a = 1, \ldots, r$ and the potential may be expanded perturbatively about the origin. Thus

$$V(\phi) = \frac{m^2}{\beta^2}\sum_0^r n_i + \frac{m^2}{2}\sum_0^r n_i\alpha_i^a\alpha_i^b\phi^a\phi^b + \frac{m^2\beta}{3!}\sum_0^r \alpha_i^a\alpha_i^b\alpha_i^c\phi^a\phi^b\phi^c + \ldots, \tag{3.2}$$

and, for any choice of roots we may compute the 'classical' masses and couplings. The first surprise is that for $e_8^{(1)}$, the affine diagram associated with E_8, the classical masses are in precisely the ratio discovered by Zamolodchikov. In other words, diagonalising the (mass)^2 matrix

$$(M^2)_{ab} = m^2\sum_0^r n_i\alpha_i^a\alpha_i^b \tag{3.3}$$

yields (in ascending mass order), the eigenvalues m_i^2/m^2:

$$\begin{aligned}
&4\sqrt{3}\sin\pi/30\sin\pi/5\\
&16\sqrt{3}\sin\pi/30\sin\pi/5\cos^2\pi/5\\
&16\sqrt{3}\sin\pi/30\sin\pi/5\cos^2\pi/30\\
&64\sqrt{3}\sin\pi/30\sin\pi/5\cos^2\pi/5\cos^2 7\pi/30\\
&4\sqrt{3}\sin 11\pi/30\sin\pi/5\\
&4\sqrt{3}\sin 7\pi/30\sin 2\pi/5\\
&4\sqrt{3}\sin 13\pi/30\sin 2\pi/5\\
&256\sqrt{3}\sin\pi/30\sin\pi/5\cos^2 2\pi/15\cos^4\pi/5.
\end{aligned} \tag{3.4}$$

An even greater surprise is that the 8-vector $\mathbf{m} = (m_1, \ldots, m_8)$ is an eigenvector of the E_8 Cartan matrix corresponding to the smallest eigenvalue $2\cos\pi/30$. This enables us to assign the masses unambiguously to the vertices of the Dynkin diagram for E_8:

(This is also in ascending order in dimension of the first eight representations of E_8.)* For the other affine theories the masses fit into a similar pattern. However, we have no satisfactory explanation for these facts.

* These masses are also numerically the same as the 'dimensions' of Ocneanu's operators associated with[12] E_8.

Once the mass matrix is diagonalised, the couplings of mass eigenstates are computed (but not by hand!), and are found to be zero except for the entries in the table at the end of this paper. The top entry in each row in the upper half of the table labels the bound state particle in ascending mass (and indicates the sign of the coupling), the lower entry is the fusion angle as a multiple of $\pi/30$. The magnitude of the couplings is given by the general formula

$$|c^{abc}| = \frac{2}{\sqrt{h}} m_a m_b \sin\theta^c_{ab} \tag{3.5}$$

where h is the Coxeter number (30 for E_8). Actually, the same formula is applicable to all simply laced Toda theories *i.e.* corresponding to the $a^{(1)}, d^{(1)}$ and $e^{(1)}$ series) and, with slight modification, to the non-simply laced cases as well. We will not discuss the signs of the couplings any further.

Using the masses and couplings we can conjecture a 'minimal' S-matrix (the part ignoring β) consistent with the bootstrap programme outlined in section (2). The result for E_8 is presented in the lower half of the table containing the couplings. The notation is quite condensed since:

$$[\alpha] = \left(\frac{\alpha}{60}\right)\left(\frac{30-\alpha}{60}\right), \tag{3.6}$$

a manifestly unitary and crossing symmetric building block for the S-matrix elements. We also note,

$$[-\alpha] = [\alpha]^{-1} \qquad [30-\alpha] = [\alpha] = [\alpha+60].$$

This S-matrix is the one obtained by Zamolodchikov[1], although he did not reveal the full table.

There are a number of interesting remarks to be made. First of all, the positions of the bound state poles are fixed by the bootstrap and for the conjectured S-matrix to actually follow from a Lagrangian like that of (3.1) there must be miracles in perturbation theory. For example, we have checked to one loop order that masses all renormalise *in the same way* so that their ratios are unaffected by the coupling constant β. To do this requires the evaluation of propagator bubbles and the result (again valid for each of the a, d, e series) is

$$\frac{\Delta m_i^2}{m_i^2} = \frac{\beta^2}{4h}\cot\left(\frac{\pi}{h}\right) \qquad i = 1, \ldots, r \tag{3.7}$$

an interesting universal formula. Obviously a proof to all orders is highly desirable.

Secondly, since the masses (and wave functions) renormalise we do not expect the couplings computed from the S-matrix to coincide with those of the bare Lagrangian. Moreover, the 'minimal' S-matrix requires adjustment to take into account the β dependence (as $\beta \to 0$ we expect the S-matrix to be unity!). In spite of this, the basic couplings in the Lagrangian appear to have topological significance in the sense that when a coupling is zero in the Lagrangian (3.1) it will not appear in the full S-matrix. Again, for this to be the case, any couplings that can be generated at higher order (e.g. a triangle vertex graph) must cancel against each other or with higher order terms in the Lagrangian. To check this for the $e_8^{(1)}$ case is a formidable

task, but we have done 'random' checks on other, simpler theories in the a, d, e series. Similarly, it is not obvious from the Lagrangian that production processes cannot occur (they are forbidden in the S-matrix by the requirement of consistency with the conserved quantities). However, we have found it to be the case wherever we have checked, at least for trees and to one loop. It is as though the higher order part of the Lagrangian is there for the express purpose of cancelling the unwanted processes, the basic structure being dictated by the masses and the three point coupling terms[13].

Finally, we may conjecture (and then attempt to verify at least to order β^4) the coupling constant dependence of the S-matrix along the lines proposed by Arinshtein et al[14]. Suppose we take the $S_{11}^{(0)}(\theta)$ matrix element from the $e_8^{(1)}$ table:

$$S_{11}^{(0)}(\theta) = -[20][12][2],$$

and modify it by adding a factor whose sole purpose is to remove the poles and zeroes as $\beta \to 0$. It must satisfy the same bootstrap and consistency conditions as $S^{(0)}$ and avoid introducing extra poles in the physical strip. We would guess that factor to be something like

$$[-20+b][-12+b][-2+b], \tag{3.8}$$

where $b = \beta^2 \sum_0^\infty b_n \beta^{2n}$. However, (3.8) fails to satisfy the bootstrap (and anyway the residue signs are wrong). To correct it needs a further factor. We set,

$$\begin{aligned} S_{11}^{(\beta)}(\theta) &= S_{11}^{(0)}(\theta)[-20+b][-12+b][-2+b][-b] \\ &\equiv S_{11}^{(0)}(\theta) F_{11}(\beta) \end{aligned} \tag{3.9}$$

which does work satisfactorily (note, $[0] = -1$). Moreover, the rest of the bootstrap appears consistent with this ansatz. Note too, at $b = 2$ the S-matrix is again unity and, further, (3.9) is invariant under the transformation $b \to 2 - b$. It is tempting to conjecture that $b \to 2$ corresponds to the large β limit of the theory. A natural assumption is to take the function b to be given by an expression of the form

$$b(\beta) = \frac{1}{2\pi}\frac{\beta^2}{1+\beta^2/4\pi}, \tag{3.10}$$

as suggested by earlier work. It satisfies

$$b(4\pi/\beta) = 2 - b(\beta),$$

displaying the symmetry between strong and weak coupling limits emphasised in the context of non-affine Toda theory by Hollowood and Mansfield[2] (who appear to have set $\hbar = 4\pi$ rather than unity). Whether (3.10) is a sensible conjecture requires checking in perturbation theory. A first non trivial check is to examine tree diagrams, this gives the small β behaviour of $S^{(\beta)}$ and verifies (3.9) and (3.10). Similar conjectures for a, d, e series agree to this order with (3.10). (An explanation for (3.5)?)

Actually, in any of the a, d or e theories with self-conjugate particles we would define similarly

$$[\alpha] = \left(\frac{\alpha}{2h}\right)\left(\frac{h-\alpha}{2h}\right), \tag{3.11}$$

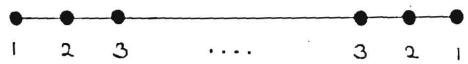
where h is the appropriate Coxeter number and analogous statements hold concerning the factor $F(\beta)$.

If one examines the S-matrix elements carefully, it is clear that the pole structure is very complex, many of the bound state poles being hidden in a multipole structure. We have not analysed these carefully, but it is expected that all these singularities are explicable in terms of singularities of Feynman loop diagrams in perturbation theory[15]. We give some examples in the next section, not for $e_8^{(1)}$ but for $d_4^{(1)}$ which is rather simpler to understand.

The detailed structure of the $e_7^{(1)}$ theory has been elucidated by Christe and Mussardo[4].

4. OTHER THEORIES: BRIEF REMARKS

In this section we shall briefly remark upon other affine Toda field theories[3]. The simplest in many ways is the $a^{(1)}$ series. There, the exponents are the set of integers $1, 2, 3, \ldots, n$ and the masses are $2\sin \pi k/(n+1)$. For n even every particle occurs with its conjugate partner, for n odd there is one self-conjugate particle (the heaviest). The particles are associated with the Dynkin diagram in mass order as before:



The couplings are given by the nice formula:

$$\begin{aligned} c^{abc} &= 0 \quad \text{if} \quad a+b+c \neq 0 \mod n+1 \\ &= \frac{1}{\sqrt{n+1}}(1-\omega^a)(1-\omega^b)(1-\omega^c) \\ &\qquad \text{if} \quad a+b+c = 0 \mod n+1 \end{aligned}$$

where ω is an $(n+1)$th root of unity. The fusing angles are given by

$$\theta_{ab}^c = \frac{a+b}{n+1}\pi, \quad a+b+c = n+1; \quad \theta_{ab}^c = 2\pi - \frac{a+b}{n+1}\pi, \quad a+b+c = 2(n+1).$$

The associated minimal S-matrix is that of Köberle and Swieca[16] with a coupling dependence conjectured by Arinshtein et al[14]. We shall not reproduce it here.

The $d_n^{(1)}$ theories are more subtle. The exponents are $1, 3, 5, \ldots, 2n-3, n-1$ with a set of masses

$$\sqrt{2},\ \sqrt{2},\ 2\sqrt{2}\sin\frac{k\pi}{2(n-1)} \qquad k = 1, 2, \ldots, n-2. \tag{4.1}$$

For n even, each particle is self-conjugate, for n odd the pair with equal mass are conjugates. Occasionally, for $n = 1 \mod 3$, there is another particle mass degenerate

with the pair. However, it is self-conjugate. Attached to the Dynkin diagram the degenerate pair correspond loosely speaking to the spinor, conjugate-spinor representations of the algebra. For even n there is also a degenerate pair of conserved charges with spin $n-1$. Each of these theories has a minimal S-matrix associated with it, but we list only $d_4^{(1)}$ to illustrate certain general features. There, the masses are $\sqrt{2}$ for each of the three light particles (l, l', l'') and $\sqrt{6}$ for the heavy particle h. The light particles are distinguished by the conserved charges of spin 3. The three non-zero couplings are:

$$c^{l,l'l''} = \sqrt{2}, \quad c^{llh} = \frac{-1}{\sqrt{2}}, \quad c^{hhh} = \frac{1}{\sqrt{2}}$$

The minimal S-matrix has the form

$$S_{ll'}(\theta) = -[4]$$

where l, l' are any two different light particles, and

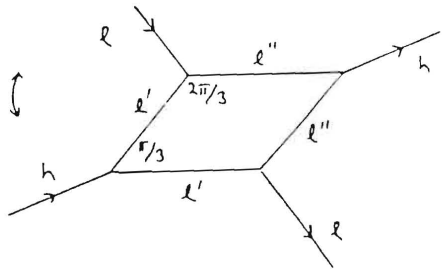
$$\begin{aligned} S_{ll}(\theta) &= -[4] \\ S_{lh}(\theta) &= [1][3] \\ S_{hh}(\theta) &= [4]^3 \end{aligned} \tag{4.2}$$

where we have used the same notation as before (3.11), with $h = 6$. The β dependence is derived consistently from the ansatz

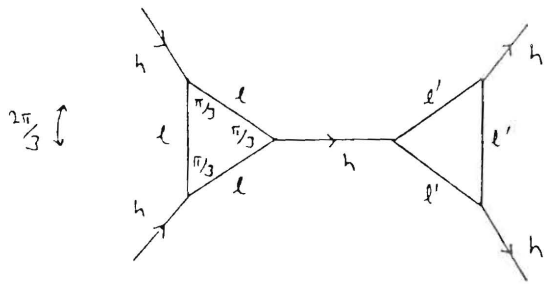
$$F_{ll'}(\beta) = [-4+b] \tag{4.3}$$

the bootstrap and the conjecture (3.10).

In this case it is relatively easy to check the claim about mass renormalisation (eq(3.7)) and the absence of production processes, at least to order β^2. It is also easy to identify the perturbative Feynman diagrams responsible for the second and third order poles in expressions (4.2). The idea, originally exploited by Coleman and Thun[17], is to identify diagrams which can be drawn as geometrical figures with internal lines contracted or on-shell. For example, the double pole in S_{lh} at $\theta = i\pi/2$ occurs in the box diagram



where the internal angles sum to 2π and hence corresponds to a planar figure with the internal light particles on their mass shells. The triple pole in S_{hh} arises differently, from a pair of vertex corrections.



Each triangle graph is singular (a simple pole) at $\theta = 2\pi i/3$, as evidenced by the internal angles summing to π when the light particles are on shell.

Obviously, the pole structure of the $e_8^{(1)}$ theory is more complex. Nevertheless, the ideas are identical although it will require some ingenuity to spot the relevant diagrams (where are the twelfth order poles in $S_{88}(\theta)$, for example?). We are confident this will work although we have not checked all possibilities explicitly.

The $d_n^{(1)}$ theories are interesting for another reason which we mention briefly. Recall the Sine-Gordon theory has a spectrum of solitons (a pair) and breathers[18]. In terms of the Lagrangian parameters,

$$V(\phi) = \frac{\alpha}{\kappa^2}\cos\kappa\phi$$

the soliton mass is given by

$$m_{soliton} \equiv m = \frac{8\sqrt{\alpha}}{\kappa^2}$$

and the spectrum of breathers by

$$m_k = 2m\sin\left(\frac{k}{16}\bar{\kappa}^2\right) \qquad \bar{\kappa}^2 = \frac{\kappa^2}{1-\kappa^2/8\pi} \tag{4.4}$$

where $k = 1, 2, \ldots < 8\pi/\bar{\kappa}^2$. For certain values of κ (actually just before a new breather enters the spectrum), there is known to be no reflection in the Sine-Gordon S-matrix. This occurs just at the moment when $8\pi/\bar{\kappa}^2$ reaches the next integer. At that point, $\bar{\kappa}^2 = 8\pi/N$ say, and the breather masses are

$$2m\sin\left(\frac{k\pi}{2N}\right) \qquad k = 1, 2, \ldots, N-1$$

i.e. precisely those of the $d_{N+1}^{(1)}$ theory (including the soliton masses). Moreover, the S-matrix is essentially identical to the 'minimal' $d_{N+1}^{(1)}$ S-matrix. For example, $\kappa^2 = 2\pi$ corresponds to $\bar{\kappa}^2 = 8\pi/3$ and a spectrum of three light and one heavy particles in the mass ratio 1 to $\sqrt{3}$. A comparison of the Faddeev-Korepin[19] S-matrix (and its extension to breathers[20]), with the expression (4.2) reveals the identity, apart from signs in some of the $l - l'$ matrix elements which are not in any case fixed by the bootstrap. Remarkably, it seems that tuning the Sine-Gordon coupling constant κ

yields successive members of a tower of $d_n^{(1)}$ minimal theories. We strongly suspect this is related to perturbed $c = 1$ conformal field theory. We note that the coset models $d_n^{(1)} \times d_n^{(1)}/d_n^{(2)}$ all have $c = 1$, and that perturbations of $c = 1$ theories have been identified with Sine-Gordon theory[21]. As with the Ising-$e_8^{(1)}$ relationship there remains the problem of understanding the affine Toda coupling constant dependence. How are two theories related if their minimal (or in the case of Sine-Gordon, actual) S-matrices happen to agree?

COMMENTS

The major problem in relating this work to perturbed conformal field theory lies in understanding the rôle of the coupling constant β. To obtain a value of the central charge c small enough to lie in one of the discrete series of coset models Hollowood and Mansfield were forced to assume an analytic continuation to imaginary β. The same conclusion has been reached by Eguchi and Yang[22], from a different starting point. It is not however clear that such a choice for β would make sense, from the straightforward Lagrangian field theory point of view, to compute an S-matrix. For example, the bootstrap proposed by Zamolodchikov provides a minimal S-matrix identical with the one we have described. However, an imaginary coupling in the postulated extra factor $F(\beta)$ would appear to alter the physical pole structure. Thus, we are not really able to confirm that the theory described by eq(3.1) actually coincides with Zamolodchikov's conjecture for any value of the coupling despite the fact that it shares the same minimal S-matrix.

On the other hand, although we were motivated to study this problem by the work on conformal field theories, the results we have obtained are independently interesting. For example, although we have not mentioned it in this article so far, the a, d, e series is rather special, here as elsewhere [3,23], in the following sense. Firstly, each of the Toda theories corresponding to a non-simply laced algebra is obtainable as a truncation of a theory in the a, d, e series. Each of the simply-laced theories enjoys a minimal S-matrix in the sense we have described but the S-matrix does not respect the truncation. In fact, the non-simply laced theories do not have minimal S-matrices; for example, the masses fail to renormalise universally. It seems that the degeneracies present in most of the a, d, e theories are actually important; if they are removed by truncation the nice features tend to disappear. In the non-simply laced theories the coupling constant dependence must play a more significant role in the bootstrap. There are many questions for the future besides elucidating the connection with conformal field theory.

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REFERENCES

1. A. B. Zamolodchikov, Integrable Field Theory from Conformal Field Theory, Proceedings of the Taniguchi Symposium, Kyoto (1988)

   —, RAL preprint 89-001.

   —, JETP Letters 46 (1987) 161

   —, Int. J. Mod. Phys. A3 (1988) 743
2. T. J. Hollowood and P. Mansfield, Oxford preprint 89-17P.
3. H. W. Braden, E. Corrigan, P.E. Dorey and R. Sasaki, Durham preprint UDCPT 89-23, to appear in Physics Letters B.
4. P. Christe and G. Mussardo, Santa Barbara preprint (revised) UCSBTH 89-19
5. S. Helgason, Differential geometry, Lie groups, and symmetric spaces (Academic Press Inc. 1978)
6. P. Bowcock and P. Goddard, Nucl. Phys B285 (1987)651
7. A. B. Zamolodchikov, Al. B. Zamolodchikov, Ann. Phys. 120 (1979) 253, and references therein.
8. A. N. Leznov and M. V. Saveliev, Group methods for the integration of nonlinear dynamical systems (Moscow Nauka 1985)
9. J-L. Gervais and A. Neveu, Nucl. Phys. B224 (1983) 329

   P. Mansfield, Nucl. Phys B222 (1983) 419

   E. Braaten, T. Curtright, G. Ghandour and C. Thorn, Phys. Letts. B125 (1983) 301
10. G. Wilson, Ergod. Th. and Dynam. Sys. 1 (1981) 361
11. A. V. Mikhailov, M. A. Olshanetsky and A. M. Perelomov, Comm. Math. Phys. 79 (1981) 473

    D. I. Olive and N. Turok, Nucl. Phys. B215[FS7] (1983) 470
12. A. Ocneanu, Lectures to this conference.
13. C. J. Goebel, Prog. Theor. Phys. Suppl. 86 (1986) 261
14. A. E. Arinshtein, V. A. Fateev and A. B. Zamolodchikov, Phys. Letts. B87 (1979) 389
15. R. J. Eden, P. V. Landshoff, D. I. Olive and J. C. Polkinghorne, The Analytic S-matrix, (Cambridge University Press 1966)
16. R. Köberle and J. A. Swieca, Phys. Letts. B86 (1979) 209
17. S. Coleman and H. Thun, Comm. Math. Phys. 61 (1978) 31
18. S. Coleman, Aspects of Symmetry, (Cambridge University Press 1985)
19. L. D. Faddeev and V. E. Korepin, Theor. Mat. Fiz. 25 (1975) 147
20. M. Karowski and H.-J. Thun, Nucl. Phys. B130 (1977) 295
21. H. Yoshii, Phys. Letts. B223 (1989) 353
22. T. Eguchi and S-K. Yang, Phys. Letts. B224 (1989) 373
23. A. Cappelli, C. Itzykson and J-B Zuber, Comm. Math. Phys. 113 (1987) 1

    V. I. Arnold, Catastrophe Theory (Springer-Verlag 1984)

TABLE: E_8 COUPLINGS AND S-MATRICES

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