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Statistical Closure and the Logistic Map

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Abstract

We consider the validity of Gaussian and Beta closure schemes as a function of time for the logistic map with stochastic initial conditions. This is done by comparing cumulants and probability distributions generated by direct simulation with predictions based on evolution equations for moments derived by assuming a certain closure approximation. All initial conditions assume a Gaussian distribution with a mean value near an unstable fixed point and a small dispersion. The control parameter for the logistic map is chosen to be in either one of two regimes: $\mu < 3$ in which case there exists a stable attracting fixed point or $\mu = 4$ which is fully chaotic. We find for either regime, the system exhibits behavior characterized by three different time periods (called initial, transient and final). The initial time interval is characterized by the Gaussian closure being very accurate and the evolution of the system dominated by the evolution of the mean. The transient time interval is characterized by rapid growth in all cumulants. For $\mu < 3$, the Gaussian closure approximation breaks down (but only weakly, as the growth of the cumulants remain perturbative to the mean evolution). We identify this transient period as a manifestation of the Kubo fluctuation enhancement theorem. The amount of enhancement and the location of the transient period is computed both theoretically and numerically. For $\mu = 4$, the Gaussian closure approximation breaks down during the transient period with rapid growth of all cumulants. Unlike the $\mu < 3$ case, the fluctuations do not remain perturbative to the evolution of the mean. We identify this period as the Suzuki scaling regime. We compare his theory with our experimental results and find good agreement. The transient period eventually evolves into the final equilibrium state. For $\mu < 3$, this is characterized by the decay of all cumulants to zero and the Gaussian closure approximation becoming accurate again. For $\mu = 4$, a non-Gaussian steady state is reached. A new closure scheme based on the Beta distribution is introduced in the chaotic regime. We find that the evolution equations for the mean and dispersion based on a Beta distribution closure give very accurate predictions over all three time periods. This new type of closure assumes nothing about the vanishing of higher order cumulants (in fact cumulants of all order are non-vanishing). The possible relevance of these results to clump kinetics is also addressed.

I. Introduction

The statistical closure problem [1] arises whenever one is attempting to solve a non-linear stochastic equation. The infinite hierarchy of moment equations generated by stochastic equations for few or many degrees of freedom (e.g. Brownian oscillator versus the stochastic Navier-Stokes equation) are only tractable if some relation exists between higher and lower order moments. The net effect being that the evolution equations governing low order moments are decoupled from the hierarchy. Of course, one can always try to solve the corresponding evolution equation for the probability distribution and thus obviate the closure problem. Unfortunately, even for stochastic systems with a few degrees of freedom where linear equations of the Fokker-Planck type govern the distribution function, time dependent solutions are hard to come by for all but the simple harmonic oscillator [2]. Evolution equations for the probability distribution in systems with infinitely many degrees of freedom (e.g. the Hopf equation in turbulence [3]) are usually complicated functionals of some field variable and hence their solution can be very difficult if not impossible to obtain. Numerical methods applied to non-linear stochastic systems are in many cases the only recourse left. Unfortunately even here there are problems. In turbulence for example, how do unresolved degrees of freedom (sub-grid scale modes) effect resolved degrees of freedom? In radiation transport, the diffusion approximation to the photon transport equation [4] arises as the result of a closure based on relating the radiation pressure tensor to the radiation energy density. The transport equation itself is usually so difficult to solve numerically that many times the diffusion approximation is invoked in regimes [4] where it ordinarily wouldn't be valid. The above examples brings up the barrier to simply solving any non-linear stochastic system numerically, computational memory and speed are many times not adequate to solve a given problem at the required resolution. Evolution equations based on closure relations offer such a drastic simplification over the full stochastic description that in many cases such as in turbulence modelling [5] they are used to provide the information on the dynamical properties of the system.

The wide use of closure relations in statistical models raises the issue of their validity as a stochastic system evolves in time. That is, given a closure relation at time zero, does the closure relation remain valid for all time? It is of course only possible to answer this question definitively for models which can be solved exactly (either numerically or if one is lucky, analytically).It is only for these class of problems that a direct comparison can be made between the true probability distribution and a distribution based on an expansion in moments derived from the closure evolution equations. Usually a set of evolution equations based on a given set of closure relations will exhibit unphysical behavior if they no longer give a faithful representation of the stochastic process. An example from turbulence modelling is the existence of negative energy spectra that appear when using the quasi-normal expansion [5]. In radiation transport it is the acausal propagation of radiation fronts, when the diffusion approximation is invoked, that signals a breakdown of the assumed closure relation [4].

Non-linear maps with either external noise [6] or stochastic initial conditions offer an excellent arena for testing the viability of closure relations. They exhibit complicated dynamics while at the same time being amenable to numerical calculation. In particular, the logistic map [7] possesses much of the structure of more complicated maps including fixed points, period doubling, and chaos. Because of some rather strong analogies with turbulence [8], non-linear maps provide an excellent testing ground for looking at the validity of closure approximations in more realistic situations such as fluid instability. In this paper we will concern ourselves with the statistical closure problem as it relates to the logistic map with stochastic initial conditions (in particular, Gaus-

sian initial conditions). The specific problem we wish to address is, given a Gaussian distribution of initial values centered near an unstable fixed point, how does this probability distribution evolve when each point of the function evolves according to the logistic map? In particular, how well do the evolution equations for the moments (which are based on a given closure relation) do in predicting the correct dynamics.? In what regimes and under what circumstances do the closure relations work? For future reference, we will call a given set of closure relations (valid at an initial time) stable if the deviations from this set of closure approximations is small as a function of time and the final state is characterized properly by the closure relations. Otherwise, the closure is called unstable. Our choice of Gaussian initial conditions is purely arbitrary and done for ease of analysis (the evolution equations for the moments are easy to derive for a Gaussian closure) and besides, there is no a priori reason for choosing any other set of stochastic initial conditions. The questions which we address in this paper constitute an admittedly restricted set of parameter space, but we feel that something can be learned by studying the closure stability issue in the context of a well understood dynamical process (i.e. the logistic map).

The well studied logistic map is defined by

$$x_{n+1} = \mu x_n (1 - x_n) \tag{1.1}$$

where μ is the usual control parameter and $x_{n=0}$ is chosen at random from a Gaussian probability distribution with a mean centered at or near an unstable fixed point and a narrow width which is an input parameter. We will sometimes refer to the iteration index n as a "time". Because this problem can be numerically solved on the computer, we can compare the exact evolution with evolutions based on closure approximations. The two we study here are based on a Gaussian closure scheme (that is further approximated by linearizing in the dispersion) and a closure based on the Beta distribution (when $\mu = 4$) [12]. In this paper, the range of the control parameter is chosen so that we are either in the stable regime ($1<\mu<3$) where the fixed point at zero is unstable but the fixed point at $1 - 1/\mu$ is stable or in the chaotic regime ($\mu = 4$) where both zero and.75 are unstable. In this paper we will not be concerned with the period doubling regimes for $\mu > 3$.

The above problem presents a nice toy model for understanding clump kinetics in turbulent plasmas and fluids [9]. A group of particles (initially close together in phase space) exposed to a turbulent flow will exhibit correlated behavior. This behavior, often referred to as clumping, is due to the fact that the turbulent potential field that the particles are moving in, is itself correlated. Hence, even though mixing forces tend to make neighboring particle trajectories diverge, there is a cooperative effect that results in the particles staying together and forming a coherent structure. Eventually however, the turbulent processes continue and the clump is destroyed. Typically, this problem is modelled by looking at the relative diffusion of a group of nearby particles which are under the influence of a turbulent flow. The logistic map provides an elementary environment in which to study not only the closure problem mentioned previously but also this problem of how a group of particles under some kind of chaotic influence evolve. Thus we can view our simple model as providing insight into how tightly packed group of Gaussian distributed particles contained in the interval [0,1] evolve under the influence of a non-linear map. Recently, Sommerer and Ott [10] have looked at the behavior of an aggregate of tracer particles on the surface of an unstable fluid. Although they were primarily interested in determining the fractal dimension of an experimentally observed process, the theoretical basis of their work relied on characterizing the evolution of a particle at a specific time in the aggregate via a random map. Therefore, although it might seem that the use of non-linear maps to describe clump kinetics in turbulent fluids and plas-

mas is merely an abstraction, there does seem to be some basis for using the mathematics developed in modern dynamical systems theory as a way of gaining insight into this problem.

Our reason for looking at both the stable and chaotic regimes is in order to differentiate the types of behavior exhibited by the relative diffusion (essentially the dispersion) and higher order cumulants. Qualitatively, the behavior of our initial Gaussian distribution and the stability of the closure relations can be addressed by looking at the Lyapunov exponent of the map (λ). In the stable regime where ($\lambda < 0$) all initial points will be attracted to the fixed point and hence we would expect the closure relation to be stable and the Gaussian clump to remain intact. However, in this paper we are also interested in the behavior of the closure relation over the whole time interval and hence we need quantitative information about the evolution of the initial Gaussian probability distribution. This is information that the Lyapunov exponent just doesn't provide. For example, as we will see, even in the stable regime there is a period when the Gaussian behavior breaks down. In the chaotic regime it is well known that there is a non-Gaussian stationary distribution associated with finding an iterate in the interval [x,x+dx] as $n \to \infty$. This distribution, the invariant distribution is given by

$$P(x) = \frac{1}{\pi\sqrt{x(1-x)}} \tag{1.2}$$

Hence, we would expect a breakdown of the initial Gaussian probability distribution. Exactly how this occurs and when can only be addressed by numerically following the time evolution of the system. Again, the Lyapunov exponent in the chaotic regime where $\lambda > 0$ provides the qualitative information regarding stability of the closure relations and the Gaussian clump of particles but it does not provide detailed temporal information that is needed if we wish address the problem of when and how the initial Gaussian closure breaks down.

A keen test of the stability of closure relations is provided by unstable systems relaxing towards equilibrium. As an introduction to the closure issues raised above, we consider the evolution of a continuous one dimensional system from an initially unstable equilibrium. Consider a particle with a single degree of freedom $x(t)$ under the influence of a potential

$$V(x) = -\frac{\gamma x^2}{2} + \frac{g x^4}{4} \tag{1.3}$$

where γ and g are both positive. Assume the initial conditions are stochastic and drawn from a Gaussian distribution with mean δ and dispersion $\sigma^2(0)$. Since x=0 is an instability point, δ is a measure of the instability of the initial state. That is, for $|\delta| \sim \sqrt{\gamma/g}$ the initial state is near equilibrium but for $|\delta| \ll \sqrt{\gamma/g}$, the initial state is highly unstable. If one performs an expansion about the deterministic path (the so called van Kampen expansion [11]) so that $x(t)$ is decomposed into a mean part plus a small remaining fluctuating part then it is found to first order in smallness that the fluctuations are Gaussian distributed about the mean trajectory. The evolution of the mean $\mu(t)$ and dispersion $\sigma^2(t)$ are given by

$$\frac{d}{dt}\mu(t) + ((-\gamma + 3g\sigma^2(t))\mu(t) + g\mu^3(t)) = 0. \tag{1.4}$$

$$\frac{d}{dt}\sigma^2(t) + (-\gamma + 3g\mu^2(t))\sigma^2(t) = 2D \tag{1.5}$$

where D is the diffusion coefficient (technically, eqn. [1.4] with the dispersion set to zero is usually called the van Kampen expansion but we will refer in this paper to moment equations linearized in the dispersion to be the van Kampen expansion). It is easily shown [12] that $\sigma^2(t)$ rises to a value proportional to $1/\delta^2$ in a time proportional to $\log(1/\delta)$ (this is an example of the so called Kubo fluctuation enhancement theorem [13]) before it drops to its stationary value. It is important to realize that as δ approaches zero (that is, as the instability point is approached), the dispersion can grow to a value larger than the mean and thus invalidate the expansion. In fact, for δ equal to zero the dispersion grows indefinitely at a rate which is exponential in time. This is known to be unphysical since the stationary state fluctuations are easily calculated and found to be equal to γ/g Analyzing Eqns. [1.4] - [1.5] further, it becomes apparent that the van Kampen expansion to first order is merely a linearized Gaussian closure scheme and the Kubo theorem is telling us that as we approach the instability point, the linearized Gaussian closure is breaking down. A better closure is one based on a Gaussian with non-linear fluctuations. That is all cumulants greater than second order are assumed to be zero.This fact allows one to relate all moments to the mean and dispersion. If for simplicity we take δ to be zero, then the equation for the dispersion becomes

$$\frac{d}{dt}\sigma^2(t) = 2(-\gamma+3g\sigma^2(t))\sigma^2(t)+2D \tag{1.6}$$

This non-linear equation for the fluctuations which is based on a full Gaussian closure yields a fluctuation which grows exponentially in time and then approaches a fixed value equal to 1/3 of the true value of the stationary state dispersion [12]. Both the van Kampen and Gaussian closure equations will eventually breakdown since the stationary state distribution function possesses double peaks each centered about $\pm\sqrt{\gamma/g}$. The breakdown of the Gaussian closure is realized by the fact that the dispersion never attains the true stationary state value. In fact, it can be shown [12] that the 1/3 value of the stationary state fluctuation can be interpreted as the largest width of the distribution function before it starts to develop double peaks. To deal with the relaxation and fluctuation near an instability point and with it the breakdown of initial Gaussian behavior, Suzuki developed what has become known as the scaling theory of transient phenomena [14. Simply put, Suzuki's method is an asymptotic technique valid for small values of the random force that can be applied to three qualitatively different time regimes that are characteristic of any nonlinear stochastic process relaxing towards equilibrium from an initially unstable state. Initially, diffusive forces dominate and the evolution of the stochastic system is approximately a Wiener process [2]. This is typically where the initial probability distribution retains a Gaussian like behavior. The intermediate regime is dominated by non-linear drift forces that give rise to exponential growth of the initial fluctuations and it is here that the Gaussian behavior breaksdown. Finally, the system settles into its stationary or long time behavior. Scaling theory interpolates the behavior between the initial diffusive and intermediate non-linear regimes. Suzuki performs a non-linear transformation of the Langevin or Fokker-Planck equation. This transformation converts the system to an Ornstein-Uhlenbeck process in the transformed variable and hence it can be solved analytically. For the above anharmonic oscillator Suzuki finds that the probability distribution P(x,t) is given by

$$P(x,t) = \frac{1}{\sqrt{2\pi j(t)}} (\gamma/g) \left(\frac{\gamma}{g} - x^2\right)^{\frac{-3}{2}} exp\left(-x^2/\left(2j(t)\left(\frac{\gamma}{g} - x^2\right)\right)\right) \quad (1.7)\ (1.8)$$

Where j(t) is the so called scaling variable and it is defined by

$$j(t) = \frac{g}{\gamma}\left(\sigma^2(0) + \frac{D}{\gamma}\right) exp(2\gamma t) \quad (1.9)$$

It is important to notice that the scaling function is merely the solution to the dispersion equation based on the van Kampen expansion (Eqn. [1.6]).The scaling variable j(t) is typically small (<<1) for the initial regime and order one in the intermediate regime. In fact where $j(t) \sim 1$ signals the appearance of double peaks in the probability distribution function.An important implication of Eqn. [1.8] for closure, is that all moments can be written as a product of the stationary state fluctuations and a function Z(j(t)) [17]. That is,

$$\langle x^{2n}(t)\rangle = (\gamma/g)^n Z_n(j(t)) \quad (1.10)$$

Hence, the evolution of an initial Gaussian closure scheme can be seen to be accurate up to a time defined by $t \sim j^{-1}(1)$. After this, the closure breaksdown. It is not apparent from Eqn. [1.9] that any closure relation is valid once j(t) is order one.Any long time behavior of this system based on closure relations, would have to be modelled on a parameterization of the steady state distribution function which is of course non-Gaussian. Therefore, we would say that the Gaussian closure scheme in this problem was unstable.It is only valid over a certain interval of time defined by the where the scaling function is small.

The Suzuki scaling theory has been applied to the fully chaotic logistic map [14,15]. In particular the relative diffusion of two nearby trajectories. If we define the relative diffusion by

$$r(n,\delta) = \langle (x_n - y_n)^2 \rangle \quad (1.11)$$

where δ is the RMS of the initial separation and $\langle \ldots \rangle$ denotes an ensemble average then the claim is, that

$$y(n,\delta) = 1 - \cos(2^n\delta) \quad (1.12)$$

provided $\delta \to 0$ and $n \to \infty$ while holding $\delta exp(n\lambda)$ fixed (λ is the Lyapunov exponent). This defines the regime in which scaling theory is to be applied. When λ is log2, this is nothing more than the scaling function mentioned earlier. That is, the relative diffusion can be written as a single function of a scaling variable $\tau = 2^n\delta$ The scaling variable is based on the fact that initially, the relative diffusion shows an exponential growth based on the Lyapunov exponent being log2. This is analogous to Eqn. [1.8] above.It should be noted that the relative diffusion of two statistically independent trajectories can be viewed as the dispersion of a single trajectory given a distribution of initial conditions. That is $r(n,\delta) = 2\sigma_n^2$ where $\sigma_n^2 = \langle x_n^2\rangle - \langle x_n\rangle^2$. Hence, scaling theory predicts

$$\sigma_n^2 \sim \frac{(1 - \cos(2^n\delta))}{2}$$

The dispersion is just the steady state value multiplied by a function of the scaling variable (see the analogous result Eqn. [1.9]). In this instance the ensemble average refers to taking different starting values for the iteration of the map. In addition, for $\delta \ll 1$ we have $\delta \sim 2\sigma_{n=0}$. The existence of a scaling regime and the application of scaling theory to relative diffusion in turbulent fluids and plasmas has been investigated by a number of people including Misguich and Balescu [9], Suzuki [15], and Grossman and Procaccia [9].

In section II, we derive the van Kampen and Gaussian closure relations for the logistic map with stochastic initial conditions in the stable regime and compare our results with solving the problem numerically. We look at both the time evolution of the probability distribution and all cumulants less than or equal to four.In addition, we compute the magnitude and time of the fluctuation enhancement.In section III, we consider the application and subsequent failure of the van Kampen and Gaussian closure relations to the logistic map with stochastic initial conditions in the fully chaotic regime. In addition, we are able to come up with a successful closure scheme which is based on the Beta distribution function. Comparisons with Suzukis' predictions for the relative diffusion in the logistic map are also made.

II.Statistical Closure and the Approach to a Fixed Point

A. Theory

Consider the logistic map Eq. [1.1] with stochastic initial conditions drawn from a Gaussian distribution. In this section we will consider how an initial Gaussian distribution evolves in the stable regime when the initial mean is near the unstable fixed point and the initial dispersion is much less than one. Because of our initial conditions, at n = 0 all moments can be related to first and second order moments by the condition that all cumulants greater than two vanish. The starting point for the analysis is to consider the ensemble average of Eq. [1.1]. That is

$$\langle x_{n+1} \rangle = \mu \langle x_n \rangle - \mu \langle x_n^2 \rangle \tag{2.1}$$

where $1 < \mu < 3$. Thus, we have an unstable fixed point at 0 and a stable fixed point at $1 - 1/\mu$. The ensemble average used here is defined in the normal sense. That is, it is merely the average of a given stochastic variable which has been sampled over its realizations. It is obvious that Eqn. [2.1] will generate an infinite hierarchy of moment equations. Because of our choice of initial

conditions we choose to close the hierarchy for all n using a Gaussian. For example, the third order and fourth order moments must obey the following equations

$$\langle x_n^3 \rangle = 3\langle x_n \rangle \langle x_n^2 \rangle - 2\langle x_n \rangle^3 \tag{2.2}$$

$$\langle x_n^4 \rangle = 3\langle x_n^2 \rangle^2 - 2\langle x_n \rangle^4 \tag{2.3}$$

It is convenient to define the mean and dispersion by M_n and σ_n^2. Using Eqns.[2.2] - [2.3] and Eqn. [2.1], we arrive at the two parameter description of the dynamics,

$$M_{n+1} = \mu M_n (1 - M_n) - \mu \sigma_n^2 \tag{2.4}$$

$$\sigma_{n+1}^2 = \mu^2 (1 - 2M_n)^2 \sigma_n^2 + 2\mu^2 \sigma_n^4 \tag{2.5}$$

This set of equations constitutes a mean field approach to the dynamics [22]. As was the situation for the non-linear potential considered in the introduction, we now have a set of non-linear equations for the mean and dispersion. This set of equations describes a two dimensional map in the space of the mean and the dispersion. In the limit of zero dispersion, this two dimensional map collapses to the usual one dimensional logistic map. It must be noted that this treatment will only remain valid so long as the dispersion remains small and the mean is not too close to the endpoints of the interval $[0, 1]$. For example, if we are too close to either zero or one and our dispersion is large, we will produce a mean for the next iterate outside the interval. Thus, the two dimensional map described above is in general unstable except in situations where the dispersion remains small for all n. Thus we expect the Gaussian closure scheme to be valid provided the fluctuations about the mean do not get too large. Hence, it is possible to simplify Eqns. [2.4] - [2.5] by ignoring non-linear terms in σ_n^2. This we call the van Kampen expansion of Eqn. [2.1]. We have found in our numerical simulations that the solutions to the equations based on a full Gaussian closure (Eqn. [2.5]) are virtually identical to the solutions based on the van Kampen expansion, provided $\mu < 3$. What happens in the chaotic regime will be mentioned in section III. The set of equations which are linearized in the fluctuations are given by,

$$M_{n+1} = \mu M_n (1 - M_n) - \mu \sigma_n^2 \tag{2.6}$$

$$\sigma_{n+1}^2 = \mu^2 (1 - 2M_n)^2 \sigma_n^2. \tag{2.7}$$

As opposed to Eqns. [2.4] - [2.5], which are a coupled set of non-linear maps, the mean and dispersion equations resulting from the van Kampen expansion (although still being two dimensional) are linear in the fluctuations and hence are amenable to analytic calculations. Of particular

interest are the fixed points and the existence of fluctuation enhancement.If we define the fixed points by $\{M_{fp}, \sigma_{fp}^2\}$ then it is easy to show that there exist two sets of solutions. The first set are given by,

$$\{1 - \frac{1}{\mu}, 0\} \tag{2.8}$$

which are just the fixed points for the ordinary logistic map. In addition we obtain a new set of fixed points given by

$$\{\frac{1}{2}(1 - \frac{1}{\mu}), -\frac{1}{2}(1 - \frac{1}{\mu})^2\} \tag{2.9}$$

$$\{\frac{1}{2}(1 + \frac{1}{\mu}), \frac{1}{4\mu^2}(\mu + 1)(\mu - 3)\} \tag{2.10}$$

The fixed point defined by Eqn. [2.9] is unphysical for all μ due to the negative dispersion, while the fixed point defined by Eqn. [2.10] is physical only for $\mu > 3$. Hence, the only physically realizable fixed point (in so far as the Gaussian closure is concerned) is the set defined by the original logistic map Eqn. [2.8].

Due to the linearity of the fluctuation equation (Eqn. [2.7]), we can immediately write the time dependent solution as

$$\sigma_n^2 = \mu^{2n} \prod_{k=0}^{n-1} (1 - 2M_k)^2 \sigma_{n=0}^2 \tag{2.11}$$

Provided the initial dispersion is non-zero and the amplification factor $\mu^2 (1 - 2M_k)^2$ is greater than one, the dispersion will continue to grow. It is interesting to note that the logarithm of Eqn [2.11], when time averaged, gives rise to the Lyapunov exponent for the logistic map. We are of course not calculating the exponent but rather we are calculating the maximum value of the dispersion at a particular time.This is the fluctuation enhancement mentioned earlier. It is important to realize that not only the dispersion grows but also all higher order moments. In this scheme where we have assumed Gaussian closure, it might seem that this is a consequence of higher order moments being related to the dispersion in a simple way. As we will see in the next section, the enhancement of all fluctuations over the same transient time interval is in fact independent of Gaussian closure and is a universal feature of the logistic map. A critical value of n (call it n_c) defined by when the dispersion reaches a maximum can be approximated by when the amplification factor is one. The mean value at n_c is then given by

$$M_{n_c} \sim \frac{1}{2}(1 - \frac{1}{\mu}) \tag{2.12}$$

We refrain from using an equal sign in the above equation. This is because while it is certainly true that the factors in Eqn. [2.11] being greater than one cause the dispersion to grow, the converse is not necessarily true. All the factors need not be greater than one for the dispersion to be large: just their product. Typically, we find that Eqn. [2.12] gives an excellent estimate of when fluctuation enhancement occurs. In the cases we have run, the exact value of n where the enhancement occurs is given by $n_c + 1$. This is because even though the amplification factor at $n_c + 1$ is less than one, its product with the amplification factors from all the previous iterates is still greater than one. Hence, Eqn. [2.12] gives a lower bound for n_c. We find that by the time iterate $n_c + 2$ is reached, the dispersion drops precipitously. Another technical point is that since n must be integral, M_n need not be exactly equal to $0.5\,(1 - 1/\mu)$. Instead, it will likely bracket this value. Hence Eqn. [2.12] tells us the approximate n value where fluctuation enhancement will occur.Unfortunately, Eqn. [2.12] can not be solved analytically except for $\mu = 2$. However, in the next section where we perform numerical calculations of the evolution of the moments exactly and via Gaussian closure, we will see that Eqn. [2.12] gives a reliable prediction of when the fluctuations reach a maximum. For $\mu = 2$, the solution to the mean equation is approximately just the solution without the fluctuations. This is because, in the stable regime of the map (as we will see in the next section), the fluctuations are only a perturbation to the mean dynamics. Therefore, our calculations may be considered first order in the fluctuations. The analytic solution is just

$$M_n \sim \frac{1}{2}\,(1 - exp\,(2^n ln\,(1 - 2M_{n=0})\,)\,) \tag{2.13}$$

where it is assumed that $M_{n=0} < 0.5$.Substituting Eqn. [2.13] into Eqn. [2.12], we obtain

$$n_c \sim \frac{1}{ln\,(2)} ln\,(\frac{(-ln2)}{ln\,(1 - 2M_{n=0})}) \tag{2.14}$$

This equation is analogous to the $ln\,(1/\delta)$ for the occurrence of fluctuation enhancement in the unstable anharmonic oscillator of section I. In the next section, we compare the predictions of Eqn. [2.14] with numerical results. An order of magnitude estimate yields $n_c \sim 5$ and 15 for $M_{n=0}$ equal to 10^{-2} and 10^{-5} respectively.Note that the critical value of n depends only on the initial mean and not the initial dispersion. This is a result of approximating the solution to the mean equation by ignoring the fluctuations. A more exact treatment would include the initial dispersion.

The magnitude of the enhanced fluctuation (that is the maximum amplitude of the dispersion) can now be estimated.First, we define the amplification exponent Λ_n by

$$ln\left(\left(\frac{\sigma_n}{\sigma_{n=0}}\right)^2\right) = \Lambda_n ln2 \tag{2.15}$$

We evaluate Eqn. [2.15] at n=n_c using Eqn. [2.14] and Eqn. [2.13]. We obtain

$$\Lambda_n \sim 2n_c + \frac{2ln\,(1-2M_{n=0})}{ln2}\,(2^{n_c}-1) \tag{2.16}$$

We can compute the amplification of the initial dispersion and find for $M_{n=0} \sim 0.01$ that σ_n reaches a maximum value of approximately 300 times its initial value. In Figs. [1] and [2] we show plots of n_c and Λ_{n_c} versus $M_{n=0}$. Care must be taken in reading amplification exponents off of Fig. [2]. At small initial mean values where the amplification is very large, it must be remembered that the width of the Gaussian must be small enough that it is effectively contained in the [0,1] interval (a quantitative description of this statement will be given in the next section). We will compare these theoretical results with our numerical experiments in the results section.

What about higher order moments and cumulants? Since we have assumed Gaussian closure, all cumulants greater than second order are zero and thus all moments greater than second order can be related simply to the mean and dispersion. We will see in the next section that by evaluating higher order cumulants numerically and locating any deviations from zero we will have a diagnostic tool for looking at the stability of the closure approximation.In conclusion, Eqns. [2.14] and [2.16] are examples of the Kubo fluctuation enhancement theorem applied to the logistic map.

In the large time limit (i.e $n \gg n_c$), the system approaches the fixed point. That is, all fluctuations go to zero and $M_n \rightarrow 0.5\,(1-1/\mu)$. Effectively, the initial distribution has contracted to a delta function centered at the fixed point.

B. Numerical

The numerical simulation of our problem is carried out in a straightforward manner. We generate independent Gaussian distributed random deviates using the Box-Muller method [23]. These deviates are characterized by their mean and standard deviation. For a given value of the control parameter μ the logistic map is iterated using these random numbers as initial conditions. Given that we are constrained not to exceed the edges of the unit interval, we always choose the mean and width of the Gaussian such that the mean is at least 5 standard deviations away from an endpoint. While in theory there is a non-zero probability of finding an initial point far from the mean, we never actually encounter this situation since the probability is vanishingly small. The map is iterated for 100 steps for each of 10^5 initial starting points. At every iterate, n, the averages of x_n^k are calculated for k=1,...,4. We then compute the first through fourth order cumulants. We note that the first order cumulant gives us the mean and the second order cumulant is simply the dispersion. The results obtained using a direct numerical simulation are then compared with those obtained by numerically solving the analytic Eqns. [2.4] and [2.5] in the preceding section. All computations were performed in double precision on an SGI workstation.

In Figs.[3a - d] - Figs.[4a - c], our numerical results for $\mu = 2$ are shown. The phenomenon of fluctuation enhancement is clearly visible. In the last section we showed that the van Kampen expansion predicted that the fluctuation enhancement should be a maximum when the mean is approximately one-half of the stable fixed point value, i.e.$M_c = 0.25$. The results presented in Figs.[3a - d] - Figs.[4a - c] are consistent with the analysis of section II A. Iterates zero through

five show a continually growing dispersion finally peaking at iterate six. Beyond iterate six, the dispersion drops rapidly and eventually reaches zero. The maximum value of the enhancement factor for an initial mean of 0.01 and an initial dispersion of $\sigma_0^2 = 4 \times 10^{-6}$ is approximately 300. For an initial mean of 10^{-5} and an initial dispersion, $\sigma_0^2 = 10^{-14}$ the maximum value of the enhancement factor is approximately 3×10^8. These results are in excellent agreement with the theoretical predictions. In the latter case the critical iterate is pushed out to approximately 15. The third order and fourth order cumulants for the above initial conditions are presented in Figs.[3c - d] and [4c]. Both cumulants are zero everywhere except for a transient regime centered about n_c. As advertised in section II A, the enhancement of fluctuations occurs not only for the dispersion but also for higher order cumulants and hence is a universal feature of the logistic equation and the specific stochastic initial conditions. Figures [5a - d] - [6a - c] show our numerical results for $\mu = 2.5$. Unlike, the above case, no analytic solution to the location and size of the fluctuation enhancement can be obtained solely in terms of the initial mean. This is because no known analytic solution exists for the value of the nth iterate of the mean of the logistic map with $\mu = 2.5$.The best prediction that can be made for when fluctuation enhancement occurs is given by where the mean has reached one - half of its fixed point value. Figures [5a - d] - [6a - c] for dispersion, third and fourth order cumulants all show fluctuation enhancement at values of n_c equal to five and twelve for $M_{n=0} = 0.01$, $\sigma_{n=0}^2 = 4 \times 10^{-6}$ and $M_{n=0} = 10^{-7}$, $\sigma_{n=0}^2 = 10^{-14}$ respectively. Both values for n_c are consistent with Eqn. [2.12].

Instead of looking at the time evolution of the moments, an alternative approach to investigating the dynamics of the logistic map with stochastic initial conditions is to look at the time evolving probability distribution $P_n(x)$ defined by

$$P_n(x) = \langle \delta(x - x_n) \rangle = \frac{1}{N_c} \sum_{i=1}^{N_c} \delta(x - x_n(i)) \tag{2.17}$$

Where as usual, the average is sampled over the number of realizations N_c. For clarity, we have explicitly inserted the realization index i in x_n. In performing the numerical evaluation of Eqn. [2.17] we approximate the delta function by a rectangle of width a and height 1/a. The interval between zero and one is partitioned into 1000 equal size bins. At each "time" step we store the values of x_n appropriate for a particular bin and average over 100,000 initial starting points. The probability is normalized to one at each "time" step.

In Fig. [7] we present the probability distribution as a function of "time" for $\mu = 2$ and $M_0 = 10^{-2}$ and $\sigma_0^2 = 4 \times 10^{-6}$. We notice that the distribution is attracted toward the stable fixed point at x=1/2. During the first several iterates the distribution spreads and maintains its Gaussian shape. An asymmetry, i.e. a deviation from Gaussian behavior, is very noticeable at iterate six where the dispersion and higher order cumulants have reached their maximum value. After iterate six, the distribution begins to narrow and a gross asymmetry appears as a result of members of the ensemble having reached the stable fixed point. By iterate nine, the distribution has become, for all intents and purposes, a delta function centered at x=1/2. The results shown in Fig. [8] for $\mu = 2.5$ with the same initial conditions are similar except, of course, for the fact that the distribution function is attracted to a fixed point value of 0.6. The numerical simulation of the evolving probability distribution function for the other initial conditions, $M_0 = 10^{-5}$ and $\sigma_0^2 = 10^{-14}$, are also consistent with the moment calculations in section II A.

C. Discussion

What are the implications of the theoretical and numerical results presented in sections II A and B for statistical closure and clump kinetics? In so far as the statistical closure problem is concerned, it is apparent that the Gaussian closure approximation of the logistic map does an excellent job of describing the dynamics provided we are approaching a stable fixed point. The only deviation from Gaussian that appears is in the transient regime defined by where fluctuation enhancement occurs. The importance of the fluctuation enhancement regime lies in the observation that it is the time regime where the Gaussian closure approximation is the weakest, as evidenced by the deviation of third and fourth order cumulants from zero. By comparing the relative magnitudes of the first through fourth order cumulants, it is clear that the Gaussian closure of our stochastic logistic map satisfies the definition of stability as discussed in the Introduction. It is only in the transient fluctuation enhanced regime that any deviations occur and these deviations are small relative to the magnitude of the mean. Hence the van Kampen expansion gives an adequate description of the mean and fluctuations about that mean for the logistic map with Gaussian distributed initial conditions. In conclusion, it can be stated that the time evolution or relaxation of a system whose dynamics are governed by the logistic map possesses three distinct time regimes. An initial regime ($n \ll n_c$) where the Gaussian closure approximation is almost exact, an intermediate transient regime ($n \sim n_c$) where all cumulants greater than first order grow and reach their maximum value. It is here that the Gaussian closure is only approximately valid. Finally, a final regime ($n \gg n_c$) where all cumulants attain their stationary values. The Gaussian closure is almost exact in this regime but almost in a trivial way since the distribution function goes to a delta function (all cumulants go to zero except for the mean).The multidimensional map defined by Eqns. [2.4] and [2.5] collapses to the standard one dimensional logistic map in the long time limit.

What do the above results mean for clump kinetics? If the time evolving probability distribution plots are interpreted as time evolving particle densities confined to the [0,1] interval it is clear what is going on. The initial conditions specify a tightly packed group of non-interacting point particles Gaussian distributed about some mean value. In the initial regime, this clump of particles gradually spreads out and drifts towards the stable fixed point. In the transient fluctuation enhanced regime, the clump has become somewhat non-Gaussian like and has spread to its maximum extent. Finally, as the clump nears the fixed point, the particle distribution recollapses, forming a tight clump that eventually arrives at the stable fixed point.

III. Statistical Closure and Chaos

A. Theory

In this section we study the approach to steady state of a stochastic system defined by the logistic map (with $\mu = 4$) whose initial conditions are drawn from a narrow Gaussian centered at

or near either one of two unstable fixed points (i.e $x_{n=0} = 0, 3/4$).This situation is similar to the approach to equilibrium from an initially unstable state for the anharmonic oscillator that was discussed in the introduction. Qualitatively, any closure scheme based on a Gaussian distribution is doomed to failure. We base this on two observations. First, the Lyapunov exponent is positive and hence any two initially nearby trajectories will diverge. The type of behavior shown in section II where the trajectories of nearby particles start to diverge but then converge on the fixed point is not present in the chaotic regime of the logistic map. Second, the steady state distribution also known as the invariant distribution has been solved analytically for this case [24] and is given by

$$P_{n \to \infty} = \frac{1}{\pi \sqrt{x(1-x)}} \tag{3.1}$$

This is the well known solution to the Perron-Frobenius equation [24]. Because we expect the Gaussian closure scheme to breakdown and hence invalidate the van Kampen expansion, the questions we wish to address are in what regime is the Gaussian closure valid, when does it breakdown and does there exist a better closure model.

The moment equations based on a Gaussian closure for the logistic map are simply given by Eqns. [2.4] and [2.5] with $\mu = 4$. Likewise, the van Kampen expansion for the Logistic map yields Eqns. [2.6] and [2.7] with $\mu = 4$. A word should be mentioned concerning the van Kampen versus Gaussian closure equations for this map. As we saw in section IIA and B, the differences between the two schemes was slight due to the fact that the fluctuations remained small for all "time". However, in the chaotic regime of the logistic map we would not expect the fluctuations to remain small due to the positive Lyapunov exponent. Hence we would expect differences between van Kampen and Gaussian closure. In fact, as we will see in the next section, the Gaussian closure yields an instability in the sense that it leads to an iterate outside of the [0,1] interval. Detailed descriptions of the breakdown of the van Kampen and Gaussian closure approximations for the chaotic logistic map will be described in the numerical section that follows. The philosophy of this section is that the Gaussian closure represents a two variable parametrization of the evolving distribution function based on the initial distribution. What if we chose a closure scheme based on parametrizing the steady state distribution? We choose to base a closure scheme on a two variable parametrization of the invariant distribution given by

$$P_n(x) = \frac{x^{p_n - 1}(1-x)^{q_n - 1}}{B(p_n, q_n)} \tag{3.2}$$

This function, called the Beta distribution, has several interesting properties. It is normalized over the [0,1] interval. The mean and dispersion of $P_n(x)$ are related to p and q by

$$M_n = \frac{p_n}{p_n + q_n} \qquad \sigma_n^2 = \frac{p_n q_n}{(1 + p_n + q_n)(p_n + q_n)^2} \tag{3.3}$$

As p and q approach 1/2, the steady state distribution is realized. For p, q >> 1/2,the distribution

looks Gaussian (see Fig. [9a - b]). In fact, as p and q get large, all cumulants greater than second order go to zero. Like the Gaussian distribution, $P_n(x)$ is parametrized by two variables. Hence we would not expect it to be an exact solution to the Perron-Frobenius equation and therefore Eqn. [3.2] would not be expected to be accurate for all cumulants for all time. However, Eqn. [3.2] should be a much better description of the dynamics for late time. In addition, because it looks Gaussian for large p and q, we would expect Eqn. [3.2] to give an approximate description of the dynamics in the initial and transition regions.

Assuming the Beta distribution accurately describes the stochastic evolution of the logistic map for $\mu = 4$ with Gaussian distributed initial conditions, the Nth order moment at iterate n is simply

$$\langle x_n^N \rangle = \frac{B(N+p_n, q_n)}{B(p_n, q_n)} \tag{3.4}$$

From Eqn. [3.3] we have

$$p_n = \frac{M^2{}_n - M_n^3 - M_n\sigma_n^2}{\sigma^2{}_n} \qquad q_n = \frac{(1-M_n)(M_n - M_n^2 - \sigma_n^2)}{\sigma_n^2} \tag{3.5}$$

From the recursive formula for the Gamma function (i.e. $\Gamma(x+1) = x\Gamma(x)$) we obtain the following closure relation based on the Beta distribution

$$\langle x_n^N \rangle = \frac{(N-1+p_n)}{(N-1+p_n+q_n)} \langle x_n^{N-1} \rangle \tag{3.6}$$

This forms the basis for what we call the Beta closure approximation. It is now straightforward to obtain a set of equations describing the evolution of the first and second order cumulants based on the Beta closure scheme (Eqn. [3.6]). For simplicity, the coefficient in Eqn. [3.6] is written as $C_N(M_n, \sigma_n^2)$. Hence,

$$\langle x_n^N \rangle = C_N C_{N-1} C_{N-2} \ldots \langle x_n^2 \rangle \tag{3.7}$$

The evolution equations for the mean and the dispersion are given by

$$M_{n+1} = \mu M_n (1 - M_n) - \mu\sigma_n^2$$

$$\sigma_{n+1}^2 = \mu^2 (1 - 2C_3(M_n, \sigma_n^2) + C_4(M_n, \sigma_n^2) C_3(M_n, \sigma_n^2)) (\sigma_n^2 + M_n^2)$$
$$- (M_n(1 - M_n) - \sigma_n^2)^2 \mu^2 \tag{3.8}$$

The above equations constitute a two parameter description of the stochastic behavior of the logistic map under study. It is important to realize that, unlike the Gaussian closure approximation discussed earlier, higher order cumulants within the Beta closure approximation are not zero. Because of Eqn. [3.7], all cumulants are defined in terms of the mean and dispersion and their dynamics is therefore known once the evolution of the mean and dispersion is known. Equation [3.8] is an admittedly more complicated set of equations than those encountered earlier (Eqn. [2.4] - [2.5]), but as we shall see, it gives a remarkably accurate description of the stochastic behavior of the logistic map.

B. Numerical

The numerical procedure used to simulate the chaotic logistic map with stochastic initial conditions, is exactly the same as described in section II B. The only exception is that now $\mu = 4$. We again study the evolution of the map with a Gaussian distributed set of initial conditions initially centered near one of the unstable fixed points (in this case, 0 and 3/4). We present in this section a comparison of the results of Eqns. [2.4] - [2.5], Eqns. [2.6] - [2.7] and Eqns. [3.8] with exact simulation for initial conditions given by $M_{n=0} = 10^{-5}$ $\sigma_{n=0}^2 = 10^{-14}$ as well as for $M_{n=0} = 0.75$ $\sigma_{n=0}^2 = 4 \times 10^{-6}$. Other sets of initial values for the mean and dispersion were all run with initial means near the unstable fixed points but with varying dispersions. The results we present here offer a representative sample.

Figures [10a - d] show the mean as a function of iterate for the initial parameters $M_{n=0} = 10^{-5}$ and $\sigma_{n=0}^2 = 10^{-14}$ as calculated both by the Beta distribution, Gaussian closure, van Kampen expansion and direct simulation. To begin with, the Gaussian closure equations are unstable (se Figs. [11a - b] for all the initial conditions we have looked at in the sense that eventually the iterate lies outside of the [0,1] interval. It is clear that the van Kampen expansion and direct simulation agree only initially. In particular, whereas direct simulation shows a definite steady state, van Kampen looks random. We can trace this observation to the fact that the fluctuations over the course of time no longer remain a small perturbation to the evolution of the mean (see Fig [12]), hence invalidating the linearization assumption used in deriving Eqn. [2.7]. Finally, the Beta closure approximation yields very accurate results for the mean and dispersion over the whole time interval. In particular, the correct steady state is reached. It is evident from the direct simulations that there are three distinct time regimes. An initial regime where the cumulants slowly grow or remain small, a transient regime where the cumulants increase rapidly and a final regime (by about iterate twenty) where the cumulants reach a steady state value. In the case of the third order cumulant it is zero while for the fourth order cumulant it is -0.023. Both values are consistent with the invariant distribution. As was seen in the Gaussian closure approximation in section II, the Beta closure approximation is weakest (for higher order cumulants) in the transient regime.

Figures [13a - d] show the mean as a function of iterate for the initial parameters $M_{n=0} = 0.75$ and $\sigma_{n=0}^2 = 10^{-10}$ as calculated via the van Kampen expansion, Beta closure and

numerical simulation. Again, we see the excellent agreement between Beta closure and direct simulation for the mean and dispersion whereas the higher order cumulants have problems in the intermediate transient regime. In addition the van Kampen expansion handles the initial regime correctly but breaksdown in the intermediate regime and looks random in the final regime. Figures [14a - d] show the mean as a function of iterate for the initial parameters $M_{n=0} = 0.75$ and $\sigma^2_{n=0} = 4 \times 10^{-6}$. The results presented are consistent with our previous calculations. Note the fact that the intermediate transient regime where the linearized Gaussian closure scheme (van Kampen) breaks down occurs earlier for larger initial dispersion.

The evolution of the exact time dependent probability distribution beautifully illustrates the approach to steady state. While this set of initial conditions would require fine binning of the unit interval, we can obtain qualitatively similar results with initial conditions $M_{n=0} = 0.01$ and $\sigma^2_{n=0} = 4 \times 10^{-6}$. The probability distribution obtained from this particular experiment clearly shows the initial narrow Gaussian distribution spreading in time as more of the phase space of the map is explored. Eventually, the distribution loses its Gaussian character as the iterates accumulate at the ends of the interval. Finally, the invariant distribution is attained as steady state has been reached. Since we have predictions for the mean and dispersion based on the Beta closure approximation as a function of time, it is possible it compute numerically p_n, q_n and get a prediction for the Beta probability distribution Eqn. [3.2]. A comparison can then be made between the probability distribution based on direct simulation versus Beta closure. Figure [15] shows the result of that comparison. Note the excellent agreement (within statistical errors) for all time.

Suzuki [19] has analyzed theoretically the behavior of the relative diffusion (in our case the dispersion) for the logistic map ($\mu = 4$) using his scaling theory. We wish to compare our numerical results with his predictions. As mentioned in the Introduction, scaling theory predicts that the dispersion should behave in the initial and transient regimes like

$$\sigma^2_n \sim \frac{(1 - \cos(2^n \delta))}{2} \tag{3.9}$$

In Fig. [16] we show comparisons between the experimental results for the dispersion versus the scaling theory prediction. The results show good qualitative agreement in the initial and transient regimes where the scaling theory applies. The agreement is completely wrong in the final steady state regime. However, this is to be expected since the theory is not relevant in the steady state. Of particular interest is the fact that the time at which the transient regime occurs and hence when the Gaussian and van Kampen expansions are breaking down is predicted to be where the scaling function is order one. That is

$$n_c \sim \frac{-ln(\delta)}{ln(2)} \tag{3.10}$$

An estimate based on Eqn. [3.10] yields times of 7 for an initial dispersion of 4×10^{-6} and 15 for an initial dispersion of 10^{-10}. Both agree with the experimental results. This expression is analogous to Eqn. [2.14] for the logistic map with a stable fixed point in that they both signify rapid growth of the initial fluctuation and hence identify the transient or fluctuation enhanced regime.

C. Discussion

Again, we ask, what are the implications of the above theoretical and numerical results for statistical closure and clump kinetics? It is obvious that the Gaussian statistical closure scheme (both in the full and linear sense) is by our definition unstable. The evolution of the logistic map with stochastic initial conditions for $\mu = 4$ can be characterized by three time regimes.The linear Gaussian closure gives the approximate early "time" ($n \ll n_c$) evolution of the logistic map but breaksdown after a certain number of iterations. The breakdown occurs when the fluctuations rapidly grow and become comparable to the mean. This is to be contrasted with the approach to a fixed point where the fluctuations grow but remain perturbative to the mean. This is the transient regime identified by Suzuki scaling theory ($n \sim n_c$). Unlike the approach to a stable fixed point, the transient regime is the first indication of a complete breakdown of the Gaussian closure scheme for all time. Finally, the stochastic system enters a steady state regime characterized by the invariant distribution ($n \gg n_c$). Interestingly enough, by basing a closure scheme on the steady state distribution instead of the initial distribution, we have been able to come up with a closure scheme, albeit not perfect, that gives very accurate results for the mean and dispersion over all three time regimes. In addition, higher order cumulants are calculated properly in the initial and final time regimes. However, any closure scheme based on a two variable parameter set is not going to get everything right and the evidence for this is the differences between theory and experiment in the transient regime of the higher order cumulants. It is important to realize that what we have introduced, is a different type of closure scheme than is typically encountered. The Beta closure scheme is non-perturbative in the sense that it assumes nothing about the relative magnitudes of higher to lower order moments. Instead, it only assumes that they can be related to the mean and dispersion in a straight forward way. In turbulence modelling for example, one usually states a priori that any cumulant beyond a certain order is zero. In a sense, this is a perturbative approach to closure where one assumes that higher order cumulants have less of an importance than lower order cumulants. This may be a particularly dangerous thing to do if the stochastic system under study is in some sense far from equilibrium. For example, in the stochastic anharmonic oscillator discussed in the Introduction, as the initial state gets closer to the instability point, the notion of fluctuations and higher order cumulants being a perturbation to the mean evolution becomes less and less accurate.For the logistic map, it is only when the fixed points are stable that the higher order cumulants behave as a perturbation to the mean evolution and hence preserve the closure scheme. In the chaotic regime, this statement is no longer accurate and we have come up with a closure scheme that successfully interpolates the cumulants between the initial and final time regimes. Whether or not these results have direct bearing on turbulence modelling is at present unknown. However, the idea of using closure schemes based on ignoring the effects of higher order cumulants especially when a stochastic system is far from equilibrium should be used with caution.

In so far as the dynamic evolution of a clump of Gaussian distributed particles is concerned, the above results can be easily interpreted. The three time regimes are characterized by the clump behaving initially like a coherent object whose dispersion or width remains small while its mean evolves according to the logistic map without fluctuations. The second time regime is characterized by the particles in the clump diverging exponentially from each other. This is the transient regime described by the Suzuki scaling theory. Finally, the clump distribution settles to a final state defined by its filling the [0,1] interval with accumulations near the end points.

IV. Conclusion

We have considered the validity of Gaussian and Beta closure schemes as a function of time for the logistic map with stochastic initial conditions. This was done by comparing cumulants and probability distributions generated by direct simulation with predictions based on evolution equations for moments derived by assuming a certain closure approximation. All initial conditions assumed a Gaussian distribution with a mean value near an unstable fixed point and a small dispersion. The control parameter for the logistic map was chosen to be in either one of two regimes: $\mu < 3$ in which case there exists a stable attracting fixed point or $\mu = 4$ which is fully chaotic. We found for either regime, that the system exhibited behavior characterized by three different time periods (called initial, transient and final). The initial time interval was characterized by the Gaussian closure being very accurate and the evolution of the system dominated by the evolution of the mean. The transient time interval was characterized by rapid growth in all cumulants. For $\mu < 3$, the Gaussian closure approximation broke down (but only weakly, as the growth of the cumulants remained perturbative to the mean evolution). We identified this transient period as a manifestation of the Kubo fluctuation enhancement theorem. The amount of enhancement and the location of the transient period was computed both theoretically and numerically. During the transient period for $\mu = 4$, the Gaussian closure approximation broke down with rapid growth of all cumulants. Unlike the $\mu < 3$ case however, the fluctuations did not remain perturbative to the evolution of the mean. We identified this period as the Suzuki scaling regime. We compared scaling theory with our experimental results and found good agreement. The transient period eventually evolved into the final equilibrium state. For $\mu < 3$, this was identified by the fluctuations decaying away and the Gaussian closure approximation becoming accurate again. For $\mu = 4$, a steady state was reached which was non-Gaussian. In order to describe the stochastic evolution of the logistic map for $\mu = 4$ using a closure scheme we introduced a closure based on the Beta distribution (basically a time dependent parametrization of the invariant distribution). We found that the evolution equations for the mean and dispersion based on a Beta distribution closure gave very accurate predictions over all three time periods. This new type of closure assumes nothing about the vanishing of higher order cumulants (in fact cumulants of all order are non-vanishing).

The relevance of these results to clump kinetics was also addressed. Similar to the results of Misguich and Balescu [18], Suzuki [19], and Grossman and Procaccia [20] we found that the clump dispersion exhibited first a coherent behavior followed by a rapid growth of the width of the clump. Quantitative estimates were made of when this rapid growth occurred. If we were in a regime of the control parameter that possessed a stable fixed point, then the maximum width of the clump never washed out the Gaussian shape (the clump as an entity was easy to identify) and we could predict the time or iterate of this maximum growth by using the van Kampen expanded stochastic equations. Numerical simulations confirmed our results. This was considered an example of the Kubo fluctuation enhancement theorem. For the chaotic regime, the dispersion grew to become comparable to the mean and the clump disintegrated. Quantitative estimates of when this growth occurred have been shown numerically to be consistent with Suzuki scaling theory. Finally, the system in both regimes settled to an equilibrium. As expected, for $\mu < 3$, the state was characterized by all particles accumulating at the fixed point. For $\mu = 4$, the equilibrium or steady state was characterized by the invariant distribution.

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Figure Captions

Fig. 1. Value of the iterate at which the fluctuations are maximum vs. the initial mean: equation 2.14.

Fig. 2. Amplification factor vs. the initial mean: equation 2.16.

Fig. 3. $\mu = 2$, $M_{n=0} = 10^{-2}$, $\sigma^2_{n=0} = 4 \times 10^{-6}$ (a) mean vs. iterate (b) dispersion vs. iterate. Circles represent the results of simulation, crosses represent the analytic results of equations 2.6 and 2.7. Lines are a guide to the eye. (c) Numerical simulation of the third order cumulant (d) Numerical simulation of the fourth order cumulant. Both (c) and (d) are identically zero in the analytic theory.

Fig. 4. $\mu = 2$, $M_{n=0} = 10^{-5}$, $\sigma^2_{n=0} = 1 \times 10^{-14}$ (a) mean vs. iterate (b) dispersion vs. iterate. Circles represent the results of simulation, crosses represent the analytic results of equations 2.6 and 2.7. Lines are a guide to the eye. (c) Numerical simulation of the third order cumulant.

Fig. 5. $\mu = 2.5$, $M_{n=0} = 10^{-2}$, $\sigma^2_{n=0} = 4 \times 10^{-6}$ (a) mean vs. iterate (b) dispersion vs. iterate. Circles represent the results of simulation, crosses represent the analytic results of equations 2.6 and 2.7. Lines are a guide to the eye. (c) Numerical simulation of the third order cumulant (d) Numerical simulation of the fourth order cumulant.

Fig. 6. $\mu = 2.5$, $M_{n=0} = 10^{-5}$, $\sigma^2_{n=0} = 1 \times 10^{-14}$ (a) mean vs. iterate (b) dispersion vs. iterate. Circles represent the results of simulation, crosses represent the analytic results of equations 2.6 and 2.7. Lines are a guide to the eye. (c) Numerical simulation of the third order cumulant.

Fig. 7. Probability distribution $P_n(x)$ vs. x for $\mu = 2$, $M_{n=0} = 10^{-2}$, $\sigma^2_{n=0} = 4 \times 10^{-6}$, equation 2.17. (A) iterate 2 (B) iterate 3 (C) iterate 4 (D) iterate 5 (E) iterate 6 (F) iterate 7 (G) iterate 8.

Fig. 8. Probability distribution $P_n(x)$ vs. x for $\mu = 2.5$, $M_{n=0} = 10^{-2}$, $\sigma^2_{n=0} = 4 \times 10^{-6}$, equation 2.17. (A) iterate 1 (B) iterate 2 (C) iterate 3 (D) iterate 4 (E) iterate 5 (F) iterate 6 (G) iterate 7.

Fig. 9. (a)Third order cumulant as a function of p and q for the Beta distribution. (b) Fourth order cumulant as a function of p and q for the Beta distribution.

Fig. 10. $\mu = 4$ $M_{n=0} = 10^{-5}$, $\sigma^2_{n=0} = 10^{-14}$, (a) mean vs. iterate (b) dispersion vs. iterate (c) third order cumulant (d) fourth order cumulant. In (c) and (d) the circles represent the results of numerical simulation and crosses represent the results of the Beta closure equations. The lines are a guide to the eye.

Fig. 11. Breakdown of the Gaussian closure. (a) mean vs. iterate: circles represent numerical simulation, crosses represent results of equations 2.4 and 2.5. The lines are a guide to the eye. (b) dispersion vs. iterate. All else is the same as in (a). The results of the Gaussian approximation lie outside the interval [0,1] and hence go off of the scales shown in the plots.

Fig. 12. Comparison of the two terms in equation 2.5 for the chaotic logistic map. The point at which these curves become equal indicates where the breakdown of the van Kampen expansion occurs.

Fig. 13. $\mu = 4$ $M_{n=0} = 0.75$, $\sigma^2_{n=0} = 10^{-10}$, (a) mean vs. iterate (b) dispersion vs. iterate (c) third order cumulant vs. iterate (d) fourth order cumulant vs. iterate. In (c) and (d) the circles represent the results of numerical simulation and crosses represent the results of the Beta closure equations. The lines are a guide to the eye.

Fig. 14. $\mu = 4$ $M_{n=0} = 0.75$, $\sigma^2_{n=0} = 4x10^{-6}$, (a) mean vs. iterate (b) dispersion vs. iterate (c) third order cumulant vs. iterate (d) fourth order cumulant vs. iterate. In (c) and (d) the circles represent the results of numerical simulation and crosses represent the results of the Beta closure equations.The lines are a guide to the eye.

Fig. 15 Probability distribution for $\mu = 4$ $M_{n=0} = 10^{-2}$, $\sigma^2_{n=0} = 4 \times 10^{-6}$: numerical simulation vs. Beta function for p and q obtained analytically (a) iterate 2 (b) iterate 4 (c) iterate 5 (d) iterate 15.

Fig. 16. Dispersion vs. iterate for various means and an initial dispersion of $4x10^{-6}$. The result of the Suzuki scaling formula is shown for comparison.

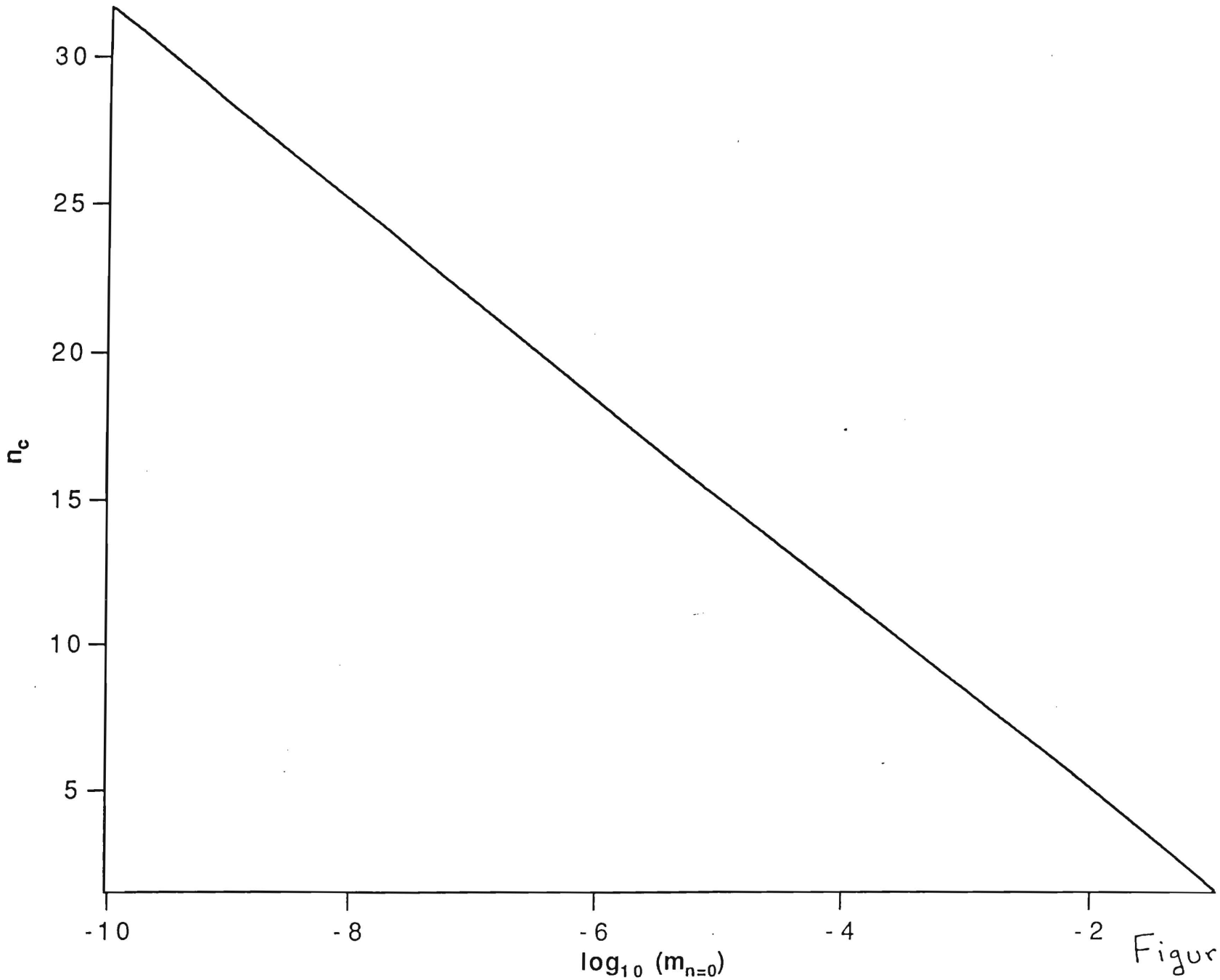


Figure 1

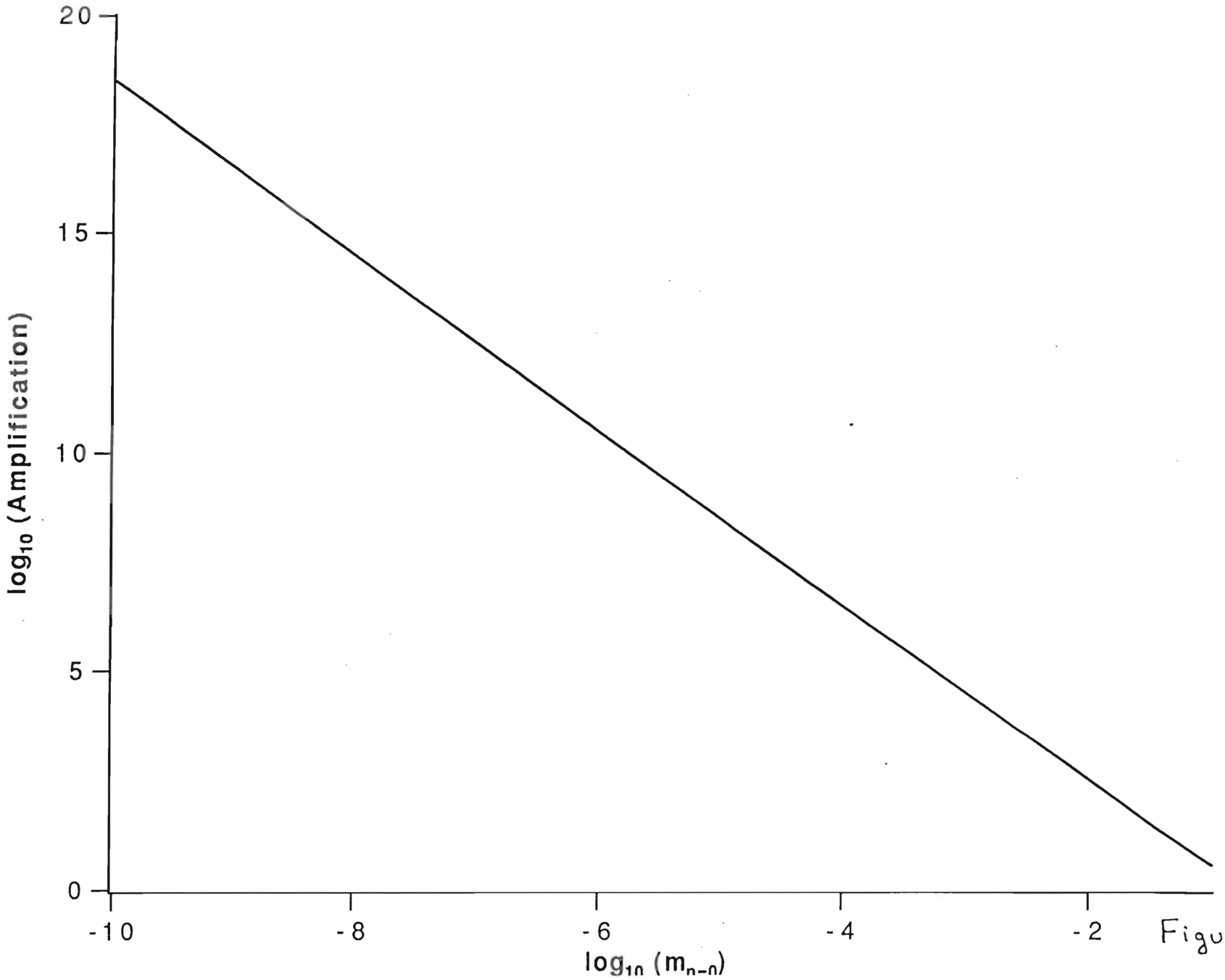


Figure 2

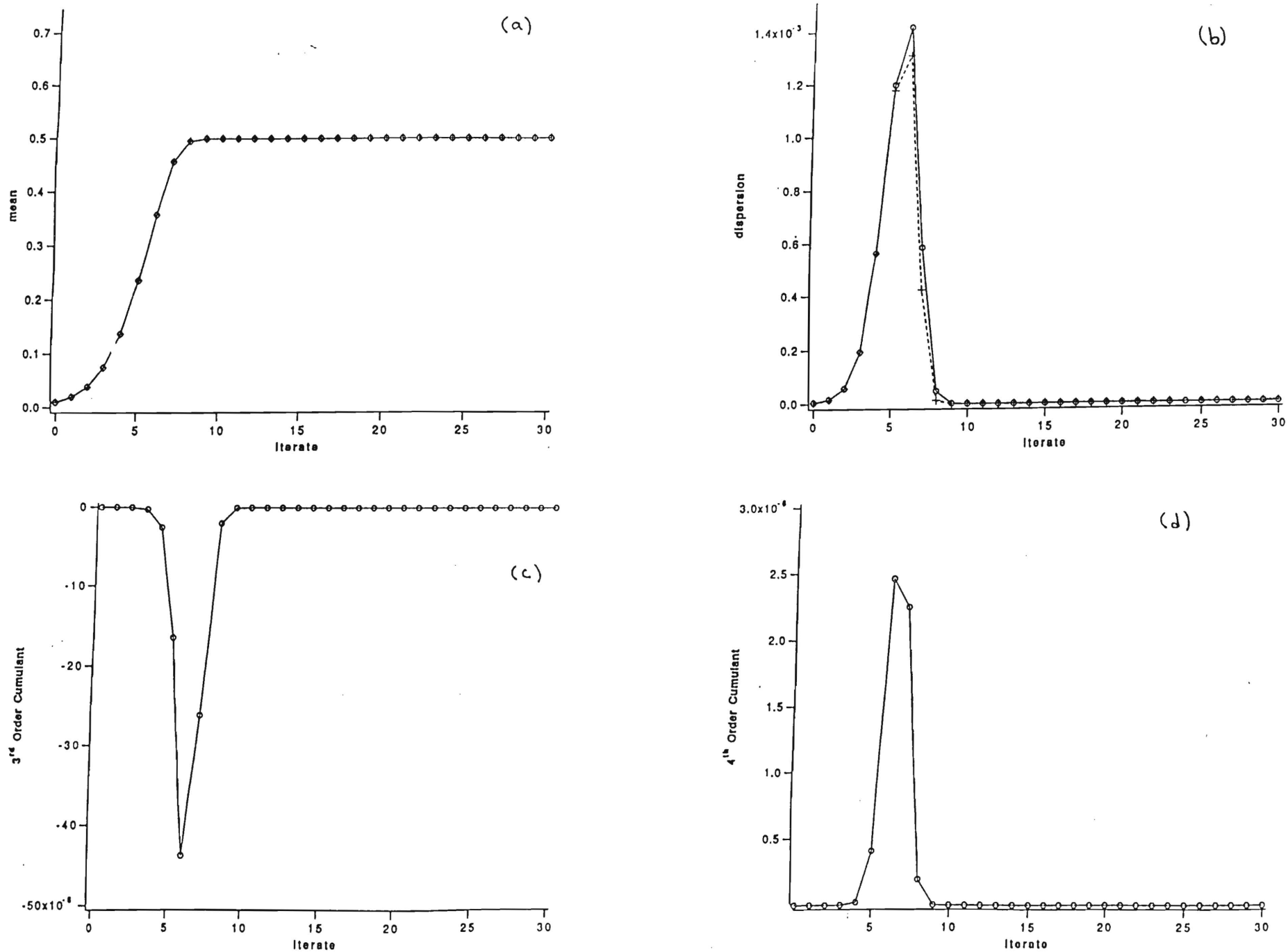

(a)
mean
Iterate
(b)
dispersion
Iterate
(c)
3rd Order Cumulant
Iterate
(d)
4th Order Cumulant
Iterate

Figure 3

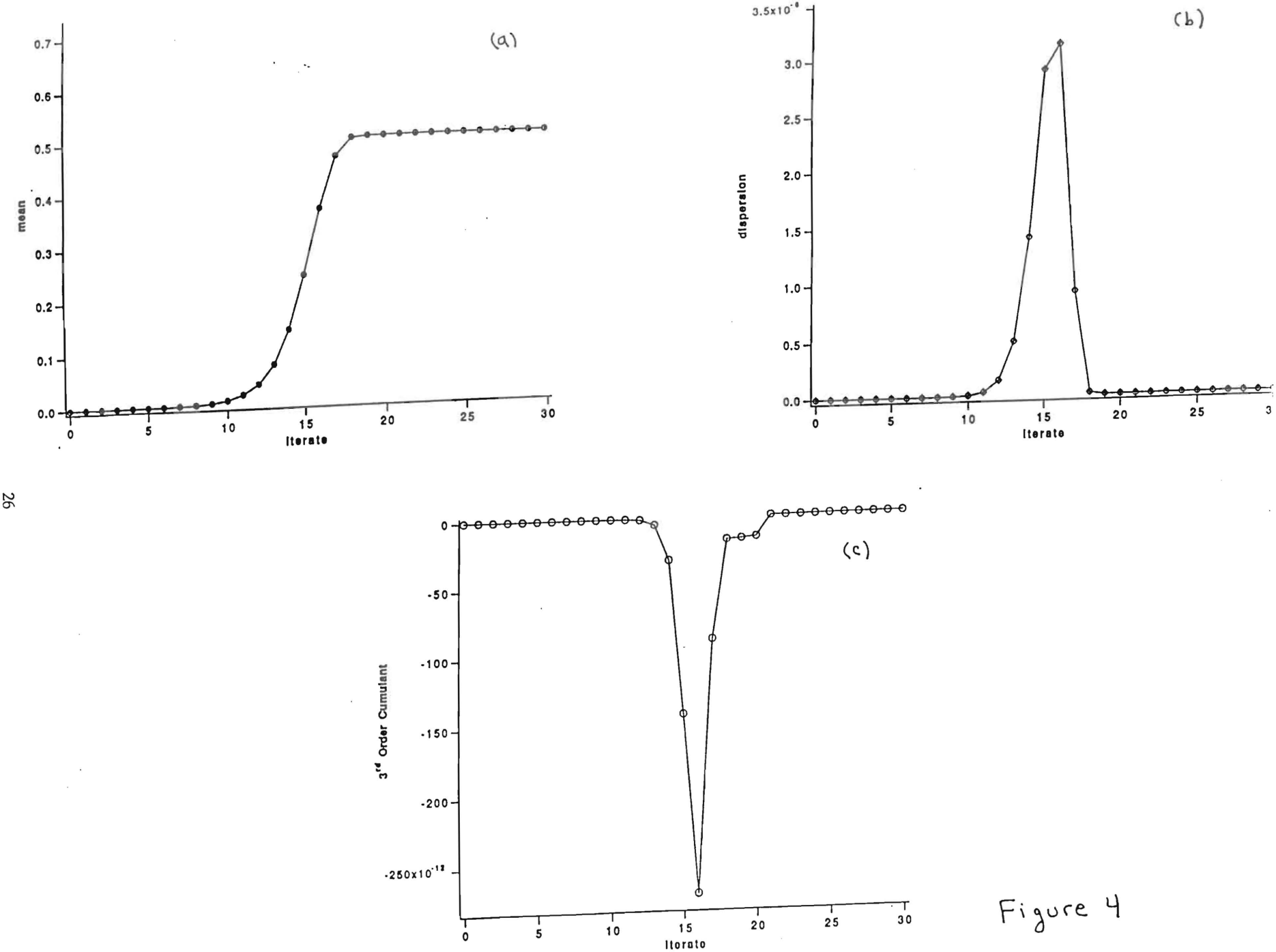

(a)
mean
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0
5
10
15
20
25
30
Iterate
(b)
dispersion
3.5x10⁻⁸
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0
5
10
15
20
25
Iterate
(c)
3rd Order Cumulant
0
-50
-100
-150
-200
-250x10⁻¹²
0
5
10
15
20
25
30
Iterate

Figure 4

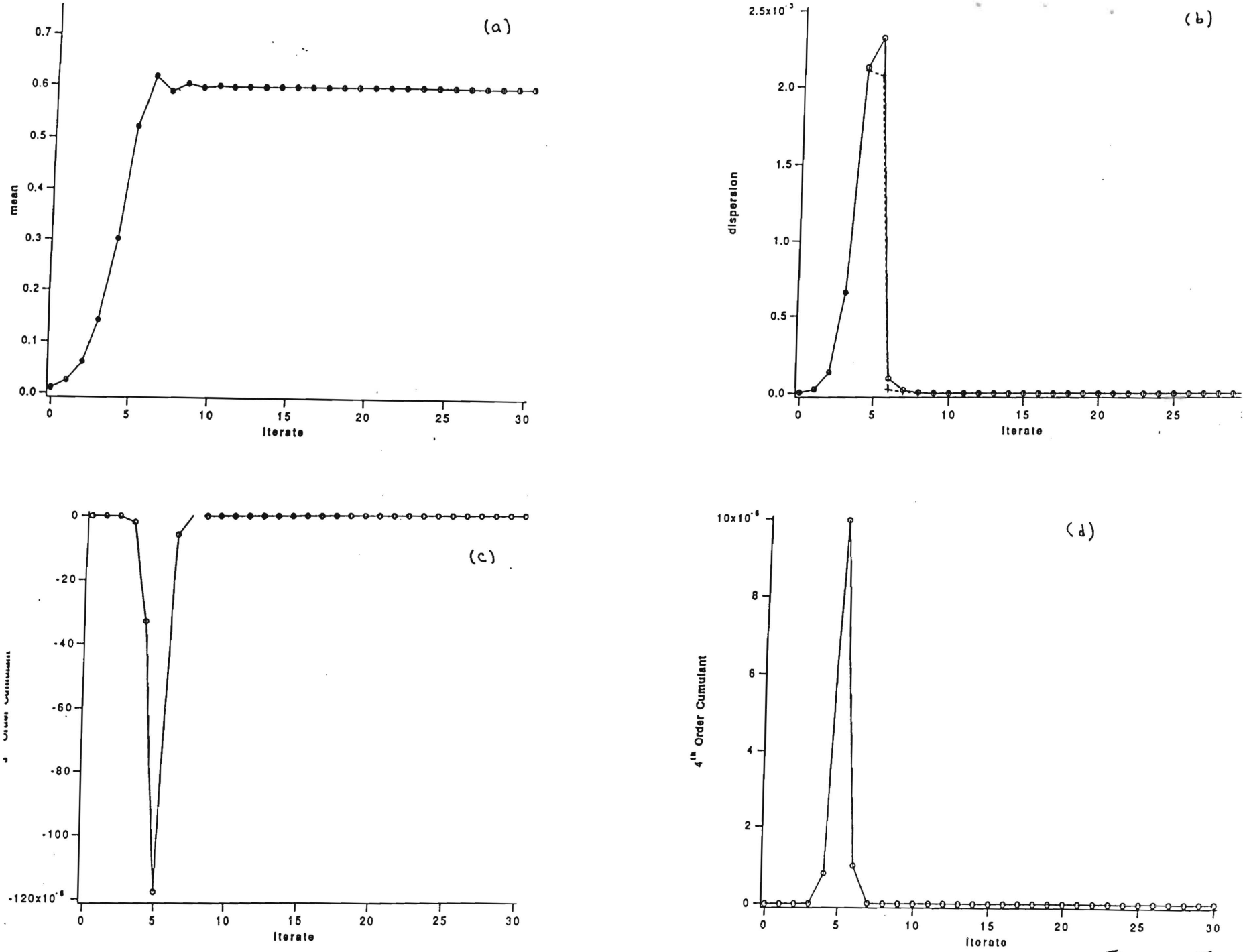

(a)
mean
Iterate
(b)
dispersion
Iterate
(c)
Iterate
(d)
4th Order Cumulant
Iterate

Figure 5

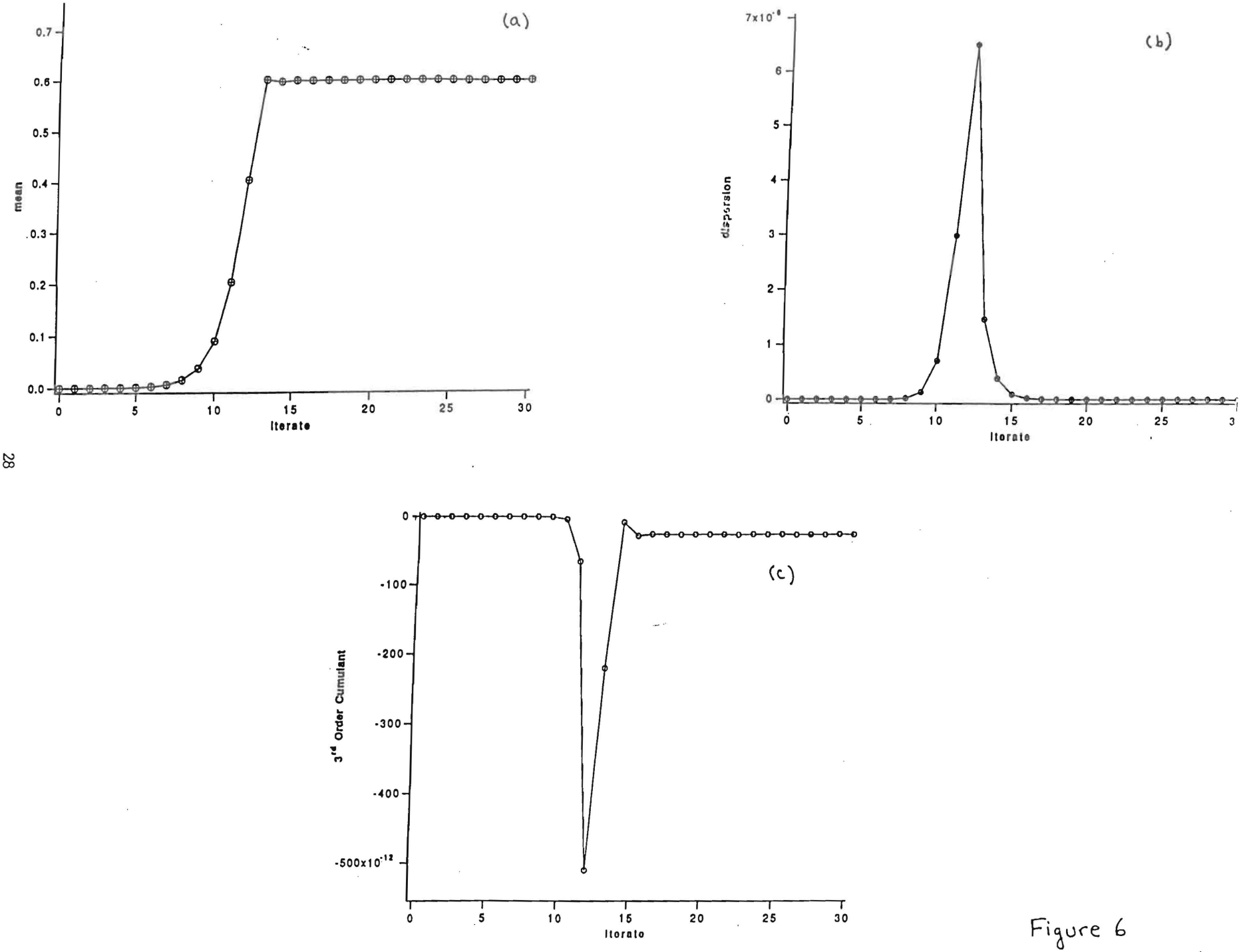

(a)
mean
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0
5
10
15
20
25
30
Iterate
(b)
dispersion
0
1
2
3
4
5
6
7x10^-6
0
5
10
15
20
25
30
Iterate
(c)
3rd Order Cumulant
0
-100
-200
-300
-400
-500x10^-12
0
5
10
15
20
25
30
Iterate

Figure 6

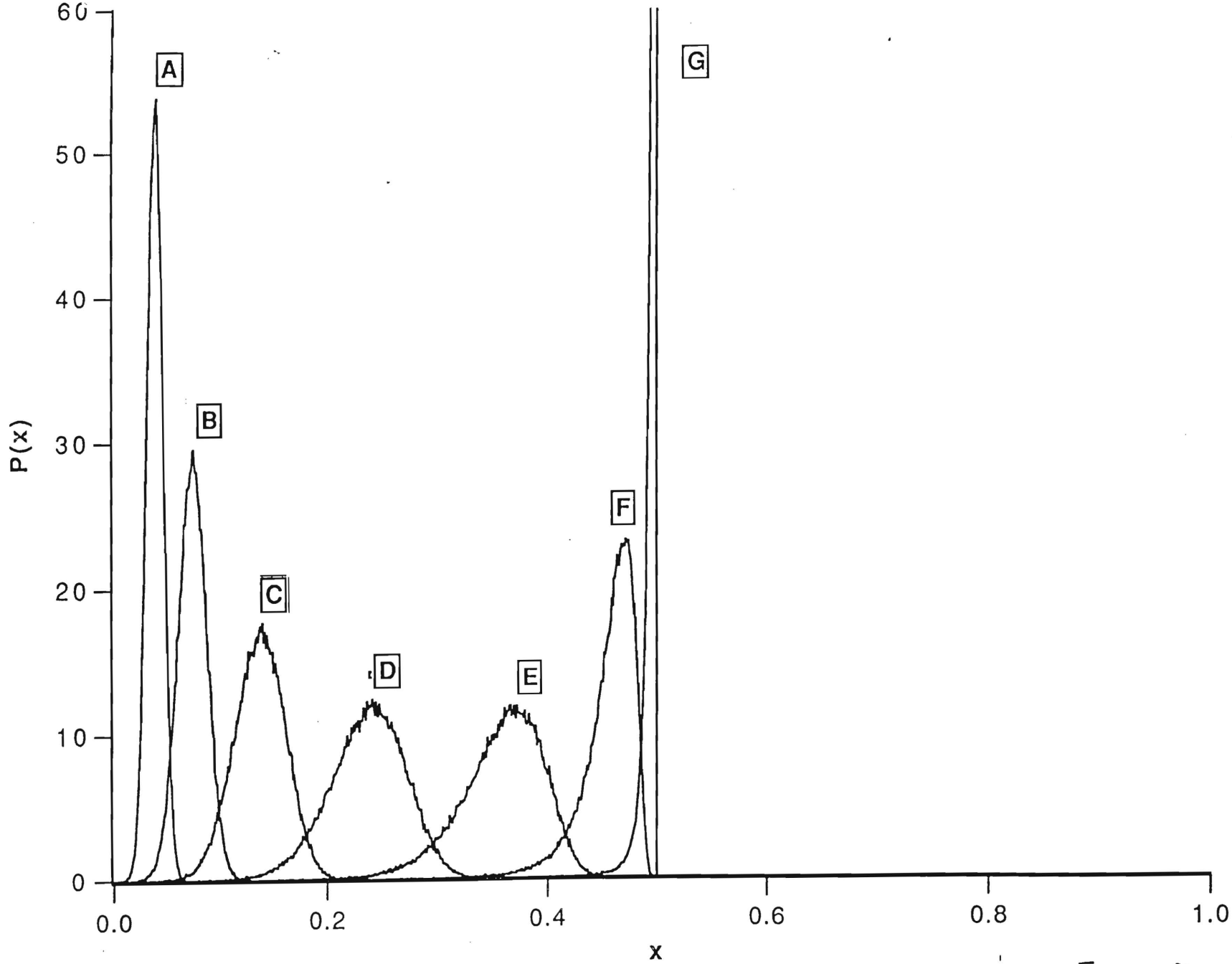


Figure 7

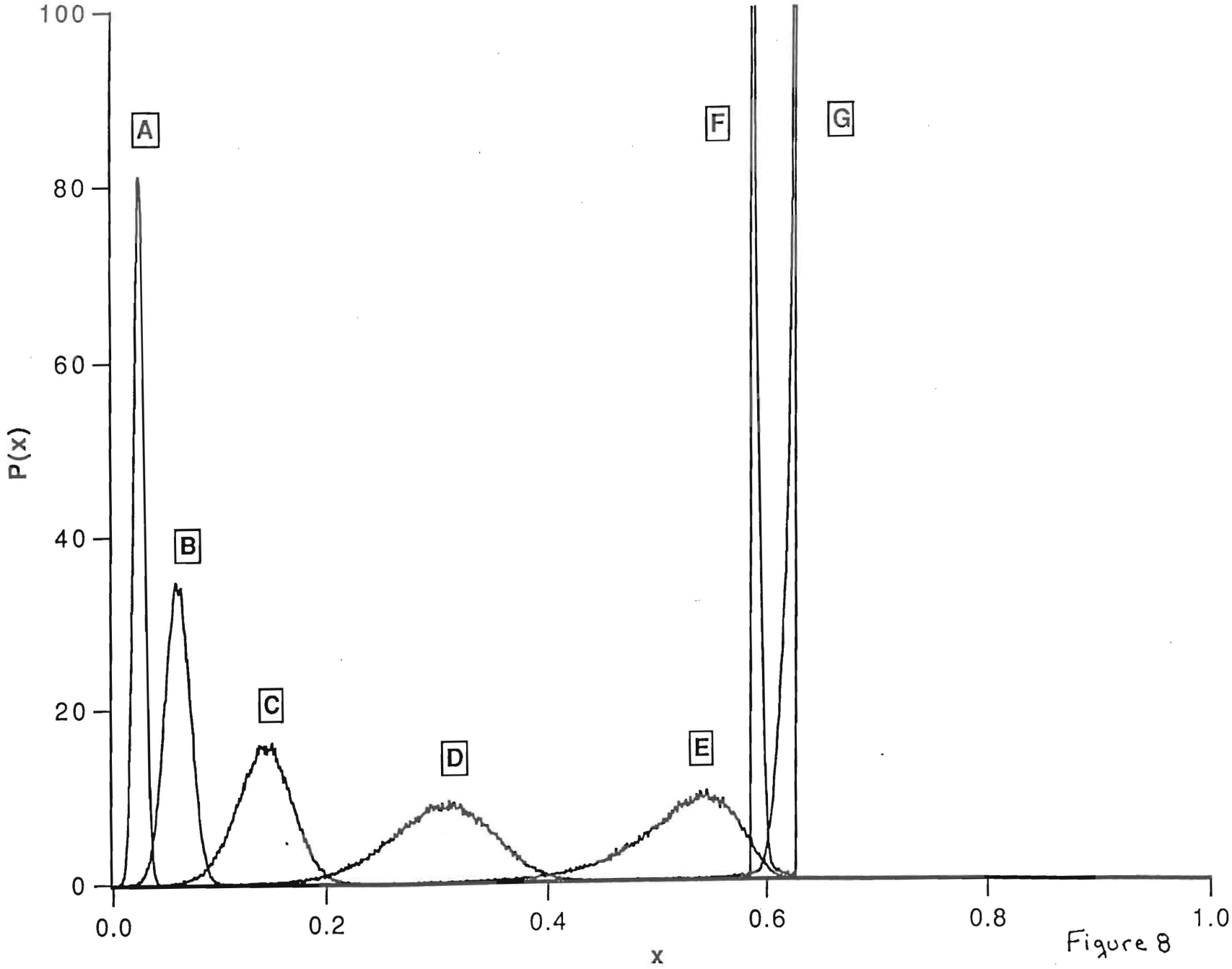
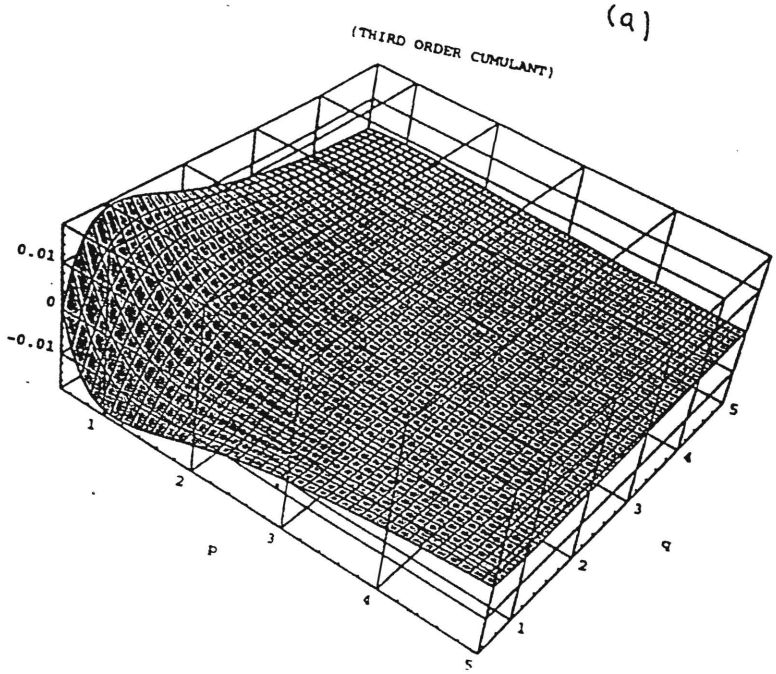


Figure 8

(a)


(THIRD ORDER CUMULANT)
0.01
0
-0.01
1
2
3
4
5
p
1
2
3
4
5
q

(b)

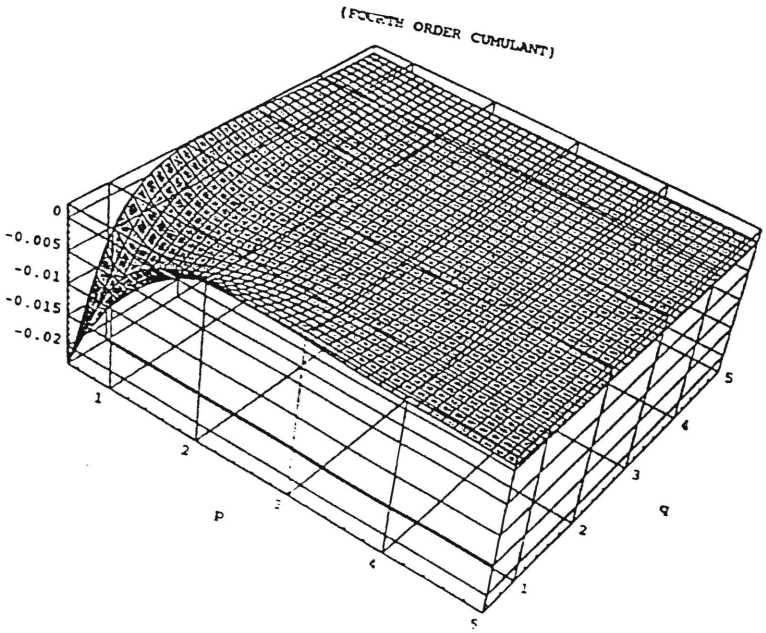
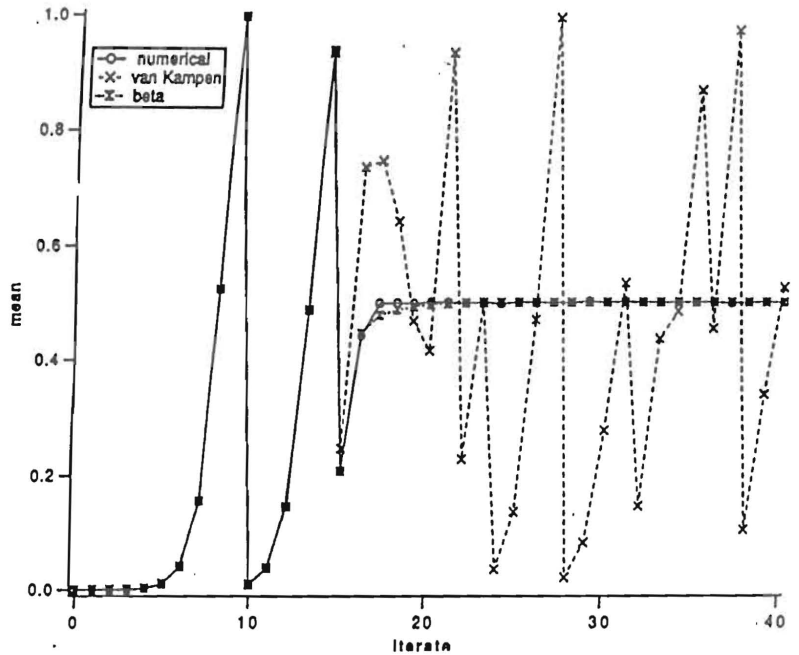
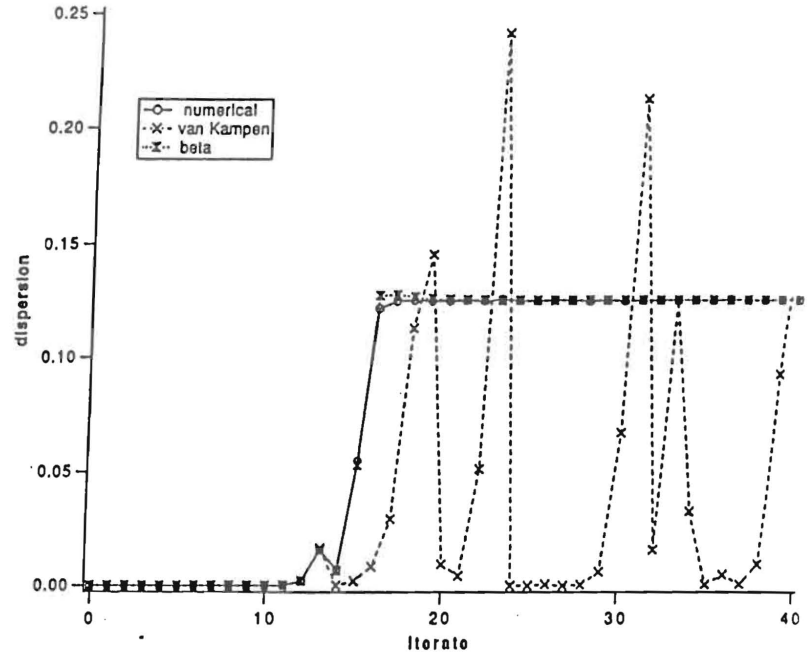
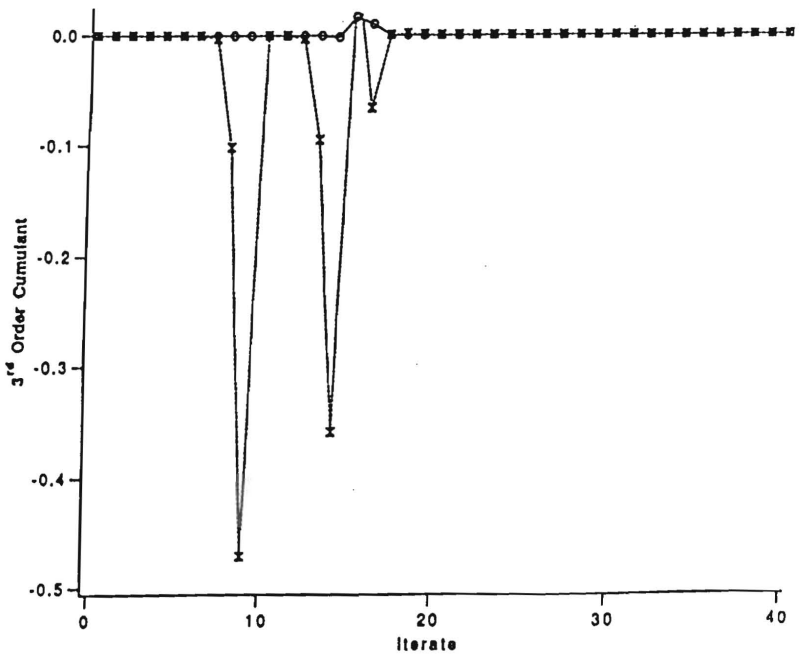

(FOURTH ORDER CUMULANT)
0
-0.005
-0.01
-0.015
-0.02
1
2
3
4
5
p
1
2
3
4
5
q

Figure 9


numerical
van Kampen
beta
mean
Iterate


numerical
van Kampen
beta
dispersion
Iterate


3rd Order Cumulant
Iterate

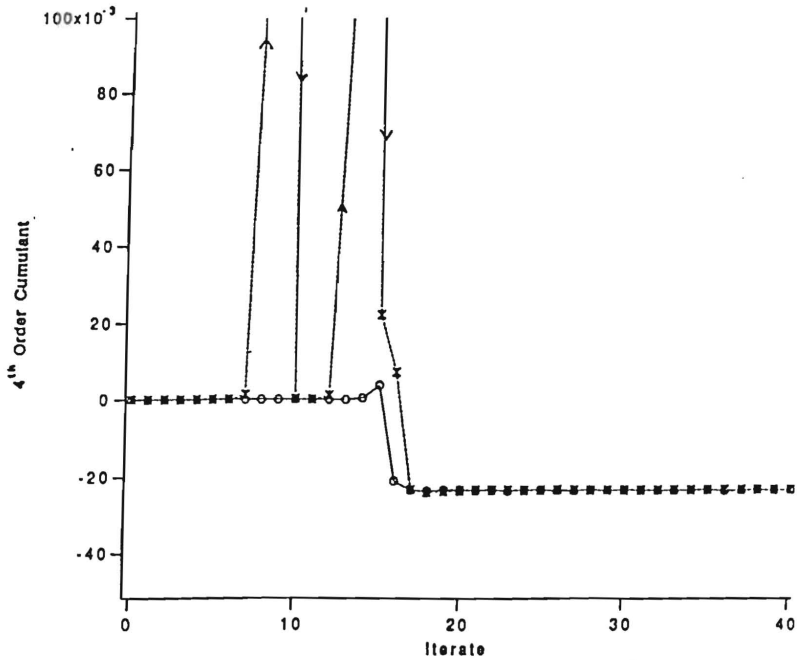
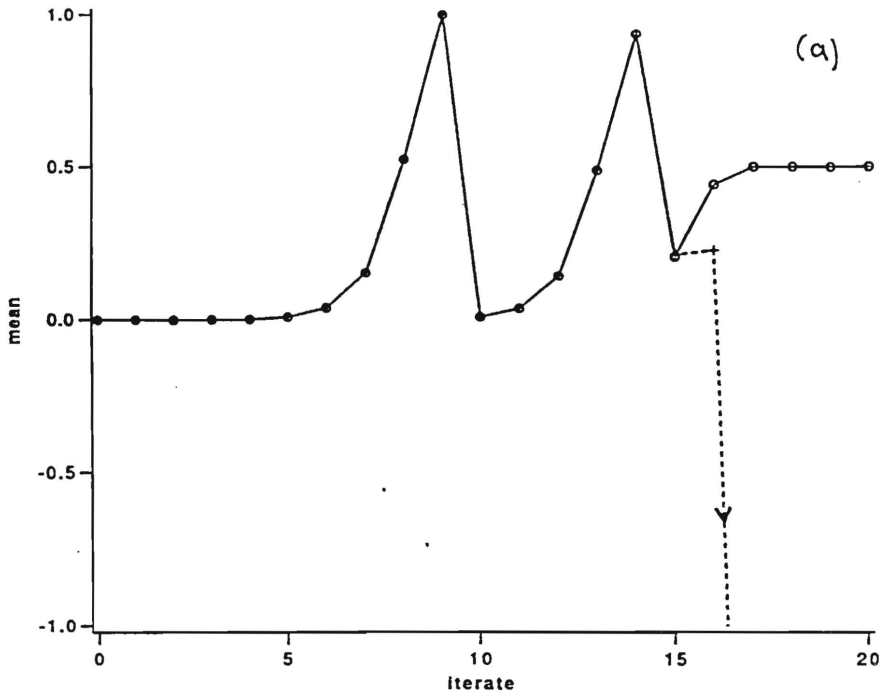

100x10^-3
4th Order Cumulant
Iterate

Figure 10


(a)
1.0
0.5
0.0
-0.5
-1.0
mean
0
5
10
15
20
Iterate

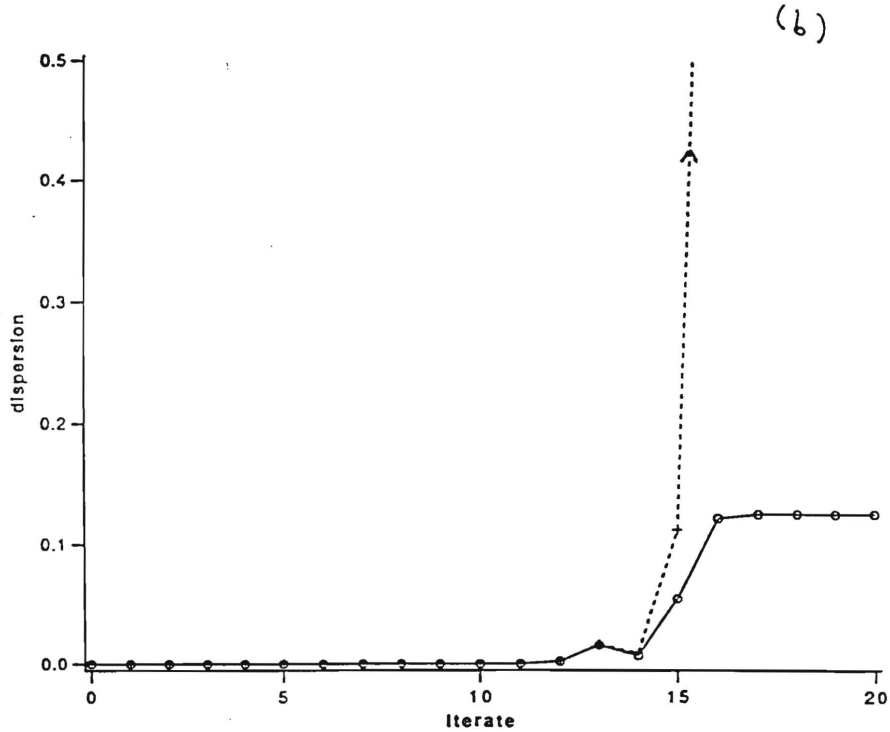

(b)
0.5
0.4
0.3
0.2
0.1
0.0
dispersion
0
5
10
15
20
Iterate

Figure 11

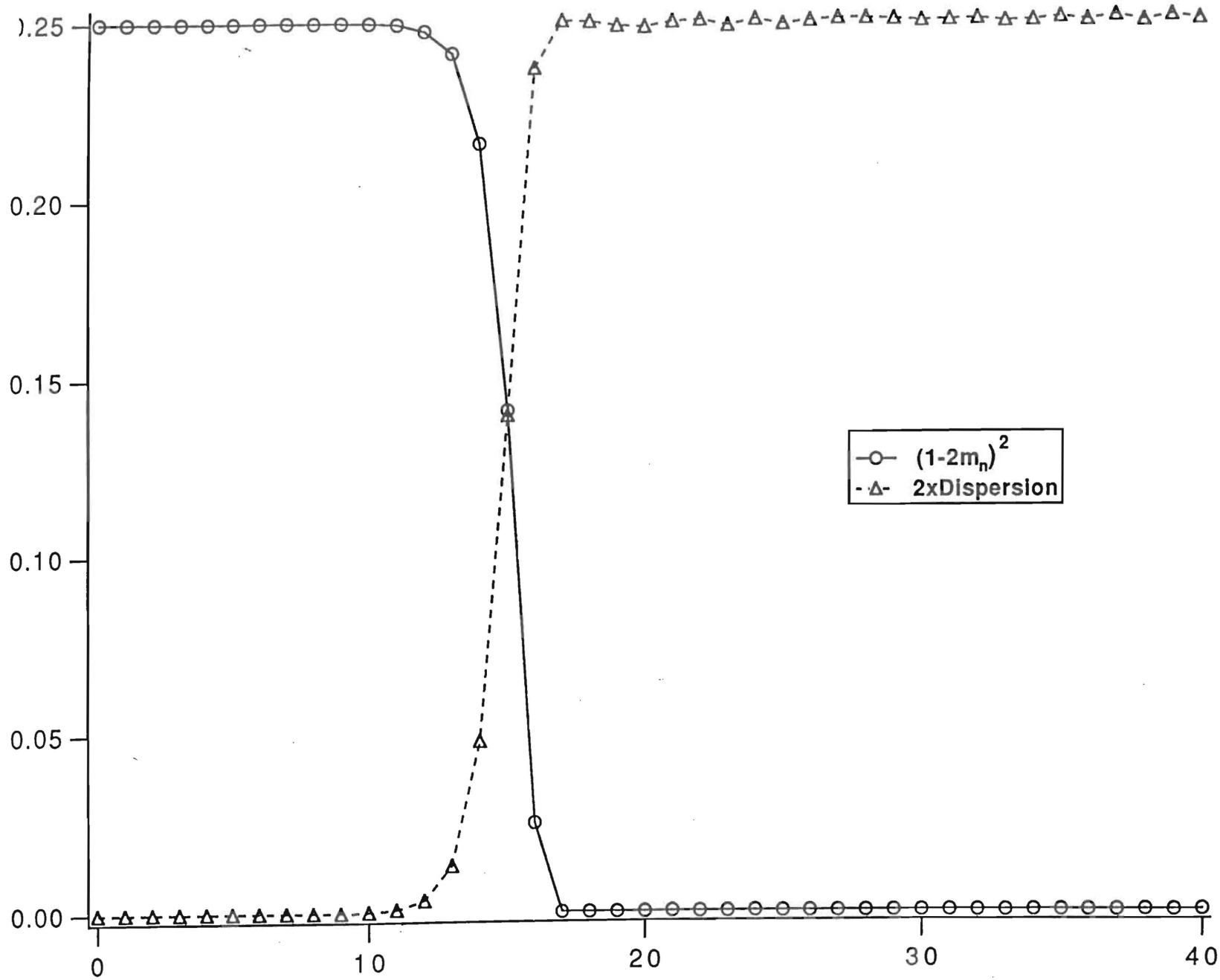
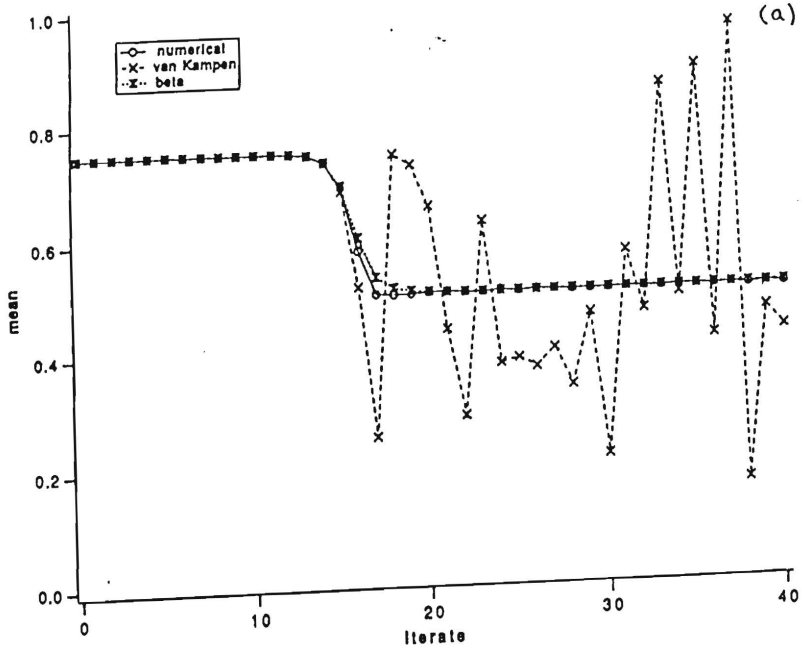
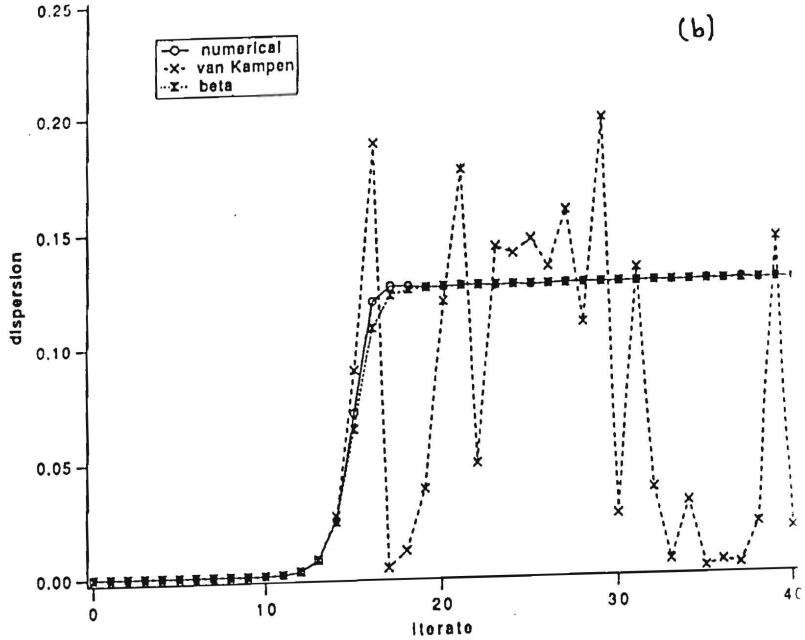
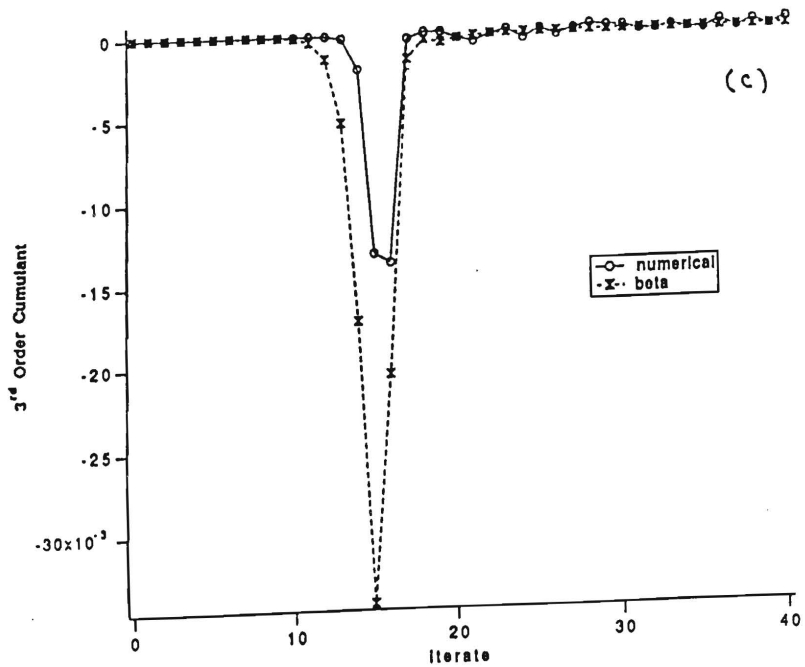

0.25
0.20
0.15
0.10
0.05
0.00
0
10
20
30
40
iterate
(1-2m_n)^2
2xDispersion

Figure 12


(a)
numerical
van Kampen
beta
mean
Iterate


(b)
numerical
van Kampen
beta
dispersion
Iterate


(c)
numerical
beta
3rd Order Cumulant
Iterate

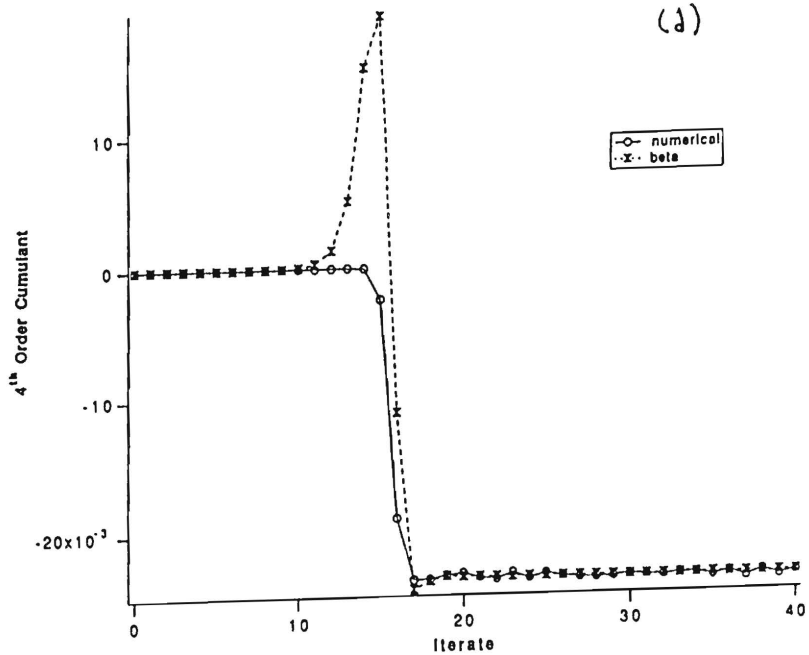

(d)
numerical
beta
4th Order Cumulant
Iterate

Figure 13

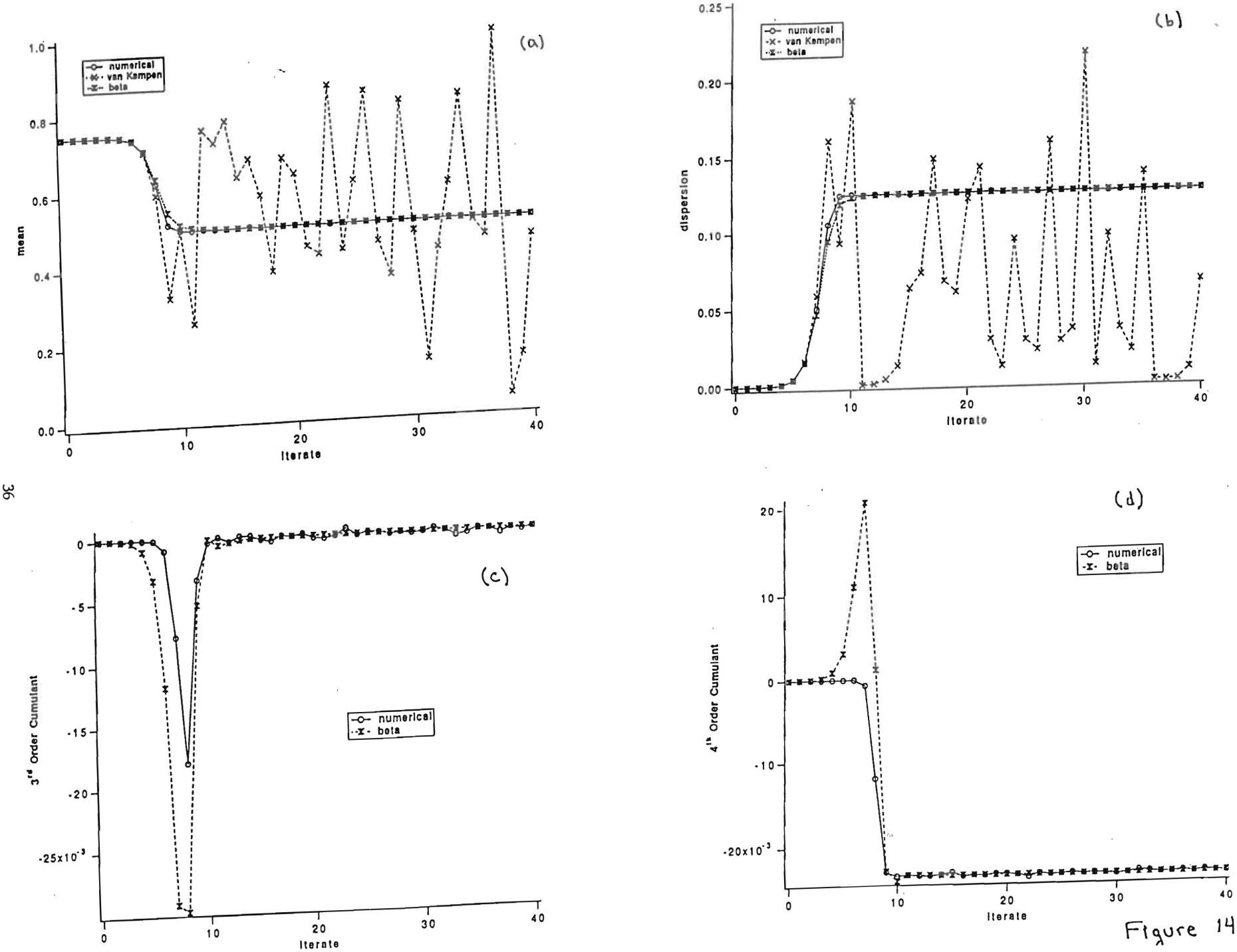

(a)
numerical
van Kampen
beta
mean
Iterate
(b)
numerical
van Kampen
beta
dispersion
Iterate
(c)
numerical
beta
3rd Order Cumulant
Iterate
(d)
numerical
beta
4th Order Cumulant
Iterate

Figure 14

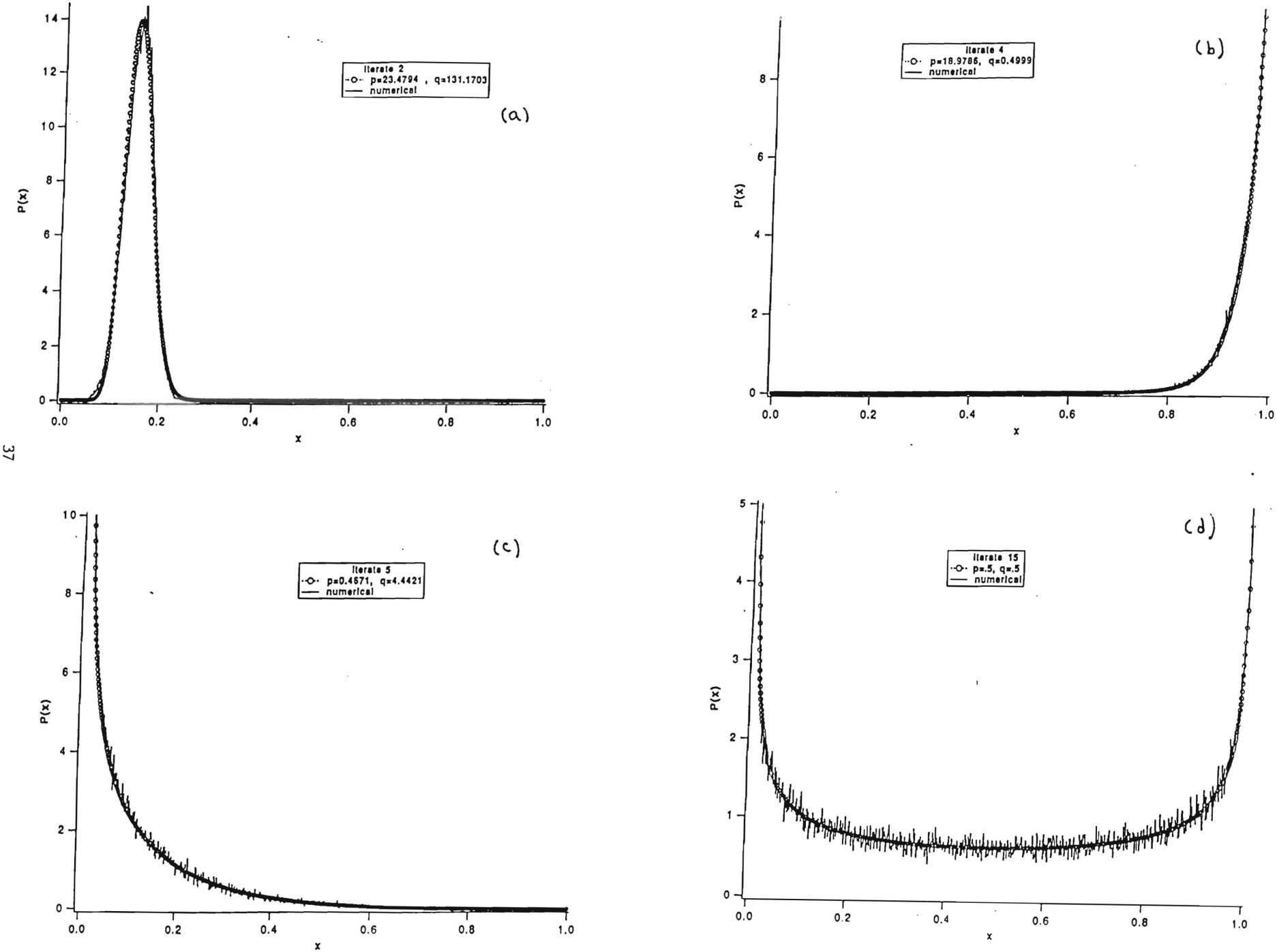

(a)
Iterate 2
p=23.4794 , q=131.1703
numerical
P(x)
x
(b)
Iterate 4
p=18.9786, q=0.4999
numerical
P(x)
x
(c)
Iterate 5
p=0.4671, q=4.4421
numerical
P(x)
x
(d)
Iterate 15
p=.5, q=.5
numerical
P(x)
x

Figure 15

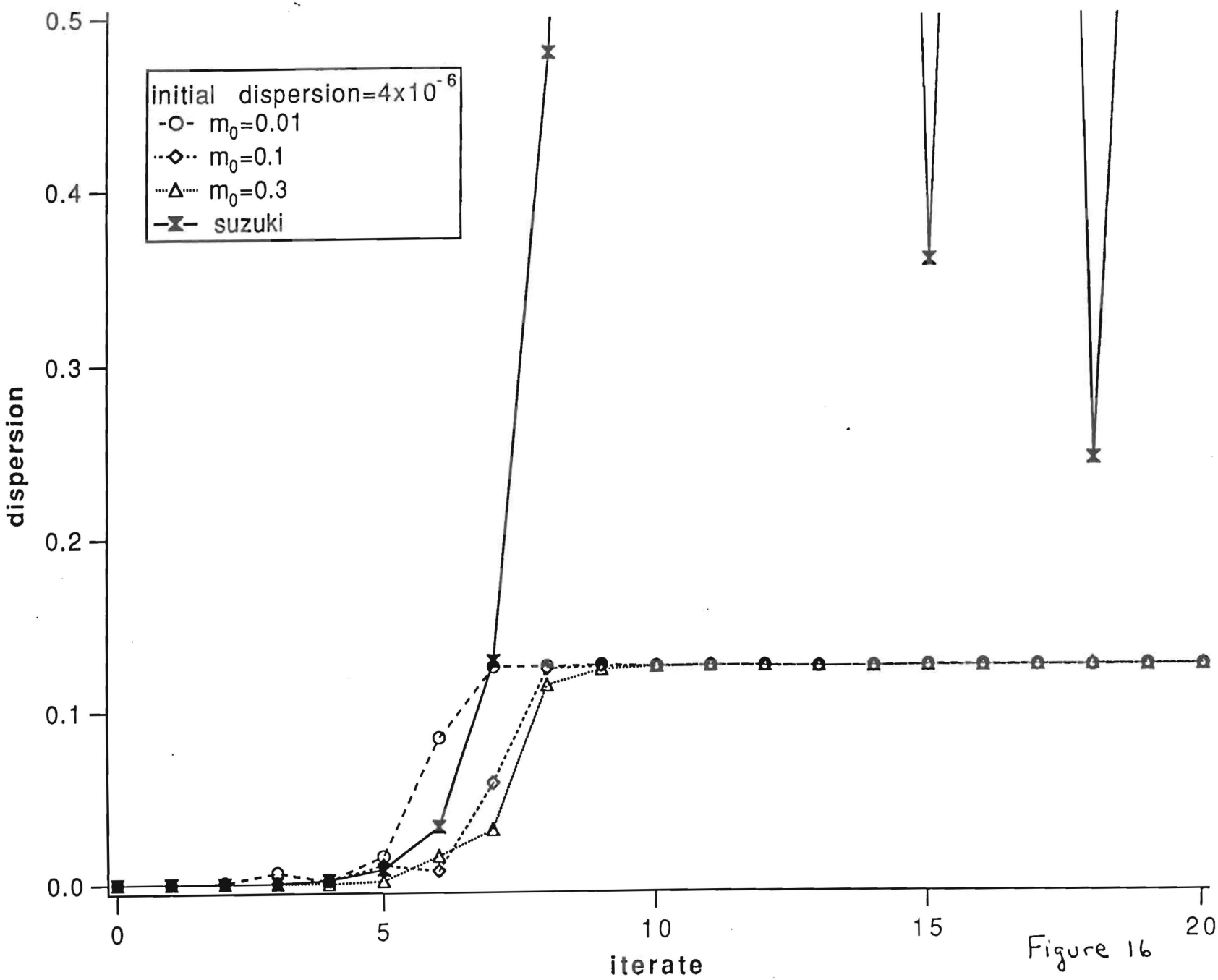

initial dispersion=4x10^-6
m0=0.01
m0=0.1
m0=0.3
suzuki
dispersion
iterate
0.0
0.1
0.2
0.3
0.4
0.5
0
5
10
15
20

Figure 16