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Anyonic states in Chern-Simons theory

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We discuss the canonical quantization of Chern-Simons theory in $2+1$ dimensions, minimally coupled to a Dirac spinor field. Gauss's law and the gauge condition, $A_0 = 0$, are implemented by embedding the formulation in an appropriate physical subspace. We find two kinds of charged particle states in this model. One kind has a rotational anomaly in the form of arbitrary phases that develop in 2π rotations; the other kind rotates "normally"—i.e., charged states only change sign in 2π rotations. The rotational anomaly has nothing to do with the implementation of Gauss's law. It is possible to inadvertently produce these anomalous states in the process of implementing Gauss's law, but it is also possible to implement Gauss's law without producing rotational anomalies. Moreover, states with or without rotational anomalies obey ordinary Fermi statistics.

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In recent work, attention has been directed to the question of how, and indeed whether, $(2+1)$-dimensional gauge theories develop anyons—i.e., particle states that have properties characteristic of neither fermions nor bosons. Examination of the literature reveals a lack of unanimity on this question. Some authors report finding anyons in $(2+1)$-dimensional gauge theories, [1, 2] others question these claims. [3–5] In some work, attention is focused on anomalies in the angular momentum operator. [6] In other work it is claimed that graded algebras develop among the gauge- invariant operators that correspond to the charged states that implement Gauss's law (or at least its long-range component). [1, 2] In earlier work, we studied the topologically massive Maxwell-Chern-Simons (MCS) theory and found that, in a canonically quantized theory in which Gauss's law and the temporal gauge ($A_0 = 0$) are implemented, the canonical angular momentum rotates charged states without anomalies, so that the state vector for an electron $|e\rangle$, returns to $-|e\rangle$ after a 2π rotation. [7] We also demonstrated that "normal" anticommutation rules govern the gauge-invariant operators that project, from the vacuum, charged fermions which obey Gauss's law. Moreover, in our work, these gauge-invariant operators arise naturally within the formalism, and do not need to be constructed *ad hoc*. Electrons in MCS theory therefore are ordinary and unexceptional fermions, albeit in $2+1$ dimensions.

In this work, we describe an investigation of Chern-Simons (CS) theory, in which the CS term is the only kinetic energy term, but the gauge field is still minimally coupled to a charged fermion field. As has been noted, such theories do not possess any observable propagating modes of the gauge field. [8] We treat this model much as we have previously treated the topologically massive MCS theory. [7] We introduce a gauge-fixing field in such a way that A_0 has a conjugate momentum and obeys canonical commutation rules. Although, as in our treatment of MCS theory, Gauss's law and the gauge condition are not primary constraints, there nevertheless are other

primary constraints in CS theory. These primary constraints relate the canonically conjugate momentum of A_1 to A_2, and vice versa, so that the gauge field A_i will be subject to Dirac rather than Poisson commutation rules. Furthermore, all components of the CS gauge field, A_1 and A_2 as well as A_0, must be represented entirely in terms of ghost operators, which can mediate interactions between charges and currents but do not carry energy-momentum, and have no probability of being observed. Neither longitudinal nor transverse components of the CS fields have any propagating particle-like excitations.

The Lagrangian for this model is given by

$$\begin{aligned}\mathcal{L} = & \tfrac{1}{4} m \epsilon_{ln}(F_{ln}A_0 - 2F_{n0}A_l) - \partial_0 A_0 G \\ & + j_l A_l - j_0 A_0 + \bar{\psi}(i\gamma^\mu \partial_\mu - M)\psi \end{aligned} \tag{1}$$

where $F_{ln} = \partial_n A_l - \partial_l A_n$ and $F_{l0} = \partial_l A_0 + \partial_0 A_l$. We follow conventions identical to those in Ref. [7].

The Euler-Lagrange equations are

$$m\epsilon_{ij}F_{j0} - j_i = 0, \tag{2}$$

$$\tfrac{1}{2} m\epsilon_{ij}F_{ij} + \partial_0 G - j_0 = 0, \tag{3}$$

$$\partial_0 A_0 = 0, \tag{4}$$

and

$$(M - i\gamma^\mu D_\mu)\psi = 0, \tag{5}$$

where D_μ is the gauge-covariant derivative $D_\mu = \partial_\mu + ieA_\mu$. Current conservation leads to

$$\partial_0 \partial_0 G = 0. \tag{6}$$

The momenta conjugate to the fields are given by

$$\Pi_0 = -G, \tag{7}$$

and

$$\Pi_i = \tfrac{1}{2} m \epsilon_{ij} A_j. \tag{8}$$

The Hamiltonian density is given by

$$\mathcal{H} = -\tfrac{1}{2} m \epsilon_{ij} F_{ij} A_0 + j_0 A_0 - j_i A_i + \mathcal{H}_{e\bar{e}} \tag{9}$$

where $\mathcal{H}_{e\bar{e}} = \psi^\dagger(\gamma^0 M - i\gamma^0\gamma^n\partial_n)\psi$ and the total derivative $\partial_j(\frac{1}{2} m \epsilon_{ij} A_i A_0)$ has been dropped.

The equal-time commutation (and anticommutation) rules (ETCR) are

$$[A_0(\mathbf{x}), G(\mathbf{y})] = -i\delta(\mathbf{x}-\mathbf{y}), \tag{10}$$

$$[A_i(\mathbf{x}), A_j(\mathbf{y})] = (i/m)\epsilon_{ij}\delta(\mathbf{x}-\mathbf{y}), \tag{11}$$

and

$$\{\psi_\alpha(\mathbf{x}), \psi_\beta{}^\dagger(\mathbf{y})\} = \delta_{\alpha,\beta}\delta(\mathbf{x}-\mathbf{y}), \tag{12}$$

where Eq. (11) is the Dirac rather than the Poisson commutation rule, and represents the influence of the constraint given in Eq. (8). [9, 10] We now construct the following momentum space expansions of the gauge fields in such a way that the ETCR given in Eqs. (10) and (11) are satisfied (all summations are over $\mathbf{k}$):

$$\begin{aligned} A_i(\mathbf{x}) = {} & (2m^{3/2})^{-1} \sum k_i [a_R(\mathbf{k}) - a_R^\star(-\mathbf{k})] e^{i\mathbf{k}\cdot\mathbf{x}} \\ & + i\sqrt{m} \sum \frac{\epsilon_{ij} k_j}{k^2} [a_Q(\mathbf{k}) + a_Q^\star(-\mathbf{k})] e^{i\mathbf{k}\cdot\mathbf{x}} \\ & + i \sum \phi(\mathbf{k}) k_i [a_Q(\mathbf{k}) + a_Q^\star(-\mathbf{k})] e^{i\mathbf{k}\cdot\mathbf{x}}, \end{aligned} \tag{13}$$

$$A_0(\mathbf{x}) = im^{-1/2} \sum [a_Q(\mathbf{k}) - a_Q^\star(-\mathbf{k})] e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{14}$$

and

$$G(\mathbf{x}) = -\tfrac{1}{2}\sqrt{m} \sum [a_R(\mathbf{k}) + a_R^\star(-\mathbf{k})] e^{i\mathbf{k}\cdot\mathbf{x}} \tag{15}$$

where $\phi(\mathbf{k})$ is some arbitrary real and even function of $\mathbf{k}$. The explicit form of $\phi(\mathbf{k})$ is immaterial to the commutation rules given in Eqs. (10) and (11); its form as well as its inclusion in Eq. (13) are therefore entirely optional. The operators $a_Q(\mathbf{k})$ and $a_R(\mathbf{k})$ and their Hermitian adjoints $a_Q^\star(\mathbf{k})$ and $a_R^\star(\mathbf{k})$ are the same ghost operators previously used for the MCS theory; [7] they obey the commutation rules

$$[a_Q(\mathbf{k}), a_R^\star(\mathbf{k})] = [a_R(\mathbf{k}), a_Q^\star(\mathbf{q})] = \delta_{\mathbf{k},\mathbf{q}}, \tag{16}$$

and

$$[a_Q(\mathbf{k}), a_Q^\star(\mathbf{q})] = [a_R(\mathbf{k}), a_R^\star(\mathbf{q})] = 0. \tag{17}$$

The Hamiltonian $H = \int d\mathbf{x}\, \mathcal{H}(\mathbf{x}) = H_0 + H_{\text{int}}$, where H_0 and H_{int} are given by

$$\begin{aligned} H_0 &= -\int d\mathbf{x}\, \tfrac{1}{2} m \epsilon_{ij} F_{ij} A_0 + H_{e\bar{e}} \\ &= im \sum [a_Q(\mathbf{k}) a_Q(-\mathbf{k}) - a_Q^\star(\mathbf{k}) a_Q^\star(-\mathbf{k})] + H_{e\bar{e}} \end{aligned} \tag{18}$$

with $H_{e\bar{e}} = \int d\mathbf{x}\, \mathcal{H}_{e\bar{e}}(\mathbf{x})$ and

$$\begin{aligned} H_{\text{int}} &= im^{-1/2} \sum [a_Q(\mathbf{k}) j_0(-\mathbf{k}) - a_Q^\star(\mathbf{k}) j_0(\mathbf{k})] \\ &\quad - (2m^{3/2})^{-1} \sum k_i [a_R(\mathbf{k}) j_i(-\mathbf{k}) + a_R^\star(\mathbf{k}) j_i(\mathbf{k})] \\ &\quad - i\sqrt{m} \sum \frac{\epsilon_{ij} k_j}{k^2} [a_Q(\mathbf{k}) j_i(-\mathbf{k}) - a_Q^\star(\mathbf{k}) j_i(\mathbf{k})] \\ &\quad - i \sum \phi(\mathbf{k}) k_i [a_Q(\mathbf{k}) j_i(-\mathbf{k}) - a_Q^\star(\mathbf{k}) j_i(\mathbf{k})]. \end{aligned} \tag{19}$$

H_0 and H_{int} operate in a Hilbert space $\{|h\rangle\}$ that very closely resembles the one used in Ref. [7]; $\{|h\rangle\}$ is based on the perturbative vacuum $|0\rangle$ annihilated by all

annihilation operators, $a_Q(\mathbf{k})$ and $a_R(\mathbf{k})$ as well as the electron and positron annihilation operators $e(\mathbf{k})$ and $\bar{e}(\mathbf{k})$, respectively. The Hilbert space $\{|h\rangle\}$ contains a subspace $\{|n\rangle\}$ that consists of all multiparticle electron-positron states of the form $|N\rangle = \bar{e}^\dagger(\mathbf{q}_1)\cdots\bar{e}^\dagger(\mathbf{q}_l)e^\dagger(\mathbf{p}_1)\cdots e^\dagger(\mathbf{p}_n)|0\rangle$, as well as all other states of the form $a_Q^\star(\mathbf{k}_1)\cdots a_Q^\star(\mathbf{k}_i)|N\rangle$. H_0 time-translates all states in $\{|n\rangle\}$ so that they remain contained within it. States in which $a_R^\star(\mathbf{k})$ operators act on a state $|n\rangle$, such as $a_R^\star(\mathbf{q}_1)\cdots a_R^\star(\mathbf{q}_i)a_Q^\star(\mathbf{k}_1)\cdots a_Q^\star(\mathbf{k}_j)|N\rangle$, are included in $\{|h\rangle\}$, but excluded from $\{|n\rangle\}$. Such states are not probabilistically interpretable.

As in all other gauge theories, Gauss's law is not an equation of motion in CS theory. The operator $\mathcal{G}(\mathbf{x})$ used to implement Gauss's law is

$$\mathcal{G}(\mathbf{x}) = j_0(\mathbf{x}) - \tfrac{1}{2}m\epsilon_{ij}F_{ij}(\mathbf{x}), \tag{20}$$

and whereas $\partial_0 G = \mathcal{G}$, $\partial_0\partial_0 G = \partial_0\mathcal{G} = 0$ is the equation of motion that governs the behavior of this model. Further measures must be taken to implement $\mathcal{G} = 0$. We can conveniently express $\mathcal{G}$ in the form

$$\mathcal{G}(\mathbf{x}) = m^{3/2}\sum[a_Q(\mathbf{k}) + a_Q^\star(-\mathbf{k}) + \frac{j_0(\mathbf{k})}{m^{3/2}}]e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{21}$$

where $j_0(\mathbf{k}) = \int d\mathbf{x}\, j_0(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}$. We can define an operator $\Omega(\mathbf{k})$ as

$$\Omega(\mathbf{k}) = a_Q(\mathbf{k}) + (2m^{3/2})^{-1}j_0(\mathbf{k}), \tag{22}$$

so that

$$\mathcal{G}(\mathbf{x}) = m^{3/2}\sum[\Omega(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} + \Omega^\star(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}]. \tag{23}$$

Similarly, we can write $A_0(\mathbf{x})$ as

$$A_0(\mathbf{x}) = im^{-1/2}\sum[\Omega(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} - \Omega^\star(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}]. \tag{24}$$

We can therefore implement Gauss's law and the gauge condition by embedding the theory in a subspace $\{|\nu\rangle\}$ of another Hilbert space. The subspace $\{|\nu\rangle\}$ consists of the states $|\nu\rangle$ which satisfy the condition

$$\Omega(\mathbf{k})|\nu\rangle = 0. \tag{25}$$

It can be easily seen from Eqs. (23) and (24) that, in the physical subspace $\{|\nu\rangle\}$, $\langle\nu'|\mathcal{G}|\nu\rangle = 0$ and $\langle\nu'|A_0|\nu\rangle = 0$, so that both Gauss's law and the gauge condition $A_0 = 0$ hold. Moreover, the condition $\Omega(\mathbf{k})|\nu\rangle = 0$, once established, continues to hold at all other times because

$$[H, \Omega(\mathbf{k})] = 0 \tag{26}$$

so that $\Omega(\mathbf{k})$ is an operator-valued constant. This demonstrates that a state initially in the physical subspace $\{|\nu\rangle\}$ will always remain entirely contained within it as it develops under time evolution.

Consider now the unitary transformation $U = e^D$ where

$$D = -i \int d\mathbf{x}\, d\mathbf{y} \sum \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{k^2} \partial_i A_i(\mathbf{x}) j_0(\mathbf{y}). \tag{27}$$

It is easy to show that

$$U^{-1}\Omega(\mathbf{k})U = a_Q(\mathbf{k}). \tag{28}$$

We can use U to establish a mapping that maps $\Omega(\mathbf{k}) \to a_Q(\mathbf{k})$ and $\{|\nu\rangle\} \to \{|n\rangle\}$, where $\{|n\rangle\}$ is the subspace described previously in the paragraph following Eq. (19). In this mapping, operators P map into $\tilde{P}$, i.e., $U^{-1}PU = \tilde{P}$. For example, $\tilde{\Omega}(\mathbf{k}) = a_Q(\mathbf{k})$, and $\tilde{H} = U^{-1}HU$ is given by

$$\begin{aligned}\tilde{H} = H_0 &- im^{-1} \sum \frac{\epsilon_{ln} k_n}{k^2} j_l(\mathbf{k}) j_0(-\mathbf{k}) \\ &- i\sqrt{m} \sum \frac{\epsilon_{ij} k_j}{k^2} [a_Q(\mathbf{k}) j_i(-\mathbf{k}) - a_Q^\star(\mathbf{k}) j_i(\mathbf{k})].\end{aligned} \tag{29}$$

If we expand D in momentum space, we get $D = D_1 + D_2$ where

$$D_1 = (2m^{3/2})^{-1} \sum [a_R(\mathbf{k}) j_0(-\mathbf{k}) - a_R^\star(\mathbf{k}) j_0(\mathbf{k})] \tag{30}$$

and

$$D_2 = i \sum \phi(\mathbf{k}) [a_Q(\mathbf{k}) j_0(-\mathbf{k}) + a_Q^\star(\mathbf{k}) j_0(\mathbf{k})]. \tag{31}$$

Since D_2 commutes with $a_Q(\mathbf{k})$, it has no role in transforming $\Omega(\mathbf{k})$ into $a_Q(\mathbf{k})$, and the operator $V = e^{D_1}$ by itself achieves the same end as U, i.e.,

$$V^{-1} \Omega(\mathbf{k}) V = a_Q(\mathbf{k}). \tag{32}$$

We can use V to establish a second mapping of this theory, in which operators map according to $P \to V^{-1} P V = \hat{P}$. $\hat{\Omega}(\mathbf{k}) = a_Q(\mathbf{k})$, so that $\hat{\Omega}$ and $\tilde{\Omega}$ are identical; under the mapping $P \to V^{-1} P V = \hat{P}$, the subspace $\{|\nu\rangle\}$ maps into the same subspace $\{|n\rangle\}$ as under the mapping $P \to U^{-1} P U = \tilde{P}$. But, in the case of other operators, $\hat{P}$ differs from $\tilde{P}$. For example, $\hat{H}$ is given by

$$\begin{aligned}
\hat{H} = H_0 &- i m^{-1} \sum \frac{\epsilon_{ln} k_n}{k^2} j_l(\mathbf{k}) j_0(-\mathbf{k}) \\
&- i m^{-3/2} \sum \phi(\mathbf{k}) k_l j_l(\mathbf{k}) j_0(-\mathbf{k}) \\
&- i \sqrt{m} \sum \frac{\epsilon_{ij} k_j}{k^2} [a_Q(\mathbf{k}) j_i(-\mathbf{k}) - a_Q^\star(\mathbf{k}) j_i(\mathbf{k})] \\
&- i \sum \phi(\mathbf{k}) k_i [a_Q(\mathbf{k}) j_i(-\mathbf{k}) - a_Q^\star(\mathbf{k}) j_i(\mathbf{k})]
\end{aligned} \tag{33}$$

Similarly, $\tilde{\psi}$ and $\hat{\psi}$ differ from each other, although both project, from the correspondingly defined vacuum states, electron states that implement Gauss's law. $\tilde{\psi}$ and $\hat{\psi}$ are given by $\tilde{\psi}(\mathbf{x}) = \exp[\mathcal{D}_{\mathrm{U}}(\mathbf{x})]\psi(\mathbf{x})$ and by $\hat{\psi}(\mathbf{x}) = \exp[\mathcal{D}_{\mathrm{V}}(\mathbf{x})]\psi(\mathbf{x})$, [11] where

$$\mathcal{D}_{\mathrm{U}}(\mathbf{x}) = -ie \int d\mathbf{y} \sum \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{k^2} \partial_i A_i(\mathbf{y}) \tag{34}$$

and

$$\begin{aligned}
\mathcal{D}_{\mathrm{V}}(\mathbf{x}) = -ie \int d\mathbf{y} \sum \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{k^2} \times \\
\left[\partial_i A_i(\mathbf{y}) + \frac{\phi(\mathbf{k})}{\sqrt{m}} k^2 \epsilon_{ij} \partial_i A_j(\mathbf{y}) \right].
\end{aligned} \tag{35}$$

In the unitarily transformed representation, $a_Q(\mathbf{k})|n\rangle = 0$ is the form taken by the constraint that implements Gauss's law and the gauge condition, when either U or V is used to carry out the transformation. $\tilde{J}$ and $\hat{J}$ are the forms into which the Noether angular momentum operator J is mapped when it is unitarily transformed by U and V, respectively. Both these forms, $\tilde{J}$ and $\hat{J}$, are therefore significant for the rotation of states in $\{|n\rangle\}$, and it is of particular importance to observe that $\tilde{J}$ and $\hat{J}$ differ from each other. J is given by

$$J = J_g + J_e, \tag{36}$$

where J_g and J_e are the angular momenta of the gauge field and the spinors, respectively. J_g and J_e are given by

$$J_g = -\int d\mathbf{x}\, \epsilon_{ln}(\Pi_i x_l \partial_n A_i - G x_l \partial_n A_0 + \Pi_l A_n) \tag{37}$$

and

$$J_e = -\int d\mathbf{x}\ (i\psi^\dagger x_l \epsilon_{ln} \partial_n \psi + \tfrac{1}{2}\psi^\dagger \gamma_0 \psi). \tag{38}$$

Under the transformation mediated by U, $J \to \tilde{J}$, and $\tilde{J} = J$, so that J remains untransformed. But, under the transformation mediated by V, $J \to \hat{J}$ where $\hat{J} = J + \mathcal{J}$ and

$$\begin{aligned}\mathcal{J} = &-\sum \epsilon_{ln} k_l \frac{\partial \phi(\mathbf{k})}{\partial k_n}[a_Q^*(\mathbf{k}) j_0(\mathbf{k}) + a_Q(\mathbf{k}) j_0(-\mathbf{k})] \\ &+ (2m)^{-3/2} \sum \epsilon_{ln} k_l \frac{\partial \phi(\mathbf{k})}{\partial k_n} j_0(\mathbf{k}) j_0(-\mathbf{k}).\end{aligned} \tag{39}$$

We can support the preceding demonstration that J transforms into itself under the unitary transformation mediated by U, whereas it transforms into $J + \mathcal{J}$ under the unitary transformation mediated by V, with the following observation: D is an integral over operators and functions which all transform as scalars under spatial

rotations. Since J is the generator of spatial rotations, the commutator $[J, D]$ must vanish. D_1 is not such an integral over scalars, and there is therefore no similar requirement that $[J, D_1]$ vanishes.

Since U and V map $\Omega(\mathbf{k})$ into $a_Q(\mathbf{k})$ in idential ways, we can conclude that the implementation of Gauss's law is not responsible for the fact that J is transformed into $J + \mathcal{J}$ when V is used to effect the mapping. In fact, we can use the Baker-Campbell-Hausdorff relation to construct an operator $W = e^{D'}$, where

$$\begin{aligned} D' = & i(2m)^{-3/2} \sum \phi(\mathbf{k}) j_0(\mathbf{k}) j_0(-\mathbf{k}) \\ & - i \sum \phi(\mathbf{k})[a_Q(\mathbf{k}) j_0(-\mathbf{k}) + a_Q^\star(\mathbf{k}) j_0(\mathbf{k})], \end{aligned} \tag{40}$$

so that $V = UW$. W has the same effect as V on J, i.e. we find that

$$W^{-1} J W = J + \mathcal{J}, \tag{41}$$

although W leaves $\Omega(\mathbf{k})$ and $\mathcal{G}(\mathbf{x})$ untransformed and does not play any role in implementing Gauss's law. $\phi(\mathbf{k})$ is arbitrary, and if we choose to set $\phi(\mathbf{k}) = 0$, U and V become identical. But if we choose $\phi(\mathbf{k}) = \sqrt{m}[\delta(k)/k] \tan^{-1}(k_2/k_1)$, then $\mathcal{J}$ becomes $\mathcal{J} = Q^2/4\pi m$, and accounts for the well- known anyonic phase in the rotation of charged states through 2π.

In comparing $\tilde{H}$ with $\hat{H}$, we note that they differ by some terms that include $a_Q(\mathbf{k})$ or $a_Q^\star(\mathbf{k})$ as factors. Since both $\tilde{H}$ and $\hat{H}$ are entirely free of $a_R(\mathbf{k})$ and $a_R^\star(\mathbf{k})$ operators, $a_Q(\mathbf{k})$ and $a_Q^\star(\mathbf{k})$ commute with every other operator that appears in $\tilde{H}$ or $\hat{H}$. The terms which include $a_Q(\mathbf{k})$ or $a_Q^\star(\mathbf{k})$ as factors therefore do not affect the time evolution of state vectors in the part of the subspace $\{|n\rangle\}$ that describes observable particles (i.e., electrons or positrons); they can neither produce projections on physical states, nor can they contribute internal loops to radiative corrections. They have no effect whatsoever on the physical predictions of the theory and if they are arbitrarily

amputated from $\tilde{H}$ or $\hat{H}$, none of the physical predictions are affected. The only other difference between $\tilde{H}$ and $\hat{H}$ is a total time derivative in $\hat{H}$, which can be expressed alternatively as $h = i[H_0, \chi]$, as $h = i[H, \chi]$, as $h = i[\tilde{H}, \chi]$, or as $h = i[\hat{H}, \chi]$ where

$$\chi = -(2m)^{-3/2} \sum \phi(\mathbf{k}) j_0(\mathbf{k}) j_0(-\mathbf{k}). \tag{42}$$

The presence of h in $\hat{H}$ is equivalent to unitarily transforming the Hamiltonian as shown by $\hat{H} = e^{i\chi} \tilde{H} e^{-i\chi}$. In earlier work, we demonstrated that two Hamiltonians that are unitarily equivalent in that fashion give rise to identical S-matrix elements under very general conditions. [7]

We observe from these results that CS theory does give rise to anyonic as well as to normal states: some states that obey Gauss's law and the gauge condition rotate like normal fermions; others show the arbitrary phase anomaly when the angular momentum operator is used to generate rotations in the plane. However, contrary to what has been suggested by other authors, [1,6] it is not the implementation of Gauss's law that is responsible for the development of anyonic properties. States can develop an anyonic angular momentum anomaly as an incidental byproduct of the process by which Gauss's law is implemented, but the change in the rotational properties of the state is not an inevitable consequence of the implementation of Gauss's law. Moreover, in corraboration of a result obtained by other means, [3] we find that regardless of whether the arbitrary rotational phase develops, the anticommutation rule that governs the electron field operator remains unchanged. And that observation applies equally to the free Dirac field and to the gauge-invariant electron field that projects electrons that obey Gauss's law. The "normal" and the "anyonic" operators are unitarily equivalent and both obey Fermi-Dirac statistics.

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REFERENCES

[1] G. W. Semenoff, in *Physics, Geometry, and Topology*, H. C. Lee, Ed., (Plenum Press, New York, 1990); G. W. Semenoff, Phys. Rev. Lett. **61**, 517 (1988); G. W. Semenoff and P. Sodano, Nucl. Phys. **B328**, 753 (1989).

[2] M. Lüscher, Nucl. Phys. **B326**, 557 (1989); R. Banerjee, Phys. Rev. Lett. **69**, 17 (1992).

[3] C. Hagen, Phys. Rev. D **31**, 2135 (1985).

[4] D. Boyanovsky, Phys. Rev. D **42**, 1179 (1990); D. Boyanovsky, E. T. Newman, and C. Rovelli, *ibid.* **45**, 1210 (1992).

[5] Y. Srivastava and A. Widom, Phys. Rep. **148**, 1 (1987).

[6] M. Swanson, Phys. Rev. D **42**, 552 (1990); G. W. Semenoff and P. Sodano, Nucl. Phys. **B328**, 753 (1989); G. W. Semenoff and L. C. R. Wijewardhana, Phys. Lett. B **184**, 397 (1987).

[7] K. Haller and E. Lim-Lombridas, Phys. Rev. D **46**, 1737 (1992).

[8] C. Hagen, Ann. Phys. (N.Y.) **157**, 342 (1984).

[9] P. A. M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva University Press, New York, 1964).

[10] L. D. Faddeev and R. Jackiw, Phys. Rev. Lett. **60**, 1692 (1988); G. V. Dunne, R. Jackiw, and C. A. Trugenberger, Ann. Phys. (N.Y.) **194**, 197 (1989).

[11] We note that the gauge-invariant operators used by Boyanovsky et al. in Ref. [4]

to project charged particle states that implement Gauss's law are similar to our $\tilde{\psi}$ (but not to our $\hat{\psi}$). Their results are in agreement with our observation that the implementation of Gauss's law does not affect the anticommutation rules for charged fermions.