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The Universal T-Matrix.

C.Fronsdal A. Galindo

ABSTRACT. The universal T-matrix of a quantum group is the Hopf algebra dual form, expressed in terms of the generators of the algebra and the generators of its dual. In the physical applications it is the familiar T-matrix of integrable models, here calculated in the structure, without specialization to a representation of the algebra of physical variables, nor to a representation of the auxiliary algebra. This article deals with some rather surprising facts that were discovered by examination of the formula for the universal T-matrix for $U_q(sl_2)$, among them the existence of a new series of quantum deformations of $U(gl_n)$ and a generalization of the quantum double. The new quantum groups have physical applications with essentially new features, principally arising from the fact that the dual Lie algebra (algebra of physical variables) is not solvable.

1 Introduction

The principal object that defines an integrable classical field theory is its "L-matrix", the spatial component of a flat Lax connection. In Drinfeld's interpretation it is a Lie bialgebra dual form. If $(l_i), i = 1, ..., n$, is a basis for a Lie bialgebra G, and (x^i) is the dual basis for the vector space dual G^*, then it is

$$L = \sum_i x^i l_i. \tag{1}$$

In practice, the l_i are matrices and the x^i are dynamical variables, but nothing so far depends on the choice of a representation; duality, in this context, is a concept with a clear structural content. After quantization, which is understood as a Hopf algebra deformation of the universal enveloping algebra $U(g)$ of G, the attention of the physicist shifts to the "T-matrix". Classically, the T-matrix is the monodromy associated with the Lax connection; after quantization one tries to preserve this interpretation. Unlike the situation in the classical theory, the structural meaning of the quantum T-matrix is not always clear; it is normally studied in a particular representation of G (the Woronowicz picture [1]) or in special representations of G^* (the Kulish-Reshetikhin picture [2]).

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Section 2 examines the Hopf algebra dual of the quantum group $U_{q,q'}(gl_2)$. It is a deformation $U_{q,q'}(gl_2{}^*)$ of the enveloping algebra of the bialgebra dual $gl(2)^*$ of $gl(2)$, with the structure of Lie algebra preserved, the deformation affecting only the coproduct. Both Hopf algebras are thus finitely generated. The Hopf algebra dual form can be obtained explicitly in terms of the generators x^i and l_i of $gl_2{}^*$ and of gl_2:

$$T = e^{x^- l_-}_{1/k} e^{x^0 l_0} e^{x^1 l_1} e^{x^+ l_+}_{k}. \tag{2}$$

Here $k = qq'$ and e^x_k is the usual twisted exponential. The result was obtained from the known structure of $U_{q,q'}(gl_2)$, both product and coproduct, using the relations that characterize and in fact determine the dual form, namely

$$T_{x,l}T_{x,l'} = T_{x,\Delta(l)}, \; T_{x,l}T_{x',l} = T_{\Delta(x),l}. \tag{3}$$

Here x and x', resp. l and l', refer to two identical copies of $U_{q,q'}(gl_2{}^*)$, resp. of $U_{q,q'}(gl_2)$. The second relation is the multiplicative property that is the essence of complete integrability. The coproducts are denoted $x^i \mapsto \Delta(x^i)$ and $l_i \mapsto \Delta(l_i)$. The calculation yields the expression for T, as well as the structure of the dual. Since Eq.(2) is structural, independent of any reference to representations of gl_2 or of $gl_2{}^*$, the term Universal T-matrix is justified. Eqs.(3) express Hopf algebra duality in a particularly succinct form.

Section 3 reports the discovery of a new quantum deformation of $U(gl_3)$. To demonstrate the practical utility of the result (2) we investigated the Hopf algebra that results from taking the x^i (not, as is usual, the l_i) in a small, faithful representation. We are thus led to

$$T = \begin{pmatrix} a & b & 0 \\ 0 & d & 0 \\ 0 & c & \hat{a} \end{pmatrix}, \tag{4}$$

where the entries are expressed in terms of the l_i and satisfy the relations

$$\begin{gathered} ab = q^{-1}ba, \; ac = qca, \; [a,\hat{a}] = 0, \\ [a,d] = 0 = [\hat{a},d], \; \hat{a}b = q^{-1}b\hat{a}, \; \hat{a}c = qc\hat{a}, \\ bd = q^{-1}db, \; cd = qdc, \; [b,c] = \lambda(a\hat{a} - dd). \end{gathered} \tag{5}$$

[The associated compatible coproduct is of course given by multiplication of matrices, this being an application of Eq.(3).] The respective roles of gl_2 and $gl_2{}^*$ have thus been reversed as compared to the usual situation. The physical significance of this is that the physical variables a, b, c, d now have the structure of quantum $gl(2)$, instead of that of its solvable dual. Applications in the form of ice models and new types of Toda field theories and nonlinear sigma models are envisaged. Because of (3), the matrix elements of (4) generate a Hopf algebra. However, since it is gl_2 and not $gl_2{}^*$ that is a coboundary Lie algebra, this Hopf algebra is not of the coboundary type. Nevertheless, a Yang-Baxter matrix for the T-matrix (4) does exist. The explanation for this lies in the fact that $U_{q,q'}(gl_2{}^*)$ is a sub-Hopf algebra of a new quantum deformation $U_{q,q'}(gl_3)$ of the enveloping algebra of $gl(3)$, and this one *is* of the coboundary type. We determined the

R-matrix using Manin's technique and from this the new structure of "esoteric quantum $gl(3)$".

In Section 4 the generalization of the formula (2) for the universal T-matrix, to standard quantum $gl(n)$, is obtained.

In Section 5 we generalize the results of Section 3 and present the structure of the new quantum groups, "esoteric quantum $gl(n)$". The most important feature of these new quantum groups is that the dual algebras are not solvable; this has a profound effect on the physical applications. The Hopf algebra dual form of these quantum groups can also be calculated; a start is made here. The essential step is a factorization that generalizes the famililar quantum double.

Section 6. Belavin and Drinfeld [3] have classified the coboundary Lie bialgebra structures associated with the simple Lie algebras. Analogous results for the quantum deformations are not yet available. To the simplest r-matrices for $gl(n)$ correspond multi-parameter deformations $U_q(gl_n)$; here q stands for a point in a multi-dimensional parameter space. The "roots of unity" points are of course exceptional, but that is not all. Investigating the rigidity of $U_q(gl_n)$ under deformations within the category of coboundary Hopf algebras, one finds rigidity in general (including the standard, one-parameter case), but nontrivial deformations in special cases, among them the new quantizations of $gl(n)$ obtained in Section 5.

2 The Hopf Algebra Dual Form for $U_{q,q'}(gl_2)$

With Woronowicz, we consider the matrix

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{6}$$

with matrix elements that satisfy the relations

$$\begin{gathered} ab = q'ba, \;\; ac = qca, \\ bd = qdb, \;\; cd = q'dc, \\ q'bc = qcb, \;\; ad - da = (q' - q^{-1})bc, \end{gathered} \tag{7}$$

involving two independent parameters q, q'. The algebra generated by a, b, c, d will be augmented by the addition of an inverse to a, to authorize the factorization

$$T = \begin{pmatrix} 1 & 0 \\ x^- & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & \hat{d} \end{pmatrix} \begin{pmatrix} 1 & x^+ \\ 0 & 1 \end{pmatrix}. \tag{8}$$

The algebra is finally completed by including logarithms of a and d, to allow the representation

$$a = e^{x^0}, \;\; \hat{d} = e^{-x^1}, \;\; q = e^h, \;\; q' = e^{h'}. \tag{9}$$

Proposition. The relations (7) are equivalent to the following structure of Lie algebra

$$\begin{gathered} [x^0, x^+] = h'x^+, \;\; [x^0, x^-] = hx^-, \;\; [x^+, x^-] = 0, \\ [x^1, x^+] = hx^+, \;\; [x^1, x^-] = h'x^-, \;\; [x^0, x^1] = 0. \end{gathered} \tag{10}$$

The compatible coproduct, defined by matrix multiplication, takes the form

$$
(11)\quad
\begin{aligned}
\Delta(x^-) &= x^- \otimes 1 + (e^{-x^1} \otimes x^-)(1 + x^+ \otimes x^-)^{-1}(e^{-x^0} \otimes 1),\\
\Delta(x^0) &= x^0 \otimes 1 + 1 \otimes x^0 - (h + h') \textstyle\sum_n (-x^+ \otimes x^-)^n / (e^{nh} - e^{-nh'}),\\
\Delta(x^1) &= x^1 \otimes 1 + 1 \otimes x^1 - (h + h') \textstyle\sum_n (-x^+ \otimes x^-)^n / (e^{nh} - e^{-nh'}),\\
\Delta(x^+) &= 1 \otimes x^+ + (1 \otimes e^{-x^0})(1 + x^+ \otimes x^-)^{-1}(x^+ \otimes e^{-x^1}).
\end{aligned}
$$

There is a sense in which the Hopf algebra generated by a, b, c, d is dual to the 2-parameter quantum group $U_{q,q'}(gl_2)$. A precise concept of duality can be formulated as follows:

Definition. Let (A, B) be an algebra A with countable basis B. The topological dual (A^*, B^*) is the vector space of linear functions on A, containing all those vanishing on all but a finite subset of B, (with B^* as the dual basis), and certain formal series [4] of elements of B^*. [When, as in the case that A is finitely generated, the choice of B is more or less natural, we speak of the dual A^* of A without referring to B.]

Proposition. The bialgebra dual of the deformed enveloping algebra $U_{q,q'}(gl_2)$ is the deformed enveloping algebra $U_{q,q'}(gl_2{}^*)$ generated by $(x^i), i = -, 0, 1, +$, with relations (10) and coproduct (11).

Proof. In $U_{q,q'}(gl_2{}^*)$ take the basis $(x^-)^m(x^0)^n(x^1)^r(x^+)^s, m, n, r, s = 0, 1, 2, \ldots$, and let (P_{mnrs}) be the dual basis, so that the dual form is

$$
(12)\qquad T = \sum (x^-)^m (x^0)^n (x^1)^r (x^+)^s \, P_{mnrs}.
$$

Direct calculation using the fundamental relations (3), the relations (10) and the coproduct (11), immediately yields the result that the dual is generated by

$$
(13)\qquad p_- = P_{1000},\ p_0 = P_{0100},\ p_1 = P_{0010},\ p_+ = P_{0001},
$$

with the relations and the coproduct that characterize $U_{q,q'}(gl_2)$, namely

$$
(14)\qquad
\begin{aligned}
[p_0, p_\pm] = \pm p_\pm = [p_1, p_\pm],\ [p_0, p_1] = 0,\\
[p_+, p_-] = (q - q'^{-1})^{-1}[e^{hp_0} e^{h'p_1} - e^{-h'p_0} e^{-hp_1}].
\end{aligned}
$$

and

$$
(15)\qquad
\begin{aligned}
\Delta(p_0) = p_0 \otimes 1 + 1 \otimes p_0,\ \Delta(p_1) = p_1 \otimes 1 + 1 \otimes p_1,\\
\Delta(p_+) = p_- \otimes e^{-h'p_0 - hp_1} + 1 \otimes p_+,\\
\Delta(p_-) = p_- \otimes 1 + e^{hp_0 + h'p_1} \otimes p_-,
\end{aligned}
$$

Proposition. The dual form (12) can be expressed in terms of the generators, as follows

$$
(16)\qquad T = e^{x^- p_-}_{1/k} e^{x^0 p_0} e^{x^1 p_1} e^{x^+ p_+}_k.
$$

Here $k = qq'$ and e^x_k is the usual twisted exponential

$$
(17)\qquad e^x_k := \sum_n \frac{x^n}{[n!]_k},\ [n!]_k := [1]_k [2]_k \ldots [n]_k,\ [n]_k = \frac{k^{n+1} - 1}{k - 1}.
$$

To summarize. The bialgebra dual of $U_{q,q'}(gl_2)$, with structure relations (14) and coproduct (15), is the solvable quantum group $U_{q,q'}(gl_2)$ with structure (10) and coproduct (11). Both are thus finitely generated and the Hopf algebra dual form can be expressed, as in (16), in terms of the generators.

3 Esoteric Quantum gl_3

All recent activity in the field of integrable models makes essential use of the Yang-Baxter matrix, in its original quantum mechanical context or in the classical form developed later [5]. The existence of the classical r-matrix was interpreted by Drinfeld [6] in terms of coboundary Lie bialgebras, and the construction of a related quantum R-matrix could then be viewed as arising from quantization, especially when quantization is interpreted as a deformation of classical structures [7]. A principal new result reported here is that Drinfeld's restriction to coboundary Lie bialgebras can be relaxed.

Basic to all applications is the construction of a Lax pair, a matrix valued, flat connection. Two Lie algebras are involved. In this section, G_x will denote the Lie algebra of basic dynamical variables. This is not the infinite dimensional Poisson algebra on phase space, but some finite subalgebra of it that is the focus of interest and that forms the basis for quantization. In the context of integrable models, G_x is generated by the dynamical variables that appear as matrix elements of the Lax connection. Typically, G_x is the Heisenberg algebra (as in the nonlinear Schroedinger model) or the Euclidean group (as in the sine-Gordon theory), and sometimes it is a simple Lie algebra (as in the case of spin systems). On the other hand, G_l will denote the auxiliary matrix algebra where the Lax connection lives. There is no *a priori* known principle that commands a particular choice of G_l, except that a preference for simplicity usually leads to the lowest possible dimension; this explains why this algebra is often simple; in the familiar examples it is $gl(2)$.

A Lax pair does not automatically imply the existence of an r-matrix. According to the insight of Drinfeld [6], the r-matrix is intimately related to the fact that G_x and G_l can be promoted to a pair of mutually dual Lie bialgebras, and more especially to the fact that G_l (but not G_x!) is a coboundary Lie bialgebra. The connection is provided by the relation $f = dr$, where f is the G_l one-form that determines the coproduct on G_l (and the Lie product on G_x), and dr is the differential of r. This then implies very strong conditions of compatibility between the two algebras. For example, if G_l is $sl(2)$, then the structure of G_x is fixed; it is not the Euclidean algebra $E(2)$, though it is similar to it (one of the signs in the commutation relations is different). Therefore, sl_2 cannot be used as the auxiliary algebra to solve the sine-Gordon model, for example, for the dynamical algebra of this model is $E(2)$. The impasse is resolved by the device of the spectral parameter. Instead of $sl(2)$, one introduces the infinite dimensional Lie algebra $sl(2)[\lambda]$, of matrices of power series in a parameter λ, the spectral parameter. This infinite dimensional Lie algebra can be endowed with a coproduct that turns it into a coboundary Lie algebra. Also, and this is the point, its (infinite dimensional) dual contains $E(2)$ as a quotient. Here are

the details.

The r-matrix that was used to solve the sine-Gordon model is [8]

$$r = (l_3 \otimes l_3 \cosh u + l_+ \otimes l_- + l_- \otimes l_+) / \sinh u, \ u = \lambda - \mu. \tag{18}$$

in which (l_i) is the usual Weyl basis for sl_2. The algebra $sl_2[\lambda]$ has the basis $(l_{ni} = \lambda^n l_i), i = 3, +, -$, with n running over an infinite set of integers to be specified. The dual form is

$$\Gamma = \sum_{n,i} \gamma^{ni} l_{ni}, \tag{19}$$

and the Lie structure of $sl(2)[\lambda]^*$ (expressed as a Poisson bracket) is determined by

$$\{\Gamma, \Gamma'\} = [r, \Gamma + \Gamma'], \ \Gamma' = \sum_{n,i} \gamma^{ni} l'_{ni}, \tag{20}$$

where l'_{ni} is the basis for another copy of $sl(2)[\lambda]$. One finds that

$$\{\gamma^{n+}, \gamma^{m-}\} = \tau_n^{n+m} \gamma^{n+m,3}, \ \{\gamma^{m,\pm}, \gamma^{n,3}\} = \sigma_m^{n+m} \gamma^{n+m,\pm}, \tag{21}$$

with the following non-zero coefficients:

$$\begin{aligned} \tau_k^n &= 1, \ k = 1-n, \ldots -3, -1, 1, 3, \ldots, n-1, \\ \sigma_k^m &= \begin{cases} -1, & k = m = 1, 3, \ldots, \\ +1, & k = m = -1, -3, \ldots, \\ -2, & k = 1, 3, 5, \ldots, m-2, \\ +2, & k = -1, -3, \ldots, 2-m. \end{cases} \end{aligned} \tag{22}$$

This dual algebra contains $E(2)$ as the quotient by the ideal generated by all γ^{ni} except $\gamma^{0,3}, \gamma^{1,+}$, and $\gamma^{-1,-}$.

Thus it seems as if, to incorporate a physically relevant dynamical algebra - $E(2)$, $sl_2, \ldots$ - into the scheme, the use of a spectral parameter is essential, so that one has to deal with infinite dimensional Lie algebras and infinite dimensional quantum groups. This does not annihilate the subject; but it is interesting to note that all recent work on quantum groups (this paper included) deals with finite dimensional Lie algebras, although this makes the connection to solvable models a little distant. It would certainly be interesting to see some more examples (besides the Liouville model and the conformal Toda models that generalize it [10]) that do not require infinite dimensional Lie algebras but instead provide direct application of q-deformations of finite dimensional Lie bialgebras to integrable field theories.

Our inspiration comes from the high degree of symmetry that exists in the dual relationship between G_x and G_l. The quantum group $U_{q,q'}(gl_2)$ is a deformation of the enveloping algebra $U(gl_2)$ and its bialgebra dual is also a deformation of an enveloping algebra. It is not too much to conjecture that, quite generally, the bialgebra dual of $U_q(G_l)$ is a deformation $U_q(G_x)$ of the enveloping algebra of G_x. This has interesting implications. Namely, by stressing the symmetry between the two Lie bialgebras, rather than the relation of duality between them, one should be able to use them interchangeably; in spite of the fact that only

one of them is a coboundary. In the simplest interesting case, that of the quantum group $U_{q,q'}(gl_2)$, the bialgebra dual form is known explicitly in terms of q-exponentials, Eq.(2). The two Lie bialgebras appear quite symmetrically in this "universal T-matrix."

Let us see what kinds of models can be built on this structure, without introducing a spectral parameter. Only $gl(2)$ is a coboundary Lie bialgebra, its dual is not; therefore, according to Drinfeld, it is $gl(2)$ that is suitable for use as matrix algebra; and $gl(2)^*$ must play the role of the dynamical algebra. But this is quite contrary to the requirements of the applications, so it is natural to persist in trying to reverse the situation. Thus, we identify $gl(2)$ with the dynamical algebra, and the solvable dual $gl(2)^*$ with the matrix algebra.

If in the formula (2) one takes the l's in the 2-dimensional representation of $gl(2)$ (which is what one normally does), then the matrix elements T_i^j of T satisfy the algebraic conditions (7) of Woronowicz's matrix pseudogroup. Instead, we shall now take the x's in a faithful representation of $gl_2{}^*$, then the matrix elements of T satisfy another set of algebraic relations. This new algebra of matrix elements is related to gl_2 and to $U_{q,q'}(gl_2)$ in the same way that the algebra of Woronowicz is related to $gl(2)^*$ and to $U_{q,q'}(gl_2{}^*)$.

The dual Lie algebra $gl(2)^*$ is solvable; the faithful representations of smallest dimension are 3-dimensional. In one such representation the universal T-matrix (2) reduces to

$$T = \begin{pmatrix} a & b & 0 \\ 0 & d & 0 \\ 0 & c & \hat{a} \end{pmatrix}, \tag{23}$$

with matrix elements

$$\begin{gathered} a = e^{h[p_1 - \eta(p_1+p_2)]}, \ \hat{a} = e^{h[p_1 + \eta(p_1+p_2)]} \\ b = hp_- d, \ c = h\hat{a}p_+, \ d = e^{hp_2} \end{gathered} \tag{24}$$

satisfying the relations

$$\begin{gathered} ab = q^{-1}ba, \ ac = qca, \ [a,\hat{a}] = 0, \ [a,d] = 0 = [\hat{a},d], \\ \hat{a}b = q^{-1}b\hat{a}, \ \hat{a}c = qc\hat{a}, \ bd = q^{-1}db, \ cd = qdc, \\ [b,c] = \lambda(a\hat{a} - dd). \end{gathered} \tag{25}$$

The two parameters q and q' of $U_{q,q'}(gl_2)$ have been replaced by $e^{h(1-\eta)}$ and $e^{h(1+\eta)}$, respectively. One regards η as a fixed parameter of the undeformed theory and h as a single deformation parameter. From now on this will uniformly be the significance of η, h and $q = e^h$.

The roles of the two underlying Lie algebras have thus been reversed. Recall that, in the applications, the matrices are used to construct the Lax connection, while the matrix elements are the dynamical variables. Our example is therefore appropriate for systems whose principal dynamical variables are those of gl_2 or sl_2 (after restriction) or $E(2)$ (after contraction); for example, spin systems and the sine-Gordon model.

In order for this to be useful one needs (according to present wisdom) an R-matrix. The existence of an R-matrix satisfying

$$R(T \otimes 1)(1 \otimes T) = (1 \otimes T)(T \otimes 1)R \tag{26}$$

would imply the "multiplication property", but the converse is not true. The multiplication property is built into (2) and is expressed by (3), but the existence of an R-matrix satisfying (26), (the "coboundary" property) must be checked directly. A matrix satisfying this condition will be constructed explicitly below, after some preparation.

Once more according to current wisdom, this matrix should have to satisfy the Yang-Baxter or braid relation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{27}$$

That this relation turns out to be satisfied in the case at hand is a surprise. Recall that the infinitesimal form of this relation is the classical Yang-Baxter relation, and that this is related to the coboundary property of a Lie bialgebra, the matrix algebra G_l. In our case that would be the solvable dual of $gl(2)$. But this Lie bialgebra is not coboundary! Nevertheless, we verify that the R-matrix does have an expansion around unity, and that the first order term satisfies a modified classical Yang-Baxter relation. This surprising and, we believe important result does not contradict what was known previously. The point is that our r-matrix does not live in $gl(2) \otimes gl(2)$, but in $M_3 \otimes M_3$.

The wider implication of this result is that, to find a useful Lax pair for integrable models in which the dynamical algebra is $E(2)$ or $sl(2)$, it may be unnecessary to introduce a spectral parameter and, with it, an infinite dimensional auxiliary algebra. A flat sl(2) connection does not exist, but one can be found that is valued in a larger, finite dimensional matrix algebra.

We try to interpret the matrix (23), with matrix elements satisfying the relations (25), in terms of automorphisms of a 3-dimensional quantum plane, an algebra with generators ("coordinates") $(\xi^i), i = 1, 2$, and quadratic relations [9]. That is, we require that the mapping given by

$$\xi \mapsto \xi T = \xi' = (\xi^1 a, \xi^1 b + \xi^2 d + \xi^3 c, \xi^3 \hat{a}), \tag{28}$$

preserve the relations. It is easy to see that this condition is satisfied if the relations among the coordinates are

$$q\xi^1\xi^2 = \xi^2\xi^1, \; q\xi^2\xi^3 = \xi^3\xi^2, \; q^2\xi^1\xi^3 = \xi^3\xi^1, \tag{29}$$

and that these are the only relations of the type $\xi^i\xi^j = q_{ij}\xi^j\xi^i$, $i < j$, that are so preserved.

Now our strategy for constructing the R-matrix calls for a similar statement about an associated exterior algebra. Let $(\theta^i), i = 1, 2, 3$, be the generators of this exterior algebra; what is preserved is the unique structure

$$\begin{gathered} \theta^1\theta^2 + q\theta^2\theta^1 = 0, \; \theta^2\theta^3 + q\theta^3\theta^2 = 0, \; \theta^1\theta^3 + \theta^3\theta^1 = 0, \\ \theta^1\theta^1 = \theta^2\theta^2 = 0, \; \theta^2\theta^2 + \lambda\theta^3\theta^1 = 0. \end{gathered} \tag{30}$$

There is nothing canonical about the relations (29-30), except for their number, 3^2; but they are all preserved when T acts from the right.

This property of the relations (29-30) has the important implication that there is a numerical 9×9 matrix U such that

$$\theta^i\theta^j U^{kl}_{ij} = 0, \; \xi^i\xi^j(U - 1 - q^2)^{kl}_{ij} = 0, \tag{31}$$

which is unique if $1+q^2 \neq 0$, namely

$$(32)\quad U = (1+q^2)\otimes\begin{pmatrix}1 & q\\ q & q^2\end{pmatrix}\oplus\begin{pmatrix}1 & -\lambda q^2 & q^2\\ 0 & 1+q^2 & 0\\ 1 & \lambda & q^2\end{pmatrix}\oplus\begin{pmatrix}1 & q\\ q & q^2\end{pmatrix}\oplus(1+q^2).$$

[The subspaces are in correspondence with $i+j = 2,3,...,6$.]

The fact that U preserves both structures is equivalent to the statement that U commutes with $(T\otimes T)$; hence U is, up to the addition of a scalar, the matrix that is conventionally denoted $\hat{R}$:

$$(33)\qquad \hat{R} = U - q^2.$$

This turns (31) into

$$(34)\qquad \theta\theta(\hat{R}+q^2) = 0,\ \xi\xi(\hat{R}-1) = 0,$$

so that $\xi^i\xi^j$ is "symmetric" (the deformed permutation $\hat{R}$ is unity on $\xi\xi$). Also, this makes $R \mapsto 1$ when $h \mapsto 0$. This construction of R does not guarantee that it satisfies the Yang-Baxter relation; nevertheless it does. In other words, we have discovered a new coboundary bialgebra deformation of $U(gl_3)$. It is related to a known but somewhat esoteric classical r-matrix [3].

4 The Dual Form for Standard Quantum $gl(n)$

Following Woronowicz [1] and Manin [9], we investigate a bialgebra A, generated by elements $(z_i^j), i,j = 1,...,n$. The relations are

$$(35)\qquad \begin{aligned} z_i^j z_i^k &= q z_i^k z_i^j,\ j<k,\\ z_i^k z_j^k &= q z_j^k z_i^k,\ i<j,\\ [z_i^j, z_k^l] &= 0, \qquad j<l,\ i>k,\\ [z_i^j, z_k^l] &= \lambda z_k^j z_i^l,\ j<l,\ i<k, \end{aligned}$$

with $\lambda := q - q^{-1}$, and the coproduct is defined by matrix multiplication,

$$(36)\qquad \Delta(z_i^j) = z_i^k \otimes z_k^j.$$

We do not fix the quantum determinant.

Define three "bialgebra quotients" of A. As algebras, $(A_i), i = -,0,+$, are quotients of A by ideals I_i that are generated respectively by

$$(37)\qquad I_- : \{z_i^j, i<j\},\ I_0 : \{z_i^j, i\neq j\},\ I_+ : \{z_i^j, i>j\}\ ;$$

the relations of A_i are obtained from A by setting the generators of I_i to zero.

The reason why these algebraic quotients deserve to be called bialgebra quotients is that a compatible coproduct is naturally induced on each from that of A; namely, the coproduct (36) has the property

$$(38)\qquad \Delta(I) \subset I\otimes A + A\otimes I,$$

for $I = I_-, I_0$ or I_+. For example, taking z_i^j in I_-, thus $i < j$, it is evident that $\Delta(z_i^j)$ has one or the other factor in I_-.

We introduce copies of A_i, with the correspondence

$$A_- : z_i^j \mapsto \hat{X}_i^j, \ A_0 : z_i^i \mapsto z_i, \ A_+ : z_i^j \mapsto \hat{Y}_i^j. \tag{39}$$

The formula

$$z_i^j = \hat{X}_i^k \hat{Y}_k^j \tag{40}$$

is an imbedding of A into $A_- \otimes A_+$, but to obtain a unique decomposition of A we need to take a factor A_0 out of A_- and A_+. The generators z_i^i of A are assumed to be invertible. Now there are unique factorizations (x_i and y_i being invertible),

$$\begin{aligned} \hat{X}_i^j &= X_i^j x_j, \ j \le i, \quad X_i^i = 1, \\ \hat{Y}_i^j &= y_i Y_i^j, \ i \le j, \quad Y_i^i = 1. \end{aligned} \tag{41}$$

The elements x_i generate a copy A_{0-} of A_0, an Abelian bialgebra (commutative and co-commutative), and the y_i's generate another copy, A_{0+}. The relations among the X_i^j (the Y_i^j) are of the same form as those that hold for the z_i^j, with the obvious specializations. These quantities generate bialgebras A_{--} and A_{++}, with the usual formula for the coproducts, but they are not bialgebra quotients of A_- and A_+.

We imbed A_0 into $A_{0-} \otimes A_{0+}$ by setting

$$z_i = x_i y_i \tag{42}$$

and notice that the factorization (40) takes the form

$$z_i^j = \sum_{k \le i,j} X_i^k z_k Y_k^j. \tag{43}$$

Proposition. This factorization is unique, $A = A_- \otimes_{A_0} A_+$.

Proof. One begins by noticing that $z_1^1 = z_1$ and proceeds by induction.

The relations satisfied by the X_i^k, the z_k and the Y_k^j are easily determined from the formulas given; note in particular that X_i^j and x_i commute with Y_k^l and y_k while for example,

$$X_i^j z_i = q z_i X_i^j, \ i > j, \quad z_i Y_i^j = q Y_i^j z_i, \ i < j. \tag{44}$$

We analyze the algebraic structure of A_{--}. Among the generators $X_i^j, i < j$, we distinguish the "simple generators"

$$X_i = X_{i+1}^i, \ i = 1, ..., N-1. \tag{45}$$

They satisfy the following relations, the commutation relations

$$[X_i, X_j] = 0, \ |i - j| > 1 \tag{46}$$

(47) $$X_iX_{i+1} - q^{-1}X_{i+1}X_i =: [X_i, X_{i+1}]_{1/q} = \lambda X_{i+2}^i,$$

and the "Serre relations"

(48) $$\begin{aligned} [X_i, [X_i, X_{i+1}]_{1/q}]_q &= 0, \\ [X_i, X_{i+1}]_{1/q}, X_{i+1}]_q &= 0. \end{aligned}$$

Recall that the generators x_i are supposed to be invertible; set $q = e^h$ and introduce new generators ρ_i by

(49) $$x_i = e^{\rho_i}.$$

The relations, expressed in terms of the X_i^j and the ρ_i, reduce in the limit $h \mapsto 0$ to the commutation relations of a parabolic subgroup of $gl(n)$. After replacing the x_i by the ρ_i we find a deformation of the enveloping algebra of this Lie algebra; by abuse of notation we still call this algebra A_-. We now calculate its dual, and the bialgebra dual form.

Consider the formal series

(50) $$T_{x,p} = \sum \prod_{i<j,m} (X_i^j)^{a_{ij}} (\rho_m)^{\alpha_m} P_{[a][\alpha]}.$$

The first two groups of factors run over a basis for A_-, the last over a basis for $A_-{}^*$. The subscript $[a]$ stands for the set of indices $(a_{ij}, i < j)$ and $[\alpha]$ for the set (α_m). What defines the bialgebra dual form is the set of relations

(51) $$T_{x,p}T_{x',p} = T_{\Delta(x),p}, \; T_{x,p}T_{x,p'} = T_{x,\Delta(p)}.$$

In the first relation, x and x' refer to two copies of A_-; the coproduct is known and the equation determines the relations of $A_-{}^*$. In the second relation, p and p' refer to two copies of $A_-{}^*$; the relations of A_- are known and the equation determines the coproduct of $A_-{}^*$.

The calculation is quite painless provided that the most convenient ordering of the X_i^j is chosen:

(52) $$X_2^1, X_3^1, ..., X_n^1, X_3^2, ..., X_n^2, X_n^3, ...$$

We find that $A_-{}^*$ is generated by

(53) $$P_i^j = P_{[a][0]}, \; a_i^j = 1, \text{ all others zero},$$

and

(54) $$P_m = P_{[0][\alpha]}, \; \alpha_m = 1, \text{ all others zero}.$$

The relations satisfied by the generators are

(55) $$\begin{aligned} [P_i, P_j] &= 0, \\ [P_i, P_j^k] &= (\delta_{ij} - \delta_{ik})P_j^k, \; j > k, \\ [P_i^j, P_j^l]_q &= P_i^l, \; i > j > l, \\ [P_i^j, P_k^l]_q &= 0, \; i > j \neq k > l. \end{aligned}$$

The expression for $T_{x,p}$ is simply

$$T_{x,p} = \prod_{i<j,m} e_k^{X_i^j P_i^j} e^{\rho_m P_m}, \; k := q^{-2}, \tag{56}$$

the factors to be ordered as above. This completes the calculation of the universal T-matrix for A_-. Similarly, for A_+, set $y_i = e^{\sigma_i}$ and

$$T_{x,p} = \prod e^{\sigma_m P_m} e_k^{Y_i^j \hat{P}_i^j}, \tag{57}$$

to discover the relations of $A_+{}^*$:

$$\begin{aligned} [P_i, P_j] &= 0, \\ [P_i, \hat{P}_j^k] &= (\delta_{ij} - \delta_{ik})\hat{P}_j^k, \; j < k, \\ [\hat{P}_i^j, \hat{P}_j^l]_q &= P_i^l, \; i < j < l, \\ [\hat{P}_i^j, \hat{P}_k^l]_q &= 0, \; i < j \neq k < l. \end{aligned} \tag{58}$$

For the complete algebra A we combine (56) and (57).

Theorem. The bialgebra dual of the standard quantum deformation of $U(gl_n)$ is generated by (X_i^j, τ_i, Y_i^j), $\tau_i = \rho_i + \sigma_i$, and with relations and coproduct inherited from (35) and (36). The bialgebra dual form can be expressed in terms of the generators,

$$T_{x,p} = \prod e_{1/k}^{X_i^j P_i^j} e^{\tau_i P_i} e_k^{Y_k^l \hat{P}_k^l}, \; k := q^2, \tag{59}$$

(The factors involving the X_i^j have to be ordered as in (52) and those involving the Y_i^j as in $Y_1^2, Y_1^3, \ldots$.)

5 Esoteric Quantum $gl(2n-1)$

We shall describe a generalization of esoteric quantum gl_3. The R-matrix takes the form

$$R = R_0 + R_1, \tag{60}$$

with

$$\begin{aligned} R_0 = \sum_{i=1}^{2n-1} M_i^i \otimes M_i^i + \sum_{i<j<2n} (1-q^2) M_j^i \otimes M_i^j + \sum_{1\neq j, i+j\neq 2n} q M_i^i \otimes M_j^j \\ + \sum_{i<n} M_i^i \otimes M_{i'}^{i'} + \sum_{i<n} q^2 M_{i'}^{i'} \otimes M_i^i, \end{aligned} \tag{61}$$

and R_1 equal to

$$\sum_{k<i<n} (\mu_i' M_i^n \otimes M_{i'}^n + \mu_i M_{i'}^n \otimes M_i^n) + \sum_{k<i<j<n} (\lambda_{ij} M_i^j \otimes M_{i'}^{j'} + q^2 \lambda_{ij}' M_{i'}^{j'} \otimes M_i^j). \tag{62}$$

The coefficients $(\mu_i \neq 0), k < i < n$ determine the others,

$$\begin{aligned} \mu'_i &= -q^{2(i-n)}\mu_i, \ k < i < n, \\ \lambda_{ij} &= (1-q^2)q^{2(i-j)}(\mu_i/\mu_j), \\ \lambda'_{ij} &= (1-q^{-2})(\mu_i/\mu_j), \ i < j < n. \end{aligned} \tag{63}$$

Here $i' := 2n - i$ and k is fixed, $0 \leq k \leq n-2$.

The bialgebra A is the algebra generated by the matrix elements $(T_i^j), i, j = 1, ..., 2n-1$ of a matrix T, with relations

$$R_{12}T_2T_1 = T_1T_2R_{12}, \tag{64}$$

and with the coproduct

$$\Delta(T_i^j) = T_i^k \otimes T_k^j. \tag{65}$$

The factorization used in the standard case does not work here.

We shall describe three bialgebra quotients, denoted A_-, A_+ and A_0; they will play the same roles as their homologues in the standard case, but their construction and their properties will be different. In particular, A_0 will not be Abelian.

We need some descriptive terms for matrices. If $m = (m_i^j)$ is any matrix we denote by (i, j) the position of m_i^j in it. Two regions are important,

$$D_+ = \{(i,j), k < i < j \leq n\}, \ D_- = \{(i,j), n \leq j < i < 2n-k\}. \tag{66}$$

Definition. A matrix (m_i^j) is called almost lower triangular if $m_i^j = 0$ whenever $i \ll j$, which means that $i < j$ and $(i,j) \notin D_+$; it is called almost upper triangular if $m_i^j = 0$ whenever $i \gg j$, by which is meant that $i > j$ and $(i,j) \notin D_-$.

As matrices, A_- is a space of almost lower triangular matrices and A_+ is a space of almost upper triangular matrices, while A_0 consists of matrices that are almost upper triangular as well as almost lower triangular; they may be said to be almost diagonal:

$$\begin{pmatrix} z & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z & z & z & 0 & 0 & 0 \\ 0 & 0 & z & z & 0 & 0 & 0 \\ 0 & 0 & 0 & z & z & 0 & 0 \\ 0 & 0 & 0 & z & z & z & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & z \end{pmatrix} \tag{67}$$

In this illustration $n = 4$ and $k = 1$. As algebras, $(A_i), i = -, +, 0$, are quotients of A by ideals I_i that are generated respectively by

$$I_- : \{z_i^j, i \ll j\}, \ I_+ : \{z_i^j, i \gg j\}, \ I_0 : \{z_i^j, i \ll j \text{ or } i \gg j\}. \tag{68}$$

The algebras A_i are generated by the complimentary sets of generators of A.

We now explain what motivated the above definitions.

It is well known, and an immediate consequence of the Yang-Baxter relation, that the mapping π_-, from A to the space of matrices of dimension $2n-1$, generated by

$$T_i^j \mapsto \pi_-(T_i^j) = R_i^j = \{(R_i^j)_a^b\} = \{R_{ia}^{jb}\} \tag{69}$$

is a representation of A as an algebra. As for the mapping

$$T_i^j \mapsto \pi_+(T_i^j) = S_i^j = \{(S_i^j)_a^b\} = \{R_{ai}^{bj}\}, \tag{70}$$

it is (almost) an anti-homomorphism. Now it is seen by inspection of R that the kernels of π_- and π_+ contain the ideals I_- and I_+, respectively, that is,

$$R_i^j = 0,\ i \ll j,\ S_i^j = 0,\ i \gg j. \tag{71}$$

Furthermore, the representatives R_i^i and S_i^i of the generators z_i^i are invertible. This shows that these generators can be taken to be invertible, as in the standard case, as generators of A_-, of A_+ or A.

There is a unique decomposition of A, similar to the decomposition (43) of the standard case, into three factors,

$$A = A_- \otimes_{A_0} A_+. \tag{72}$$

The factors are defined in the same way as before, starting with the new ideals defined in (67). This decomposition is useful for understanding the structure of esoteric quantum $gl(2n-1)$ and suitable for evaluating the associated universal T-matrix. Complete results will be published elsewhere.

We list all the relations Eq.(65). When $a < i, b < j$, and $a \neq i'$, $b \neq j'$,

$$\begin{aligned}
[z_a^j, z_a^b]_q &= 0,\\
[z_a^{b'}, z_a^b]_{q^2} &= q^2 \textstyle\sum_l \lambda_{lb} z_a^l z_a^{l'} = -\sum_l \lambda'_{lb} z_a^{l'} z_a^l,\\
[z_i^b, z_a^b]_q &= \delta_n^b \textstyle\sum_l (-\mu'_l z_i^{l'} z_a^l - \mu_l z_i^l z_a^{l'}) = \delta_{bn} q^{-1} \sum_l (\mu_l z_a^l z_i^{l'} + \mu'_l z_a^{l'} z_i^l),\\
[z_a^b, z_{a'}^b] &= \delta_n^b q^{-2} \textstyle\sum_l (-\mu'_l z_a^{l'} z_{a'}^l - \mu_l z_a^l z_{a'}^{l'}) + \sum_l \lambda_{al} z_{l'}^b z_l^b + q^{-2} \mu_a z_n^b z_n^b\\
&= \delta_n^b \textstyle\sum_l (\mu_l z_{a'}^l z_a^{l'} + \mu'_l z_{a'}^{l'} z_a^l) - \mu'_a z_n^b z_n^b - \sum_l \lambda'_{al} z_l^b z_{l'}^b,\\
[z_a^b, z_i^j] &= 0,\\
[z_a^{b'}, z_i^b]_q &= -\textstyle\sum_l \lambda'_{lb} z_a^{l'} z_i^l,\\
[z_a^j, z_{a'}^b]_q &= q^{-1} \mu_a z_n^b z_n^j + q \textstyle\sum_l \lambda_{al} z_{l'}^b, z_l^j,\\
[z_a^{b'}, z_{a'}^b]_{q^2} &= q^2 \textstyle\sum_l \lambda_{al} z_{l'}^b z_l^{b'} - \sum_l \lambda'_{ib} z_a^{l'} z_{a'}^l + \mu_a z_n^b z_n^{b'},\\
[z_a^b, z_i^j] &= (q^{-1} - q) z_a^j z_i^b,\\
[z_i^{b'}, z_a^b]_q &= (q - q^{-1}) z_a^{b'} z_i^b + q \textstyle\sum_l \lambda_{lb} z_a^l z_i^{l'},\\
[z_a^b, z_{a'}^j]_q &= (q^{-1} - q) z_a^j z_{a'}^b + q^{-1} \mu_a z_n^j z_n^b + q \textstyle\sum_l \lambda_{al} z_{l'}^j z_l^b,\\
[z_a^b, z_{a'}^{b'}] &= (q^{-2} - 1) z_a^{b'} z_{a'}^b - \textstyle\sum_l \lambda_{lb} z_a^l z_{a'}^{l'} + q^{-2} \mu_a z_n^{b'} z_n^{b'} + \sum_l \lambda_{al} z_{l'}^{b'} z_l^b.
\end{aligned} \tag{73}$$

It is easy to see that the relations are just enough to allow every element of the algebra to be expressed in the form of a standard ordered polynomial.

Now we pass to the quotient A_0 by setting

$$z_i^j = 0, \text{ unless } k < i \leq j \leq n \text{ or } n \leq j \leq i < 2n - k. \tag{74}$$

Again one verifies that the standard ordered monomials span the algebra. The

relations of A_0 are a little simpler; with $a < i, b < j$ and $a' \neq i, b' \neq j$,

$$
\begin{aligned}
&[z_a^j, z_a^b]_q &&= 0,\\
&[z_i^b, z_a^b]_q &&= \delta_n^b \textstyle\sum_l(-\mu_l' z_i^{l'} z_a^l - \mu_l z_i^l z_a^{l'}) = \delta_n^b q^{-1} \sum_l(\mu_l z_a^l z_i^{l'} + \mu_l' z_a^{l'} z_i^l),\\
&[z_a^b, z_{a'}^b] &&= \delta_n^b(-q^{-2}\textstyle\sum_l \mu_l z_a^l z_{a'}^{l'} + \sum_l \lambda_{al} z_{l'}^n z_l^n + q^{-2}\mu_a z_n^n z_n^n)\\
& &&= \delta_n^b(\textstyle\sum_l \mu_l' z_{a'}^{l'} z_a^l - \mu_a' z_n^n z_n^n - \sum_l \lambda_{al}' z_l^n z_{l'}^n),\\
(75)\quad &[z_a^j, z_i^b] &&= 0,\\
&[z_a^b, z_i^j] &&= (q^{-1} - q) z_a^j z_i^b,\\
&[z_i^{b'}, z_a^b]_q &&= q \textstyle\sum_l \lambda_{lb} z_a^l z_i^{l'},\\
&[z_a^b, z_{a'}^j]_q &&= q \textstyle\sum_l \lambda_{al} z_{l'}^j z_l^b,\\
&[z_a^b, z_{a'}^{b'}] &&= -\textstyle\sum_l \lambda_{lb} z_a^l z_{a'}^{l'} + \sum_l \lambda_{al} z_{l'}^{b'} z_l^b.
\end{aligned}
$$

The positive and negative simple roots are

$$e_i = z_i^{i+1}, \text{ and } f_i = z_{i'}^{i'-1}, \ k < i < n; \tag{76}$$

the former commute with the latter except that

$$
\begin{aligned}
[e_{n-1}, f_{n-1}] &= q^{-2}\mu_{n-1}(h_n h_n - h_{n-1}h_{n+1}), \ h_i = z_i^i,\\
[e_i, f_i] &= \lambda_{i,i+1}(h_{i+1}h_{i'-1} - h_i h_{i'}), \ i < n-1.
\end{aligned} \tag{77}
$$

This algebra has a nontrivial center that includes $\{h_a - h_{a'}, a = 1, ..., n-1\}$.

6 Deformations of Standard Multiparameter $U(gl_n)$.

The structure of esoteric quantum $gl(3)$ was found by accident, the generalization to $gl(n)$ by making a general ansatz for the R-matrix and demanding that the quantum Yang-Baxter relation be satisfied. A more satisfactory approach is through the use of deformation theory [11]. Deformations around the classical theory gives nothing in the first order; the classical Yang-Baxter relation appears only in the second order. Perturbations around multiparameter $U_q(gl_n)$, $q \in \mathbf{R}^N$, are more rewarding. We find rigidity at general position in parameter space and interesting, essential deformations on certain algebraic surfaces. We have determined the equivalence classes of essential deformations, and we have related these results to the classification, by Belavin and Drinfeld [3], of classical Yang-Baxter matrices. We describe, briefly, a very interesting special case.

Taking $n = 3$, for simplicity, we note that standard multiparameter quantum $gl(3)$ [12] is characterized by three q-parameters and, in addition, the Hecke parameter; that is, the second eigenvalue of $\hat{R}$, the first one being unity. A very special kind of deformation exists under some conditions on the q's, provided the fourth parameter is a cubic root of -1. The R-matrix for this unusual case has the form (61), with

$$R_1 = \mu M_1^3 M_2^3. \tag{78}$$

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Physics Department, University of California, Los Angeles CA 90024.
Departamento de Física Teórica, Universidad Complutense, Madrid.
E-mail address: Bitnet fronsdal@uclaph.