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IRREDUCIBLE REPRESENTATIONS OF $SL_q(2)$

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Abstract

The irreducible representations of $SL_q(2)$ are obtained by following the familiar procedures for $SL(2)$.

1. Two-Dimensional Representations.

Although the representation theory of quantum groups has been extensively discussed,[1,2,3] we would like to describe a very elementary approach based on the following characterization of the 2-dimensional representations of $SL_q(2)$:

$$T^t \epsilon T = T \epsilon T^t = \epsilon \leftrightarrow T \epsilon SL_q(2) \tag{1.1}$$

where T^t is the transposed matrix and

$$\epsilon = \begin{pmatrix} 0 & q^{-1/2} \\ -q^{1/2} & 0 \end{pmatrix}. \tag{1.2}$$

The statement (1.1) implies the usual $SL_q(2)$ commutation rules between the matrix elements of T and exhibits ϵ as the fundamental invariant of $SL_q(2)$. Thus ϵ plays the same role for $SL_q(2)$ as the identity does for the rotation group.

From (1.1) closure immediately follows:

$$(TT')^t \epsilon (TT') = \epsilon \tag{1.3}$$

if all elements of T commute with all elements of T'.

In general there is a Borel factorization of any 2×2 matrix:

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \tag{1.4}$$

but (1.4) and the commutation rules required by (1.1) further imply

$$T = e^{\beta\sigma_+} e^{\lambda\theta\sigma_3} e^{\gamma\sigma_-} \tag{1.5}$$

where

$$\begin{aligned} (\beta, \gamma) &= 0 \\ (\theta, \beta) &= \beta \\ (\theta, \gamma) &= \gamma \end{aligned} \tag{1.6}$$

and

$$\lambda = \ln q \ .$$

Let us now introduce the vector basis whose components satisfy the quantum plane relation:

$$\chi = (\bar{z}, z) \tag{1.7}$$

$$\bar{z} z = q_1 z \bar{z} \tag{1.8}$$

where z and $\bar{z}$ still commute with all components of T. Then

$$\chi^t \epsilon \chi = q^{-1/2} \bar{z} z - q^{1/2} z \bar{z} \tag{1.9}$$

and by (1.1) $\chi^t \epsilon \chi$ is invariant under $T \epsilon SL_q(2)$:

$$(\chi^t \epsilon \chi))' = \chi^t \epsilon \chi \tag{1.10}$$

where

$$\chi' = T\chi \tag{1.11}$$

$$\chi^{t'} = \chi^t T^t. \tag{1.12}$$

If $q = q_1$, $\chi^t \epsilon \chi$ vanishes and defines an invariant cone.

2. Higher Dimensional Invariants and Representations.

The two terms of (1.9) commute:

$$\bar{z}z \cdot z\bar{z} = z\bar{z} \cdot \bar{z}z. \tag{2.1}$$

Then by the binomial theorem for commuting objects

$$(\chi^t \epsilon \chi)^{2j} = (-)^j \sum \binom{2j}{j+m} (z\bar{z})^{j+m} (\bar{z}z)^{j-m} (-q)^m \tag{2.2}$$

and

$$\frac{(i\chi^t \epsilon \chi)^{2j}}{(2j)!} = \sum \tilde{V}(jm) V(jm) (-q)^m \tag{2.3}$$

$$= 0 \quad \text{if} \quad q = q_1 \tag{2.4}$$

where

$$V(jm) = \frac{z^{j+m} \bar{z}^{j-m}}{[(j+m)!(j-m)!]^{1/2}} \; \epsilon(q_1^{-1} | j+m) \tag{2.5a}$$

$$\tilde{V}(jm) = \frac{\bar{z}^{j+m} z^{j-m}}{[(j+m)!(j-m)!]^{1/2}} \; \epsilon(q_1 | j-m) \tag{2.5b}$$

and

$$\epsilon(q|n) = q^{n(n+1)/2} \tag{2.6}$$

with the following exponential generating function:

$$e^{ix^t \epsilon x} = \sum_{jm} \tilde{V}(jm) V(jm) q^{-m} \tag{2.7}$$

$$= 1 \quad \text{if} \quad q = q_1 . \tag{2.8}$$

The invariants (2.3) may be expressed as follows:

$$Q(j) = \tilde{V}(j) C^j(-q) V(j) \tag{2.9}$$

where $C^j(q)$ is the diagonal matrix:

$$C^j(q) = \begin{pmatrix} q^j & & \\ & \ddots & \\ & & q^{-j} \end{pmatrix} \tag{2.10}$$

The invariants $Q(j)$ lie in a $2j+1$-dimensional space.

The 2-dimensional transformation (1.12) induces on $\tilde{V}$ the following transformation:

$$\tilde{V}'(jm) = \frac{(\bar{z}a + zc)^{j+m} (\bar{z}b + zc)^{j-m}}{[(j+m)!(j-m)!]^{1/2}} \, \epsilon(q_1^{-1} | j+m). \tag{2.11}$$

To expand the right-hand side, note

$$\begin{aligned} (\bar{z}a)(zc) &= q_1 q (zc)(\bar{z}a) \\ (\bar{z}b)(zd) &= q_1 q (zd)(\bar{z}b) \end{aligned} \tag{2.12}$$

Then by the q-binomial theorem

$$\tilde{V}(jm)' = \sum \tilde{V}(jm') \tilde{D}^j(m', m) \tag{2.13}$$

where

$$\begin{aligned} \tilde{D}^j(m', m) = {} & N^{jm}_{jm'} \frac{\epsilon(q_1 | j-m)}{\epsilon(q_1 | j-m')} \, q_1^{-j^2} \\ & \times \sum_t q_1^{\sigma} \left\langle \begin{matrix} j+m \\ j-m'-t \end{matrix} \right\rangle_{qq_1} \left\langle \begin{matrix} j-m \\ t \end{matrix} \right\rangle_{qq_1} c^{j-m'-t} a^{m+m'+t} d^t b^{j-m-t} \end{aligned} \tag{2.14}$$

and

$$N_{jm'}^{jm} = \left[\frac{j+m)!(j-m)!}{(j+m'!)!(j-m'!)!}\right]^{1/2}. \tag{2.15}$$

These are the $2j+1$-dimensional representations. Here

$$\sigma = (t+m')^2 + t(m-m') \tag{2.16}$$

and

$$\left\langle \begin{matrix} n \\ p \end{matrix} \right\rangle_q = \frac{< n >_q!}{< p >_q! < n-p >_q!} \tag{2.17}$$

$$< n >_q = \frac{q^n - 1}{q-1}. \tag{2.17a}$$

The corresponding transformation on $V(jm)$ is

$$V(jm)' = \sum D^j(m,m')V(jm') \tag{2.18}$$

where

$$D^j(m,m') = q^{(m'-m)(2j+1)}\tilde{D}^j(-m',-m). \tag{2.19}$$

The product of two $2j+1$-dimensional transformations will preserve the invariant (2.9). Let the set $(abcd)$ be denoted by α. Then closure is expressed by

$$D^j(\alpha)D^j(\alpha') = D^j(\alpha'') \tag{2.20}$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} \tag{2.21}$$

One may exhibit the inverse transformation by rewriting the basic invariant as

$$Q = \sum \tilde{u}(jm)\ u(jm) \tag{2.22}$$

where

$$
\begin{aligned}
u(jm) &= \lambda^m v(jm) \\
\tilde{u}(jm) &= \lambda^m \tilde{v}(jm) \quad \lambda = iq^{1/2}
\end{aligned}
\tag{2.23}
$$

Let

$$
\begin{aligned}
u'(jm) &= \sum \mathcal{D}^j(m,m')u(jm') \\
\tilde{u}'(jm) &= \sum \tilde{u}(jm')\tilde{\mathcal{D}}^j(m',m)
\end{aligned}
\tag{2.24}
$$

Then

$$
\mathcal{D}^{-1} = \tilde{\mathcal{D}} \tag{2.25}
$$

and

$$
[(\mathcal{D}^j)^{-1}]_{m'm} = (-)^{m-m'} q^{2j(m'-m)} \mathcal{D}^j(-m,-m'). \tag{2.26}
$$

3. Generators of Quantum Group.

For simplicity consider the special case $q_1 = q$ and the differently normalized basis functions

$$
W(jm) = \frac{z^{j+m}\bar{z}^{j-m}}{[< j+m >_q! < j-m >_q!]^{1/2}} \epsilon(q^{-1}|j+m) \tag{3.1}
$$

$$
\tilde{W}(jm) = \frac{\bar{z}^{j+m} z^{j-m}}{[< j+m >_q! < j-m >_q!]^{1/2}} \epsilon(q|j-m). \tag{3.2}
$$

Then

$$
\tilde{D}^j(m',m) = N^{jm}_{jm'} \sum_t q^{-j^2+\sigma} \left\langle \begin{matrix} j+m \\ j-m'-t \end{matrix} \right\rangle_{q^2} \left\langle \begin{matrix} j-m \\ t \end{matrix} \right\rangle_{q^2} c^{j-m'-t} a^{m+m'+t} d^t b^{j-m-t}. \tag{3.3}
$$

By (1.4)

$$
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha + \beta\delta\gamma & \beta\delta \\ \delta\gamma & \delta \end{pmatrix} \tag{3.4}
$$

Define the generators $J_\pm, J_3$ by

$$\tilde{\mathcal{D}}^j(\beta,\theta,\gamma) = \tilde{\mathcal{D}}^j(0,0,0) + \beta J_-^j + 2\lambda\theta J_3^j + \gamma J_+^j + \cdots \tag{3.5}$$

Then the only non-vanishing elements are

$$< m-1|J_-^j|m >= [< j-m+1 >_{q^2} < j+m >_{q^2}]^{1/2} q^{-j-m+1} \tag{3.6}$$

$$< m+1|J_+^j|m >= [< j+m+1 >_{q^2} < j-m >_{q^2}]^{1/2} q^{-j+m+1} \tag{3.7}$$

$$< m|J_3|m >= m \tag{3.8}$$

leading to the commutation rules

$$(J_+, J_-) = [2J_3]_q \tag{3.9}$$

$$(J_\pm, J_3) = J_\pm \tag{3.10}$$

where

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}. \tag{3.11}$$

4. Orthogonality.

Following the classical argument construct

$$\sum_t D^1(t) A D^2(t)^{-1} = B \tag{4.1}$$

where D^1 and D^2 belong to non-equivalent irreducible representations and A is the appropriate rectangular matrix. Here the sum is over the complete set for which closure holds. Then if s is an arbitrary member of the set one has

$$D^1(s) B D^2(s)^{-1} = B \tag{4.2}$$

by closure. But (4.2) implies

$$B = 0 \tag{4.3}$$

or

$$\sum_t \sum_{k\kappa} D^1_{\ell k}(t) A_{k\kappa} (D^2(t)^{-1}_{\kappa\lambda} = 0. \tag{4.4}$$

Let

$$A_{k\kappa} = \delta_{km}\delta_{\kappa\mu}. \tag{4.5}$$

Then

$$\sum_t D^1_{\ell m}(t) D^2(t)^{-1}_{\mu\lambda} = 0. \tag{4.6}$$

For $SU(2)$

$$\sum_t D^1_{\ell m}(t) D^2(t)^{+}_{\mu\lambda} = 0 \tag{4.7}$$

where D^+ is the adjoint matrix. For $SU_q(2)$

$$\sum_t D^1_{\ell m}(t) D^2(t)^{\ddagger}_{\mu\lambda} = 0 \tag{4.8}$$

where by (2.27)

$$D^{\ddagger}_{\mu\lambda} = (-)^{\mu-\lambda} q^{2j(\lambda-\mu)} D(-\lambda, -\mu) \tag{4.9}$$

Here $D^{\ddagger}$ replaces the adjoint matrix.

It is necessary to interpret $\sum_t$ in these equations as an integral and at the same time to provide an invariant volume element $d\tau$. For $SU(2)$ one has

$$d\tau = \sqrt{g}\ d\theta d\beta d\gamma \tag{4.10}$$

where g is the determinant of the group metric. For $SU_q(2)$ let us represent the multiplication $t \to st$ by

$$\overset{st}{\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}} = \overset{s}{\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}} \overset{t}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \tag{4.11}$$

or in terms of the parameters (θ, β, γ) of Eq. (1.5)

$$\begin{aligned} e^{-\lambda\theta''} &= c'(\beta e^{-\lambda\theta}) + d'(e^{-\lambda\theta}) \\ \beta'' e^{-\lambda\theta''} &= a'(\beta e^{-\lambda\theta}) + b'(e^{-\lambda\theta}) \\ e^{-\lambda\theta''}\gamma'' &= c'(e^{\lambda\theta} + \beta e^{-\lambda\theta}\gamma) + d'(e^{-\lambda\theta}\gamma) \end{aligned} \tag{4.12}$$

One now finds

$$d\theta'' d\beta'' d\gamma'' = J\left(\frac{\theta''\beta''\gamma''}{\theta\beta\gamma}\right) d\theta d\beta d\gamma \tag{4.13}$$

$$= (a'd' - b'c')e^{2\lambda(\theta''-\theta)} d\theta d\beta d\gamma \tag{4.14}$$

and

$$\sqrt{g''}\; d\theta'' d\beta'' d\gamma'' = (a'd' - b'c')\sqrt{g}\; d\theta d\beta d\gamma \tag{4.15}$$

where we have computed $\sqrt{g}$ for $SU_q(2)$ as one would for $SU(2)$ but with proper regard for the commutation rules (1.6). In this way we obtain

$$\sqrt{g} = \frac{1}{2}\frac{\lambda}{q} e^{-2\lambda\theta}. \tag{4.16}$$

Then, if we define $d\tau$ for $SU_q(2)$ as in (4.10) for $SU(2)$, we have

$$d\tau'' = (a'd' - b'c')d\tau. \tag{4.17}$$

To obtain an invariant volume element one may now replace $d\tau$ by the relative volume element $d\tau/d\tau_o$ where $d\tau_o$ is a reference element that transforms in the same way as $d\tau$. Then (4.6) may be written

$$\int D^1_{\ell m}(t) D^2_{\mu\lambda}(t^{-1}) d\omega(t) = 0 \tag{4.18}$$

where

$$d\omega = d\tau / d\tau_o. \tag{4.19}$$

For equivalent representations one has by the usual argument through Schur's lemma

$$\int D^j_{\ell m}(t) D^j_{\mu\lambda}(t^{-1}) d\omega(t) = c_{m\mu} \delta_{\ell\lambda}. \tag{4.20}$$

By putting $\ell = \lambda$ and summing one determines $c_{m\mu}$ in the familiar way.

Finally

$$\int D^j_{\ell m}(t) D^{j'}_{m'\ell'}(t^{-1}) d\omega(t) = \frac{\Omega}{2j+1} \delta^{jj'} \delta_{mm'} \delta_{\ell\ell'} \tag{4.21}$$

where

$$\Omega = \int d\omega \tag{4.22}$$

and

$$D^j_{m'\ell'}(t^{-1}) = (-)^{m'-\ell'} q^{2j(\ell'-m')} D(-\ell', -m') \tag{4.23}$$

by (4.9).

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