Fermilab Library
0 1160 0006244 2

UCLA-92-TEP4

UCLA/92/TEP/4
January 1992

GRASSMANNIAN LEGENDRE FUNCTIONS

FERMILAB
FEB 1 1992
LIBRARY

Fuk-Lok Chan

Kim-Ming Lau

Robert J. Finkelstein

Department of Physics

University of California, Los Angeles, CA 90024-1547

ABSTRACT. The Grassmannian analogues of the Legendre polonomials are obtained from the corresponding Grassmannian Hermite multinomials. The generating function, recurrence relations and differential equation are given. In contrast to the Hermitian case there is no Berezin weight function that orthogonalizes these functions. Generalizations to the Tchebyscheff and Gegenbauer cases are also possible.

ROUTE TO
NAME
LOCATION
ROUTE TO FERMILAB LIBRARY

1. Introduction.

In an earlier investigation of the representations of the super-Poincaré algebra, it was found useful to introduce a Grassmannian version of the Hermite functions.[1] In that context these functions appear as eigenfunctions of the second Casimir in all but 2 mod 8 dimensions. In ten dimensions they appear as eigenfunctions of the operators of the Cartan subalgebra.[2] Not surprisingly they also appear as eigenfunctions of the Grassmannian oscillator.[3] They are simply related to the Hermite multinomials in many variables that were introduced by Hermite.[4] One may then naturally ask about similar Grassmannian constructions that correspond to other classical polynomials. In particular we are here interested in the Grassmannian analogues of the Legendre polynomials.

If the Grassmann elements are interpreted as coordinates on superspace, then these multinomials provide a basis in superspace. In the Hermite case they are orthogonal in the Berezin sense. If these anticommuting elements are alternatively identified with fermionic emission and absorption operators, then the same formal expressions, in which the individual terms represent clusters of fermions, may serve as a basis in Fock space where orthogonality would be defined in the Fock sense. There is also a geometric interpretation according to which these Grassmannian constructions may be regarded as simplicial complexes or as the corresponding sums of differential forms.

2. Grassmannian Hermite Multinomials.

The extension to many variables of the familiar Hermite polynomials was introduced by Hermite and may be defined as follows:[3]

$$\bar{H}_{m_1,\ldots,m_n}(\vec{x}) = (-)^m \exp\left[\frac{1}{2}\phi(\vec{x})\right] \frac{\partial^m}{\partial x_1^{m_1} \ldots \partial x_n^{m_n}} \exp\left[-\frac{1}{2}\phi(\vec{x})\right], \tag{2.1}$$

where $(-) \equiv (-1)$ and

$$m = \sum_{i=1}^{n} m_i \tag{2.1a}$$

$$\phi(\vec{x}) = \sum_{i,j=1}^{n} C_{ij} x^i x^j . \tag{2.1b}$$

Here C_{ij} is a real symmetric matrix and the x^i are commuting variables.

The Grassmannian generalization is[1,2]

$$H^{\alpha_{m_1},\ldots,\alpha_{m_n}}(q) = \exp\left[\frac{1}{2}\bar{q}q\right] \frac{\partial}{\partial \bar{q}_{\alpha_{m_1}}} \ldots \frac{\partial}{\partial \bar{q}_{\alpha_{m_n}}} \exp\left[-\frac{1}{2}\bar{q}q\right] \tag{2.2}$$

where n is even and

$$\bar{q}_\beta = q^\alpha C_{\alpha\beta} \tag{2.2a}$$

$$\bar{q}q = \sum_{\alpha,\beta=1}^{n} q^\alpha C_{\alpha\beta} q^\beta . \tag{2.2b}$$

Here the q^α are Grassmann arguments

$$(q^\alpha, q^\beta)_+ = 0 \; ; \quad (q^\alpha)^2 = 0 \; ; \quad \alpha,\beta = 1,\ldots,n \tag{2.3}$$

and C is a hermitian but antisymmetric matrix.

It is assumed that C is invertible and therefore of even order. Then

$$q^{\beta} = \bar{q}_{\alpha}(C^{-1})^{\alpha\beta} \tag{2.4}$$

$$\frac{\partial q^{\beta}}{\partial \bar{q}_{\alpha}} = (C^{-1})^{\alpha\beta} \tag{2.5}$$

and

$$H^{\alpha_1 \ldots \alpha_m}(q) = (C^{-1})^{\alpha_1 \beta_1} \ldots (C^{-1})^{\alpha_m \beta_m} \bar{H}_{\beta_1 \ldots \beta_m}(q) \ , \tag{2.6}$$

where

$$\bar{H}_{\beta_1 \ldots \beta_m}(q) = \exp\left[\frac{1}{2}\bar{q}q\right] \frac{\partial}{\partial q^{\beta_1}} \ldots \frac{\partial}{\partial q^{\beta_m}} \exp\left[-\frac{1}{2}\bar{q}q\right] \ . \tag{2.7}$$

One may establish a generating function, orthogonality, recursion relations and a partial differential equation for both the commuting and anticommuting cases. We consider just the anticommuting case.

The generating function is

$$\exp[-\bar{t}t+2\bar{t}q] = \sum \frac{\varepsilon(m)}{m!} \bar{t}_{\alpha_1} \ldots \bar{t}_{\alpha_m} H^{\alpha_1 \ldots \alpha_m}(q) \tag{2.8a}$$

$$= \sum \frac{\varepsilon(m)}{m!} t^{\beta_1} \ldots t^{\beta_m} \bar{H}_{\beta_1 \ldots \beta_m}(q) \ , \tag{2.8b}$$

where the t_{α} also are Grassmannian and also anticommute with the q_{α}

$$(t^{\alpha}, t^{\beta})_{+} = 0 \qquad (q^{\alpha}, t^{\beta})_{+} = 0 \ . \tag{2.9}$$

Here

$$\varepsilon(m) = (-1)^{m/2\ (m-1)} \ . \tag{2.10}$$

Orthogonality is expressed by

$$\int [dq]\ \exp\left[-\frac{1}{2}\bar{q}q\right] H^{\alpha_1 \ldots \alpha_p}(q)\ \bar{H}_{\beta_1 \ldots \beta_m}(q) = \delta^p_m\ \varepsilon(m)\ \varepsilon^{\alpha_1 \ldots \alpha_m}_{\beta_1 \ldots \beta_m}\ B_n \tag{2.11}$$

where

$$B_n = (-1)^{n/2}\ \mathrm{Pf}C \ . \tag{2.12}$$

Here a Berezin integration is understood and PfC is the Pfaffian of C.

3. Recursion Relations and Differential Equations.

Starting with the generating function (2.8b) one obtains by differentiating with respect to q and t respectively

$$\frac{\partial}{\partial q^\lambda}\ \bar{H}_{\beta_1 \ldots \beta_m}(q) = -2m\ C_{\lambda[\beta_1}\ \bar{H}_{\beta_2 \ldots \beta_m]}(q) \tag{3.1}$$

$$\bar{H}_{[\lambda\beta_1 \ldots \beta_m]}(q) + 2\bar{q}_{[\lambda}\ \bar{H}_{\beta_1 \ldots \beta_m]}(q) - 2mC_{[\lambda\beta_1}\ H_{\beta_2 \ldots \beta_m]}(q) = 0 \ . \tag{3.2}$$

These may be combined to give the second order equation

$$\partial_{[\mu\lambda}\ \bar{H}_{\beta_1 \ldots \beta_m]}(q) + \bar{q}_{[\mu}\partial_\lambda\ \bar{H}_{\beta_1 \ldots \beta_m]} - 2mC_{[\mu\lambda}\ \bar{H}_{\beta_1 \ldots \beta_m]} = 0 \ . \tag{3.3}$$

We also know the differential equation satisfied by the complete Hermite

function:[2]

$$\left[(C^{-1})^{\alpha\beta}\,\partial_\alpha\partial_\beta + C_{\alpha\beta}q^\alpha q^\beta\right]\psi^{\lambda_1\ldots\lambda_n} = \left(n - \frac{1}{2}N\right)\psi^{\lambda_1\ldots\lambda_n} \tag{3.4}$$

where

$$\psi^{\lambda_1\ldots\lambda_n} = \exp\left[-\frac{1}{2}\bar{q}q\right] H^{\lambda_1\ldots\lambda_n}(q) \tag{3.5}$$

from which it follows that

$$(C^{-1})^{\alpha\beta}\,\partial_\alpha\partial_\beta\, H^{\lambda_1\ldots\lambda_n}(q) + 2q^\alpha\partial_\alpha H^{\lambda_1\ldots\lambda_n}(q) + 2n\, H^{\lambda_1\ldots\lambda_n}(q) = 0\ . \tag{3.6}$$

4. Grassmannian Legendre Multinomials.

One may pass from the Hermite to the Legendre polynomials by the following rescaling procedure beginning with the usual expression for the generating function for the Hermite polynomials

$$\exp\left[2xt-t^2\right] = \sum_{n=0}^{\infty} H_n(x)\,\frac{t^n}{n!}\ . \tag{4.1}$$

After rescaling both x and t and applying the operator

$$\int_{-\infty}^{\infty} d\lambda\, e^{-\lambda^2} \tag{4.2}$$

one finds

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\int_{-\infty}^{\infty} d\lambda\,\lambda^n e^{-\lambda^2} H_n(\lambda x) = \int_{-\infty}^{\infty} e^{-A^2\lambda^2}\,d\lambda \qquad = \sqrt{\pi}/A \tag{4.3}$$

where

$$A = (1-2xt+t^2)^{1/2} \quad . \tag{4.3a}$$

But $(1-2xt+t^2)^{-1/2}$ is the generating function for the Legendre polynomials. Therefore

$$P_n(x) = \frac{1}{\sqrt{\pi}} \frac{1}{n!} \int_{-\infty}^{\infty} d\lambda \, \lambda^n \, e^{-\lambda^2} \, H_n(\lambda x) \quad . \tag{4.4}$$

As this procedure is primarily a rescaling it may also be applied to the many variable functions. In the commuting case we have

$$\sum \frac{z_1^{m_1}}{m_1!} \cdots \frac{z_n^{m_n}}{m_n!} H_{\vec{m}}(\vec{x}) = \exp\left[2C_{ij}z_ix_j - C_{ij}z_iz_j\right] \quad . \tag{4.5}$$

Then

$$\sum \frac{z_1^{m_1}}{m_1!} \cdots \frac{z_n^{m_n}}{m_n!} \int_{-\infty}^{\infty} d\lambda \, e^{-\lambda^2} \, \lambda^m \, H_{\vec{m}}(\lambda\vec{x}) = \int_{-\infty}^{\infty} d\lambda \, \exp\left[-A^2\lambda^2\right] \tag{4.6}$$

where

$$m = \sum_1^n m_i \tag{4.6a}$$

and

$$A^2 = 1 - 2C_{ij}z_ix_j + C_{ij}z_iz_j \; , \tag{4.6b}$$

where the repeated indices are summed. Since A is the many variable form of

(4.3a) let us define the corresponding many variable form of the Legendre polynomials by

$$P_{\vec{m}}(\vec{x}) = \int_{-\infty}^{\infty} d\lambda \, \exp\left[-\frac{1}{2}\lambda^2\right] \lambda^m \, H_{\vec{m}}(\lambda\vec{x}) \ , \quad \sum m_i = m \ . \tag{4.7}$$

Then

$$\sum \frac{z_1^{m_1}}{m_1!} \cdots \frac{z_n^{m_n}}{m_n!} P_{m_1 \ldots m_n}(\vec{x}) = (1 - 2\, C_{ij} z_i x_j + C_{ij}\, z_i z_j)^{-1/2} \ . \tag{4.8}$$

The extension to Grassmann variables is now immediate. Rescaling the t and q elements of (2.8) by the commuting element, λ, one finds

$$\sum \frac{\varepsilon(m)}{m!} t^{\beta_1} \ldots t^{\beta_m} \int_{-\infty}^{\infty} d\lambda \, e^{-\lambda^2} \lambda^m \, \bar{H}_{\beta_1 \ldots \beta_m}(\lambda q) = \int_{-\infty}^{\infty} e^{-A^2\lambda^2} \, d\lambda \tag{4.9}$$

and

$$\sum \varepsilon(m) \, t^{\beta_1} \ldots t^{\beta_m} \, \bar{P}_{\beta_1 \ldots \beta_m}(q) = (1 - 2\, \bar{t}q + \bar{t}t)^{-1/2} \ , \tag{4.10}$$

where

$$\bar{P}_{\beta_1 \ldots \beta_m}(q) = \frac{1}{\sqrt{\pi}} \frac{1}{m!} \int_{-\infty}^{\infty} d\lambda \, e^{-\lambda^2} \lambda^m \, \bar{H}_{\beta_1 \ldots \beta_m}(q) \ . \tag{4.11}$$

Again

$$P^{\alpha_1 \ldots \alpha_m}(q) = (C^{-1})^{\alpha_1 \beta_1} \ldots (C^{-1})^{\alpha_m \beta_m} \, \bar{P}_{\beta_1 \ldots \beta_m}(q) \ . \tag{4.12}$$

The generating function is

$$(1 - 2\bar{t}q + \bar{t}t)^{-1/2} \quad . \tag{4.13}$$

5. The Explicit Multinomials.

We shall obtain these multinomials by expanding their generating function. Both the Hermite and the Legendre expressions depend upon the bilinears $\bar{t}q$ and $\bar{t}t$ which are simply c-numbers. Let us first consider the Hermite case

$$\exp\left[2\bar{t}q-\bar{t}t\right] = \sum_{p=0}^{\infty} \frac{(2\bar{t}q-\bar{t}t)^p}{p!} \tag{5.1}$$

$$= \sum_{p=0}^{\infty} \sum_{s=0}^{p} \frac{1}{p!} \binom{p}{s} (-\bar{t}t)^s \, (2\bar{t}q)^{p-s} \tag{5.2}$$

or

$$\exp\left[2\bar{t}q-\bar{t}t\right] = \sum_{n=0}^{\infty} \sum_{s=0}^{[n/2]} \frac{(-)^s}{(n-s)!} \binom{n-s}{s} (\bar{t}t)^s \, (2\bar{t}q)^{n-2s} \ , \tag{5.3}$$

where

$$n = s + p \ . \tag{5.4}$$

The generating function for the Legendre multinomials may be expanded in the same way

$$(1+\bar{t}t-2\bar{t}q)^{-1/2} = \sum_{p=0}^{\infty} \frac{(-)^p}{2^{2p}} \binom{2p}{p} (\bar{t}t-2\bar{t}q)^p = \tag{5.5}$$

$$= \sum_{p=0}^{\infty} \sum_{s=0}^{p} \frac{(-)^s}{2^{p+s}} \binom{2p}{p}\binom{p}{s} (\bar{t}t)^s \, (\bar{t}q)^{p-s} \tag{5.6}$$

by a repeated use of the binomial expansion. Again let

$$n = s+p \tag{5.7}$$

then

$$\binom{2p}{p}\binom{p}{s} = \binom{2n-2s}{n}\binom{n}{s} \tag{5.8}$$

and

$$(1+\bar{t}t-2\bar{t}q)^{-1/2} = \sum_{n=0}^{\infty} \sum_{s=0}^{[n/2]} \frac{(-)^s}{2^n} \binom{2n-2s}{n}\binom{n}{s} (\bar{t}t)^s (\bar{t}q)^{n-2s} . \tag{5.9}$$

By comparing (5.3) and (5.9) one sees that the two expansions are both functions of $(\bar{t}t)^s (\bar{t}q)^r$. Written out in detail this common product is

$$(\bar{t}t)^s (\bar{t}q)^r = \varepsilon(r)(G_{\alpha_1\beta_1\ldots\alpha_s\beta_s} t^{\alpha_1}\ldots t^{\beta_s})(G_{\gamma_1\delta_1\ldots\gamma_r\delta_r} t^{\gamma_1}\ldots t^{\gamma_r} q^{\delta_1}\ldots q^{\delta_r}) \tag{5.10}$$

where

$$\varepsilon(r) = (-)^{r(r-1)/2} \tag{5.11}$$

and

$$G_{\alpha_1\beta_1\ldots\alpha_s\beta_s} = \prod_{k=1}^{s} C_{\alpha_k\beta_k} . \tag{5.12}$$

Here $\varepsilon(r)$ is the sign induced when the t and q factors are reordered.

In Eq. (5.9) r is n-2s. Then

$$(-)^s\ (\bar{t}t)^s\ (\bar{t}q)^{n-2s} = (-)^s\ \varepsilon(n-2s)\ G_{\lambda_1\lambda_2\ldots\lambda_{2s-1}\lambda_{2s}}G_{\lambda_{2s+1}\delta_1\ldots\lambda_n\delta_{n-2s}}$$

$$\times\ t^{\lambda_1}\ldots t^{\lambda_n}\ q^{\delta_1}\ldots q^{\delta_{n-2s}} \tag{5.13}$$

$$= (-)^n\ \varepsilon(n)\ t^{\lambda_1}\ldots t^{\lambda_n}\ G_{\lambda_1\ldots\lambda_{2s}}\bar{q}_{\lambda_{2s+1}}\ldots\bar{q}_{\lambda_n}\ . \tag{5.14}$$

Since the t's anticommute, (5.14) may be reexpressed as

$$(-)^s\ (\bar{t}t)^s\ (\bar{t}q)^{n-2s} = \varepsilon(n)\ t^{\lambda_1}\ldots t^{\lambda_n}\ G^s{}_{\lambda_1\ldots\lambda_n}(q)\ , \tag{5.15}$$

where

$$G^s{}_{\lambda_1\ldots\lambda_n}(q) = (-)^n\ G_{[\lambda_1\ldots\lambda_{2s}}\ \bar{q}_{\lambda_{2s+1}}\ldots\bar{q}_{\lambda_n]} \tag{5.16}$$

is completely antisymmetrized.

We may now return to (5.3) and (5.9). The Hermite generating function becomes

$$e^{2\bar{t}q-\bar{t}t} = \sum_{n=0}^{\infty}\ \sum_{s=0}^{[n/2]}\ \frac{\varepsilon(n)}{(n-s)!}\begin{pmatrix}n-s\\ s\end{pmatrix}\ t^{\lambda_1}\ldots t^{\lambda_n}\ G^s{}_{\lambda_1\ldots\lambda_n}(2q) \tag{5.17}$$

and the corresponding Legendre expansion is

$$(1+\bar{t}t-2\bar{t}q)^{-1/2} = \sum_{n=0}^{\infty}\ \sum_{s=0}^{[n/2]}\ \frac{\varepsilon(n)}{2^n}\begin{pmatrix}2n-2s\\ n\end{pmatrix}\begin{pmatrix}n\\ s\end{pmatrix}\ t^{\lambda_1}\ldots t^{\lambda_n}\ G^s{}_{\lambda_1\ldots\lambda_n}(q)\ . \tag{5.18}$$

The expansions (5.17) and (5.18) may be compared with (2.8) and (4.10). We

then find

$$\overline{H}_{\lambda_1 \ldots \lambda_n}(q) = n! \sum_{s=0}^{[n/2]} \frac{1}{(n-2s)!} \frac{1}{s!} G^s_{\lambda_1 \ldots \lambda_n}(2q) \tag{5.19}$$

$$\overline{P}_{\lambda_1 \ldots \lambda_n}(q) = \frac{1}{2^n} \sum_{s=0}^{[n/2]} \binom{n}{s} \binom{2n-2s}{n} G^s_{\lambda_1 \ldots \lambda_n}(q) \ . \tag{5.20}$$

One may check the transform

$$\int_{-\infty}^{\infty} dy\ e^{-y^2}\ y^n\ H_{\lambda_1 \ldots \lambda_n}(yq) = n! \sum_{s=0}^{[n/2]} \frac{1}{(n-2s)!\,s!} \int_{-\infty}^{\infty} dy\ e^{-y^2}\ y^n$$

$$\times\ G^s_{\lambda_1 \ldots \lambda_n}(2yq) \ . \tag{5.21}$$

But

$$\int_{-\infty}^{\infty} dy\ e^{-y^2}\ y^n\ G^s_{\lambda_1 \ldots \lambda_n}(2yq) = \int_{-\infty}^{\infty} dy\ e^{-y^2}\ y^n\ (2y)^{n-2s}$$

$$\times\ G^s_{\lambda_1 \ldots \lambda_n}(q) \tag{5.22}$$

$$= \frac{\sqrt{\pi}}{2^n} \frac{(2n-2s)!}{(n-s)!} G^s_{\lambda_1 \ldots \lambda_n}(q) \ . \tag{5.23}$$

Therefore

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dy\ e^{-y^2}\ y^n\ \overline{H}_{\lambda_1 \ldots \lambda_n}(yq) = n!\ \overline{P}_{\lambda_1 \ldots \lambda_n}(q) \tag{5.24}$$

as required.

The formulas (5.19) and (5.20) are useful for computing the individual multinomials. In these formulas $G^s_{\lambda_1 \ldots \lambda_n}(q)$ is given by (5.16) or

$$G^{s}_{\lambda_1 \ldots \lambda_n}(q) = (-)^n C_{[\lambda_1 \lambda_2} \ldots C_{\lambda_{2s+1} \lambda_{2s}} \bar{q}_{\lambda_{2s+1}} \ldots \bar{q}_{\lambda_n]} \quad . \tag{5.25}$$

Here s counts the number of C factors. Except for alternating signs the numerical coefficients are the same as for the usual Hermite and Legendre polynomials.

The basis functions $G^{s}_{\lambda_1 \ldots \lambda_n}(q)$ satisfy the following simple relations

$$C_{\alpha\lambda_1} G^{s}_{\lambda_2 \ldots \lambda_n}(q) = G^{s+1}_{\alpha\lambda_1 \ldots \lambda_n} \tag{5.26}$$

$$\bar{q}_{\alpha} G^{s}_{\lambda_1 \ldots \lambda_n}(q) = -G^{s}_{\alpha\lambda_1 \ldots \lambda_n}(q) \tag{5.27}$$

$$q^{\sigma}\partial_{\sigma} G^{s}_{\lambda_1 \ldots \lambda_n} = (n-2s)\, G^{s}_{\lambda_1 \ldots \lambda_n} \quad . \tag{5.28}$$

Example: System of 4 anticommuting elements

(a) Hermitian system

$$\bar{H}_{\lambda_1} = -2\bar{q}_{\lambda_1}$$

$$\bar{H}_{\lambda_1 \lambda_2} = 2C_{\lambda_1 \lambda_2} + 4\bar{q}_{\lambda_1} \bar{q}_{\lambda_2}$$

$$\bar{H}_{\lambda_1 \lambda_2 \lambda_3} = -12C_{[\lambda_1 \lambda_2} \bar{q}_{\lambda_3]} - 8\bar{q}_{\lambda_1} \bar{q}_{\lambda_2} \bar{q}_{\lambda_3}$$

$$\bar{H}_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} = 12C_{[\lambda_1 \lambda_2} C_{\lambda_3 \lambda_4]} + 48C_{[\lambda_1 \lambda_2} \bar{q}_{\lambda_3 \lambda_4]} + 16\bar{q}_{\lambda_1} \bar{q}_{\lambda_2} \bar{q}_{\lambda_3} \bar{q}_{\lambda_4} \quad .$$

(b) Legendre system

$$\bar{P}_{\lambda_1} = -\bar{q}_{\lambda_1}$$

$$\bar{P}_{\lambda_1\lambda_2} = \frac{1}{2} C_{\lambda_1\lambda_2} + \frac{3}{2} \bar{q}_{\lambda_1} \bar{q}_{\lambda_2}$$

$$\bar{P}_{\lambda_1\lambda_2\lambda_3} = -\frac{3}{2} C_{[\lambda_1\lambda_2} \bar{q}_{\lambda_3]} - \frac{5}{2} \bar{q}_{\lambda_1} \bar{q}_{\lambda_2} \bar{q}_{\lambda_3}$$

$$\bar{P}_{\lambda_1\lambda_2\lambda_3\lambda_4} = \frac{1}{16} \left[6C_{[\lambda_1\lambda_2} C_{\lambda_3\lambda_4]} + 60C_{[\lambda_1\lambda_2} \bar{q}_{\lambda_3} \bar{q}_{\lambda_4]} + 70\bar{q}_{\lambda_1} \bar{q}_{\lambda_2} \bar{q}_{\lambda_3} \bar{q}_{\lambda_4} \right] .$$

6. Recursion Relations for Legendre Multinomials.

We have

$$(1+\bar{t}t-2\bar{t}q)^{-1/2} = \sum_{p=0}^{\infty} \binom{-1/2}{p} (\bar{t}t-2\bar{t}q)^p \tag{6.1}$$

$$(1+\bar{t}t-2\bar{t}q)^{-3/2} = \sum_{p=0}^{\infty} \binom{-3/2}{p} (\bar{t}t-2\bar{t}q)^p \tag{6.2}$$

$$= \sum_{p=0}^{\infty} (2p+1) \binom{-1/2}{p} (\bar{t}t-2\bar{t}q)^p \tag{6.3}$$

Then if we complete the expansion of (6.1) and (6.3) as in (5.18) and abbreviate as follows,

$$(1+\bar{t}t-2\bar{t}q)^{-1/2} = \sum_{n=0}^{\infty} \sum_{s=0}^{[n/2]} A^{(-1/2)}(n,s)\, t^{\lambda_1} \ldots t^{\lambda_n}\, G^s_{\lambda_1 \ldots \lambda_n}(q) \tag{6.4}$$

$$(1+\bar{t}t-2\bar{t}q)^{-3/2} = \sum_{n=0}^{\infty} \sum_{s=0}^{[n/2]} A^{(-3/2)}(n,s)\, t^{\lambda_1} \ldots t^{\lambda_n}\, G^s_{\lambda_1 \ldots \lambda_n}(q) \tag{6.5}$$

we have

$$A^{(-3/2)}(n,s) = (2n-2s+1)\ A^{(-1/2)}(n,s)\ , \tag{6.6}$$

where

$$A^{(-1/2)}(n,s) = 2^{-n}\ \varepsilon(n) \binom{2n-2s}{n} \binom{n}{s} \quad . \tag{6.7}$$

In rewriting (6.1) as (6.4) we have replaced p by n-s. We also have by (5.20) and (6.7)

$$\bar{P}_{\lambda_1 \ldots \lambda_n}(q) = \varepsilon(n) \sum_{s=0}^{[n/2]} A^{(-1/2)}(n,s)\ G^s{}_{\lambda_1 \ldots \lambda_n}(q) \quad . \tag{6.8}$$

Then differentiating (4.10) with respect to q^α

$$C_{\alpha\beta} t^\beta (1+\bar{t}t-2\bar{t}q)^{-3/2} = \sum_{n=0}^{\infty} \varepsilon(n)\ (-)^n\ t^{\lambda_1} \ldots t^{\lambda_n} \frac{\partial \bar{P}_{\lambda_1 \ldots \lambda_n}(q)}{\partial q^\alpha} \tag{6.9}$$

and by (6.5)

$$\begin{aligned} C_{\alpha\beta} \sum_{n=0}^{\infty} \sum_{s=0}^{[n/2]} & A^{(-3/2)}(n,s)\ t^\beta t^{\lambda_1} \ldots t^{\lambda_n}\ G^s{}_{\lambda_1 \ldots \lambda_n}(q) \\ &= \sum_{n=1}^{\infty} \varepsilon(n+1)\ t^{\lambda_1} \ldots t^{\lambda_n}\ \partial_\alpha P_{\lambda_1 \ldots \lambda_n}(q) \\ &= \sum_{n=0}^{\infty} \varepsilon(n+2)\ t^{\lambda_1} \ldots t^{\lambda_{n+1}}\ \partial_\alpha P_{\lambda_1 \ldots \lambda_{n+1}}(q) \\ &= -\sum_{0}^{\infty} \varepsilon(n)\ t^\beta t^{\lambda_1} \ldots t^{\lambda_n}\ \partial_\alpha P_{\beta\lambda_1 \ldots \lambda_n}(q) \end{aligned} \tag{6.10}$$

or

$$C_{\alpha[\beta} \sum_{s=0}^{[n/2]} A^{(-3/2)}(n,s)\ G^{s}{}_{\lambda_1\ldots\lambda_n]}(q) = -\varepsilon(n)\ \partial_\alpha\ \bar{P}_{\beta\lambda_1\ldots\lambda_n}(q)\ . \tag{6.11}$$

By (6.6)

$$C_{\alpha[\beta} \sum_{s=0}^{[n/2]} (2n-2s+1)\ A^{-1/2}(n,s)\ G^{s}{}_{\lambda_1\ldots\lambda_n]}(q) = -\varepsilon(n)\ \partial_\alpha \bar{P}_{\beta\lambda_1\ldots\lambda_n}(q) \tag{6.12}$$

and by (6.8)

$$(n+1)\ \varepsilon(n)\ \varepsilon_{\alpha[\beta}\ \bar{P}_{\lambda_1\ldots\lambda_n]}(q) + C_{\alpha[\beta} \sum_{s=0}^{[n/2]} (n-2s)\ A^{-1/2}(n,s)$$

$$\times\ G^{s}{}_{\lambda_1\ldots\lambda_n]}(q) = -\varepsilon(n)\ \partial_\alpha\ P_{\beta\lambda_1\ldots\lambda_n}(q)\ . \tag{6.13}$$

But

$$q^\sigma \frac{\partial}{\partial q^\sigma}\ G^{s}{}_{\lambda_1\ldots\lambda_n}(q) = (n-2s)\ G^{s}{}_{\lambda_1\ldots\lambda_n}(q) \tag{6.14}$$

since $G^{s}{}_{\lambda_1\ldots\lambda_n}$ is homogeneous of degree n-2s and therefore

$$\sum (n-2s)\ A^{(-1/2)}(n,s)\ G^{s}{}_{\lambda_1\ldots\lambda_n}(q) = q^\sigma\partial_\sigma \sum A^{(-1/2)}(n,s)\ G^{s}{}_{\lambda_1\ldots\lambda_n}(q)$$

$$= \varepsilon(n)\ q^\sigma\partial_\sigma\ \bar{P}_{\lambda_1\ldots\lambda_n}(q)\ . \tag{6.15}$$

Then by (6.13)

$$\partial_\alpha \bar{P}_{\beta\lambda_1\ldots\lambda_n}(q) + [q^\sigma\partial_\sigma + (n+1)]\ C_{\alpha[\beta}\ \bar{P}_{\lambda_1\ldots\lambda_n]}(q) = 0\ . \tag{6.16}$$

This last equation expresses the derivative of the $(n+1)^{st}$ multinomial in terms of the nth multinomial and its derivative.

Next by differentiating (4.10) with respect to t we obtain, with the aid of (6.5)

$$\sum_n n\varepsilon(n)\ t^{\lambda_2}\ldots t^{\lambda_n}\ \bar{P}_{\alpha\lambda_2\ldots\lambda_n}(q)$$

$$= C_{\alpha\gamma} \sum_n \sum_s \left[A^{(-3/2)}(n,s)\ q^{\gamma} t^{\lambda_1}\ldots t^{\lambda_n} - A^{(-3/2)}(n,s)\ t^{\gamma} t^{\lambda_1}\ldots t^{\lambda_n}\right]$$

$$\times\ G^{s}{}_{\lambda_1\ldots\lambda_n}(q) \tag{6.17}$$

or

$$\sum (n+1)\ \varepsilon(n+1)\ t^{\lambda_1}\ldots t^{\lambda_n}\ P_{\alpha\lambda_1\ldots\lambda_n}(q)$$

$$= C_{\alpha\beta} q^{\beta} \sum_n \sum_s A^{(-3/2)}(n,s)\ t^{\lambda_1}\ldots t^{\lambda_n}\ G^{s}{}_{\lambda_1\ldots\lambda_n}(q)$$

$$- C_{\alpha\gamma} \sum_n \sum_s A^{(-3/2)}(n-1,s)\ t^{\gamma} t^{\lambda_1}\ldots t^{\lambda_{n-1}}\ G^{s}{}_{\lambda_1\ldots\lambda_n}(q)\ . \tag{6.18}$$

Then

$$-(n+1)\ \varepsilon(n+1)\ P_{\alpha\lambda_1\ldots\lambda_n}(q) = (-)^n \sum_s A^{(-3/2)}(n,s)\ \bar{q}_{[\alpha}\ G^{s}{}_{\lambda_1\ldots\lambda_n]}(q)$$

$$+ \sum_s A^{(-3/2)}(n-1,s)\ C_{[\alpha\lambda_1}\ G^{s}{}_{\lambda_2\ldots\lambda_n]}(q)\ . \tag{6.19}$$

By (6.6)

$$\sum_s A^{(-3/2)}(n,s)\ G^s{}_{\lambda_1\ldots\lambda_n}(q) = (n+1)\sum_s A^{(-1/2)}(n,s)\ G^s{}_{\lambda_1\ldots\lambda_n}(q)$$

$$+\sum_s (n-2s)\ A^{(-1/2)}(n,s)\ G^s{}_{\lambda_1\ldots\lambda_n}(q)$$

and according to (6.8) and (6.14)

$$= \varepsilon(n)\left[q^\sigma \frac{\partial}{\partial q^\sigma} + n + 1\right] \bar{P}_{\lambda_1\ldots\lambda_n}(q)\ . \tag{6.20}$$

Then (6.19) becomes

$$-(n+1)\ \varepsilon(n+1)\ \bar{P}_{\alpha\lambda_1\ldots\lambda_n}(q)$$

$$= (-)^n\ \bar{q}_{[\alpha}\left\{\varepsilon(n)\,[q^\sigma\partial_\sigma+n+1]\ \bar{P}_{\lambda_1\ldots\lambda_n]}(q)\right\}$$

$$+ C_{[\alpha\lambda_1}\left\{\varepsilon(n-1)\,[q^\sigma\partial_\sigma+n]\ \bar{P}_{\lambda_2\ldots\lambda_n]}(q)\right\}\ .$$

where $q^\sigma\partial_\sigma$ is excluded from the antisymmetrization. We then have the following equation which connects three successive orders

$$(n+1)\ \bar{P}_{\alpha\lambda_1\ldots\lambda_n}(q) + \bar{q}_{[\alpha}(q^\sigma\partial_\sigma+n+1)\ \bar{P}_{\lambda_1\ldots\lambda_n]}(q) - (q^\sigma\partial_\sigma+n)\ C_{[\alpha\lambda_1}\bar{P}_{\lambda_2\ldots\lambda_n]}(q)$$

$$= 0\ . \tag{6.21}$$

This equation may be checked for the explicit forms (5.20) with the aid of (5.26)-(5.28). One has

$$\bar{q}_{[\alpha}(q^\sigma\partial_\sigma+n+1)\ \bar{P}_{\lambda_1\ldots\lambda_n]}(q) =$$

$$= -2^{-n} \sum_{s=0}^{[n/2]} \binom{n}{s}\binom{2n-2s}{n} (2n-2s+1)\ G^{s}{}_{\alpha\lambda_1\ldots\lambda_n}(q) \tag{6.22}$$

$$(q^{\sigma}\partial_{\sigma}+n)\ C_{[\alpha\lambda_1}\ \bar{P}_{\lambda_2]\ldots\lambda_n]}(q) =$$

$$= 2^{-n+1} \sum_{s=1}^{[n+1/2]} \binom{n-1}{s-1}\binom{2n-2s}{n-1} (2n-2s+1)\ G^{s}{}_{\alpha\lambda_1\ldots\lambda_n}(q)\ . \tag{6.23}$$

Then

$$(q^{\sigma}\partial_{\sigma}+n)\ C_{[\alpha\lambda_1}\ \bar{P}_{\lambda_2\ldots\lambda_n]}(q) - \bar{q}_{[\alpha}(q^{\sigma}\partial_{\sigma}+n+1)\ \bar{P}_{\lambda_1\ldots\lambda_n]}(q)$$

$$= \frac{n+1}{2^{n+1}} \sum_{s=0}^{[(n+1)/2]} \binom{n+1}{s}\binom{2n-2s+2}{n+1}\ G^{s}{}_{\alpha\lambda_1\ldots\lambda_n}$$

$$= (n+1)\ \bar{P}_{\alpha\lambda_1\ldots\lambda_n}(q)\ . \tag{6.24}$$

The second order differential equation for $P_{\lambda_1\ldots\lambda_n}(q)$ may be obtained by combining (6.16) and (6.21) which may be reexpressed as

$$(n+1)\ \bar{P}_{\beta\lambda_1\ldots\lambda_n} + \bar{q}_{[\beta}(D+1)\ P_{\lambda_1\ldots\lambda_n]} = D_n C_{[\beta\lambda_1}\bar{P}_{\lambda_2\ldots\lambda_n]} \tag{6.21'}$$

and

$$\partial_{\alpha}\ \bar{P}_{\beta\lambda_1\ldots\lambda_n} + (D_n+1)\ C_{\alpha[\beta}\ \bar{P}_{\lambda_1\ldots\lambda_n]} = 0 \tag{6.17'}$$

where

$$D_n = q^{\sigma}\partial_{\sigma} + n\ . \tag{6.18}$$

By (6.16)'

$$\partial_\alpha \bar{P}_{\lambda_1 \ldots \lambda_n} + C_{\alpha[\lambda_1} D \bar{P}_{\lambda_2 \ldots \lambda_n]} = 0 \qquad (6.17)''$$

and by (6.21)'

$$(n+1) \bar{P}_{\beta\lambda_1 \ldots \lambda_n} + \bar{q}_{[\beta} (D_n+1) P_{\lambda_1 \ldots \lambda_n]} = -\partial_{[\beta} P_{\lambda_1 \ldots \lambda_n]} \ . \qquad (6.19)$$

Then

$$(n+1) \partial_\alpha P_{\beta\lambda_1 \ldots \lambda_n} + \partial_\alpha \bar{q}_{[\beta} (D_n+1) P_{\lambda_1 \ldots \lambda_n]} = -\partial_\alpha \partial_{[\beta} P_{\lambda_1 \ldots \lambda_n]} \ . \qquad (6.20)$$

Again by (6.17)'

$$-(n+1) C_{\alpha[\beta} (D_n+1) P_{\lambda_1 \ldots \lambda_n]} + \partial_\alpha (\bar{q}_{[\beta} (D_n+1) P_{\lambda_1 \ldots \lambda_n]}) = -\partial_\alpha \partial_{[\beta} P_{\lambda_1 \ldots \lambda_n]}$$

or

$$\partial_\alpha \partial_{[\beta} \bar{P}_{\lambda_1 \ldots \lambda_n]}(q) - [nC_{\alpha[\beta} + \bar{q}_\beta \partial_{\alpha]} (D_n+1) \bar{P}_{\lambda_1 \ldots \lambda_n]}(q) = 0 \ . \qquad (6.21)$$

where ∂_α is not included in the antisymmetrization. This is the required second order differential equation for $\bar{P}_{\lambda_1 \ldots \lambda_n}(q)$.

7. Rodrigues' Formula.

The natural transcription of Rodrigues' formula to Grassmannian elements is

$$2^n\ n!\ \bar{P}_{\lambda_1 \ldots \lambda_n}(q) = \varepsilon(n) \frac{\partial}{\partial q^{\lambda_1}} \ldots \frac{\partial}{\partial q^{\lambda_n}} (\bar{q}q-1)^n \ . \tag{7.1}$$

Then

$$(\bar{q}q-1)^n = \sum_{s=n/2}^{n} \binom{n}{s} (\bar{q}q)^s \ (-)^{n-s} \tag{7.2}$$

where

$$(\bar{q}q)^s = C_{\alpha_1\beta_1} \ldots C_{\alpha_s\beta_s} q^{\alpha_1} q^{\beta_1} \ldots q^{\alpha_2\beta_2}$$

$$= \hat{G}_{\alpha_1 \ldots \alpha_{2s}} \ q^{\alpha_1} \ldots q^{\alpha_{2s}} \tag{7.3}$$

and

$$\hat{G}_{\alpha_1 \ldots \alpha_{2s}} = C_{[\alpha_1\alpha_2} \ldots C_{\alpha_{2s-1}\ 2s]} \ . \tag{7.4}$$

Then

$$2^n\ n!\ \bar{P}_{\lambda_1 \ldots \lambda_n}(q) = \varepsilon(n) \sum_{s=0}^{2s \geq n} \binom{n}{s} (-)^{n-s} \ \hat{G}_{\alpha_1 \ldots \alpha_{2s}} \ \partial_{\lambda_1} \ldots \partial_{\lambda_n} (q^{\alpha_1} \ldots q^{\alpha_{2s}}) \ . \tag{7.5}$$

But

$$\hat{G}_{\alpha_1 \ldots \alpha_{2s}} \partial_{\lambda_1} \ldots \partial_{\lambda_n} (q^{\alpha_1} \ldots q^{\alpha_{2s}}) = (2s)(2s-1) \ldots (2s-n+1)$$

$$\times \hat{G}_{\lambda_n \ldots \lambda_1 \alpha_{n+1} \ldots \alpha_{2s}} q^{\alpha_{n+1}} \ldots q^{\alpha_{2s}} =$$

$$= \varepsilon(n) \binom{2s}{n} n! \; \hat{G}_{\lambda_1 \ldots \lambda_n \alpha_{n+1} \ldots \alpha_{2s}} \; q^{\alpha_{n+1}} \ldots q^{\alpha_{2s}} \tag{7.6}$$

or

$$\bar{P}_{\lambda_1 \ldots \lambda_n}(q) = 2^{-n} \sum_{t=0}^{[n/2]} \binom{n}{t} \binom{2n-2t}{n} (-)^t \; \hat{G}^t_{\lambda_1 \ldots \lambda_n}(q) \; , \tag{7.7}$$

where

$$\hat{G}^t_{\lambda_1 \ldots \lambda_n}(q) = \hat{G}_{[\lambda_1 \ldots \lambda_n \alpha_{n-1} \ldots \alpha_{2n-2t}]} \; q^{\alpha_{n+1}} \ldots q^{\alpha_{2n-2t}} \tag{7.8}$$

and $t = n-s$. Note that $\hat{G}$ is completely antisymmetric.

In (7.7) the numerical coefficients are exactly the same as for the Legendre polynomials but differ by the alternating sign from (5.20). Also $G^s_{\lambda_1 \ldots \lambda_n}$ is not the same as $\hat{G}^s_{\lambda_1 \ldots \lambda_n}$.

To compare (7.7) with (5.20) note that the coefficient in (7.8) is

$$\hat{G}_{[\lambda_1 \ldots \lambda_r, \lambda_{r+1} \ldots \lambda_n, \alpha_{n+1} \ldots \alpha_{n+r}]}$$

$$= (-)^{r(n-r)} \; \hat{G}_{[\lambda_1 \ldots \lambda_r, \alpha_{n+1} \ldots \alpha_{n+r}, \lambda_{r+1} \ldots \lambda_n]} \tag{7.9}$$

$$= (-)^{r(n-r)} \; \varepsilon(r) \; \hat{G}_{[\lambda_1 \alpha_{n+1} \ldots \lambda_r \alpha_{n+r}, \lambda_{r+1} \ldots \lambda_n]} \; , \tag{7.10}$$

where

$$r = n-2t \; . \tag{7.11}$$

Then

$$(-)^{r(n-r)}\ \varepsilon(r) = \varepsilon(n-2t) = \varepsilon(n)\ (-)^{t}\ . \tag{7.12}$$

By (7.8)

$$\hat{G}^{t}_{\lambda_1\ldots\lambda_n}(q)$$

$$= \varepsilon(n)\ (-1)^{t}\ C_{\lambda_1\alpha_{n+1}}\ldots C_{\lambda_r\alpha_{n+r}}C_{\lambda_{r+1}\lambda_{r+2}}\ldots C_{\lambda_{n-1}\lambda_n}q^{\alpha_{n+1}}\ldots q^{\alpha_{n+r}} \tag{7.13}$$

$$= \varepsilon(n)\ (-)^{t+r}\ C_{\lambda_{r+1}\lambda_{r+2}}\ldots C_{\lambda_{n-1}\lambda_n}\ \bar{q}_{\lambda_1}\ldots\bar{q}_{\lambda_r}$$

$$= \varepsilon(n)\ (-)^{t+r}\ \hat{G}_{\lambda_{r+1}\ldots\lambda_n}\ \bar{q}_{\lambda_1}\ldots\bar{q}_{\lambda_{n-2t}}\ . \tag{7.14}$$

By (7.7)

$$\bar{P}_{\lambda_1\ldots\lambda_n}(q) = 2^{-n}\ \varepsilon(n)\ \sum_{t=0}^{[n/2]}\binom{n}{t}\binom{2n-2t}{n}\ \tilde{G}^{t}_{\lambda_1\ldots\lambda_n}(q)\ , \tag{7.15}$$

where

$$\tilde{G}^{t}_{\lambda_1\ldots\lambda_n}(q) = (-)^{n}\ \hat{G}_{\lambda_{r+1}\ldots\lambda_n}\ \bar{q}_{\lambda_1}\ldots\bar{q}_{\lambda_{n-2t}}\ . \tag{7.16}$$

Then the numerical coefficients in (7.15) and (5.20) agree except for the normalization $\varepsilon(n)$. The functions $\tilde{G}^{t}_{\lambda_1\ldots\lambda_n}(q)$ and $G^{s}_{\lambda_1\ldots\lambda_n}(q)$ in (5.25) also agree since the number of C factors in $\tilde{G}$ is $\frac{1}{2}(n-r) = t$ and the number of q factors is $r = n-2t$, while the corresponding numbers in G are s and n-2s.

Therefore Rodrigues' rule leads to the same functions as our earlier definition in terms of the generating function.

8. Orthogonality.

An examination of the Legendre examples in paragraph 5 reveals that these multinomials are not all Berezin orthogonal. It remains to be decided whether a weight function can be found for which the complete Legendre set is orthogonal.

Therefore consider

$$\int (dq)\ P^{\lambda_1 \ldots \lambda_n}(q)\ \bar{P}_{\mu_1 \ldots \mu_m}(q)\ \rho(q) = \delta^n_m\ \varepsilon^{\lambda_1 \ldots \lambda_n}_{\mu_1 \ldots \mu_m}\ R\ , \tag{8.1}$$

where $\rho(q)$ and R remain to be determined. Here $\rho(q)$ is the general weight function which has the representation

$$\rho(q) = \sum_{j=0}^{N} \rho^{\beta_1 \ldots \beta_j}\ \bar{q}_{\beta_1} \ldots \bar{q}_{\beta_j}\ . \tag{8.2}$$

By introducing the corresponding expansions for $P^{\lambda_1 \ldots \lambda_n}(q)$ and $\bar{P}_{\mu_1 \ldots \mu_n}(q)$ [see (5.20) for example] one obtains conditions on the coefficients $\rho^{\beta_1 \ldots \beta_j}$. Unfortunately one then finds that $\rho^{\beta_1 \cdots \beta_j}$ depends on n and m and for this reason is not satisfactory as a weight function. In contrast to this case the Grassmann Hermite multinomials are Berezin orthogonal with the usual Gaussian weights.[1]

9. Generalizations.

In both the Hermite and Legendre cases we have followed the simple rule of replacing terms like x^2, xz, and z^2 by the symplectic forms $q^m C_{mn} q^n$, $q^m C_{mn} t^n$, and $t^m C_{mn} t^n$, where q^m and t^m are anticommuting objects. The possibility of this construction depends only on the structure of the generating function. For example, the same Grassmannian extension is possible

for Tchebyscheff polynomials since the generating function again depends on xz and z^2:

$$\frac{1-z^2}{1-2xz+z^2} = 1 + 2 \sum_{n=1}^{\infty} T_n(x)\ z^n \tag{9.1}$$

$$\frac{1}{1-2xz+z^2} = \sum_{n=0}^{\infty} U_n(x)\ z^n\ , \tag{9.2}$$

where T_n and U_n are the two kinds of Tchebyscheff polynomials.

The Gegenbauer polynomials are closely related to the Tchebyscheff polynomials and are also derivable from a generating function such as we have discussed

$$(1-2xz+z^2)^{-m} = \sum_{n=0}^{\infty} C^m_n(x)\ z^n\ , \tag{9.3}$$

where the $C^m_n(x)$ are the Gegenbauer polynomials for integral powers of n.

On the other hand Laguerre polynomials for example cannot be treated by this simple rule.

References.

1. R. Finkelstein and M. Villasante, J. Math Phys. **27**, 1595 (1986).
2. P. Kwon and M. Villasante, J. Math. Phys. **29**, 560 (1988); **30**, 201 (1989).
3. R. Finkelstein and M. Villasante, Phys. Rev. D**33**, 1666 (1986).
4. Bateman Manuscript Project, Higher Transcendental Functions Volume 2, 1953, McGraw Hill.