UCLA-92-TEP-37

Fermi National Accelerator Laboratory

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FERMILAB-Pub-92/281-A
UCLA/92/TEP/37
October 1992

HOW EFFECTIVE IS THE EFFECTIVE POTENTIAL ?

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ABSTRACT

Motivated by bubble nucleation in first order phase transitions, we question the validity of the effective potential for inhomogeneous configurations. In an attempt to get some insight into the importance of derivative terms, we analyze a simple model, a kink in (1+1) dimensions and zero temperature. We evaluate the energy shift from the quantum fluctuations about the non-uniform background (*i.e.*, the effective action) and compare it to the energy from the effective potential. Our results clearly show that for inhomogeneous configurations it may be inadequate to omit derivative terms and confine oneself to the effective potential. We then couple the kink field to an additional scalar field and perform the same comparison. The addition of the second field allows us to vary the mass of the fluctuations and their coupling to the underlying kink. If the mass of the second field is large, it does not feel the inhomogeneities in the kink field and consequently does not give rise to important derivative corrections in the effective action. In contrast, if the mass is small, derivative terms are significant and the effective potential fails. In the latter regime we can, however, rely on the Born approximation to calculate the effective action.

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Operated by Universities Research Association Inc. under contract with the United States Department of Energy

1. Introduction

Phase transitions have played a crucial role in the early evolution of the universe. From a possible inflationary epoch at the GUT scale, through the electroweak and quark-hadron transitions, phase transitions took a major part in shaping our universe. The significance and observable consequences of a specific transition depend on its detailed nature, and in particular, on whether it was first or second order. To mention only two examples in this connection: If the QCD phase transition [1] was first order (which at present seems unlikely), Big Bang nucleosynthesis may have taken place in a very inhomogeneous setting [2] and the elemental abundances could differ significantly from the predictions of the standard scenario. If the electroweak phase transition was first order [3]†, the resulting non-equilibrium period may have generated the observed baryon asymmetry in the universe [7].

In a second order phase transition, the field evolves smoothly from its high temperature 'false' vacuum state to its low temperature 'true' vacuum, whereas in a first order transition the field is trapped in a local minimum. In the latter case, the transition eventually proceeds (at zero temperature) via tunneling through the energy barrier, which separates the true from the false vacuum. (At finite temperature, thermal fluctuations 'push' the field over the barrier in *free* energy.) This corresponds, in the overall picture, to nucleation of true vacuum bubbles in the surrounding false vacuum sea. The bubbles expand and coalesce, thereby completing the transition. In general, first order phase transitions are associated with supercooling, out-of-equilibrium processes, and the formation of shock waves in front of the expanding bubbles. The typical time scale for the first two effects is given by the inverse of the bubble nucleation rate or decay rate of the false vacuum. To understand a given phase transition, we clearly have to calculate the nucleation rate accurately.

Coleman [8] derived the following expression for the tunneling rate per unit

† See ref. 4 for a clear, analytic treatment which appears to be justified by more recent treatments accounting for higher order effects [5]. For a slightly different point of view see [6].

volume per unit time[‡],

$$\Gamma = \left(\frac{S(\bar{\phi})}{2\pi\hbar}\right)^2 \left|\frac{\det(-\partial_\mu\partial_\mu + V''(\phi_f))}{\det'(-\partial_\mu\partial_\mu + V''(\bar{\phi}))}\right|^{1/2} e^{-S(\bar{\phi})/\hbar} , \tag{1.1}$$

where ϕ_f is the value of the field in the false vacuum; the Euclidean index μ runs from 0 to D, where D is the number of spatial dimensions; S is the Euclidean action, a functional of ϕ; V is the potential; and $\bar{\phi}$ is a non-trivial solution to the Euclidean equations of motion subject to the boundary conditions: $\lim_{|\vec{x}|,\tau\to\infty} \bar{\phi}(\tau,\vec{x}) = \phi_f$. The prime on the determinant in the denominator indicates that we exclude the (four) zero modes corresponding to translations. Eq. (1.1) tells us that the leading contribution to the imaginary part of the energy, or the decay rate, comes from the virtual configuration $\bar{\phi}(\vec{x},\tau)$. The ratio of the determinants represents quantum fluctuations about this configuration.

Our interest in this work is in the quantum corrections, represented by the determinant in eq. (1.1). Nominally one expects corrections from the determinant to be small, so it is often sufficient to simply evaluate the argument of the exponential and estimate the determinant on dimensional grounds. Nonetheless, it is precisely these quantum corrections which sometimes transform a second order phase transition to a first order one, or *vice versa.* For example, present consensus [4,5] is that the electroweak phase transition is first order because of quantum corrections from gauge bosons.

A standard method for estimating the quantum corrections is to first integrate out quantum fluctuations about a constant background. This gives an effective potential for ϕ which is then used in the equations of motion determining $\bar{\phi}(\vec{x},\tau)$. This approach is clearly inconsistent, as the configuration $\bar{\phi}$ is not constant; one should really integrate out fluctuations about a general *inhomogeneous* configuration. Put differently, one should evaluate the effective *action*, including all derivative terms, not just an effective potential, which drops these terms[♮]. In the present work we question the usefulness of the effective potential

‡ For reviews, see refs. [9] and [10].

♮ In the context of the electroweak phase transition, several groups have recently estimated the "derivative" corrections to the tunnelling rate, see Dine, *et al.* and Brahm and Hsu in ref. [5]. There is a wealth of literature on the derivative terms in the effective action. See ref. [11] for a partial listing.

for inhomogeneous configurations. Specifically we ask: How accurately can the effective action be approximated by the effective potential, how important are derivative terms?

In section 2 we briefly review the standard technique for calculating decay rates and formalize the above questions. If we eventually want to make contact with cosmological phase transitions, we will have to consider a bubble at finite temperature in a (3+1) dimensions. As we are trying to get as much insight into the problem as possible, we will restrict ourselves to a very simple system: a kink at zero temperature in (1+1) dimensions. Unlike a nucleating bubble, a kink is, of course, stable but evaluating the quantum corrections to its energy is analogous to the problem of interest. (After all, a decay rate is really (the imaginary part of) an energy.) The advantage of this simple system lies in the fact that the determinant can be evaluated exactly [12]; that is, it is straightforward to compute all the derivative terms in the effective action. This gives us a mean of correctly checking the effective potential technique. In section 3 we therefore consider the fluctuations $\delta\phi$ in the background of a kink $\bar{\phi}$ and see how the resulting change in energy compares with the effective potential approach. In section 4 we couple a second scalar field to the kink field and again see how the two approaches compare. At the end of section 4 we introduce a new approximation, which succeeds in the regime where the effective potential fails. Section 5 contains our conclusions. We reserve technical details to two appendices.

2. Quantum Corrections to an Inhomogeneous Configuration

Consider a field in an unstable configuration ϕ_f with energy E_f. For definiteness we take ϕ to be a scalar field. In complete analogy to quantum mechanics, the decay rate is given by [8,10]

$$\begin{aligned} \Gamma &= -\frac{2}{\hbar}\,\mathrm{Im}E_f \\ &= -2\,\mathrm{Im}\lim_{T\to\infty}\frac{1}{T}\ln\,\langle\phi_f|e^{-HT/\hbar}|\phi_f\rangle\ , \end{aligned} \tag{2.1}$$

where H is the Hamiltonian. We can rewrite the matrix element in (2.1) in terms

of a functional integral,

$$\langle \phi_f | e^{-HT/\hbar} | \phi_f \rangle \;=\; N \int \mathcal{D}\phi \; e^{-S(\phi)/\hbar} \;, \tag{2.2}$$

with ϕ subject to the condition $\phi(T/2) = \phi(-T/2) = \phi_f$; S denotes the Euclidean action. The standard procedure is to evaluate the functional integral in the WKB approximation, or equivalently to one-loop. Expanding $S(\phi)$ about a solution to the equations of motion, $\bar{\phi}$, and keeping only terms quadratic in the fluctuations $\delta\phi = \phi - \bar{\phi}$, we obtain

$$\begin{aligned} N \int \mathcal{D}\phi \; e^{-S(\phi)/\hbar} \;&\simeq\; N e^{-S(\bar{\phi})/\hbar} \big[\det \big(-\partial_\mu \partial_\mu + V''(\bar{\phi}) \big) \big]^{-1/2} \\ &\equiv \exp \Big[-S_{\text{eff}}(\bar{\phi})/\hbar \Big], \end{aligned} \tag{2.3}$$

where $\bar{\phi}$ (the "bounce") satisfies the boundary conditions $\lim_{\tau \to \pm\infty} = \phi_f$ and $\lim_{|\vec{x}| \to \infty} = \phi_f$, and S_{eff} is the effective action which includes all one-loop quantum corrections. We can determine the coefficient N in eq. (2.2) by performing a dilute-gas approximation, based on the bounce solution [10]. This, together with eqs. (2.1) and (2.3), then leads to the final expression for the decay rate given in eq. (1.1).

The two different expressions on the right hand side of eq. (2.3) represent two alternative methods for calculating the quantum corrections. If the effective action approach is chosen, it is often convenient to expand S_{eff} about a *constant* ϕ, *i.e.*, in powers of momentum about a point with zero external momenta. In position space this reads

$$S_{\text{eff}}(\phi) \;=\; \int d^4x \; \big[-V_{\text{eff}}(\phi) + \frac{1}{2} Z(\phi) \; \partial_\mu \phi \partial^\mu \phi + \mathcal{O}((\partial_\mu \phi)^4) \big] \;. \tag{2.4}$$

For constant ϕ only the first term survives and the effective action is entirely given by the effective potential. Although different techniques for computing the additional derivative corrections have been designed [11], it is computationally elaborate to go beyond the leading terms. Alternatively we can evaluate the

quantum corrections in the form of the determinant, eq. (2.3), and thereby include *all* derivative terms. However, we only know of one specific higher dimensional system (and this system is non-renormalizable) where the determinant has been found analytically [13]. In every other case one has to rely on numerical methods [14]. This is also true for most lower dimensional systems [15]. One of the rare exceptions is the kink in (1+1) dimensions, where the determinant can be evaluated analytically [12].

We therefore use the kink to probe the effective potential approximation. It is our goal to give a clear exposé on the evaluation of the determinant about an *inhomogeneous* background and compare it to the estimates of the effective potential. Although our motivation stems from the rate of bubble nucleation, and the bubble is unstable while the kink is stable, the difference between stability and instability is irrelevant. (The only difference is that we do not expect a negative eigenvalue, which would signal instability.) The procedure outlined above can equally well be used to calculate the energy of the kink to one-loop.

Before proceeding to the calculation, two points are in order. First, if other fields coupled to ϕ are added to the problem, then the result (2.3) is easily extended to

$$\begin{aligned}\int \mathcal{D}\phi\mathcal{D}\chi\; e^{-S(\phi,\chi)/\hbar} \simeq e^{-S(\bar{\phi})/\hbar}&\left[\det\left(-\partial_\mu\partial_\mu + \left.\frac{d^2V}{d\phi^2}\right|_{\phi=\bar{\phi};\chi=0}\right)\right]^{-1/2} \\ &\times\left[\det\left(-\partial_\mu\partial_\mu + \left.\frac{d^2V}{d\chi^2}\right|_{\phi=\bar{\phi};\chi=0}\right)\right]^{-1/2}.\end{aligned} \tag{2.5}$$

Second, it is perhaps more convenient to rewrite the determinant as[♯]

$$\left[\det\left(-\partial_\mu\partial_\mu + \left.\frac{d^2V}{d\phi^2}\right|_{\phi=\bar{\phi}}\right)\right]^{-1/2} = \exp\left[-\frac{T}{2}\sum_n \omega_n\right], \tag{2.6}$$

where the sum is over the eigenfrequencies of the time-independent Schrödinger

♯ To derive this equation we apply $\det O = \exp(\mathrm{Tr}\ln O)$ and integrate over the time-component of the wavenumber.

equation

$$\left[-\partial_i \partial_i + \frac{d^2V}{d\phi^2} \right] \psi_n(x) = \omega_n^2 \psi_n(x). \tag{2.7}$$

Here the index i runs over the number of spatial dimensions; T has dimensions of time.

3. Quantum fluctuations about a kink — the single scalar field case

3.1 Effective Action

We focus on the classic example of a quantized soliton, first analysed by Dashen, Hasslacher and Neveu [12] (henceforth DHN; for a review, see also ref. [16]). It involves the following Lagrangian in $1+1$ dimensions:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V_0(\phi) \; ; \quad V_0(\phi) = \frac{\lambda}{4}\left(\phi^2 - \frac{m_0^2}{\lambda}\right)^2 . \tag{3.1}$$

The subscript on the mass m_0 will be used to differentiate the bare mass from the physical, renormalized value, m. (According to our renormalization scheme (see Appendix A) there is no difference between the bare and the physical coupling constant λ.) The subscript on V_0 specifies that this is the zero order or tree level potential. There are two degenerate vacuum states, $\phi = \text{constant} = \pm m_0/\sqrt{\lambda}$. The kink is an inhomogeneous field configuration which interpolates between these two vacua at $x = \pm\infty$. Specifically,

$$\phi_{\text{kink}}(x) = \frac{m_0}{\sqrt{\lambda}} \tanh\left(\frac{m_0 x}{\sqrt{2}}\right) . \tag{3.2}$$

For topological reasons, the kink is completely stable against decay to the vacuum.

The classical energy of the kink is

$$\begin{aligned} E_{cl} &= \int_{-\infty}^{\infty} dx \left[\frac{1}{2}\left(\frac{d\phi_{\text{kink}}(x)}{dx}\right)^2 + V_0\big(\phi_{\text{kink}}(x)\big) \right] \\ &= \frac{2\sqrt{2}}{3}\,\frac{m_0^3}{\lambda} . \end{aligned} \tag{3.3}$$

Within our renormalization scheme, when the bare parameter m_0 is reexpressed

in terms of the physical mass m and the ultraviolet cut-off Λ, there are no "finite" corrections to E_{cl}. The only new term diverges as Λ goes to infinity, but is eventually cancelled by an appropriate counterterm, arising from quantum fluctuations.

We now calculate the energy shift due to the quantum fluctuations about the zero order kink field. To one-loop order this corresponds to evaluating the determinant in eq. (2.3). As in eq. (2.6), the quantum correction to the energy is half the sum of all eigenvalues, ω_n, of the following Schrödinger equation:

$$\left[-\frac{d^2}{dx^2} + \frac{\partial^2 V_0}{\partial \phi^2}\left(\phi_{\rm kink}(x)\right) \right] \psi_n(x) \;=\; \omega_n^2 \psi_n(x). \tag{3.4}$$

As shown by DHN, eq. (3.4) has two bound (discrete) states with eigenfrequencies 0 (the 'zero mode') and $\sqrt{3/2}m$[†]. In addition there is a continuous spectrum of states with frequencies $\omega_{\rm cont}(k_n) = \sqrt{k_n^2 + 2m^2}$. These states are very similar to the corresponding plane waves in the absence of the kink, with one significant difference: they are shifted in phase by an amount $\delta(k_n)$ relative to each other. DHN were able to derive an analytic form for the phase shift,

$$\delta(k) = 2\pi - 2\tan^{-1}\left(\sqrt{2}k/m\right) - 2\tan^{-1}\left(\sqrt{2}k/2m\right). \tag{3.5}$$

To determine the allowed k_n, we quantize the wavefunction in a box of length L (taken to ∞ at the end of the calculation) and impose periodic boundary conditions. Then the allowed values of k_n satisfy

$$k_n = \frac{2\pi n}{L} - \frac{\delta(k_n)}{L}\ . \tag{3.6}$$

Eq. (3.6) tells us that the wavenumber, and therefore the frequency, associated with a given mode n is modified in the presence of a kink. This leads to a change in the ultraviolet cut-off needed to regulate the theory. Let us call the ultraviolet cut-off in the vacuum Λ. In the vacuum the phase shift is of course zero, so eq.

† In calculating the corrections of order $\hbar$, we can replace m_0 by m; any error is of $\mathcal{O}(\hbar^2)$.

(3.6) shows that imposing the ultraviolet cut-off is equivalent to including only those states with $n \leq n_{\max} \equiv L\Lambda/2\pi$. In the presence of the kink we must count the exact same states, *i.e.*, only those with $n \leq n_{\max}$. Therefore, denoting the ultraviolet cut-off in this case as $k_{\max}$, we see from eq. (3.6) that

$$\begin{aligned} k_{\max} &\equiv \frac{2\pi n_{\max}}{L} - \frac{\delta(k_{\max})}{L} = \Lambda - \frac{\delta(k_{\max})}{L} \\ &\simeq \Lambda - \frac{6m}{\sqrt{2}\Lambda L} \end{aligned} \quad , \tag{3.7}$$

where the last line results from applying $\tan^{-1} x \simeq (\pi/2) - (1/x)$ to eq. (3.5). The effect of the kink on the wavenumbers and on the cut-off is illustrated in fig. 1.

Having catalogued the eigenfrequencies of eq. (3.4), we need only sum them up to find the total energy shift. We turn the sum over the continuous frequencies into an integral by using the differential form of eq. (3.6), $dn = (dkL + d\delta)/2\pi$. Consequently, the correction to the energy due to quantum fluctuations of ϕ is

$$\begin{aligned} E_{\delta\phi} &= \frac{\hbar}{2}\sum \omega_{\text{discrete}} + \frac{\hbar}{2}\sum \omega_{\text{cont}} \\ &= \frac{\hbar}{2}\sum \omega_{\text{discrete}} + \frac{\hbar}{2}\int_{-k_{\max}}^{+k_{\max}} \frac{dk}{2\pi}\left(L + \frac{d\delta}{dk}\right)\omega_{\text{cont}} \ . \end{aligned} \tag{3.8}$$

Eq. (3.8) is a general expression for the quantum corrections about a background field. How involved is this computation in general? The continuum frequencies are completely equivalent to those in the absence of the background field. Thus, the only parts requiring detailed knowledge of the solution to the Schrödinger equation are the frequencies of the discrete states and the phase shifts. If we go back momentarily to the problem which motivated us, that of quantum corrections to tunneling rates in $3+1$ dimensions, we cannot generally expect to solve the corresponding Schrödinger equation analytically. We anticipate, though, that if there are no bound states and if the phase shift can easily be approximated, then the quantum corrections will be relatively easy to evaluate. In fact, in the next section we argue that these two conditions – no bound states

and phase shift easy to approximate – go hand in hand. This is particularly interesting, since it is precisely this regime where the derivative terms are important and the effective potential a poor approximation to the full action.

Returning to the problem at hand, we can insert the phase shift (3.5) into eq. (3.8) and evaluate the integral over the continuous eigenvalues,

$$\begin{aligned}\frac{\hbar}{2}\int_{-k_{\max}}^{+k_{\max}}\frac{dk}{2\pi}\left(L+\frac{d\delta}{dk}\right)\omega_{\rm cont} = \hbar\frac{L}{4\pi}\left[\,k_{\max}^2+m^2\left(1+\ln\frac{2k_{\max}^2}{m^2}\right)\right] \\ -\ \hbar m\left[\,\frac{3}{2\sqrt{2}\pi}\,\ln\frac{2k_{\max}^2}{m^2}+\frac{1}{\sqrt{6}}\right].\end{aligned} \tag{3.9}$$

After inserting k_{max} from eq. (3.7) and including the bound state energy, we obtain

$$\begin{aligned}E_{\delta\phi} &= \hbar m\left[\frac{1}{2\sqrt{6}}-\frac{3}{\sqrt{2}\pi}\right] \\ &\quad+\hbar\frac{L}{4\pi}\left[\,\Lambda^2+m^2\left(\,1+\ln\frac{2\Lambda^2}{m^2}\right)\right]\ -\ \hbar m\,\frac{3}{2\sqrt{2}\pi}\,\ln\frac{2\Lambda^2}{m^2}\,.\end{aligned} \tag{3.10}$$

The second line here consists of terms that go to infinity when $\Lambda, L \to \infty$. These infinities, though, are exactly cancelled by the infinities in E_{cl}, when we express the bare mass in terms of the physical one and introduce the induced "cosmological constant". Adding up all these contributions, the total kink energy then becomes

$$\begin{aligned}E_{\rm kink}^{action} &= \frac{2\sqrt{2}}{3}\frac{m^3}{\lambda}+\hbar m\left(\frac{1}{2\sqrt{6}}-\frac{3}{\sqrt{2}\pi}\right)\ +\ \mathcal{O}(\hbar^2) \\ &\simeq \frac{2\sqrt{2}}{3}\frac{m^3}{\lambda}\left[\,1\ -\ 0.4997\,\frac{\hbar\lambda}{m^2}\ +\ \mathcal{O}(\hbar^2)\right],\end{aligned} \tag{3.11}$$

where the superscript *action* indicates that the effective action was used, *i.e.*, that the quantum fluctuations were evaluated about the true inhomogeneous background field.

3.2 Effective Potential Method

Let us now recalculate the shift in the kink energy within the effective potential approach. The recipe is simple: First evaluate the effective potential (*i.e.*, integrate out the ϕ fluctuations about a *homogeneous* background), then solve the equations of motion for the kink with this "improved" potential, and finally derive the energy of this modified kink. To lowest order in the coupling constant (or $\hbar$) there is, however, a shortcut to this procedure. From basic quantum mechanics we recall that the first order change in the energy due to a perturbation V_1 is simply $\langle\psi|V_1|\psi\rangle$, where ψ is the zero order wave function. Consequently, we only have to separate out the perturbation of order $\hbar$ due to fluctuations of ϕ, $V_1^{\delta\phi}(\phi)$, from the effective potential,

$$V_{\rm eff} \;=\; V_0(m,\lambda) \;+\; V_1^{\delta\phi}\;. \tag{3.12}$$

Note that $V_0(m,\lambda)$ is in terms of the physical mass, $V_0 = (\lambda/4)(\phi^2 - (m^2/\lambda))^2$, so it is finitie. In $V_1^{\delta\phi}$ we can replace the bare mass by the physical one, only making an error of $\mathcal{O}(\hbar^2)$. According to eq. (A9) of Appendix A,

$$V_1^{\delta\phi} \;=\; \frac{3\hbar}{8\pi}\Big(\lambda\phi^2 - m^2\Big) \;-\; \frac{\hbar}{8\pi}\Big(3\lambda\phi^2 - m^2\Big)\;\ln\left(\frac{3\lambda\phi^2 - m^2}{2m^2}\right)\;. \tag{3.13}$$

From here we can calculate the quantum corrections to the kink energy,

$$E_{\delta\phi}^{potential} = \int dx\; V_1^{\delta\phi}\left(\phi_{\rm kink}(x)\right)\;, \tag{3.14}$$

where $\phi_{\rm kink}$ is the zero order (tree-level) kink solution, eq. (3.2). The total kink energy is then

$$\begin{aligned} E_{\rm kink}^{potential} \;&=\; \frac{2\sqrt{2}}{3}\frac{m^3}{\lambda} \;-\; \hbar m\,\frac{3\sqrt{2}}{4\pi} \\ &\quad -\; \hbar m\frac{\sqrt{2}}{8\pi}\int_{-\infty}^{\infty} dz\Big(3\tanh^2 z - 1\Big)\;\ln\left|\frac{3\tanh^2 z - 1}{2}\right| \;+\; \mathcal{O}(\hbar^2) \\ &=\; \frac{2\sqrt{2}}{3}\;\frac{m^3}{\lambda}\left[1 \;-\; 0.3165\;\frac{\hbar\lambda}{m^2} \;+\; \mathcal{O}(\hbar^2)\right]. \end{aligned} \tag{3.15}$$

How "effective" is the effective potential technique? By comparing $E_{\rm kink}^{action}$ with

$E_{\rm kink}^{potential}$, eqs. (3.11) and (3.15), we see that there is a difference in the quantum corrections of about 50 percent. This is a significant discrepancy and clearly demonstrates the insufficiency of the effective potential for inhomogeneous configurations.

4. Coupled fields

In section 3 we focused on a single scalar field ϕ and computed the quantum fluctuations about its own non-homogeneous configuration. As there exists only a single coupling constant, λ, this system is rather restrictive; λ (together with m) fixes the kink, but at the same time controls the coupling of the fluctuations to the background. As a result, the Schrödinger equation (3.4) turns out to be λ-independent. (By rescaling the coordinate x, the dependence on m becomes trivial.) There is, therefore, not much of an interesting parameter space to cover. But there is a more basic problem with this simplified model. A non-interacting scalar field may not adequately reflect the physical situation. As we are eventually interested in the tunneling rate within, *e.g.*, the electroweak phase transition, we have to treat the Higgs field with all its interactions.

In the present section we challenge the effective potential in a slightly more physical framework and couple ϕ to a second scalar field χ (still confined to (1+1) dimensions). Although this falls short of the real situation, in which the Higgs field is coupled to gauge bosons and quarks in (3+1) dimensions, we hope that it will give us some hint about the sufficiency or insufficiency of the effective potential. In sections 4.1 and 4.2 we will again compute the total kink energy to one-loop in the same two approaches discussed in section 3, and compare the results. We will see how the introduction of a second scalar field allows us to vary the coupling of the χ-fluctuations to the underlying soliton and therefore to find a regime where derivative terms are indeed negligible.

There is also a regime in which derivative terms are important. In section 4.3 we introduce a different approximation, the Born approximation, for use in this regime. We recalculate the kink energy within the Born approximation and compare it with the accurate result of section 4.1.

4.1 Effective Action for two fields

Let us focus on the Lagrangian

$$\mathcal{L} \;=\; \frac{1}{2}\partial_\mu\phi\partial^\mu\phi+\frac{1}{2}\partial_\mu\chi\partial^\mu\chi-V_0(\phi,\chi)\,;\quad V_0 \;=\; \frac{\lambda}{4}\left(\phi^2-\frac{m_0^2}{\lambda}\right)^2+\frac{1}{2}\,g\,\phi^2\chi^2\,. \tag{4.1}$$

This model again permits a kink configuration $\phi_{\rm kink}$, eq. (3.2), along with $\chi \equiv 0$. The classical energy of the kink is unchanged from eq. (3.3) and is *g-independent.* Now consider the quantum fluctuations about the kink. To one-loop, the ϕ and χ fluctuations are independent, and therefore yield two separate contributions to the energy shift. The contributions from the ϕ field were calculated in the previous section, here we concentrate on the χ-fluctuations. Their contribution to the kink energy again amounts to half the sum of the eigenfrequencies ω_n^χ of a Schrödinger equation, in this case of

$$\left[\;-\frac{d^2}{dx^2}+\frac{g}{\lambda}m^2\,\tanh^2\!\left(\frac{mx}{\sqrt{2}}\right)\;\right]\psi_n^\chi(x) \;=\; \left(\omega_n^\chi\right)^2\psi_n^\chi(x)\,. \tag{4.2}$$

The coupling constants do not cancel out, but appear in the combination g/λ and can, in principle, attain *any* value within our perturbation expansion. (Note that our expansion parameters are $\hbar\lambda/m^2$ and $\hbar g/m^2$.) At times it will prove more convenient to use the dimensionless variables, $\epsilon_n \equiv 2\,(\omega_n^\chi)^2/m^2$ and $z \equiv mx/\sqrt{2}$. In terms of these variables, eq. (4.2) becomes

$$\left[\;\frac{d^2}{dz^2}\;+\;\epsilon_n\;-\;2\frac{g}{\lambda}\,\tanh^2 z\;\right]\psi_n^\chi(z) \;=\; 0\,. \tag{4.3}$$

First let us consider the bound states of Eq. (4.3). The "potential" in this Schrödinger equation is $2(g/\lambda)\tanh^2 z$, its strength governed by the dimensionless ratio g/λ. Large g/λ corresponds to deep potentials; we expect these to have many bound states. Small values of g/λ should have only one bound state.‡ In

‡ Recall the peculiarity of one spatial dimension, in which there always exists at least one bound state, no matter how 'shallow' the potential.

fact, as shown in ref. [17], eq. (4.3) has N bound states with energies

$$\epsilon_n = 2\,\frac{g}{\lambda} - \left(\gamma - n - \frac{1}{2}\right)^2 \;;\quad n = 0, 1, ..., N-1 < \gamma - 1/2\;, \tag{4.4}$$

where

$$\gamma \equiv \sqrt{2g/\lambda + 1/4}. \tag{4.5}$$

Next we find the continuous eigenmodes of eq. (4.3). These have frequencies equal to those in the vacuum: $\omega^\chi_{\rm cont} = \sqrt{k^2 + (g/\lambda)m^2}$. Corresponding to each frequency are two states with definite (and opposite) values of parity and generally also different phase shifts. (Since the potential is invariant under parity, we can use a basis in which all states have definite parity.) As we saw in the previous section, it is necessary to find the phase shift in order to sum up the energy of the continuous modes. To this end, we note that the problem at hand is equivalent to a scattering problem. Assuming an incident wave from $z = -\infty$, the asymptotic form of the scattered wave is [17]

$$\psi^\chi \;\to\; \mathcal{N}\begin{cases} ae^{iqz} + be^{-iqz} & ;\quad \mathrm{z} \to -\infty \\ e^{iqz} & ;\quad \mathrm{z} \to \infty \end{cases} \tag{4.6}$$

where

$$a \;=\; \frac{\Gamma(1-iq)\;\Gamma(-iq)}{\Gamma(\frac{1}{2}+\gamma-iq)\;\Gamma(\frac{1}{2}-\gamma-iq)}\;,\quad b \;=\; -i\,\frac{\cos(\pi\gamma)}{\sinh(\pi q)}\;. \tag{4.7}$$

$\mathcal{N}$ is an irrelevant (complex) normalization factor and $q = \sqrt{2}k/m$ is the dimensionless wavenumber. In Appendix B we derive the phase shifts for the two parity eigenstates (the symmetric and antisymmetric parts of the scattering wave),

$$\delta^\pm_\chi \;=\; i\,\ln\left(\frac{a}{1\pm b}\right)\;,\quad \text{and hence}\quad \delta_\chi \;\equiv\; \delta^+_\chi + \delta^-_\chi \;=\; 2i\,\ln\left(\frac{a}{|a|}\right)\;. \tag{4.8}$$

For γ a half-integer, $b = 0$, implying that the even and odd parity phase shifts are identical: $\delta^+_\chi = \delta^-_\chi$. (In terms of scattering theory, the potential is reflectionless.) This was the case for the single scalar field of section 3 (there $\gamma = 5/2$), and it was, therefore, unnecessary to consider even/odd parity states separately.

In fig. 2 we show the phase shifts as a function of q for fixed g/λ. The behavior of the phase shift at low momenta is of particular interest and we therefore plot in fig. 3 the phase shifts at zero wavenumber as a function of g/λ. $\delta^+(0)$ ($\delta^-(0)$) changes discontinuously as γ passes through $2.5, 4.5, 6.5, \ldots$ ($1.5, 3.5, 5.5, \ldots$). To be specific, let us focus on the regime around $\gamma = 2.5$, when $\delta^+(0)$ changes from π to 3π. According to eq. (4.4), at this γ a new bound state appears. In fact each time a new bound state appears, $\delta^+(0)$ or $\delta^-(0)$ changes by 2π. The connection between the phase shift and the number of bound states is, of course, no accident, but a manifestation of Levinson's theorem [18,19].[♮]

As in section 3, we quantize the continuous states in a box and constrain k (separately for the even and odd wavefunctions) by imposing periodic boundary conditions,

$$k_n \;=\; \frac{2\pi n}{L} \;-\; \frac{\delta_\chi^\pm(k)}{L}\;. \tag{4.9}$$

Requiring $n \leq n_{max}$ leads again to an ultraviolet cut-off,

$$k \;\leq\; k_{\rm max} \;\equiv\; \Lambda \;-\; \frac{\delta_\chi^\pm(k_{\rm max})}{L}\;. \tag{4.10}$$

Because $\delta^+(k) \simeq \delta^-(k)$ for large k, the ultraviolet cut-off for the symmetric and anti-symmetric states are identical.

In summing up the continuous eigenmodes we again replace the sum by an integral, but this time separate out the even parity states from the odd ones,

$$\frac{\hbar}{2}\sum \omega_{\rm cont}^\chi = \frac{\hbar}{2}\int_{k_{\rm min}^+}^{k_{\rm max}} \frac{dk}{2\pi}\left(L + \frac{d\delta_\chi^+}{dk}\right)\omega_{\rm cont}^\chi \;+\; \frac{\hbar}{2}\int_{k_{\rm min}^-}^{k_{\rm max}} \frac{dk}{2\pi}\left(L + \frac{d\delta_\chi^-}{dk}\right)\omega_{\rm cont}^\chi\;. \tag{4.11}$$

We have also introduced lower cutoffs $k_{\rm min}^\pm$, reflecting the fact that below a minimum n all states have dropped into the potential well and are counted as bound

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[♮] We can count the number of bound states by means of Levinson's theorem for one-dimensional potential scattering. The number of even and odd states amounts to

$$N_\pm = \frac{1}{2\pi}\,\delta_\chi^\pm(0) \mp \frac{1}{4}\left(\,\exp[i\delta_\chi^\pm(0)] - 1\,\right).$$

For a clear discussion of Levinson's theorem in quantum field theory, see ref. 20.

states. Due to the periodic boundary conditions, we require

$$k \;>\; k^{\pm}_{\rm min} \;\equiv\; \frac{2\pi n^{\pm}_{\rm min}}{L} \;-\; \frac{\delta^{\pm}_{\chi}(k_{\rm min})}{L}\;. \tag{4.12}$$

$n^{\pm}_{\rm min}$ can be identified with the number of even and odd bound states, $n^{\pm}_{\rm min} = n^{\pm}_{\rm discrete}$. One might think that the lower limit in momentum is irrelevant as it goes like $1/L$. This is true except for the term in eq. (4.11) which diverges with L. Neglecting $k^{\pm}_{\rm min}$ in this term could lead to a doublecounting of the discrete states. It was merely a numerical coincidence that in the one-field case of section 3 we could set $k_{\rm min} = 0$. This is due to the fact that for γ half-integer δ_χ is a multiple of 2π, and, together with the exact number of bound states, leads to $k^{\pm}_{\rm min} \equiv 0$.♯ With the help of $k_{\rm min}$, eq. (4.12), we can recast the sum over the continuous eigenmodes,

$$\begin{aligned}
\frac{\hbar}{2}\sum \omega^{\chi}_{\rm cont} &= \frac{\hbar}{2}\int_0^{k_{\rm max}} \frac{dk}{2\pi}\left[\, 2L + \frac{d}{dk}\left(\delta^{+}_{\chi} + \delta^{-}_{\chi}\right)\,\right]\omega^{\chi}_{\rm cont} \\
&\qquad - \frac{\hbar}{2}\left[\, n^{+}_{\rm min} + n^{-}_{\rm min} - \frac{1}{2\pi}\left(\delta^{+}_{\chi}(0) + \delta^{-}_{\chi}(0)\right)\,\right]\omega^{\chi}_{\rm cont}(k=0) \\
&= \frac{\hbar}{4\pi}\int_0^{k_{\rm max}} dk\left[\, 2L\;\omega^{\chi}_{\rm cont} - \frac{k}{\omega^{\chi}_{\rm cont}}\;\delta_{\chi}(k)\,\right] \\
&\qquad + \frac{\hbar}{4\pi}\;\delta_{\chi}(k_{max})\;\omega^{\chi}_{\rm cont}(k_{max}) \;-\; \frac{\hbar}{2}\;n_{\rm min}\;\omega^{\chi}_{\rm cont}(0)\;,
\end{aligned} \tag{4.13}$$

where we have made an integration by parts and defined $n_{\rm min} = n^{+}_{\rm min} + n^{-}_{\rm min}$ (recall also that $d\omega/dk = k/\omega$). The last term above takes care of the previously mentioned doublecounting: As the integral for the continuum in eq. (4.13) extends over all states, we have to subtract off the contributions from the bound states ($n_{\rm min}$ in total).

Finally note that the second term in the integral of eq. (4.13) is logarithmically divergent. When we replace the bare mass with the physical one in the

♯ In section 3 we have integrated from $-k_{max}$ to k_{max} (see eq. (3.8)), which is the equivalent of integrating the even and odd states independently from 0 to k_{max} (the integrand is symmetric).

classical energy, we induce a correction $\hbar\frac{\sqrt{2}m}{4\pi}\frac{g}{\lambda}\ln\left(\frac{4\Lambda^2}{m^2g/\lambda}\right)$, which exactly cancels this divergence. (The same is true for the term proportional to L, which gets canceled by the induced "cosmological constant".)

Collecting all the contributions to the kink energy, *i.e.*, the classical energy, the contributions from the renormalization, and the shifts due to the quantum fluctuations of the ϕ and the χ fields about the inhomogeneous background, we finally obtain (to order $\hbar$)

$$E_{\rm kink}^{action} = E_0 + E_{\delta\phi}^{action} + E_{\delta\chi}^{action} , \tag{4.14}$$

where

$$\begin{aligned} E_0 &= \frac{2\sqrt{2}}{3}\frac{m^3}{\lambda} , \\ E_{\delta\phi}^{action} &= \hbar m\left(\frac{1}{2\sqrt{6}} - \frac{3}{\sqrt{2}\pi}\right) , \\ E_{\delta\chi}^{action} &= \frac{\hbar}{2}\sum\omega_{\rm discrete}^{\chi} - \frac{\hbar}{2}\,n_{\rm discrete}\sqrt{g/\lambda}\,m \\ &\quad - \frac{\hbar}{4\pi}\int_0^{\Lambda} dk\,\frac{k\delta_\chi(k)}{\sqrt{k^2+m^2g/\lambda}} + \frac{\sqrt{2}}{4\pi}\hbar m\frac{g}{\lambda}\,\ln\frac{4\Lambda^2}{m^2g/\lambda}. \end{aligned} \tag{4.15}$$

In $E_{\delta\chi}^{action}$ we have set $n_{\rm min} = n_{\rm discrete}$.† Fig. 4 shows the energy shift due to χ fluctuations, $E_{\delta\chi}^{action}$, as a function of the dimensionless ratio of the couplings, g/λ.

† The integral in the χ-correction is logarithmically divergent; the divergence is cancelled by by the $\ln\Lambda$ term in eq. (4.15). Numerically, we can evaluate the sum of these two terms by adding and subtracting the term $-\int\frac{dk}{k+c} = -\ln(\frac{\Lambda+c}{c})$, c an arbitrary positive constant.

4.2 EFFECTIVE POTENTIAL METHOD FOR TWO FIELDS

Let us now compare the above energy shift, induced by the χ field, with the energy within the effective potential approach. We again separate out the one-loop corrections in the effective potential from the classical potential,

$$V_{\rm eff} \;=\; V_0(m,\lambda) + V_1^{\delta\phi}(m,\lambda) + V_1^{\delta\chi}(m,\lambda,g)\;. \tag{4.16}$$

The energy shift due to $V_1^{\delta\phi}$ has already been calculated in section 3.2. Here we focus on the χ contribution (see eq. (A9)),

$$V_1^{\delta\chi}(\phi) \;=\; \frac{\hbar g}{8\pi}\left(\,\phi^2 - \frac{m^2}{\lambda} \;-\; \phi^2\,\ln\frac{g\phi^2}{m^2 g/\lambda}\,\right). \tag{4.17}$$

In complete analogy to eq. (3.14), the energy shift is now given by

$$\begin{aligned} E_{\delta\chi}^{potential} &= \int_{-\infty}^{\infty} dx\; V_1^{\delta\chi}(\phi_{\rm kink}(x)) \\ &= \frac{\sqrt{2}}{8\pi}\frac{g}{\lambda}\hbar m \int_{-\infty}^{\infty} dz\left[\tanh^2 z \;-\; 1 \;-\; \tanh^2 z\,\ln\left(\tanh^2 z\right)\right] \\ &= \frac{\sqrt{2}}{8\pi}\,\frac{g}{\lambda}\hbar m\left(\,\frac{\pi^2}{2} - 6\,\right) \;\simeq\; -0.0599\;\hbar m\,\frac{g}{\lambda}\;, \end{aligned} \tag{4.18}$$

where we have used $\phi_{\rm kink}$ from eq. (3.2). By adding these χ-corrections to the ϕ-terms, eq. (3.15), we obtain the total kink energy within the effective potential approach,

$$\begin{aligned} E_{\rm kink}^{potential} &= E_{cl} \;+\; E_{\delta\phi}^{potential} \;+\; E_{\delta\chi}^{potential} \\ &\simeq \frac{2\sqrt{2}}{3}\frac{m^3}{\lambda}\left[\,1 \;-\; 0.3165\,\frac{\hbar\lambda}{m^2} \;-\; 0.0636\,\frac{\hbar g}{m^2}\,\right]. \end{aligned} \tag{4.19}$$

In fig. 4 we compare $E_{\delta\chi}^{action}$ with $E_{\delta\chi}^{potential}$ for varying g/λ. For small g/λ, $E_{\delta\chi}^{potential}$ differs considerably from $E_{\delta\chi}^{action}$ and the effective potential is only a

poor approximation to the effective action. However, for $g/\lambda \to \infty$, $E_{\delta\chi}^{action}$ asymptotically approaches $E_{\delta\chi}^{potential}$. Qualitatively we can understand this behavior in the following way: The χ-field has a mass $m_\chi \simeq m\sqrt{g/\lambda}$ and hence varies typically on a scale m_χ^{-1}. On the other hand, the kink extends over a region m^{-1}. If $m_\chi^{-1} \ll m^{-1}$ $(g/\lambda \gg 1)$ then the χ-field oscillates vehemently along the kink profile and is not affected by the background variation. Here the effective potential is valid. In the opposite limit $(g/\lambda \ll 1)$, the χ-field changes at least as slowly as the kink and therefore probes the underlying configuration. In this case derivative terms are non-negligible.

Another way to understand the fact that $E_{\delta\chi}^{potential}$ is a good approximation when g/λ is large is to compute the relative contributions of the next terms in the derivative expansion. Let us consider the Feynman diagram in fig. 5. Its contribution to the effective action is weighted by the Feynman integral, which can be expanded in powers of the external momentum $p = p_1 + p_2$:

$$\int \frac{d^2k}{(k^2 - m_\chi^2)((p+k)^2 - m_\chi^2)} = \int \frac{d^2k}{(k^2 - m_\chi^2)^2} + \int \frac{d^2k}{(k^2 - m_\chi^2)^3}\left(p^2 + \frac{4(p\cdot k)^2}{k^2 - m_\chi^2}\right) + \dots \tag{4.20}$$

The first term on the right hand side is of order $1/m_\chi^2$, while the second $\sim p^2/m_\chi^4$. Since the kink field varies on spatial scales of about m^{-1}, the factors of p are $\mathcal{O}(m)$, so that the second integral is of order m^2/m_χ^4. Therefore, the first derivative correction is a factor of $m^2/m_\chi^2 = \lambda/g$ smaller (or larger) than the non-derivative terms in the effective potential. The derivative expansion is therefore an expansion in the ratio of Compton wavelengths of the background field and the fluctuating field. This explains why in the one field case analyzed in section 3, the corrections from derivative terms are significant: the expansion parameter is of order one.

The derivative terms are down by at least $\mathcal{O}(\lambda/g)$ and can therefore be neglected for $g/\lambda \to \infty$. As already pointed out, in the other limit they are important and the full effective action has to be evaluated. As we will demonstrate now, it is precisely in this regime where one can rely on additional approximations and thereby often simplify the computations.

4.3 BORN APPROXIMATION

We confined ourselves to a kink in (1+1) dimensions due to its simplicity. Most realistic models do not allow for a closed solution (within perturbation theory) and one has to rely on approximations. The problem usually pins down to solving the Schrödinger equation. In the worst case, the eigenvalues of the discrete levels have to be found numerically (after all, there are only a finite number of bound states). It may, however, not be that simple to sum up the eigenfrequencies of the continuum and one would like to have a way of approximating their contribution. Recall that to sum up the continuum eigenfrequencies, one need only know the phase shift (the frequencies themselves are trivial). Fortunately, there are circumstances where one can approximate the phase shift without exactly solving the Schrödinger equation. This is especially true when the fluctuations are only weakly coupled to the underlying kink, or in other words, the "potential" in the Schrödinger equation is "shallow". In our model, this corresponds to $g/\lambda \ll 1$. In this limit we can use the Born approximation.

Let us rewrite eq. (4.3). Since for continuum states, $\epsilon_n = q^2 + 2g/\lambda$, we find

$$\left[\frac{d^2}{dz^2} + q^2 + u(z) \right] \psi_n^{\chi}(z) = 0 , \quad \text{where } u(z) = \frac{2}{\cosh^2 z} \frac{g}{\lambda} ; \tag{4.21}$$

$u(z)$ is the relevant "potential."

Next we expand the wavefunction around the unperturbed wave $\psi_0(z) = \exp(iqz)$, incident from the left (we take $\mathcal{N} = 1/a$ in eq. (4.6)). The total wavefunction then reads

$$\psi^{\chi}(z) = \psi_0(z) + \psi_1(z) . \tag{4.22}$$

For perturbative "potentials" ($g/\lambda \ll 1$), ψ_1 can be evaluated within the Born approximation,

$$\psi_1(z) = -\int_{-\infty}^{\infty} dy \; G(z,y) \; u(y) \; \psi_0(y) , \tag{4.23}$$

where $G(z,y)$ denotes the one-dimensional Green's function,

$$G(z,y) = -\frac{i}{2q} e^{iq|z-y|} . \tag{4.24}$$

Specifically, for $z \to \infty$ we obtain

$$\psi_1(z \to \infty) \;=\; \frac{i}{2q}\, e^{iqz} \int_{-\infty}^{\infty} u(y)dy \;=\; \frac{2i}{q}\frac{g}{\lambda}\, e^{iqz} \;. \tag{4.25}$$

By adding this to the unperturbed wave and comparing to eq. (4.6), we can fix a,

$$a \;=\; \left(1 + \frac{2i}{q}\frac{g}{\lambda} \right)^{-1} , \tag{4.26}$$

and hence the Born approximated phase shift,

$$\begin{aligned} \delta_\chi^{Born} \;&=\; 2i\, \ln\left(\frac{a}{|a|}\right) \;=\; i \ln\left(\frac{a}{a^*}\right) \\ &=\; -i\, \ln\left(\frac{1 + i\frac{2g/\lambda}{q}}{1 - i\frac{2g/\lambda}{q}}\right) \;=\; 2\tan^{-1}\left(\frac{\sqrt{2}mg/\lambda}{k}\right) . \end{aligned} \tag{4.27}$$

With the help of δ_χ^{Born} we can calculate the contribution from the continuous eigenfrequencies, and specifically the integral

$$\begin{aligned} \frac{\hbar}{4\pi} \int_0^{k_{max}} dk\, \frac{d\delta_\chi^{Born}}{dk}\, \omega_{\rm cont}^{\chi} \;&=\; -\frac{\hbar m}{4\pi} \left[\sqrt{2}\, \frac{g}{\lambda}\, \ln\frac{4\Lambda^2}{m^2 g/\lambda} \right. \\ &\left. + \sqrt{\frac{g}{\lambda}\left(1 - 2\frac{g}{\lambda}\right)} \left(\sin^{-1}\left(1 - 4\frac{g}{\lambda}\right) + \frac{\pi}{2} \right) \right] . \end{aligned} \tag{4.28}$$

(This equation only holds in its present form for $g/\lambda < 1/2$.) For the (single) discrete eigenfrequency we will use the exact result according to eq. (4.4). (In most cases one will have to evaluate the discrete eigenvalue(s), if there are any, numerically.‡) After collecting also the contributions from the renormalization,

‡ In more than one spatial dimension there are likely to be no discrete states for shallow potentials.

the total energy shift due to the χ field adds up to

$$E_{\delta\chi}^{Born} = \frac{\hbar m}{2\sqrt{2}}\left[2\,\frac{g}{\lambda} - \left(\sqrt{2\frac{g}{\lambda}+\frac{1}{4}}-\frac{1}{2}\right)^2\right]^{1/2} - \hbar m\,\frac{\sqrt{2}}{2\pi}\,\frac{g}{\lambda} - \frac{\hbar m}{4\pi}\sqrt{\frac{g}{\lambda}\left(1-2\frac{g}{\lambda}\right)}\left(\sin^{-1}\left(1-4\frac{g}{\lambda}\right)+\frac{\pi}{2}\right) - \frac{\hbar m}{4}\sqrt{\frac{g}{\lambda}}\,. \tag{4.29}$$

The last term again prevents us from doublecounting the bound state.

In fig. 4 we plot the energy shift from the Born approximation and compare it with the previous results. The Born approximation does rather well for very small g/λ, but starts to deviate significantly for $g/\lambda \gtrsim 0.01$. Note, however, that the effective potential approach is always worse than the Born approximation in the small g/λ regime.

4.4 Conclusions

One of the prerequisites of understanding a first order phase transition is the calculation of the bubble nucleation rate, or the decay rate of the false vacuum. This involves, by definition, inhomogeneous configurations. In evaluating the decay rate, one has to consider quantum fluctuations. The question then arises, whether it is adequate to drop the derivative terms in the effective action and only retain the effective potential, or whether these derivative corrections are important. In other words, do we have to integrate out quantum fluctuations about the *inhomogeneous* configuration, or is it sufficient to consider the fluctuations about a *constant* background? In the attempt to answer this question, we confined ourselves to a simple model, a kink in (1+1) dimensions and zero temperature. We first computed the energy shift from quantum fluctuations about the inhomogeneous background and then compared it to the energy from the effective potential. The discrepancy in the two results clearly demonstrates the insufficiency of the effective potential. Note that although quantum effects are small by definition, it is precisely these corrections, which may fix the order of a given phase transition. Any error may therefore obscure the real situation. We next coupled the kink field to a second scalar field and also included its quantum fluctuations in the energy calculation. The addition of the second field allowed us

to vary the mass of the fluctuations and their coupling to the kink and thereby to span a larger parameter space. In the limit where the mass of the additional fluctuations is large, the fluctuations do not probe the kink profile efficiently. This is the regime, where the effective potential represents a valid approximation to the full action. In the limit of small mass, however, the derivative terms are important and one cannot rely on the effective potential. It is interesting to note that it is precisely in this regime, where one can resort to additional approximations, as for example the Born approximation. In more realistic situations it may be impossible to derive an exact result and one is forced to rely on such approximations. We therefore recalculated the energy shift within the Born approximation and compared it to the exact result. Although, in our case, the approximated energy falls somewhat short of the exact result, its estimate is significantly more accurate than the effective potential result.

In conclusion we should point out, that the current calculations have been carried out in a very simple system and their results may not necessarily be extrapolated to the analysis of a first order phase transition in, *e.g.*, the electroweak theory. Higher dimensions, finite temperature effects and the introduction of fermions and gauge fields can possibly change the conclusions. In light of our results, it must, however, be kept in mind, that there is no *a priori* justification to neglect derivative terms, but that the validity of the effective potential for inhomogeneous configurations must be checked from case to case.

ACKNOWLEDGEMENTS

It is a pleasure to thank Erick Weinberg for many helpful discussions about effective potentials and derivative terms. We also acknowledge valuable discussions with Robert Brandenberger, Brian Greene, Chris Hill, Dallas Kennedy, and Esteban Roulet. This work was supported in part by the DOE and NASA grant NAGW-2381 at Fermilab. B.G. also acknowledges DOE grant #DE-FG03-91ER (40662 Task C) at UCLA.

APPENDIX A

In this Appendix we outline our renormalization scheme for the two field case.

We start with the tree level potential of eq. (4.1),

$$V_0 \;=\; \frac{\lambda}{4}\left(\phi^2 - \frac{m_0^2}{\lambda}\right)^2 + \frac{1}{2} g\, \phi^2 \chi^2 \;. \qquad (A1)$$

Following standard procedure, the effective potential to one-loop becomes

$$\begin{aligned} V_{\rm eff} \;=\; V_0 \;+\; & \frac{\hbar}{8\pi}\left(3\lambda\phi^2 - m_0^2\right)\left[\, 1 + \ln\frac{4\Lambda^2}{3\lambda\phi^2 - m_0^2}\,\right] \\ & + \;\frac{\hbar}{8\pi} g\phi^2\left[\, 1 + \ln\frac{4\Lambda^2}{g\phi^2}\,\right] \;+\; 2\,\frac{\hbar\Lambda^2}{4\pi}\;, \end{aligned} \qquad (A2)$$

with minimum at ϕ_M,

$$\phi_M^2 \;=\; \frac{m_0^2}{\lambda} - \frac{\hbar}{4\pi}\left[\, 3\ln\frac{2\Lambda^2}{m_0^2} + \frac{g}{\lambda}\ln\frac{4\Lambda^2}{m_0^2 g/\lambda}\,\right] \;+\; \mathcal{O}(\hbar^2)\;. \qquad (A3)$$

The regularization parameter Λ is the ultraviolet momentum cutoff. (We can trivially retrieve the effective potential for the single scalar field model of section 3 by setting $g = 0$ and neglecting one of the terms $\hbar\Lambda^2/(4\pi)$ in eq. (A2).)

We define the physical (renormalized) parameters by "minimal subtraction", which 'pins down' to the identification of the bar coupling constant λ and the physical one,

$$\lambda_{phys} \;\equiv\; \lambda\;. \qquad (A4)$$

Note that this identity holds to all orders in perturbation theory. (As the χ-field has vanishing expectation value and we are only working to one-loop, we do not have do worry about the renormalization of g.) We define the renormalized mass by

$$\begin{aligned} \frac{1}{2}\,\frac{d^2 V_{\rm eff}}{d\phi^2}(\phi = \phi_M) \;&=\; m_0^2 \;-\; \lambda\,\frac{\hbar}{4\pi}\left(\,\frac{9}{2} + \frac{g}{\lambda}\,\right) \\ &\quad -\; \lambda\frac{\hbar}{4\pi}\left(\,3\ln\frac{2\Lambda^2}{m^2} + \frac{g}{\lambda}\,\ln\frac{4\Lambda}{m^2 g/\lambda}\,\right) \\ &\equiv\; m^2 \;-\; \lambda\,\frac{\hbar}{4\pi}\left(\,\frac{9}{2} + \frac{g}{\lambda}\,\right), \end{aligned} \qquad (A5)$$

where m without the subscript denotes the physical mass. From here,

$$m_0^2 \;=\; m^2 \;+\; \lambda\,\frac{\hbar}{4\pi}\left(\,3\ln\frac{2\Lambda^2}{m^2} \;+\; \frac{g}{\lambda}\,\ln\frac{4\Lambda}{m^2 g/\lambda}\,\right). \qquad (A6)$$

The virtue of this renormalization scheme lays in the absence of any *finite* quantum corrections to the mass. Consequently, the tree-level energy of the kink, if written in terms of the physical parameters, contains only infinite corrections and no additional finite terms.

The requirement that $V_{\rm eff}(\phi_M) = 0$ generates a vacuum energy ("cosmological constant"),

$$\begin{aligned}\Gamma \;&\equiv\; \Gamma^{\phi} \;+\; \Gamma^{\chi} \\ &=\; -\frac{\hbar}{4\pi}\left[\, m^2\left(1+\ln\frac{2\Lambda^2}{m^2}\right) \;+\; \Lambda^2\,\right] \\ &\quad -\; \frac{\hbar}{4\pi}\left[\,\frac{m^2}{2}\frac{g}{\lambda}\left(1+\ln\frac{4\Lambda^2}{m^2 g/\lambda}\right) \;+\; \Lambda^2\,\right] \;+\; \mathcal{O}(\hbar^2)\,.\end{aligned} \qquad (A7)$$

(For the single scalar field case we omit the whole second term.)

Inserting (A6) and (A7) into eq. (A2) yields the effective potential in terms of the renormalized parameters,

$$V_{\rm eff} \;=\; V_0(m,\lambda) \;+\; V_1^{\delta\phi}(m,\lambda) \;+\; V_1^{\delta\chi}(m,\lambda,g)\;, \qquad (A8)$$

where V_0 is the tree-level potential (A1) in terms of the physical mass, and

$$\begin{aligned} V_1^{\delta\phi} \;&=\; \frac{3\hbar}{8\pi}\left(\,\lambda\phi^2 - m^2\,\right) \;-\; \frac{\hbar}{8\pi}\left(3\lambda\phi^2 - m^2\right)\,\ln\left(\frac{3\lambda\phi^2 - m^2}{2m^2}\right), \\ V_1^{\delta\chi} \;&=\; \frac{\hbar g}{8\pi}\left(\,\phi^2 - \frac{m^2}{\lambda} - \phi^2\,\ln\frac{g\phi^2}{m^2 g/\lambda}\,\right). \end{aligned} \qquad (A9)$$

APPENDIX B

In this Appendix we evaluate the phase shifts, associated with χ-waves scattering off the kink.

The "potential" in the Schrödinger equation (4.2) is symmetric under the reflection $x \to -x$. This allows us to write the wavefunction ψ as a linear combination of parity eigenstates, *i.e.*, in terms of symmetric and antisymmetric functions,

$$\psi^{\chi}(x) \;=\; \psi_{+}(x) \;+\; \psi_{-}(x) \quad \text{with} \quad \psi_{\pm}(-x) = \pm\psi_{\pm}(x)\;. \tag{B1}$$

For simplicity, let us define 1-dimensional "polar" coordinates $r = |x|$ and $\mu = \mathrm{sgn}(x)$ and recast the symmetric and antisymmetric waves as [21]

$$\psi_{+} \;=\; D_{+}\, e^{ikr} \;+\; C_{+}\, e^{-ikr} \;\;,\quad \psi_{-} \;=\; \mu\left(D_{-}\, e^{ikr} \;+\; C_{-}\, e^{-ikr}\right)\;, \tag{B2}$$

where the coefficients are given in terms of a and b of eq. (4.7),

$$D_{\pm} \;=\; \frac{\mathcal{N}}{2}\left(\,1 \pm b\,\right) \;\;,\quad C_{\pm} \;=\; \pm\frac{\mathcal{N}}{2}\,a\;. \tag{B3}$$

The inflection invariance of the potential implies [21] that

$$|D_{+}|^2 \;=\; |C_{+}|^2 \;\text{ and }\; |D_{-}|^2 \;=\; |C_{-}|^2\;, \tag{B4}$$

and hence D_{+} (D_{-}) can at most differ from C_{+} (C_{-}) by a phase,

$$D_{+} \;=\; C_{+}\, e^{i\delta_{\chi}^{+}} \;\;,\quad D_{-} \;=\; C_{-}\, e^{i\delta_{\chi}^{-}}\;. \tag{B5}$$

This, together with eqs. (B3), yields

$$\delta_{\chi}^{\pm} \;=\; i\,\ln\!\left(\frac{a}{1\pm b}\right) \quad \text{and} \quad \delta_{\chi} \;\equiv\; \delta_{\chi}^{+} + \delta_{\chi}^{-} \;=\; 2i\,\ln\!\left(\frac{a}{|a|}\right)\;. \tag{B6}$$

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FIGURE CAPTIONS

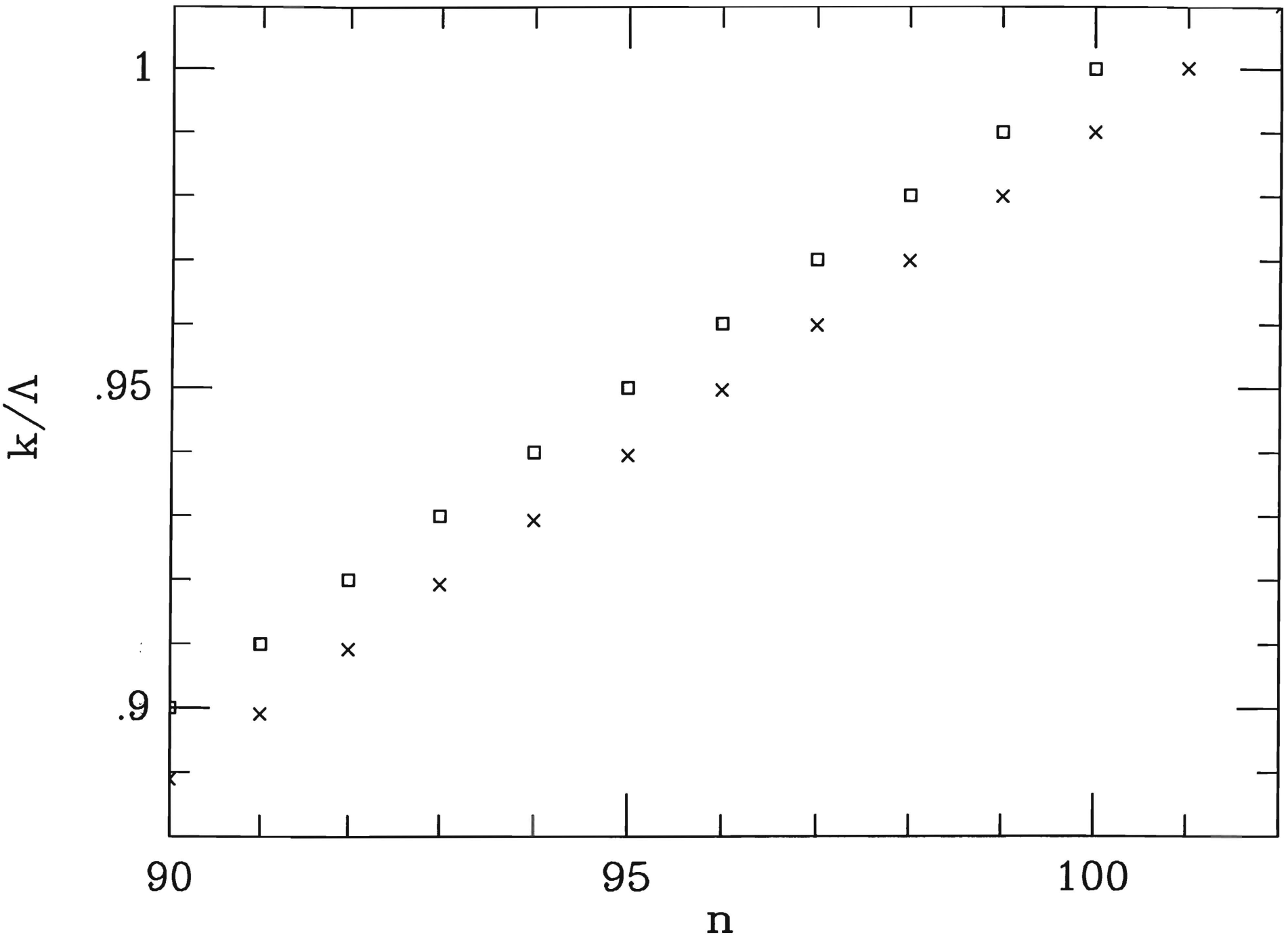
1) The effect of the kink on the frequencies and on the cut-off. Open squares represent wavevectors k in the absence of the kink. In this artificial example, the momentum cut-off $k = \Lambda$ corresponds to including only those states with $n \leq 100$. In the presence of the kink (crosses), the same value of n has a different wavenumber. In particular, for $n = 100$, k is less than Λ. Blindly imposing the ultraviolet cut-off $k \leq \Lambda$ in the presence of the kink leads to the counting of an extra state, the one with $n = 101$.

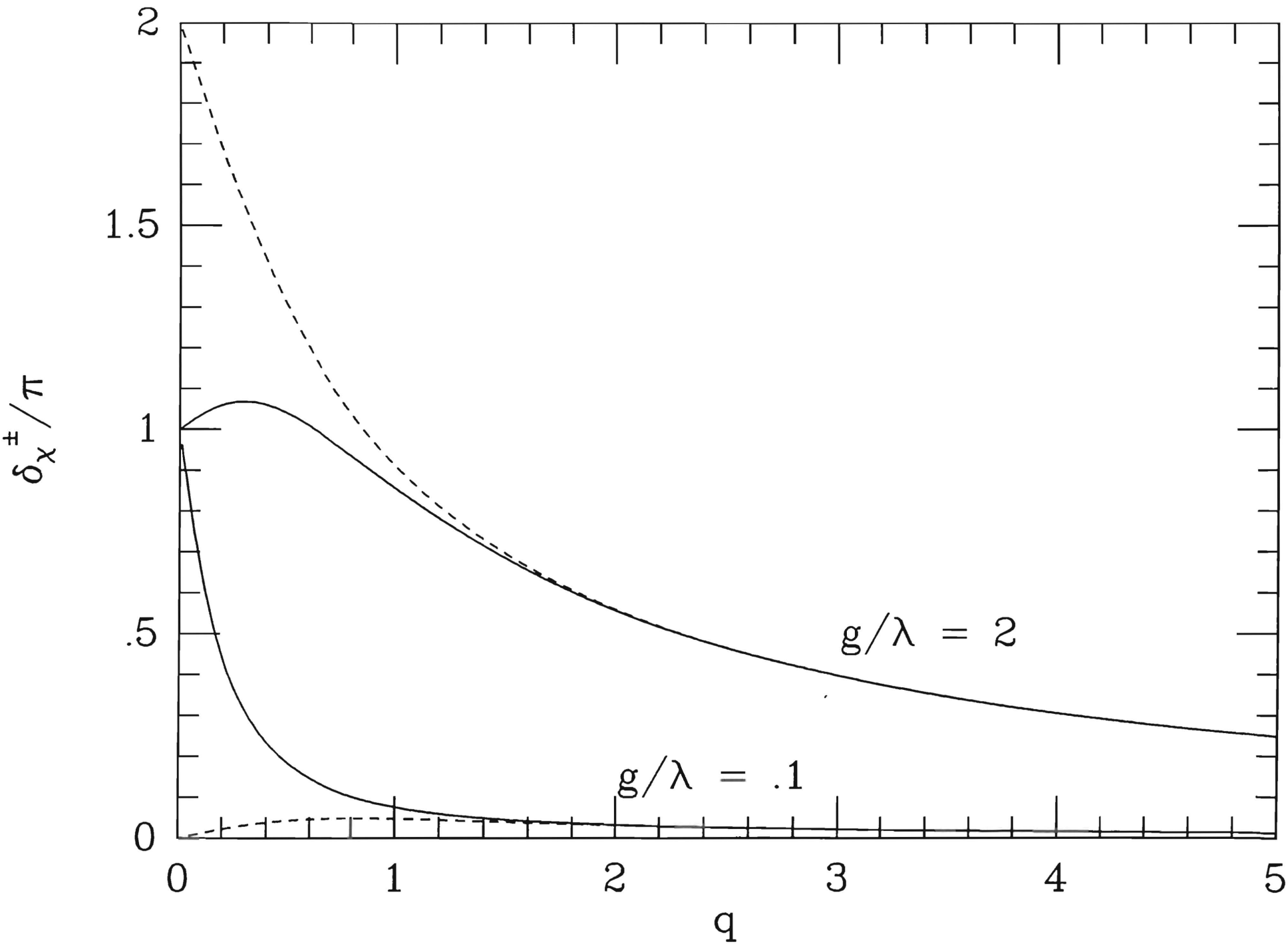
2) Even (solid line) and odd (dashed line) parity phase shifts of the χ continuum, as a function of wavenumber q for fixed g/λ. At large q, the phase shift goes to zero; only at low q does the presence of the kink lead to a significant phase shift. Larger values of g/λ correspond to deeper potentials and hence larger phase shifts.

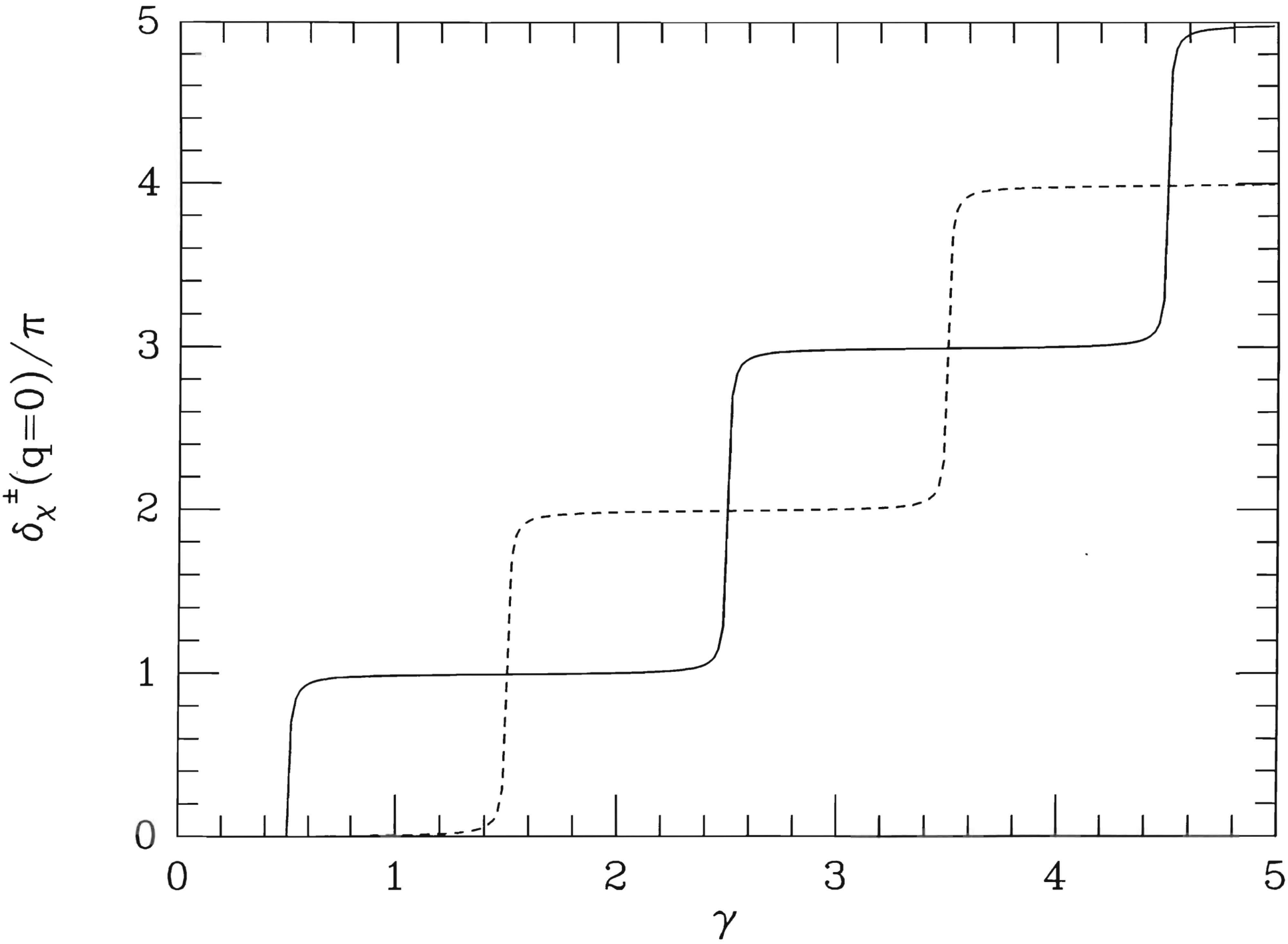
3) Even (solid line) and odd (dashed line) phase shifts for the continuous χ states at zero momentum, as a function of g/λ.

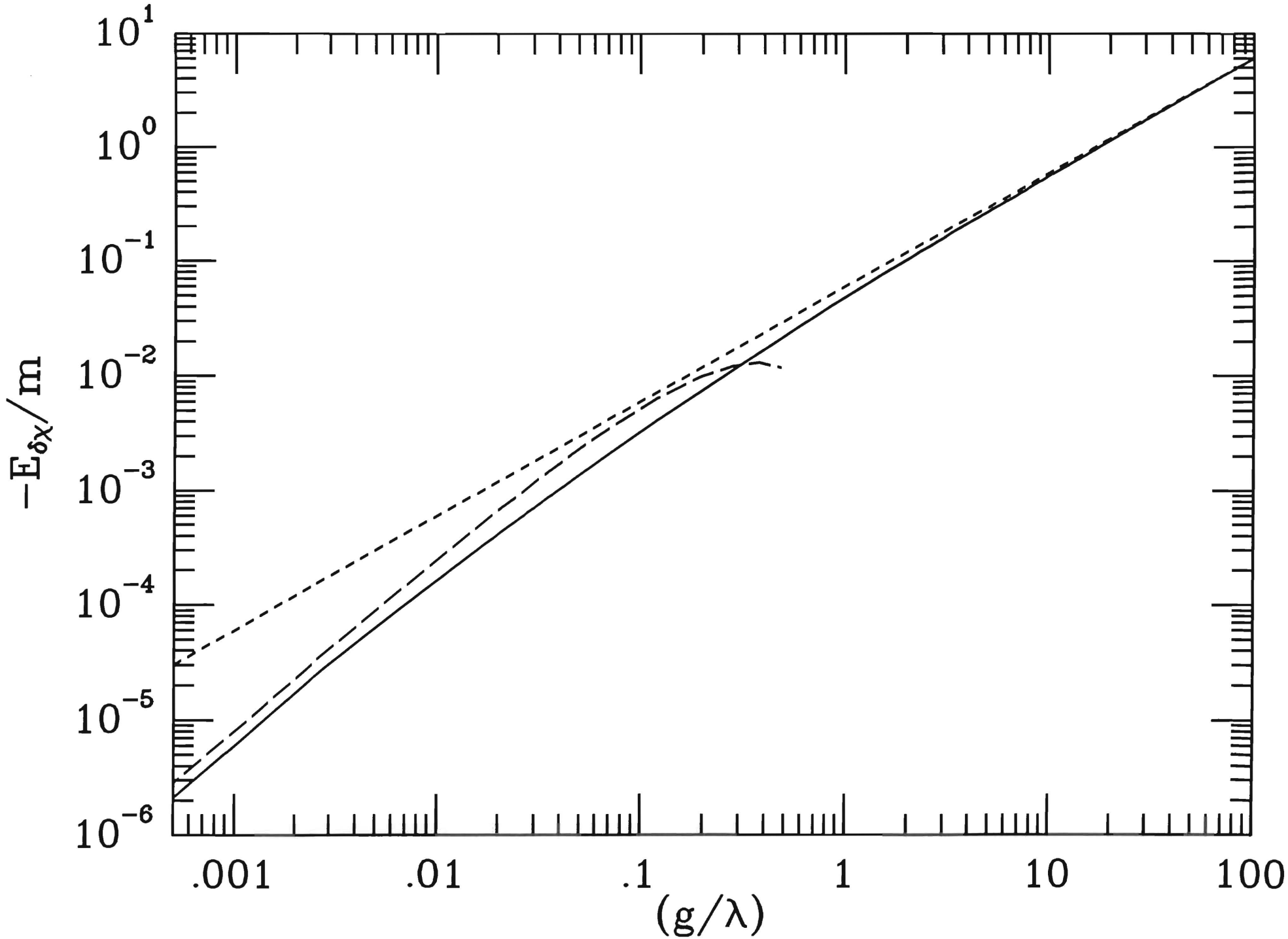
4) A comparison of the energy shift from the χ-fluctuations, as a function of g/λ, calculated within the effective action (solid line) and the effective potential (short dashed line) approaches. For large g/λ the derivative terms are unimportant and the effective potential becomes a valid approximation to the full action. For small g/λ, however, the effective potential is clearly inadequate for calculating the energy of an inhomogeneous configuration. We also present the energy shift, computed with in the Born approximation (long dashed line).

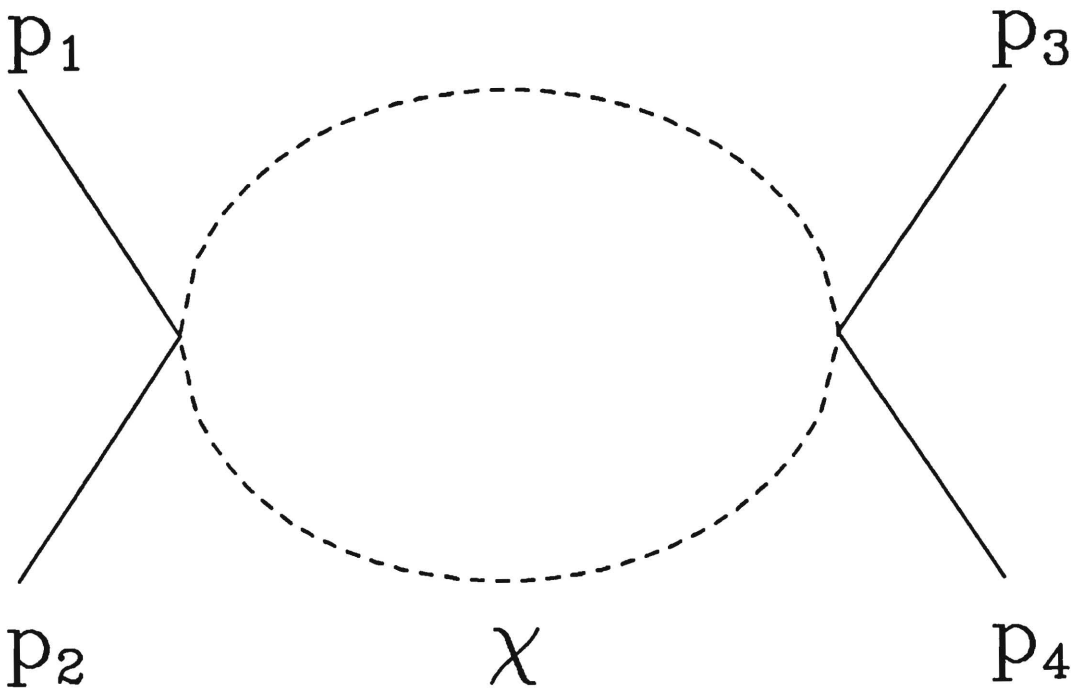
5) Feynman diagram leading to terms $\phi^4, \phi^2(\partial_\mu\phi)^2, \ldots$ in the effective action. Dotted line represents the internal χ propagator.


1
.95
.9
k/λ
90
95
100
n


2
1.5
1
.5
0
$\delta_\chi^\pm/\pi$
0
1
2
3
4
5
q
g/λ = 2
g/λ = .1


5
4
3
2
1
0
0
1
2
3
4
5
γ
$\delta_\chi^{\pm}(q=0)/\pi$


10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
$-E_{\delta\chi}/m$
.001
.01
.1
1
10
100
(g/λ)


p1
p3
p2
χ
p4