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q-Generating Functions and the q-Oscillator*

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Abstract

We construct generating functions which interpolate between the generating functions for Hermitian and Grassmann multinomials. The new functions which arise in this way may be related to state functions of the q-deformed harmonic oscillator.

1. Introduction

We extend the quantum plane relation[1] to two spaces of indeterminates $\{x^\alpha\}$ and $\{t^\alpha\}$:

$$x^\alpha t^\beta = q\, t^\beta x^\alpha \quad \text{all} \quad \alpha,\beta \tag{1.1}$$

In terms of these two sets $\{x^\alpha\}$ and $\{t^\alpha\}$ one may construct a generating function for multinomials in the $\{x^\alpha\}$.

One may also generalize to three spaces. Denoting the α-coordinate of the ith space by z_i^α we consider

$$z_i^\alpha z_j^\beta = R_{ij}^{\alpha\beta} z_j^\beta z_i^\alpha , \qquad j = 1,2,3 \quad \text{all} \quad \alpha,\beta \tag{1.2}$$

where the R_{ii} are restricted by a braid algebra and the R_{ij} $(i\neq j)$ satisfy a Yang-Baxter equation. With these three sets one may again construct new generating functions.

In this framework one obtains new functions which are generalizations of the classical special functions in many variables. We shall here restrict our discussion to (1.1) and the Hermite case. The commuting case (q=1) had been introduced long ago by Hermite[2] while the Grassmannian case (q=-1) naturally turns up in a supersymmetric context.[3,4] Although our work will be based on the Hermitian generating function, the method is general and has been extended to other classical functions.[5]

2. Expansion of Generating Function[6]

Let

$$G = \exp(-A_{\alpha\beta} t^\alpha t^\beta + 2\, B_{\alpha\beta} t^\alpha x^\beta) \tag{2.1}$$

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where

$$
\begin{aligned}
{}_t\alpha_t\beta &= q^{tt}\,{}_t\beta_t\alpha & q^{tt} &= \pm 1 \\
{}_x\alpha_x\beta &= q^{xx}\,{}_x\beta_x\alpha & q^{xx} &= \pm 1 \\
{}_x\alpha_t\beta &= q^{xt}\,{}_t\beta_x\alpha & q^{xt} &= q
\end{aligned}
\tag{2.2}
$$

Then

$$G = \sum_{p=0}^{\infty} \frac{(-A+B)^p}{p!} \tag{2.3}$$

By the q-binomial theorem

$$G = \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{s=0}^{p} \langle {}^{p}_{s} \rangle_{q^2} \, (-A)^s \, (B)^{p-s} \tag{2.4}$$

where

$$\langle {}^{p}_{s} \rangle_q = \frac{\langle p \rangle_q !}{\langle s \rangle_q ! \; \langle p-s \rangle_q !} \tag{2.5}$$

and

$$\langle p \rangle_q = \frac{q^p - 1}{q-1} \tag{2.6}$$

We also introduce the corresponding symmetric forms:

$$\begin{bmatrix} p \\ s \end{bmatrix}_q = \frac{[p]_q !}{[s]_q ! \; [p-s]_q !} \tag{2.7}$$

where

$$[p]_q = \frac{q^p - q^{-p}}{q - q^{-1}} \tag{2.8}$$

After some rearrangements one finds

$$G = \sum_{n=0}^{\infty} \sum_{\lambda_1 \ldots \lambda_n} t^{\lambda_1} \ldots t^{\lambda_n} \, H_{\lambda_1 \ldots \lambda_n}(\bar{x}_1 \ldots \bar{x}_n) \tag{2.9a}$$

where

$$
\begin{aligned}
H_{\lambda_1 \ldots \lambda_n}(\bar{x}_1 \ldots \bar{x}_n) = \sum_{s=0}^{[n/2]} & \frac{(-)^s}{(n-s)!} \begin{bmatrix} n-s \\ s \end{bmatrix}_q \frac{\varepsilon_q(n-s)}{\varepsilon_q(s)} \\
& \times A_{\lambda_1 \lambda_2} \ldots A_{\lambda_{2s-1} \lambda_{2s}} (2\bar{x})_{\lambda_{2s+1}} \ldots (2\bar{x})_{\lambda_n}
\end{aligned}
\tag{2.9b}
$$

and

$$
\begin{aligned}
\bar{x}_\gamma &= B_{\gamma\delta} x^\delta \\
\varepsilon_q(s) &= q^{s(s-1)/2}
\end{aligned}
\tag{2.9c}
$$

In the derivations of (2.9) no assumption has been made about $q^{(xx)}$. However, (2.9a) implies that the λ-symmetry is determined by the $\{t^\alpha\}$ and consequently the symmetry of the $\{t^\alpha\}$ is passed on to the $\{x^\alpha\}$. Therefore $q^{xx} = q^{tt} = \pm 1$ as anticipated in (2.2).

We are particularly interested in the special case:

$$q^{xx} = q^{tt} = 1 \ , \quad |q^{xt}| \neq 1$$

$$A_{\lambda_1\lambda_2} = B_{\lambda_1\lambda_2} = \delta_{\lambda_1\lambda_2} \quad \text{and} \quad \lambda_1 = \lambda_2 = \ldots \lambda_n \tag{2.10}$$

Then

$$H_n^{\,q}(x) = \sum_{s=0}^{[n/2]} \frac{(-)^s}{(n-s)!} \begin{bmatrix} n-s \\ s \end{bmatrix}_q \frac{\varepsilon_q(n-s)}{\varepsilon_q(s)} (2x)^{n-2s} \tag{2.11}$$

If q=1, (2.11) becomes the usual Hermite polynomial. We now turn to the relation of these q-polynomials to the state functions of the q-deformed oscillator.

3. The Quantum Group and the q-Quantum System

We shall here define a q-deformed quantum mechanical system by an unchanged Hamiltonian and the q-modified Heisenberg commutator:

$$Dx - qxD = 1 \ , \quad p = -i\hbar D \tag{3.1}$$

Eq. (3.1) may be rewritten

$$\tilde{z}Qz = 1 \tag{3.2}$$

where

$$Q = \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} D \\ x \end{pmatrix} \tag{3.3}$$

Let $T \in GL_q(2,C)$. Then under $z \to Tz$ (3.2) becomes

$$\tilde{z}Qz = \Delta \tag{3.4a}$$

or

$$Dx - qxD = \Delta \tag{3.4b}$$

where Δ is the quantum determinant.

Since Δ is central, one may put $\Delta = 1$ and if $T \in SL_q(2,C)$, the q-commutators are invariant.

One may satisfy (3.4) by the following choices of D:

$$\text{(a)} \quad q = 1 \quad \Delta = 1 \quad D = \frac{d}{dx} \tag{3.5}$$

$$\text{(b)} \quad q \neq 1 \quad \Delta = 1 \quad D = x^{-1} \langle\theta\rangle_q \tag{3.6}$$

$$\text{(c)} \quad q \neq 1 \quad \Delta \neq 1 \quad D = x^{-1} [\theta]_q \tag{3.7}$$

where

$$\theta = x \frac{d}{dx} \tag{3.8}$$

Here the symbols < > and [ ] are defined in (2.6) and (2.8). In cases (b) and (c) D is a difference operator. In case (c) one finds

$$\Delta = q^{-\theta} \tag{3.9}$$

The wave function of the q-deformed free particle is in case (b)

$$e_q(ipx) \equiv \sum \frac{(ipx)^n}{\langle n\rangle_q!} \tag{3.10}$$

and in case (c) $\langle n\rangle_q \to [n]_q$.

If the q-deformed oscillator is defined by

$$H = \frac{1}{2}\,(a\bar{a} + \bar{a}a) \tag{3.11a}$$

$$a\bar{a} - q\bar{a}a = q^{-\theta} \tag{3.11b}$$

then in this representation the energy spectrum is

$$E_n = \frac{1}{2}\left([n]_q + [n+1]_q\right) \tag{3.12}$$

while the usual monomial state functions are normalized by $([n]_q!)^{-1/2}$ instead of $(n!)^{-1/2}$.

4. The q-Oscillator in Configuration Space

If the q-oscillator is defined by an unchanged Hamiltonian and a q-commutator, one has

$$H = \frac{1}{2}\,(p^2+x^2) \tag{4.1}$$

$$Dx - qxD = 1 \tag{4.2}$$

where (4.2) corresponds to (3.1) rather than (3.4). Define

$$D_\pm = D \pm q^{\mp 1/2}\,x \tag{4.3}$$

then

$$H - \frac{1}{2}\,q^{-1/2} = -\frac{1}{2}\,D_-D_+ \tag{4.4}$$

If we regard D_+ and D_- as lowering and raising operators, then the ground state may be defined by

$$D_+\psi_o = 0 \tag{4.5}$$

and the lowest energy is then

$$E_o = \frac{1}{2}\,q^{-1/2} \tag{4.6}$$

Now (4.5) is a functional (difference) equation for the ground state. The exact solution of (4.5), which may be obtained by iteration, is the convergent infinite product:

$$\psi_o(x) = \prod_{s=0}^{\infty}\left[1 + q^{2s}(q^{1/2} - q^{-1/2})\,x^2\right]^{-1}\psi_o(0)\ , \quad |q| < 1 \tag{4.7}$$

and an approximate solution, approximating the usual Gaussian as $q \to 1$, is

$$\psi_o(x) = \exp\left[-f(q)\,\frac{x^2}{2}\right] \tag{4.8}$$

$$f(q) = \left(q^{1/2} - q^{-1/2}\right) (\ell n\ q)^{-1} \tag{4.9}$$

One would like to find a complete set of exact solutions of the deformed oscillator that approach the known eigensolutions of the undeformed system. Part of the difficulty, however, is that there are several reasonable formulations of the deformed problem besides the one defined by Eqs. (4.1) and (4.2). Here are some possibilities for the deformed state functions.

1. The state functions are the Hermitian q-functions (2.11) defined by the generating function (2.1).

2. If one chooses (3.6) then it also is natural to choose $e_q(x)$, given by (3.10), instead of the exponential as the generating function. The new G, replacing (2.1), is then either

$$G = \sum \frac{(-A+B)^n}{\langle n\rangle_q!} \tag{4.10}$$

$$= e_q(-A+B) \tag{4.11}$$

or a similar construction depending on $[n_q]!$ Here q is left free. If, for example, one chooses the new generating function to be $e_{q^2}(-A+B)$ then the new Hermitian q-function $H_n^q(x)'$ is

$$H_n^q(x)' = \sum_{s=0}^{[n/2]} \frac{(-)^s}{\varepsilon_q(s)} \frac{1}{[s]_q!\ [n-2s]_q!} (2x)^{n-2s} \tag{4.12}$$

3. The deformed state functions are solutions of the following integral equation

$$\varphi_n^{(q)}(p) = \lambda_n \int dx\ e_q(ipx)\ \varphi_n^{(q)}(x) \tag{4.13}$$

As $q\to 1$, e_q approaches the usual exponential and $\varphi_n^q(x)$ approaches the familiar Hermite function (which is its own Fourier transform).

All of these q-functions approach the usual Hermite functions in the limit $q\to 1$.

In the q-space of the oscillator there is a complete solution of the dynamical problem not only at q=1 but also at the Grassmann limit, q=-1.[4] In addition, since there is also a complete solution for arbitrary q in the (a,a) representation, it would be of interest if there were a satisfactory q-transformation theory leading from the (a,a) to the (x,p) representation There is of course no physical support for a q-quantum mechanics--or any similar modification of quantum theory--at the present time.[7]

5. References

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