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New Quantum Groups, Quantum Planes and Applications

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Having previously determined the full dual form or universal T-matrix for the quantum group $U_{q,q'}(\mathfrak{gl}_2)$, we can reverse the roles usually played by the solvable dynamical algebra and the dual matrix algebra. As a bialgebra, $\mathfrak{gl}_2$ is coboundary and therefore admits a Yang-Baxter matrix. The Lie bialgebra dual is not coboundary, in the strict sense, nevertheless it too admits a Yang-Baxter matrix. The price that one has to pay for this is cheap, and the scope for applications is thereby greatly expanded. We do all the necessary calculations for $U_{q,q'}(\mathfrak{gl}_2)$ and also for $U_q(\mathfrak{gl}_3)$; whence the generalization to $U_q(\mathfrak{gl}_n)$ is expected to be straightforward. We calculate the structure of the new Woronowicz pseudo-groups that are related (directly, not by duality!) to these quantum groups. We interpret them as automorphisms of novel quantum planes, in the sense of Manin. This leads to a quantum Yang-Baxter matrix and *a fortiori* to a classical one. The latter is one of the more esoteric r-matrices found by Belavin and Drinfeld. Applications may include new integrable models without spectral parameters.

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I. INTRODUCTION

All recent activity in the field of integrable models makes essential use of the Yang-Baxter matrix, in its original quantum mechanical context[1] or in the classical form developed later.[2] The existence of the classical r-matrix was interpreted by Drinfeld in terms of coboundary Lie bialgebras,[3] and the construction of a related quantum R-matrix could then be viewed as arising from quantization, especially when quantization is interpreted as a deformation of classical structures.[4]

A principal new result reported here is that Drinfeld's restriction to coboundary Lie bialgebras can be relaxed. We begin by attempting to show that this would have important applications.

Basic to all applications is the construction of a Lax pair, a matrix valued, flat connection on phase space. One sees immediately that two Lie algebras are involved; for convenience we shall assign names to them, to be maintained throughout the paper. Thus $\mathfrak{g}_x$ will denote the Lie algebra of basic dynamical variables. This is not the infinite dimensional Poisson algebra on phase space, but some finite subalgebra of it that is the focus of interest and that forms the basis for quantization.[4] Typically, $\mathfrak{g}_x$ is the Heisenberg algebra (as in the nonlinear Schrödinger model) or the Euclidean group (as in the sine-Gordon theory), and sometimes it is a simple Lie algebra (as in the case of spin systems). On the other hand, $\mathfrak{g}_l$ will denote the auxiliary matrix algebra where the Lax connection lives. There is no *a priori* known principle that commands a particular choice of $\mathfrak{g}_l$, except that a preference for simplicity usually leads to the lowest possible dimension; this explains why this algebra is normally simple; in the familiar examples it is $\mathfrak{gl}_2$.

A Lax pair does not automatically imply the existence of an r-matrix. According to the insight of Drinfeld, the r-matrix is intimitely related to the fact that $\mathfrak{g}_x$ and $\mathfrak{g}_l$ can be promoted to a pair of mutually dual Lie bialgebras, and more especially to the fact that $\mathfrak{g}_l$ (but not $\mathfrak{g}_x$!) is a coboundary Lie bialgebra.[3] The connection is provided by the relation $f = \mathrm{d}r$, where f is the $\mathfrak{g}_l$ one-form that determines the coproduct on $\mathfrak{g}_l$ (and the Lie product on $\mathfrak{g}_x$), and $\mathrm{d}r$ is the differential of r. This then implies very strong conditions of compatibility between the two algebras. For example, if $\mathfrak{g}_l$ is $\mathfrak{sl}_2$, then the structure of $\mathfrak{g}_x$ is fixed; it is not the Euclidean algebra $E(2)$, though it is similar to it (one of the signs in the commutation relations is different). Therefore, $\mathfrak{sl}_2$ cannot be used as the auxiliary algebra to solve the sine-Gordon model, for example, for the dynamical algebra of this model is $E(2)$. The impasse is resolved by a device known as the spectral parameter. Instead of $\mathfrak{sl}_2$, one introduces the infinite dimensional Lie algebra $\mathfrak{sl}_2[\lambda]$, of matrices of power series in a parameter λ, the spectral parameter. This infinite dimensional Lie algebra can be endowed with a coproduct that turns it into a coboundary Lie bialgebra. Also, and

this is the point, its (infinite dimensional) dual contains $E(2)$ as a quotient. Thus it seems as if, to incorporate a physically relevant dynamical algebra —$E(2)$, $\mathfrak{sl}_2$, ... — into the scheme, the use of a spectral parameter is essential, so that one has to deal with infinite dimensional Lie algebras and infinite dimensional quantum groups. This does not annihilate the subject; but it is interesting to note that all recent work on quantum groups deals with finite dimensional Lie algebras, although this makes the connection to solvable models a little distant. It would certainly be interesting to see some examples that do not require infinite dimensional Lie algebras but instead provide direct applications of q-deformations of finite dimensional Lie bialgebras.

Our inspiration comes from the high degree of symmetry that exists in the dual relationship between $\mathfrak{g}_x$ and $\mathfrak{g}_l$. The quantum group $U_q(\mathfrak{gl}_2)$ is a deformation of the enveloping algebra $U(\mathfrak{gl}_2)$ and its bialgebra dual is also a deformation of an enveloping algebra.[5] It is not too much to conjecture that, quite generally, the bialgebra dual of $U_q(\mathfrak{g}_l)$ is a deformation $U_q(\mathfrak{g}_x)$ of the enveloping algebra of $\mathfrak{g}_x$. This has been verified in some cases, and it carries interesting implications. Namely, by stressing the symmetry between the two Lie bialgebras, rather than the relation of duality between them, one should be able to use them interchangeably; in spite of the fact that only one of them is a coboundary.

In the simplest interesting case, that of the quantum group $U_{q,q'}(\mathfrak{gl}_2)$, the bialgebra dual form is known explicitly in terms of q-exponentials,[5]

$$T = \mathrm{e}_{1/k}^{hzp_-}\mathrm{e}^{h(xp_1-\hat{x}p_2)}\mathrm{e}_k^{hyp_+}, \ k := qq'. \tag{1.1}$$

Here the p's generate $U_{q,q'}(\mathfrak{gl}_2)$ and $z, x, \hat{x}, y$ generate (linearly) the Lie bialgebra dual $\mathfrak{gl}_2^*$ of $\mathfrak{gl}_2$. Nonlinearly these latter quantities generate a quantum group $U_{q,q'}(\mathfrak{gl}_2^*)$ that is the bialgebra dual of $U_{q,q'}(\mathfrak{gl}_2)$. The two Lie bialgebras thus appear quite symmetrically in this "universal T-matrix".

That (1.1) is the bialgebra dual form implies the crucial "multiplication property" of transfer matrices; more precisely, two such properties. Let there be given two copies of each algebra, one of each distinguished by a prime. Then

$$T(x,p)T(x,p') = T(x,\Delta_1(p)), \tag{1.2}$$

$$T(x,p)T(x',p) = T(\Delta_2(x),p). \tag{1.3}$$

Here Δ_i are the coproducts, both are homomorphisms of the respective Lie algebras. If the x's, that is, $z, x, \hat{x}, y$ are dynamical variables and the p's are matrices, then (1.3) is the relation that is normally made use of. But both relations are valid, so it is natural to expect that the roles of the two algebras can be interchanged.

Let us see what kinds of models can be built on this structure, without introducing a spectral parameter. Only $\mathfrak{gl}_2$ is a coboundary Lie bialgebra, its dual is not; therefore, according to Drinfeld, it is the one that is suitable for use as matrix algebra; so that $\mathfrak{gl}_2^*$ must play the role of the dynamical algebra. This is quite contrary to the requirements of the applications, so it is natural to attempt to reverse the situation. Thus, in this paper we identify $\mathfrak{gl}_2$ (or, according to need, a restriction or a contraction of it) with the dynamical algebra, and the solvable dual $\mathfrak{gl}_2^*$ with the matrix algebra.

If in the formula (1.1) one takes the p's in the 2-dimensional representation of $\mathfrak{gl}_2$ (which is what one normally does), then the matrix elements T_i^j of T satisfy the algebraic conditions of Woronowicz' matrix pseudogroup.[6] Instead, we shall take $z, x, \hat{x}, y$ in a faithful representation of $\mathfrak{gl}_2^*$, then the matrix elements of T satisfy another set of algebraic relations. This new algebra of matrix elements is related to $\mathfrak{gl}_2$ and to $U_{q,q'}(\mathfrak{gl}_2)$ in the same way that the algebra of Woronowicz is related to $\mathfrak{gl}_2^*$ and to $U_{q,q'}(\mathfrak{gl}_2^*)$.

The lowest dimension of a faithful representation of this solvable Lie algebra is 3. But we are not dealing with an algebra of 9 generators, for the number of generators is always the dimension of the Lie algebra, hence four. The expression for T investigated in this paper, here simplified by setting $q = q'$, is

$$T = \begin{pmatrix} a & b & 0 \\ 0 & d & 0 \\ 0 & c & a \end{pmatrix} \tag{1.4}$$

and the algebraic relations satisfied by the matrix elements are

$$\begin{aligned} &ab = q^{-1}ba, ac = qca, ad = da, \\ &bd = q^{-1}db, cd = qdc, \\ &bc - qcb = \lambda(aa - dd), \end{aligned} \tag{1.5}$$

where λ is a parameter. The contrast between these and the well known relations of the simplest Woronowicz pseudogroup,[6] which are traceable to the commutation relations of a solvable Lie algebra,[5] is not striking, although (1.5) express the structure of a simple quantum group (after fixing the determinant).

The relationship between a, b, c, d and the more conventional generators of $U_{q,q'}(\mathfrak{gl}_2)$ is given in the beginning of the next section.

It is evident from the construction that T must have the multiplication property; it is easily verified directly.

The roles of the two underlying Lie algebras have thus been reversed. Recall that, in the applications, the matrices are used to construct the Lax connection, while the matrix elements are the dynamical variables. Our example is therefore appropriate for

systems whose principal dynamical variables are those of $\mathfrak{gl}_2$ or $\mathfrak{sl}_2$ (after restriction) or $E(2)$ (after contraction); for example, spin systems and the sine-Gordon model. An important result is that the T-matrix, and hence also the matrices of the Lax connection must necessarily be of dimension at least 3.

In order for this to be useful one needs (according to the present wisdom) an R-matrix. The existence of an R-matrix satisfying

$$R(T \otimes 1)(1 \otimes T) = (1 \otimes T)(T \otimes 1)R \tag{1.6}$$

would imply the multiplication property, but the converse is no true. Here the multiplication property is built into (1.1) and expressed by (1.2) and (1.3), but the existence of an R-matrix satisfying (1.6) (the coboundary property) must be checked directly. Indeed a matrix satisfying this condition will be constructed explicitly in Section 4, after we have prepared for the task in Section 3.

Once more according to current wisdom, this matrix should have to satisfy the Yang-Baxter or braid relation,

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{1.7}$$

That this relation turns out to be satisfied in the case at hand is a surprise. Recall that the infinitesimal form of this relation is the classical Yang-Baxter relation, and that this is related to the coboundary property of a Lie bialgebra, the matrix algebra $\mathfrak{g}_l$. In our case that would be the solvable dual of $\mathfrak{gl}_2$. But this Lie bialgebra is not coboundary! Nevertheless, we verify that the R-matrix does have an expansion around unity, and that the first order term does the work of a classical Yang-Baxter matrix. This surprising and, we believe important result does not contradict what was known previously. The point is that our r-matrix does not live in $\mathfrak{gl}_2^* \otimes \mathfrak{gl}_2^*$, but in $M_3 \otimes M_3$. We return to a discussion of this point in the end of Section 4 and again in the conclusions.

Before turning to the applications, which we defer to the concluding section and to another paper, we would like to investigate some additional problems of structure. Namely, is it possible to adapt Manin's elegant construction,[7] where the matrix pseudogroups of Woronowicz are interpreted in terms of automorphisms of quantum planes, to the new algebras? That this works well is shown in Section 3. However, in one important respect we depart from Manin's philosophy. Following Manin, we require the conservation of bilinear relations among the "coordinates" of the quantum plane, with T acting from the right. In addition, to get a complete characterization of T, we demand the preservation of an associated exterior algebra. (This way of tying down all the relations was considered but then rejected by Manin.[8]) In our case

this is just the right thing to do, for it leads directly to the R-matrix, as is shown in Section 4.

So far, all our calculations have been very painless. Unfortunately, we are not smart enough to guess how all this should be generalized. For this reason we found it necessary to repeat the calculations for the next more complicated case, when $\mathfrak{gl}_2$ is replaced by $\mathfrak{gl}_3$. The main difficulty was that we did not possess the formula for the universal T-matrix, the analogue of Eq. (1.1), in this case. (Indeed, this is another important outstanding problem.) We had to make a guess as to the correct expression for T, not in general but in the simplest (5-dimensional) faithful representation of $\mathfrak{g}_x$. An educated guess for the relations involved about 50 coefficients to be determined. But it all works out, and we now realize that the generalization to $\mathfrak{gl}_n$ is feasible.

One of the minor discoveries of this paper is an R-matrix associated with one of the more esoteric classical r-matrices found by Belavin and Drinfeld for $\mathfrak{sl}_3$ and $\mathfrak{sl}_5$.[9]

The last section of the paper represents what we understand, at this time, about the outlook for physical applications of these results.

II. NEW MATRIX PSEUDOGROUP FROM $\mathfrak{gl}_2$

The starting point is in this case the universal T-matrix in Eq. (1.1), slightly generalized below. The generators p_i of the 2-parameter deformation $U_{q,q'}(\mathfrak{gl}_2)$ satisfy the following commutation relations,

$$\begin{aligned} &[p_+, p_-] = -(\lambda/h^2)\mathrm{e}^{-h\eta(p_1+p_2)}\left[\mathrm{e}^{h(p_1-p_2)} - \mathrm{e}^{h(p_2-p_1)}\right], \\ &[p_1, p_\pm] = \pm p_\pm = -[p_2, p_\pm], [p_1, p_2] = 0, q := \mathrm{e}^h, \end{aligned} \tag{2.1}$$

and the Lie algebra dual form is the sum of the exponents in (1.1). One arrives at this form of the commutation relations by a small change of notation relative to our paper.[5] The two parameters q and q' of $U_{q,q'}(\mathfrak{gl}_2)$ have been replaced by $\mathrm{e}^{h(1-\eta)}$ and $\mathrm{e}^{h(1+\eta)}$, respectively. One regards η as a fixed parameter of the undeformed theory and h as a single deformation parameter. From now on this will uniformly be the significance of η, h and $q = \mathrm{e}^h$. In addition, we have replaced p_0, p_1 by $p_1, -p_2$.

The dual Lie algebra $\mathfrak{gl}_2^*$ is solvable; the faithful representations of smallest dimension are 3-dimensional. In one such representation the universal T-matrix (1.1), generalized to allow q not equal to q', reduces to

$$T = \begin{pmatrix} a & b & 0 \\ 0 & d & 0 \\ 0 & c & \hat{a} \end{pmatrix} \tag{2.2}$$

$$
\begin{aligned}
&a = \mathrm{e}^{h[p_1 - \eta(p_1+p_2)]}, \hat{a} = \mathrm{e}^{h[p_1 + \eta(p_1+p_2)]}, \\
&b = hp_- d, c = h\hat{a}p_+, d = \mathrm{e}^{hp_2}.
\end{aligned} \tag{2.3}
$$

The commutation relations (2.1) are expressed by the relations

$$
\begin{aligned}
&ab = q^{-1}ba, ac = qca, [a, \hat{a}] = 0, [a, d] = 0 = [\hat{a}, d], \\
&\hat{a}b = q^{-1}b\hat{a}, \hat{a}c = qc\hat{a}, bd = q^{-1}db, cd = qdc, \\
&[b, c] = \lambda(a\hat{a} - dd).
\end{aligned} \tag{2.4}
$$

We have not made a deep study of this algebra (the tensor algebra generated by $a, \hat{a}, b, c$ and d, modulo these relations). The fundamental algebra is $U_{q,q'}(\mathfrak{gl}_2)$; representations of (2.4) that do not allow for the generators to be expressed as in (2.3) will not be considered. For this reason we shall suppose that $a, \hat{a}$ and d are invertible.

We have investigated only one of the two equivalence classes of faithful, 3-dimensional representations of $\mathfrak{gl}_2^*$; and only one element of this class. If algebras that are obtained by taking two equivalent representations of $\mathfrak{g}_x$ are "equivalent algebras", than this has to be understood in some very particular sense. Some additional clarification would be desirable, both to understand "equivalence" and to investigate the other equivalence class.

If T' has matrix elements $a', \ldots, d'$ that satisfy the same relations, and commute with $a, \ldots, d$, then the matrix elements $a'', \ldots, d''$ of $T'' := TT'$ also satisfy the same relations. This is as much expected as it is fundamental.

The next task, that of finding an R-matrix, is greatly facilitated by some preparation, to which we turn.

III. A NEW QUANTUM PLANE

We try to interpret the matrix (2.2), with matrix elements satisfying the relations (2.4), in terms of automorphisms of a 3-dimensional quantum plane with generators ("coordinates", or from now on, coordinates) $\xi^i, i = 1, 2, 3$. That is, we require that the mapping given by

$$
\xi \mapsto \xi T = \xi' = (\xi^1 a, \xi^1 b + \xi^2 d + \xi^3 c, \xi^3 \hat{a}), \tag{3.1}
$$

preserve the relations. It is very easy to see that this condition is satisfied if the relations among the coordinates are

$$
q\xi^1\xi^2 = \xi^2\xi^1, q\xi^2\xi^3 = \xi^3\xi^2, q^2\xi^1\xi^3 = \xi^3\xi^1, \tag{3.2}
$$

and that these are the only relations of the type $\xi^i\xi^j = q_{ij}\xi^j\xi^i, i < j$, that are so preserved.

Now our strategy for constructing the R-matrix (which will be explained in more detail elsewhere) calls for a similar statement about an associated exterior algebra (in a way "dual" to the coordinates, but we avoid using that word in yet another sense). No set of relations is canonical, indeed they are determined by the structure of T. Let $\theta^i, i = 1,2,3$, be the generators of this exterior algebra; then one would expect relations of the kind (3.2), with the signs reversed, and in addition $\theta^i\theta^i = 0$ for $i = 1,2,3$. One easily finds that what is preserved is the unique structure

$$\begin{aligned} &\theta^1\theta^2 + q\theta^2\theta^1 = 0, \theta^2\theta^3 + q\theta^3\theta^2 = 0, \theta^1\theta^3 + \theta^3\theta^1 = 0, \\ &\theta^1\theta^1 = 0, \theta^2\theta^2 = \lambda\theta^1\theta^3, \theta^3\theta^3 = 0. \end{aligned} \tag{3.3}$$

We repeat that there is nothing canonical about the relations (3.2, 3.3), except for their number, 3^2; but they are all preserved when T acts from the right.

IV. YANG-BAXTER MATRIX

This property of the relations (3.2, 3.3) has the important implication that there is a numerical 9×9 matrix U such that

$$\theta^i\theta^j U^{kl}_{ij} = 0, \xi^i\xi^j(U - 1 - q^2)^{kl}_{ij} = 0, \tag{4.1}$$

which is unique if $1 + q^2 \neq 0$. To be precise, what is clear is that there is a matrix, unique up to scale, such that it annihilates $\theta\theta$ and has $\xi\xi$ as eigenvector. When we actually construct this matrix we find that the eigenvalue 0 is 3-fold degenerate and that there is just one other (6-fold) non-zero eigenvalue. We fix the scale by choosing it to be $1 + q^2$. To write down the result we choose a basis in the 3^2-dimensional direct product space indicated by an ordered sequence of pairs, as follows,

$$U = (1+q^2) \oplus \begin{pmatrix} 1 & q \\ q & q^2 \end{pmatrix} \oplus \begin{pmatrix} 1 & -\lambda q^2 & q^2 \\ 0 & 1+q^2 & 0 \\ 1 & \lambda & q^2 \end{pmatrix} \oplus \begin{pmatrix} 1 & q \\ q & q^2 \end{pmatrix} \oplus (1+q^2). \tag{4.2}$$

$$\begin{array}{lccccccccc} \text{Basis}: & 11 & 12 & 21 & 13 & 22 & 31 & 23 & 32 & 33 \end{array}$$

The fact that U preserves both structures is equivalent to the statement that U commutes with $(T \otimes 1)(1 \otimes T)$; hence U is, up to the addition of a scalar, the matrix that is conventionally denoted $\hat{R}$ or RP, where R is a Yang-Baxter matrix and P is the permutation of factors in the direct product of spaces. The suitable scalar substraction is q^2; that is,

$$RP = U - q^2. \tag{4.3}$$

This turns (4.1) into

$$\theta\theta(RP + q^2) = 0, \xi\xi(RP - 1) = 0, \tag{4.4}$$

so that $\xi^i\xi^j$ is "symmetric" (the deformed permutation RP is unity on $\xi\xi$). Also, and this is important, this makes R go to 1 when h goes to 0.

This construction of R does not guarantee that it satisfies the Yang-Baxter relation; nevertheless it does. In other words, we have discovered a new structure of coboundary bialgebra on $\mathfrak{gl}_3$. It is related to a known but somewhat esoteric classical r-matrix.

Define the classical Yang-Baxter matrix by expanding R as a series in h, with λ/h fixed,

$$R = 1 + hr + \mathcal{O}(h^2); \tag{4.5}$$

then one finds

$$r = 0 \oplus \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 2 & \mu & -2 \\ 0 & 0 & 0 \\ 0 & -\mu & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \oplus 0, \tag{4.6}$$

$$\begin{aligned} r = {} & M_1^1 \otimes M_2^2 + M_2^2 \otimes M_1^1 + M_2^2 \otimes M_3^3 + M_3^3 \otimes M_2^2 + 2M_1^1 \otimes M_3^3 \\ & -2M_1^2 \otimes M_2^1 - 2M_1^3 \otimes M_3^1 - 2M_2^3 \otimes M_3^2 - \mu(M_3^2 \otimes M_1^2 - M_1^2 \otimes M_3^2). \end{aligned} \tag{4.7}$$

Here μ appears as the limit, as $h \to 0$, of $-\lambda/h$. The second formula is more convenient for calculations; here M_i^j is the matrix with 1 in position i, j and zero elsewhere. We defer comments on this r-matrix to the end of this section.

Define a Lie coproduct on this matrix algebra $\mathfrak{gl}_3$ by

$$\delta(M) := \mathrm{d}r(M) = [M \otimes 1 + 1 \otimes M, r]. \tag{4.8}$$

Restriction to $\mathfrak{gl}_2^*$ gives a Lie coproduct that is compatible with the Lie structure and thus it gives $\mathfrak{gl}_2^*$ the structure of a Lie bialgebra. Of course, this is the Lie bialgebra that we started with. To verify this we first note that the 3-dimensional representation used for $\mathfrak{gl}_2^*$ is

$$\begin{aligned} & x \mapsto (1-\eta)M_1^1 + (1+\eta)M_3^3, \\ & \hat{x} \mapsto \eta M_1^1 - M_2^2 - \eta M_3^3, \\ & y \mapsto M_3^2, z \mapsto M_1^2. \end{aligned} \tag{4.9}$$

Eq.s (4.7-4.9) give us

$$\begin{aligned}\delta(x) &= -y \otimes z + z \otimes y = \delta(\hat{x}),\\ \delta(y) &= -(x+\hat{x}) \otimes y + y \otimes (x+\hat{x}),\\ \delta(z) &= (x+\hat{x}) \otimes z - z \otimes (x+\hat{x}).\end{aligned} \tag{4.10}$$

The Lie algebra dual form is $xp_1 - \hat{x}p_2 + yp_+ + zp_-$, and this then gives us a Lie structure for the p's that is the same as structure (2.1) in the limit as h goes to zero.

The original coproduct on $\mathfrak{gl}_2^*$ is thus given by the Lie algebra 1-form $\mathrm{d}r$, which is as good as to say that it is a coboundary Lie bialgebra. But Drinfeld has reserved this term for the case that r is an element of $\mathfrak{gl}_2^* \otimes \mathfrak{gl}_2^*$, which is not true here. Instead, we have imbedded $\mathfrak{gl}_2^*$ in $\mathfrak{gl}_3$, as a sub-bialgebra, and found the classical r-matrix sitting in $\mathfrak{gl}_3 \otimes \mathfrak{gl}_3$.

It is not yet clear what price has to be paid for going outside the conventional framework of coboundary Lie bialgebras, in the strict sense, but it does seem as though it is well worth it. The bottom line is determined by the new applications that it opens up.

We return briefly to the r-matrix (4.6). It satisfies the classical Yang-Baxter relation but it is not antisymmetric. A simpler and more transparent formula is obtained by adding the invariant $t := M_i^j M_j^i$; this does not change the bialgebra defined by r since $\mathrm{d}t = 0$. We also drop a term $I \otimes I$ where $I := M_i^i$ generates the center of $\mathfrak{gl}_3$. The result is a matrix

$$r' = -\sum_{i<j} M_i^j \wedge M_j^i + M_1^1 \wedge M_3^3 + \mu(M_1^2 \wedge M_3^2). \tag{4.11}$$

This does not satisfy the Yang-Baxter relation; it is quasi-triangular. The first term is the most commonly seen r-matrix for $\mathfrak{sl}_3$. The second term represents a simple modification that is allowed according to the general discussion of Belavin and Drinfeld.[9] If $\mu \neq 0$ we have one of the esoteric r-matrices in the list of Belavin and Drinfeld.[9] We find it worth while to emphasize that the inclusion of the μ-term significantly alters the structure of the bialgebra. If μ is zero then the dual Lie algebra is solvable; if μ is not zero then it contains $\mathfrak{sl}_2$.

V. THE CASE OF $\mathfrak{gl}_3$

We would deal here with the next more complicated case, applying exactly the same strategy. However, the universal T-matrix for $\mathfrak{gl}_q(3)$ is not known, so a different approach is required. We shall deal with the conventional quantum group, determined by the most familiar Yang-Baxter matrix; the dual Lie algebra $\mathfrak{gl}_3^*$ is again solvable,

and the smallest faithful representations are 5-dimensional. Taking our clue from the shape of the T-matrix (2.2) we shall represent $\mathfrak{gl}_3^*$ by matrices of similar form, taking

$$T = \begin{pmatrix} a_1 & P_1^2 & P_1^3 & & \\ & a_2 & P_2^3 & & \\ & & a_3 & & \\ & & Q_3^2 & a_2 & \\ & & Q_3^1 & Q_2^1 & a_1 \end{pmatrix}, \tag{5.1}$$

empty spaces to be read as zero. If the entries are complex numbers, then this is a representation of $\mathfrak{gl}_3^*$. Instead, the entries will be taken to have relations that reflect the structure of a deformation of a subalgebra of $U(\mathfrak{gl}_3)$.

Unlike the case of $U_q(\mathfrak{sl}_2)$, we do not know precisely what this means, on the other hand it is possible to make a very good guess on the basis of the preceeding experience. The key consideration is, of course, to incorporate the multiplication property. A little trial and error shows that the P's and the Q's must be related to the positive and negative roots of $\mathfrak{sl}_3$, and that undeformed commutators are those suggested by the notation, for example, $[P_1^2, P_2^3] \propto P_1^3$ and $[P_1^2, Q_2^1] \propto a_1 a_1 - a_2 a_2$.

On this basis we postulate algebraic relations in the form of q-ommutators, with all q-factors arbitrary, with more than 50 free parameters. Requiring the multiplication property reduces this freedom to 6 parameters. The result is

$$[a_i, P_1^2]_{q_i} = [a_i, P_2^3]_{p_i} = [a_i, Q_2^1]_{r_i} = [a_i, Q_3^2]_{s_i} = 0, \tag{5.2}$$

where

$$\begin{aligned} &[A, B]_\alpha := AB - \alpha BA, \\ &p_1 = q_3, p_2 = q_1, p_3 = q_2, \\ &s_1 = r_3, s_2 = r_1, s_3 = r_2, \\ &r_1/q_2 = r_2/q_1 = r_3/q_3 =: k, \end{aligned} \tag{5.3}$$

and

$$
\begin{aligned}
&[a_i, P_1^3]_{p_i q_i} = [a_i, Q_3^1]_{r_i s_i} = 0, \\
&[P_1^2, Q_3^2]_k = [P_2^3, Q_2^1]_k = 0, \\
&[P_1^2, Q_2^1]_k = \rho(a_1 a_1 - a_2 a_2), \\
&[P_2^3, Q_3^2]_k = \rho(a_2 a_2 - a_3 a_3), \\
&[P_1^2, P_1^3]_{q_1} = [P_2^3, P_1^3]_{q_2} = 0, \\
&[Q_2^1, Q_3^1]_{r_1} = [Q_3^2, Q_3^1]_{r_2} = 0, \\
&[P_1^2, P_2^3] = (q_2^{-1} - q_1^{-1}) a_2 P_1^3, \\
&[Q_2^1, Q_3^2] = (r_2^{-1} - r_1^{-1}) a_2 Q_3^1, \\
&[P_1^2, Q_3^1]_{kr_1} = -\rho a_2 Q_3^2, \\
&[P_2^3, Q_3^1]_{kr_2} = \rho a_2 Q_2^1, \\
&[P_1^3, Q_2^1]_{k/q_1} = -\rho P_2^3 a_2, \\
&[P_1^3, Q_3^2]_{k/q_2} = \rho P_1^2 a_2, \\
&[P_1^3, Q_3^1] = \rho(P_1^2 Q_2^1 - k Q_3^2 P_2^3) + \rho'(a_1 a_1 - a_3 a_3).
\end{aligned}
\tag{5.4}
$$

Next, we apply the strategy of the quantum plane. The relations

$$
[\xi^i, \xi^j]_{\mu_{ij}} = 0, i < j, \text{ except } \quad [\xi^2, \xi^4]_{\mu_{24}} = \lambda \xi^1 \xi^5, \tag{5.5}
$$

among the five ξ's involve 11 parameters that have to satisfy 31 relations for which there is precisely one solution, namely $\lambda = \rho$ and

$$
\begin{aligned}
&\mu_{12} = q_1, \quad \mu_{13} = q_1 q_3, \quad \mu_{14} = q_1 q_3 / r_3, \mu_{15} = q_1 q_3 / r_1 r_3, \\
&\mu_{23} = q_1, \quad \mu_{24} = q_1 / r_1, \quad \mu_{25} = q_3 / r_1 r_3, \\
&\mu_{34} = 1/r_1, \mu_{35} = 1/r_1 r_3, \\
&\mu_{45} = 1/r_1.
\end{aligned}
\tag{5.6}
$$

Similarly, the generators of the exterior algebra satisfy

$$
\begin{aligned}
&[\theta^i, \theta^j]_{\nu_{ij}} = 0, \quad \text{except} \quad [\theta^2, \theta^4]_{-1} = \rho \theta^1 \theta^5, \\
&\theta^i \theta^i = 0, \quad \text{except} \quad \theta^3 \theta^3 = \rho \theta^2 \theta^4 + \rho' \theta^1 \theta^5, \\
&\nu_{ij} := -(q_2/q_1) \mu_{ij}, i < j.
\end{aligned}
\tag{5.7}
$$

The U-matrix is 25-dimensional, but our method is quite efficient and in any case it is clear that this matrix reduces to block form,

$$
\begin{aligned}
U \quad &= \sigma \oplus \ldots (4 \text{ times}) \oplus \sigma \oplus V \oplus \ldots (8 \text{ times}) \oplus V \oplus W, \sigma := 1 + q_2/q_1. \\
\text{Basis}: \quad &11, 22, 44, 55 \qquad ij, i \neq j, i + j \neq 6
\end{aligned}
\tag{5.8}
$$

This matrix has generically two distinct eigenvalues, 0 and σ, with multiplicities $5\cdot 4/2$ and $5\cdot 6/2$ as is canonical for a deformed "symmetrizer". The first four terms are one-dimensional, the next eight are two-dimensional and each of them has the form

$$V = \begin{pmatrix} 1 & 1/\mu \\ -\nu & -\nu/\mu \end{pmatrix}. \tag{5.9}$$

We calculate the 5-dimensional matrix W in the case of the simplest parameterization; when

$$q_1 = r_2 =: q^{-1}, q_2 = r_1 = q, q_3 = r_3 = 1, \tag{5.10}$$

$$\sigma := 1 + q^2, V = \begin{pmatrix} 1 & q \\ q & q^2 \end{pmatrix}, \tag{5.11}$$

$$\begin{array}{cc} W = & \begin{pmatrix} 1 & 0 & -\lambda^2 q^4 \tau & -\lambda q^2 & q^2 \\ 0 & 1 & -\lambda q^2 & q^2 & 0 \\ 0 & 0 & \sigma & 0 & 0 \\ 0 & 1 & \lambda & q^2 & 0 \\ 1 & \lambda & \lambda^2 \tau & 0 & q^2 \end{pmatrix}, \tau := (1-q^2)^{-1}. \\ \text{Basis} & \quad 15 \quad 24 \quad 33 \quad 42 \quad 51 \end{array} \tag{5.12}$$

We define $RP := U - q^2$ and thus

$$R = 1^{\oplus 4} \oplus \hat{V}^{\oplus 8} \oplus \hat{W}, \tag{5.13}$$

where

$$\hat{V} = \begin{pmatrix} q & 1-q^2 \\ 0 & q \end{pmatrix}, \tag{5.14}$$

and

$$\hat{W} = \begin{pmatrix} q^2 & -\lambda q^2 & -\lambda^2 q^4 \tau & 0 & 1-q^2 \\ 0 & q^2 & -\lambda q^2 & 1-q^2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda^2 \tau & \lambda & 1 \end{pmatrix}. \tag{5.15}$$

Finally we expand $R = 1 + hr + \mathcal{O}(h^2), (q := \mathrm{e}^h)$, to obtain

$$r = 0^{\oplus 4} \oplus \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}^{\oplus 8} \oplus \begin{pmatrix} 2 & \mu & \mu^2/2 & 0 & -2 \\ 0 & 2 & \mu & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mu & 0 & 0 \\ 0 & 0 & -\mu^2/2 & -\mu & 0 \end{pmatrix} \tag{5.16}$$

As in the simpler case treated in more detail above, one now verifies that this gives a coproduct on the matrices that appear when T is expanded to first order in h, and that these matrices have the structure of a bialgebra. It is precisely the Lie bialgebra dual of $\mathfrak{sl}_3$. The latter is coboundary, but the solvable dual is not, in the strict sense at least. Nevertheless we see that the coproduct structure tensor $f = \mathrm{d}r$ is exact as a Lie algebra one-form valued in a space of 25-dimensional matrices.

VI. APPLICATIONS

The inverse scattering method can handle the Liouville equation without invoking a spectral parameter. The equation in lightcone coordinates is

$$\phi_{tx} + \mathrm{e}^{2\phi} = 0. \tag{6.1}$$

A Lax pair for this equation is

$$L = \phi_t \sigma_3 + 2\mathrm{e}^{\phi}\sigma_+, A = \phi_x \sigma_3 + 2\mathrm{e}^{\phi}\sigma_-. \tag{6.2}$$

The dynamical algebra $\mathfrak{g}_x$ is generated by ϕ_t and e^{ϕ}, the dual algebra $\mathfrak{g}_l$ by σ_3 and σ_+, and L is the Lie bialgebra dual form. The two bialgebras are isomorphic and neither has the coboundary property.

What this means is that there is no r-matrix in $\mathfrak{g}_l \otimes \mathfrak{g}_l$ that solves the following problem: to express the Poisson brackets of $\mathfrak{g}_x$ in the form

$$\{L \otimes, L\} = [r, L \otimes 1 + 1 \otimes L]. \tag{6.3}$$

Nevertheless, it is well known that this problem is solved by

$$r = \sigma_+ \otimes \sigma_- - \sigma_- \otimes \sigma_+. \tag{6.4}$$

This is the simplest example of the phenomenon amply illustrated in this paper: if $\mathfrak{g}_l$ is not of requisite coboundary form then an r-matrix (and a quantum R-matrix) can sometimes be found by embedding $\mathfrak{g}_l$ in a larger, coboundary Lie bialgebra.

In the present case the structure is very clear. The Lie bialgebra $\mathfrak{g}_l$ is a sub-Lie bialgebra of $\mathfrak{sl}_2$, this being generated by σ_3, σ_+ and σ_-. The dual form for $\mathfrak{sl}_2$ can be expressed as

$$\phi_t\sigma_3 + \mathrm{e}^{\phi}\sigma_+ + \psi\sigma_-; \tag{6.5}$$

the dual is generated by $\phi_t, \mathrm{e}^{\phi}$ and ψ, with new relations

$$[\phi_t, \psi] = \psi, [\mathrm{e}^{\phi}, \psi] = 0. \tag{6.6}$$

The example of the Liouville model shows that field theories that can be treated without recourse to a spectral parameter need not be entirely trivial. The results of this paper encourage research for interesting applications in this direction.

Now we ask whether models without a spectral parameter can be interesting in the context of vertex models. The simplest of all such theories is based on the familiar

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, ab = qba, \text{ etc.} \tag{6.7}$$

Here we take the interpretation that is common among people who like to compare vertex models with solvable field theories. The matrices in these expression act on the horizontal bonds of a square lattice and generate the "auxiliary" algebra $\mathfrak{gl}_2$. The quantities a, b, c, d are matrices that act on the vertical bonds; they are the "dynamical variables" and we have seen that they can be constructed in any representation of the solvable algebra generated by $x(= \hat{x}), y$ and z.

In the Liouville model one takes a unitary representation, setting $x = \phi_t, y = \mathrm{e}^{\phi}$ and $z = 0$. There may be some vertex models where this is appropriate but it is more germaine to our ideas to consider a finite dimensional representation. We thus interpret a, b, c, d as matrices of a finite, non-decomposable matrix representation π. As we remarked near the end of Section II, we suppose a and d invertible. Then there is a natural number k such that

$$b^k = c^k = 0. \tag{6.8}$$

This implies, in particular, that there is at least one state $|0\rangle$ on which c vanishes. We shall choose a state such that

$$c|0\rangle = 0, a|0\rangle = \alpha|0\rangle, d|0\rangle = \delta|0\rangle, \tag{6.9}$$

where α and β are positive numbers. Then a representation π_0 is generated by applying b to $|0\rangle$, and the states

$$|n\rangle := b^n|0\rangle, n = 0, 1, \ldots, k-1 \tag{6.10}$$

form a basis for the representation space of π_0. Note that Ker π_0 is generated by c and that this is a sub-bialgebra; therefore c vanishes on the tensor product of π_0.

It is evident that choosing $|0\rangle$ as we have done effectively reduces c to zero. This tends to simplify a model that appears to be nearly trivial in the first place. It is our immediate purpose to find out whether such models are as trivial as they seem to be.

Thus we proceed in the usual way. We have a matrix T_i for each site $i = 1, \ldots, N$ of a lattice row, and we are interested in

$$\mathcal{T} := \prod_{i=1}^{N} T_i =: \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{6.11}$$

The operators A, B, C, D act on the space $V^{\otimes N}$, where V is the representation space of π_0. The "reference state" is

$$\Psi_0 := |0\rangle \otimes |0\rangle \otimes \ldots \otimes |0\rangle, (N \text{ factors}). \tag{6.12}$$

On this state the operator C is zero and

$$A\Psi_0 = \alpha^N \Psi_0, D\Psi_0 = \delta^N \Psi_0, \mathrm{tr}\mathcal{T}\Psi_0 = (\alpha^N + \delta^N)\Psi_0. \tag{6.13}$$

So $\alpha^N + \delta^N$ is an eigenvalue of the horizontal transfer matrix $\mathrm{tr}\mathcal{T}$.

In the thermodynamic limit $N \to \infty$ the partition function is determined by the largest eigenvalue of this transfer matrix. For a single site we have

$$a|n\rangle = \alpha q^n |n\rangle, d|n\rangle = \delta q^{-n} |n\rangle. \tag{6.14}$$

Now consider a state Ψ for which n_i is the excitation number of the ith site, then

$$\begin{gathered} A\Psi = \alpha^N q^M \Psi, D\Psi = \delta^N q^{-M} \Psi, M := \sum n_i, \\ \mathrm{tr}\mathcal{T}\Psi = (\alpha^N q^M + \delta^N q^{-M})\Psi. \end{gathered} \tag{6.15}$$

The problem is to maximize this with respect to the total excitation number M, given α, δ. The value $M = 0$ is favored when $\alpha/\delta < q^{1-k}$. The largest value of M is $N(k-1)$; it dominates when the inequality is reversed.

Let us suppose that $\alpha/\delta < q^{1-k}$, so our reference state is the favored one. Later we can increase α and look for a phase transition. (In this connection it should be understood that the matrix elements of T are Boltzmann weights and q—along with α and δ—depends on the temperature.)

But this is not all that this simple model has to offer. The vertices allowed in it when $k = 2$ are 5. There is a sink of arrows but not a source. With periodic boundary conditions the number of sinks must be the same as the number of sources, so we

cannot allow periodic boundary conditions. The ground state has all arrows pointing right or up. The first excited states have a sink at some site (i, j). The arrows are reversed in row i to the right of the sink and in column j above it. The number of such odd arrows is $\mathcal{O}(N)$, so the excitation energy is large, of order N. What comes to mind here is a cracked crystal, the result of a hammer blow at the site of the sink. As α/δ is increased, more and more cracks appear until the crystal becomes amorphous; then it freezes again, and the cracks go in the oposite sense until they disappear at high values of the parameter.

Incidentally, q has a direct physical interpretation. Writing (when $k = 2$)

$$T = \begin{pmatrix} q\alpha & & & 1 \\ & \alpha & & \\ & & \delta/q & \\ & & & \delta \end{pmatrix}, \tag{6.16}$$

the four principal relative weights are $q\alpha = \mathrm{e}^{-\beta\epsilon_1}$, $\alpha = \mathrm{e}^{-\beta\epsilon_2}$, $\delta/q = \mathrm{e}^{-\beta\epsilon_3}$, $\delta = \mathrm{e}^{-\beta\epsilon_4}$, so that $q\alpha/\delta = \mathrm{e}^{-\beta(\epsilon_1-\epsilon_4)}$. It goes through 1 as the ϵ's change but not with the temperature β^{-1}. To see a phase transition one has to change the forces, not β.

What we learn from this simple example is that "triviality" is a result of the solvable nature of the dynamical algebra—namely, the nilpotent nature of some of the generators when taken in a finite dimensional representation. Possible remedies are: 1/ take an infinite dimesional representation, as in the Liouville model; 2/ introduce a spectral parameter; and 3/ look for applications of T-matrices such as (2.2) that incorporate non-solvable "dynamical" algebras. A direct application of (2.2) in place of (6.7) should already be of some interest.

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