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q-MULTINOMIALS AND QUANTUM GROUPS

Robert J. Finkelstein

Department of Physics

University of California, Los Angeles, CA 90024-1547

Abstract. Utilizing an extension of the quantum plane relation, we construct a new class of functions defined by a generating function built out of either two or three sets of non-commuting variables. The commutators of variables within a single set may be related by a braid algebra, while the commutators between variables in the three different sets may be related to operators that satisfy a Yang-Baxter relation. The new functions are constructed out of q-functions, which depend on the commutation relations between the different sets, while the symmetry properties of the new functions are determined by the commutation rules within the separate sets.

1. Introduction.

The quantum groups appear to be related to the basic or q-functions[1] much as the Lie groups are related to the familiar special functions. For example, the little q-Jacobi polynomials may be identified as the matrix elements of representations of $SU_q(2)$.[2] We are here interested in q- multinomials defined by a generating function either closely related to or invariant under a quantum group.

One may begin with the quantum plane relation[3]

$$xy = qyx \tag{1.1}$$

which is invariant under the quantum general linear group $GL_q(2,C)$. Here x and y are the noncommuting components of the position vector $\vec{x}$. If this relation is preserved by the action of the matrix T on the column vector $\vec{x}$: $\vec{x} \to T\vec{x}$ and also by its action on the row vector $\vec{x}$: $\vec{x} \to \vec{x}T$, then T belongs to the quantum general linear group $GL_q(2,C)$.

The connection of Eq. (1.1) with $GL_q(2,C)$ may also be expressed by requiring the invariance under T and its transpose, $\tilde{T}$, of the cone

$$\vec{x}Q\vec{x} = 0 \tag{1.2a}$$

where

$$Q = \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix} . \tag{1.2b}$$

Then

$$\tilde{T}QT = TQ\tilde{T} = D_q Q \tag{1.3}$$

and therefore

$$\vec{x}' \; Q \; \vec{x}' = D_q \; \vec{x}Q\vec{x} = 0 \tag{1.4}$$

where

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1.5}$$

$$\begin{aligned} ab &= qba & bc &= cb \\ cd &= qdc & ad-da &= (q-q^{-1})bc \\ ac &= qca & D_q &= ad-qbc \\ bd &= qdb \end{aligned} \tag{1.6}$$

and D_q is the q-determinant.

Since D_q commutes with all elements of the algebra it may be set equal to unity. If $D_q = 1$, then $T \in SL_q(2,C)$ and $\vec{x}Q\vec{x}$ is invariant and not merely conformally invariant, under $SL_q(2,C)$.

If T and T' both satisfy (1.3) so does TT' and therefore $TT' \in GL_q(2C)$ as well.

The relations (1.6) may be summarized by the matrix equation

$$RT_1T_2 = T_2T_1R \tag{1.7}$$

where

$$T_1 = 1 \otimes T \ , \quad T_2 = T \otimes 1 \tag{1.7a}$$

and

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \delta & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \qquad \delta = q - \frac{1}{q} \tag{1.7b}$$

If the matrix elements of T commute (q=1) then of course $(T_1, T_2) = 0$.

One may rewrite (1.1) by introducing the following representation of $\vec{x}$:

$$\vec{x} = \begin{pmatrix} e^{a} \\ e^{\lambda \bar{a}} \end{pmatrix} \tag{1.8}$$

where a and $\bar{a}$ are absorption and emission operators:

$$(a, \bar{a}) = 1 \ . \tag{1.9}$$

Then (1.1) becomes

$$xy = yx \ e^{(a, \lambda \bar{a})} \tag{1.10}$$

or

$$xy = q \ yx$$

where

$$q = e^{\lambda} \ . \tag{1.11}$$

Now the quantum plane relation (1.1) has been replaced by the commutation rule of the normal oscillator (1.9).

Let us next consider a second representation of the two-dimensional "quantum vector":

$$\vec{A} = \begin{pmatrix} a \\ \bar{a} \end{pmatrix} . \tag{1.12}$$

Then

$$\tilde{A}'QA' = \tilde{A}QA \tag{1.13}$$

if Q has its previous meaning and

$$A' = SA , \qquad S \in SL_q(2C) . \tag{1.14}$$

Define the invariant operator N by

$$\tilde{A}QA = q^{-N} . \tag{1.15}$$

Then

$$a\bar{a} - q\bar{a}a = q^{-N} . \tag{1.16}$$

These are the commutation relations for the deformed oscillator algebra.[4,5] Thus the quantum plane relation may be rewritten in terms of the commutator of either the normal (1.9) or the deformed oscillator (1.16).

Adopting (1.16) one may introduce, as an ansatz, a complete set of states such that

$$\bar{a}|n\rangle = [n+1]^{1/2}|n+1\rangle \tag{1.17a}$$

$$a|n\rangle = [n]^{1/2}|n-1\rangle \tag{1.17b}$$

$$a|0\rangle = 0 \tag{1.17c}$$

where

$$[n] \equiv \frac{q^n - q^{-n}}{q - q^{-1}} \ . \tag{1.18}$$

Using this representation one finds

$$\bar{a}a|n\rangle = [n]|n\rangle \tag{1.19a}$$

$$a\bar{a}|n\rangle = [n+1]|n\rangle \ . \tag{1.19b}$$

Then by (1.16) the $|n\rangle$ are eigenstates of N:

$$N|n\rangle = n|n\rangle \ . \tag{1.20}$$

Also

$$(N,a) = -a \tag{1.21a}$$

$$(N,\bar{a}) = \bar{a} \ . \tag{1.21b}$$

One may also relate back to the algebra of $SU_q(2)$ by setting[6]

$$J_+ = -a^2/[2] \tag{1.22a}$$

$$J_- = \bar{a}^2/[2] \tag{1.22b}$$

$$J_z = \frac{1}{4}\ (2N+1) \ . \tag{1.22c}$$

Then with the representation described by (1.19) one finds the algebra of $SU_q(2)$.

$$(J_z, J_\pm) = \pm J_\pm \tag{1.23a}$$

$$(J_+, J_-) = [4J_z]/[2] = \frac{q^{4J_z} - q^{-4J_z}}{q^2 - q^{-2}} \tag{1.23b}$$

$$\lim_{q\to 1} (J_+, J_-) = 2J_z \ . \tag{1.23c}$$

Although we have just described the deformed oscillator algebra, we shall not in this paper base our development of the deformed Hermite functions on that algebra. Instead we shall make use of the suitably generalized quantum plane relation (1.1) from which the deformed oscillator algebra was obtained.

The first generalization of (1.1) we consider here is to two spaces of indeterminates $\{x^\alpha\}$ and $\{t^\beta\}$:

$$x^\alpha t^\beta = q\ t^\beta\ x^\alpha \qquad \text{all } \alpha,\beta \ .$$

In terms of these two sets $\{x^\alpha\}$ and $\{t^\alpha\}$ one may construct a generating function for multinomials in the $\{x^\alpha\}$.

We shall then make the following generalization to three spaces. Denoting the α-coordinate of the ith space by z^α_i, we consider

$$z^\alpha_i z^\beta_j = R^{\alpha\beta}_{ij} z^\alpha_i z^\beta_j \qquad i,j = 1,2,3 , \quad \text{all } \alpha,\beta \tag{1.24}$$

where the $R^{\alpha\beta}_{ii}$ are restricted by a braid algebra and the $R^{\alpha\beta}_{ij}$ $(i\neq j)$ satisfy a Yang-Baxter equation. With these three sets $\{z^\alpha_i\}$ one may now construct a new generating function.

In this framework we obtain a new class of functions which may be looked upon a generalization of the classical special functions in many variables. We shall here restrict our discussion to the Hermite case. The commuting case had been introduced long ago by Hermite[7] while the Grassmannian case naturally turns up in a supersymmetric context.[8-11] Although our work will be based on the Hermitian generating function, the method is general and has been extended to other classical functions.[12]

It turns out that the new structures are dependent on new functions related to the mixed operators R_{ij} while the symmetry properties of the new structures are determined by the pure operators R_{ii}.

2. q-Hermite Multinomials.

We take the fundamental generating function to be a Gaussian exponential defining the Hermite multinomials as follows:

$$e^{-A+B} = \sum_{0}^{\infty} t^{\alpha_1} \ldots t^{\alpha_m} H_{\alpha_1 \ldots \alpha_m}(x) \tag{2.1}$$

where

$$A = A_{\alpha\beta} t^{\alpha} t^{\beta} \tag{2.1a}$$

$$B = 2B_{\alpha\beta} t^{\alpha} x^{\beta} \ . \tag{2.1b}$$

If $(t^{\alpha}, t^{\beta})_- = (t^{\alpha}, x^{\beta})_- = 0$ and $(A_{\alpha\beta})$ and $(B_{\alpha\beta})$ are symmetric then the $H_{\alpha_1 \ldots \alpha_m}(x)$ are the Hermite multinomials introduced by Hermite.[4] Then also $(x^{\alpha}, x^{\beta})_- = 0$.

If $(t^{\alpha}, t^{\beta})_+ = (t^{\alpha}, x^{\beta})_+ = 0$ and $(A_{\alpha\beta})$ and $(B_{\alpha\beta})$ are antisymmetric then the $H_{\alpha_1 \ldots \alpha_m}(x)$ are the Grassmann-Hermite multinomials.[8,9]

Let us now consider the more general case

$$t^{\alpha} t^{\beta} = q^{(t,t)} \ t^{\beta} t^{\alpha} \tag{2.2}$$

$$x^{\alpha} x^{\beta} = q^{(x,x)} \ x^{\beta} x^{\alpha} \tag{2.3}$$

$$x^{\alpha} t^{\beta} = q^{(t,x)} \ t^{\beta} x^{\alpha} \quad \text{all } \alpha, \beta \tag{2.4}$$

where

$$q^{(tt)} = \pm 1 \tag{2.2a}$$

$$q^{(xx)} = \pm 1 \tag{2.3a}$$

$$q^{(tx)} = q \ . \tag{2.4a}$$

Then the $\{t^{\alpha}\}$ and $\{x^{\alpha}\}$ are separately either all commuting or all anti-commuting ($q^{(tt)} = q^{(xx)}$) but the t and x spaces are related by (2.4) and (2.4a) where q is unrestricted.

3. Expansion of Generating Function.

For the calculation of $H_{\alpha_1 \ldots \alpha_m}(x)$ we need the q-binomial theorem: If

$$xy = qyx \tag{3.1}$$

then

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q y^k \, x^{n-k} \tag{3.2}$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q|q)_n}{(q|q)_k \, (q|q)_{n-k}} \tag{3.3}$$

and

$$(a|q)_n = (1-a)(1-aq) \ \ldots \ (1-aq^{n-1}) \tag{3.4}$$

$$(q|q)_n = (1-q)(1-q^2) \ \ldots \ (1-q^n) \ . \tag{3.5}$$

Then

$$\lim_{q \to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k} \ . \tag{3.6}$$

The generating function (2.1) is

$$G \equiv e^{-A+B} \tag{3.7}$$

$$= \sum_{p=0}^{\infty} \frac{(-A+B)^p}{p!} ,$$

where A and B are non-commuting. We have

$$BA = B_{\gamma\delta} A_{\alpha\beta} \; t^{\gamma} x^{\delta} t^{\alpha} t^{\beta}$$

$$BA = (q^{tx})^2 \; AB$$

or

$$BA = q^2 \; AB \; . \tag{3.8}$$

Then

$$G = \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{s=0}^{p} \begin{bmatrix} p \\ s \end{bmatrix}_Q (-A)^s \; B^{p-s} \; , \tag{3.9}$$

where

$$Q = q^2 \tag{3.10}$$

or

$$G = \sum_{n=0}^{\infty} \sum_{s=0}^{[n/2]} \frac{(-)^s}{(n-s)!} \begin{bmatrix} n-s \\ s \end{bmatrix}_Q A^s \; B^{n-2s} \; , \tag{3.11}$$

where

$$n = s+p \; . \tag{3.12}$$

One needs to reorder the factors in A^sB^r to bring all the t to the left of the x. We have

$$A^sB^r = 2^r(A_{\alpha\beta}t^\alpha t^\beta)^s\ (B_{\gamma\delta}t^\gamma x^\delta)^r$$

$$= 2^r(A_{\alpha_1\beta_1}\ldots A_{\alpha_s\beta_s})(B_{\gamma_1\delta_1}\ldots B_{\gamma_r\delta_r})(t^{\alpha_1}t^{\beta_1}\ldots t^{\alpha_s}t^{\beta_s})(t^{\gamma_1}x^{\delta_1}\ldots t^{\gamma_r}x^{\delta_r}) \tag{3.13}$$

But

$$t^{\gamma_1}x^{\delta_1}\ldots t^{\gamma_r}x^{\delta_r} = \varepsilon_q(r)\prod_1^r t^{\gamma_s}\prod_1^r x^{\delta_s} \tag{3.14}$$

where the original order of the t's and x's is preserved

$$\varepsilon_q(r) = q^{\frac{r}{2}(r-1)}\ . \tag{3.15}$$

By (3.13) and (3.14)

$$A^sB^r = \varepsilon_q(r)2^r\left[\prod_{\alpha=1}^s A_{\alpha_j\beta_j}\right]\left[\prod_{k=1}^r B_{\gamma_k\delta_k}\right]$$

$$\times\ (t^{\alpha_1}t^{\beta_1}\ldots t^{\alpha_s}t^{\beta_s})(t^{\gamma_1}\ldots t^{\gamma_r})(x^{\delta_1}\ldots x^{\delta_r})\ , \tag{3.16}$$

where r = n-2s. Let

$$\tilde{x}_\gamma = B_{\gamma\delta}x^\delta\ . \tag{3.17}$$

Then

$$2^{-r} A^s B^r \equiv \varepsilon_q(r) \left[\left(\prod_{j=1}^{s} A_{\alpha_j \beta_j} \right) t^{\alpha_1} \ldots t^{\beta_s} \right] \left[t^{\gamma_1} \ldots t^{\gamma_r} \tilde{x}_{\gamma_1} \ldots \tilde{x}_{\gamma_r} \right] . \tag{3.18}$$

By (3.11)

$$G = \sum_{n=0}^{\infty} \sum_{s=0}^{[n/2]} 2^r \frac{(-)^s}{(n-s)!} \begin{bmatrix} n-s \\ s \end{bmatrix}_Q \varepsilon_q(r) \sum_{\alpha_1 \beta_1} \ldots \sum_{\alpha_s \beta_s} \sum_{\gamma_1} \ldots \sum_{\gamma_r}$$

$$\times\, t^{\alpha_1} \ldots t^{\beta_s} t^{\gamma_1} \ldots t^{\gamma_r} \prod_{j=1}^{s} A_{\alpha_j \beta_j} \tilde{x}_{\gamma_1} \ldots \tilde{x}_{\gamma_r} \tag{3.19}$$

where $r = n-2s$. Let

$$A^s_{\alpha_1 \beta_1 \ldots \alpha_s \beta_s \gamma_1 \ldots \gamma_r}(2\tilde{x}) = A_{\alpha_1 \beta_1} \ldots A_{\alpha_s \beta_s} (2\tilde{x}_{\gamma_1}) \ldots (2\tilde{x}_{\gamma_r}) \tag{3.20}$$

and

$$(\lambda_1 \ldots \lambda_{2s} \lambda_{2s+1} \ldots \lambda_n) = (\alpha_1 \beta_1 \ldots \alpha_s \beta_s \gamma_1 \ldots \gamma_{n-2s}) . \tag{3.21}$$

Then

$$G = \sum_{n=0}^{\infty} \sum_{s=0}^{[n/2]} \frac{(-)^s}{(n-s)!} \begin{bmatrix} n-s \\ s \end{bmatrix}_Q \varepsilon_q(n-2s) \sum_{\lambda_1 \ldots \lambda_n} t^{\lambda_1} \ldots t^{\lambda_{2s}} t^{\lambda_{2s+1}} \ldots t^{\lambda_n}$$

$$\times\, A^s_{\lambda_1 \ldots \lambda_{2s} \lambda_{2s+1} \ldots \lambda_n}(2\tilde{x}) . \tag{3.22}$$

Therefore

$$G = \sum_{n=0} \sum_{\lambda_1 \ldots \lambda_n} t^{\lambda_1} \ldots t^{\lambda_n} H_{\lambda_1 \ldots \lambda_n}(2\tilde{x}) \tag{3.23}$$

where finally

$$H_{\lambda_1 \ldots \lambda_n}(2\tilde{x}) = \sum_{s=0}^{[n/2]} \frac{(-)^s}{(n-s)!} \begin{bmatrix} n-s \\ s \end{bmatrix}_Q \varepsilon_q(n-2s)\ A^s{}_{\lambda_1 \ldots \lambda_{2s} \lambda_{2s+1} \ldots \lambda_n}(2\tilde{x}) \tag{3.24}$$

or

$$H_{\lambda_1 \ldots \lambda_n}(2\tilde{x}) = \sum_{s=0}^{[n/2]} \frac{(-)^s}{(n-s)!} \begin{bmatrix} n-s \\ s \end{bmatrix}_{q^2}$$

$$\times\ \varepsilon_q(n-2s)\ A_{\lambda_1 \lambda_2} \ldots A_{\lambda_{2s-1} \lambda_{2s}} (2\tilde{x})_{\lambda_{2s+1}} \ldots (2\tilde{x})_{\lambda_n} \tag{3.25}$$

The separate factors are defined by (3.3), (3.15) and $\tilde{x}_\lambda$ is given by (3.17). The following special cases of (3.25) may be noted

(a) $q^{(xt)} = 1,\ q^{(xx)} = q^{(tt)} = 1$

$$H_{\lambda_1 \ldots \lambda_n}(2\tilde{x}) = \sum_{s=0}^{[n/2]} \frac{(-)^s}{(n-s)!} \binom{n-s}{s} A_{\lambda_1 \lambda_2} \ldots A_{\lambda_{2s-1} \lambda_{2s}} (2\tilde{x})_{\lambda_{2s+1}} \ldots (2\tilde{x})_{\lambda_n} \tag{3.26}$$

This is the Hermite multinomial.

(a)' $q^{(xt)} = 1,\ A_{\lambda_1 \lambda_2} = B_{\lambda_1 \lambda_2} = \delta_{\lambda_1 \lambda_2}$

$$\lambda_1 = \lambda_2 = \ldots \lambda_{2s} \quad \text{and} \quad x_{2s+1} = \ldots = x_n$$

$$H_n(2x) = \sum_{s=0}^{[n/2]} \frac{(-)^s}{(n-s)!} \binom{n-s}{s} (2x)^{n-2s} \ . \tag{3.27}$$

This is the Hermite polynomial.

(b) $q^{(xt)} = -1,\ q^{(xx)} = q^{(tt)} = -1,$

$$H_{\lambda_1 \ldots \lambda_n}(2\tilde{x}) = (-)^{\frac{n}{2}(n-1)} \sum_{s=0}^{[n/2]} \frac{1}{(n-s)!} \binom{n-s}{s}$$

$$\times A_{\lambda_1\lambda_2} \ldots A_{\lambda_{2s-1}\lambda_{2s}} (2\tilde{x})_{\lambda_{2s+1}} \ldots (2\tilde{x})_{\lambda_n} \ . \tag{3.28}$$

This is the Grassmann Hermite multinomial.

In the derivation of (3.25) no assumption has been made about $q^{(xx)}$. One observes however that the form of the expansion (3.23) implies that the permutation properties of the $\{\lambda^\alpha\}$ are determined by the commutation properties of the $\{t^\alpha\}$ and that the same properties are passed on to the $\{x^\alpha\}$ according to (3.25). Therefore

$$q^{(xx)} = q^{(tt)} \quad (= \pm 1) \tag{3.29}$$

as anticipated in (2.2a) and (2.3a).

4. Invariance of the Generating Function.

The invariance of the relation (1.1) under $SL_q(2,C)$ may be promoted to the same invariance of the generating function (2.1) by replacing $B_{\alpha\beta}t^{\alpha}x^{\beta}$ by

$$\hat{B}^{ij}_{\alpha\beta}\, z^{\alpha}{}_{i} z^{\beta}{}_{j} = B_{\alpha\beta} M^{ij} z^{\alpha}{}_{i} z^{\beta}{}_{j} \ , \tag{4.1}$$

where M is a numerical matrix invariant under $SL_q(2,C)$:

$$M = \tilde{T}MT \ , \quad T \in SL_q(2,C) \tag{4.2}$$

and where

$$\begin{aligned} z_1{}^{\alpha} &= x^{\alpha} \\ z_2{}^{\alpha} &= t^{\alpha} \end{aligned} \tag{4.3a}$$

satisfying

$$z^{\alpha}{}_{i} z^{\beta}{}_{j} = q_{ji} z^{\beta}{}_{j} z^{\alpha}{}_{i} \ . \tag{4.3b}$$

Note that $q_{21} = q_{tx}$ is unrelated to the q characterizing SL_q.

It may be checked that M must be of the following form

$$M = \begin{pmatrix} 0 & M^{12} \\ -qM^{12} & 0 \end{pmatrix} = M^{12} Q \ , \tag{4.4}$$

where q is the parameter of SL_q and Q is given by (1.2). Then

$$\hat{B}^{ij}_{\alpha\beta} Z^{\alpha}{}_{i} Z^{\beta}{}_{j} = M^{12} (B_{\alpha\beta} Z^{\alpha}{}_{1} Z^{\beta}{}_{2} - B_{\alpha\beta} q Z^{\alpha}{}_{2} Z^{\beta}{}_{1})$$

$$= M^{12} (B_{\alpha\beta} q_{21} Z^{\beta}{}_{2} Z^{\alpha}{}_{1} - B_{\alpha\beta} q Z^{\alpha}{}_{2} Z^{\beta}{}_{1})$$

$$= M^{12} B_{\alpha\beta} (q_{tx} - q) \; Z^{\alpha}{}_{2} Z_{1}{}^{\beta} \qquad \text{if} \qquad B_{\alpha\beta} = B_{\beta\alpha}$$

$$= M^{12} B_{\alpha\beta} (-q_{tx} - q) \; Z^{\alpha}{}_{2} Z^{\beta}{}_{1} \qquad B_{\alpha\beta} = -B_{\beta\alpha} \quad .$$

With respect to the Greek indices $\hat{B}^{ij}_{\alpha\beta} Z^{\alpha}{}_{i} Z^{\beta}{}_{j}$ is either an orthogonal or symplectic form and with respect to the Latin indices it is invariant under $SL_q(2,C)$.

5. Generalization to Three Sets.

We may consider two quadratic forms, A and B, as before, where A is unchanged:

$$A = A_{\alpha\beta}{}^{t}{}^{\alpha}{}_{t}{}^{\beta} \ . \tag{5.1}$$

In B however replace t by y.

$$B = B_{\alpha\beta}{}^{y}{}^{\alpha}{}_{x}{}^{\beta} \tag{5.2}$$

In addition consider the new commutation relations

$$y^{\alpha}{}_{t}{}^{\beta} = q_{ty}{}^{t}{}^{\beta}{}_{y}{}^{\alpha} \tag{5.3}$$

$$x^{\alpha}{}_{t}{}^{\beta} = q_{tx}{}^{t}{}^{\beta}{}_{x}{}^{\alpha} \tag{5.4}$$

$$x^{\alpha}{}_{y}{}^{\beta} = q_{yx}{}^{y}{}^{\beta}{}_{x}{}^{\alpha} \tag{5.5}$$

Then there will be the following changes in the previous development

$$BA = B_{\gamma\delta}A_{\alpha\beta}{}^{y}{}^{\gamma}{}_{x}{}^{\delta}{}_{t}{}^{\alpha}{}_{t}{}^{\beta} \tag{5.6}$$

$$= B_{\gamma\delta}A_{\alpha\beta}{}^{t}{}^{\alpha}{}_{t}{}^{\beta}{}_{y}{}^{\gamma}{}_{x}{}^{\delta}(q_{tx}q_{ty})^{2} \tag{5.7}$$

$$= QAB \tag{5.8}$$

where the new Q is

$$Q = (q_{tx} q_{ty})^2 \ . \tag{5.9}$$

In addition

$$B^r = \prod_{j=1}^{r} B_{\alpha_j \beta_j} \, y^{\alpha_1} x^{\beta_1} \ldots y^{\alpha_r} x^{\beta_r} \tag{5.10}$$

$$= \prod_{j=1}^{r} B_{\alpha_j \beta_j} \, \varepsilon_{q_{yx}}(r) \prod_{k=1}^{r} y^{\alpha_k} \prod_{\ell=1}^{r} x^{\beta_\ell} \ . \tag{5.11}$$

Then one finds instead of (3.22) for $G(t|x)$ a new equation for $G(t|x,y)$:

$$G(t|x,y) = \sum_{n=0}^{\infty} \sum_{s=0}^{[n/2]} \frac{(-)^s}{(n-s)!} \begin{bmatrix} n-s \\ s \end{bmatrix}_Q \varepsilon_{q_{yx}}(n-2s) \; t^{\lambda_1} \ldots t^{\lambda_{2s}} y^{\lambda_{2s+1}} \ldots y^{\lambda_n}$$

$$\times \, A^s_{\lambda_1 \ldots \lambda_{2s} \lambda_{2s+1} \ldots \lambda_n}(2\tilde{x}) \ , \tag{5.12}$$

where

$$Q = (q_{ty})^2 \; (q_{tx})^2 \tag{5.13}$$

and instead of (3.25)

$$H_{\lambda_1 \ldots \lambda_{2s} \lambda_{2s+1} \ldots \lambda_n}(2\tilde{x}) = \sum_{s=0}^{[n/2]} \frac{(-)^s}{(n-s)!} \begin{bmatrix} n-s \\ s \end{bmatrix}_Q \varepsilon_{q_{yx}}(n-2s)$$

$$\times \, A_{\lambda_1 \lambda_2} \ldots A_{\lambda_{2s-1} \lambda_{2s}}(2\tilde{x})_{\lambda_{2s+1}} \ldots (2\tilde{x})_{\lambda_n} \ . \tag{5.14}$$

Then the properties of the first 2s indices under permutations will be determined by the commutation of the $t^{\lambda_1} \ldots t^{\lambda_{2s}}$ while the rules for permuting the remaining indices are determined by the corresponding rules for permuting

the $y^{\lambda_{2s+1}}\ldots y^{\lambda_n}$. The factors $\begin{bmatrix} n-s \\ s \end{bmatrix}_Q \varepsilon_q(n-2s)$ depend on the three mixed parameters q_{yt}, q_{xt}, and q_{yx}, but the index symmetry of the $H_{\lambda_1\ldots\lambda_{2s}\lambda_{2s+1}\ldots\lambda_n}(2\tilde{x})$ is determined by q_{tt} and q_{yy}. The permutation rules for the $\tilde{x}^{\alpha}$ are the same as for the last n-2s indices and are therefore also determined by q_{yy}. Therefore $q_{yy} = q_{xx}$.

Instead of defining H(x), dependent on a single set {x} of variables, one may define H(x,y), dependent on two sets of variables {x} and {y}, in terms of the same generating function G(t|x,y) in (5.12).

6. The Braid Algebra and the Yang-Baxter Equation.

In (5.14) use has been made only of the mixed parameters q_{yt}, q_{xt}, and q_{yx}. Let us now consider the q_{xx}, q_{yy}, and q_{tt} algebras.

$$t^{\alpha}t^{\beta} = q_{tt}^{\alpha\beta}\, t^{\beta}t^{\alpha} \tag{6.1}$$

$$y^{\alpha}y^{\beta} = q_{yy}^{\alpha\beta}\, y^{\beta}y^{\alpha} \tag{6.2}$$

$$x^{\alpha}x^{\beta} = q_{xx}^{\alpha\beta}\, x^{\beta}x^{\alpha} \ . \tag{6.3}$$

Let us restrict $q^{\alpha\beta}$ as follows:

$$q^{\alpha,\alpha\pm1} = q^{\pm1} \tag{6.4}$$

$$q^{\alpha\beta} = 1 \quad , \quad |\beta-\alpha| \geq 2 \quad . \tag{6.5}$$

With this choice of $q^{\alpha\beta}$ define

$$R^{\alpha} = q^{\alpha\beta}P^{\alpha\beta} \quad , \quad \beta = \alpha+1 \ , \tag{6.6}$$

where $P^{\alpha\beta}$ is the transposition operator. Then

$$R^{\alpha}R^{\alpha+1}R^{\alpha} = R^{\alpha+1}R^{\alpha}R^{\alpha+1} \tag{6.7}$$

$$R^{\alpha}R^{\beta} = R^{\beta}R^{\alpha} \quad , \quad |\alpha-\beta| \geq 2 \ . \tag{6.8}$$

Then these equations (6.7), (6.8), define a braid algebra.[13]

We may assume that the operators in all three spaces, (tt), (yy), and (xx) obey braid algebras.

We may also introduce

$$R_{xy} = q_{xy}P_{xy} \tag{6.9}$$

$$R_{yt} = q_{yt}P_{yt} \tag{6.10}$$

$$R_{xt} = q_{xy}P_{xt} \ . \tag{6.11}$$

These operators act on the tensor product of the three spaces and satisfy the Yang-Baxter relation[14,15]

$$R_{xy}R_{xt}R_{yt} = R_{yt}R_{xt}R_{xy} \ . \tag{6.12}$$

Both the Yang-Baxter relation and the braid relation (6.7) depend on the same identity

$$aba = bab = c \ , \tag{6.13}$$

where a, b, and c are the three transpositions of S_3.

The new parameters q_{tt}, q_{yy}, and q_{xx} are independent of the mixed parameters q_{xt}, q_{yx}, and q_{yt}.

In generalizing from the commuting and anticommuting cases, one cannot consistently allow an arbitrary choice of $R_{ij}^{\alpha\beta}$. The choice just made of the braid algebra to fix $R_{ii}^{\alpha\beta}$, together with the Yang-Baxter equation to determine $R_{ij}^{\alpha\beta}$ $(i\neq j)$, is consistent and relates the $R_{ii}^{\alpha\beta}$ to the $R_{ij}^{\alpha\beta}$ matrices simply by

implementing the same identity (6.13) in two ways: by (6.7) for $R^{\alpha\beta}$ and by (6.12) for R_{ij}. (The second braid relation (6.8) is not relevant in the comparison since there are only three classes of noncommuting variables.)

The Yang-Baxter equation including the spectral parameter is

$$R_{xy}(u)\ R_{xt}(u+v)\ R_{yt}(v) = R_{yt}(v)\ R_{xt}(u+v)\ R_{xy}(u)\ . \tag{6.14}$$

There is also a trivial solution of this equation of the form (6.9)-(6.11) where

$$\begin{aligned} q_{xy} &= f_{xy}(u) \\ q_{yt} &= f_{yt}(v) \\ q_{xt} &= f_{xt}(u+v) \end{aligned} \tag{6.15}$$

In this case the generating function and the multinomials depend on the arbitrary functions $f_{ij}(u)$.

7. Symmetry of the Multinomials and Invariance of the Generating Function.

The permutation symmetries of the first 2s indices of H(x) are contragradiently determined by q_{tt} and the commutation symmetries of the x are similarly determined by the q_{yy}. For example, consider

$$\Sigma\, t^{\alpha} t^{\beta} \ldots \; H_{\alpha\beta} \ldots \; . \tag{7.1}$$

This expression is unchanged if $H_{\alpha\beta\ldots}$ changes contragradiently to $t^{\alpha} t^{\beta} \ldots$ in the usual way:

$$t^{\alpha} t^{\beta} = q^{\alpha\beta} t^{\beta} t^{\alpha}$$

$$H_{\alpha\beta\ldots} = q^{\beta\alpha} H_{\beta\alpha\ldots} \tag{7.2}$$

where

$$q^{\alpha\beta} q^{\beta\alpha} = 1 \; . \tag{7.3}$$

The repeated index is not summed in these equations.

Let us next consider the invariance of the new generating function under the new rules (6.4) and (6.5) of the braid algebra.

We have

$$A = A_{\alpha\beta} t^{\alpha} t^{\beta}$$

$$= A_{\alpha\beta} q^{\alpha\beta}_{tt} t^{\beta} t^{\alpha}$$

$$= A_{\beta\alpha} q^{\beta\alpha}_{tt} \alpha_t \beta_t$$

and

$$A_{\alpha\beta} = A_{\beta\alpha}\, q^{\beta\alpha}_{tt} \ . \tag{7.4}$$

If the q_{tt} are now given by (6.4) and (6.5), the matrix $(A_{\alpha\beta})$ will have the following structure

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} & \\ qA_{12} & A_{22} & A_{23} & A_{24} & \cdot \\ A_{13} & qA_{23} & A_{33} & A_{34} & \vdots \\ A_{14} & A_{24} & qA_{34} & A_{44} & \\ & \cdot & \cdot & \cdot & \end{pmatrix} \tag{7.5}$$

This matrix may be resolved into three components

$$A = E + O + S \ , \tag{7.6}$$

where

$$S = (A_{\alpha\beta}) \qquad |\beta-\alpha| \geq 2 \tag{7.7}$$

or

$$\beta-\alpha = 0 \ .$$

This is a symmetric matrix and the other two parts, E and O, are

$$E_{\alpha\beta} = \left[\delta(\alpha,\beta-1)\ O(\alpha) + q\delta(\alpha,\beta+1)\ E(\alpha)\right] A_\alpha \tag{7.8}$$

$$O_{\alpha\beta} = \left[\delta(\alpha,\beta-1)\ E(\alpha) + q\ \delta(\alpha,\beta+1)\ O(\alpha)\right] A_\alpha\ . \tag{7.9}$$

Here $O(\alpha)$ and $E(\alpha)$ project onto odd and even integers

$$O(\alpha) = \frac{1-(-)^\alpha}{2} \tag{7.10a}$$

$$E(\alpha) = \frac{1+(-)^\alpha}{2}\ . \tag{7.10b}$$

Then

$$(E+O)_{\alpha\beta} = \left[\delta(\alpha,\beta-1) + q\ \delta(\alpha,\beta+1)\right] A_\alpha \tag{7.11}$$

and E+O is the matrix

$$\begin{pmatrix} O & A_1 & & & & . \\ qA_1 & O & A_2 & & & . \\ & qA_2 & O & A_3 & & . \\ & & qA_3 & O & A_4 & . \\ & . & . & . & & \end{pmatrix} \tag{7.12}$$

while

$$O = \begin{pmatrix} O & A_1 & & & & \\ qA_1 & 0 & & & & \\ & & O & A_3 & & \\ & & qA_3 & O & & \\ & & & & & . \\ & & & & & & . \end{pmatrix} \qquad E = \begin{pmatrix} O & O & & & & \\ O & O & A_2 & & & \\ & qA_2 & O & & & \\ & & & O & A_4 & \\ & & & qA_4 & O & . \\ & & & & & & . \end{pmatrix} \tag{7.13}$$

Thus O and E, the odd and even parts, have the same structure.

Clearly O is invariant under the transformations

$$M = \begin{pmatrix} a_1 & b_1 & & & \\ c_1 & d_1 & & & \\ & & a_3 & b_3 & \\ & & c_5 & d_5 & \\ & & & & \ddots \\ & & & & & \ddots \end{pmatrix} \tag{7.14}$$

or

$$M = M_1 \oplus M_3 \oplus \ldots \tag{7.15}$$

where

$$M_i \in SL_q(2,C) \tag{7.16}$$

while E is invariant under

$$M = M_2 \oplus M_4 \oplus \ldots \ . \tag{7.17}$$

Thus if the "braid matrix" $(A_{\alpha\beta})$ is restricted to be $(O_{\alpha\beta})$ or $(E_{\alpha\beta})$ then the corresponding quadratic form and the associated generating function will be invariant under $SL_q(2,C)$ as expressed by (7.15) and (7.17) respectively.

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