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GROUPOIDS AND LIE BIGEBRAS IN GAUGE AND STRING THEORIES *

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ABSTRACT. This talk deals with two seemingly disparate mathematical concepts: Lie groupoids and algebroids, and Lie bigebras (bialgebras), also known as Lie-Hopf-algebras (LHA), which have recently found to have applications in quantum field theory, gauge theory, and string theory. Although many of the mathematical concepts have been around for some time, the interest in groupoids lies in potential application to string theories, whereas Lie bigebras appear in recent work by Drinfel'd and others on "quantum groups" and their relation to the classical Yang-Baxter equation.

Introduction

At one of the early of these conferences (Bonn 1975) [5a], I talked about attempts at formulating gauge field theories in terms of a "representation theory of the holonomy groupoid". My interest in groupoids was rekindled by a recognition that open bosonic strings, and the Witten composition law of strings [6] can be viewed, in part, as groupoid operations. I wish to emphasize the natural way in which some aspects of groupoid theory enter into the formulation of gauge theories and some models of string theory. I will start out with a review of basic definitions. While these notes were being written, I received the recent monograph by Mackenzie [18], which deals with the relation between Lie groupoids and algebroids and principal bundle theory, and contains a treatment of connections in groupoids based on the Atiyah sequence [19]. I was very pleased to see this, since I have been using this approach to the definition of connections in principal bundles in my forthcoming book [20]. Since connections in Lie groupoids turn out to be equivalent to these, this furnishes a further argument in favor of a groupoid approach to gauge theories. The existence of [18] book allowed me to reduce the amount of background material included in this lecture. The reader is advised to consult this book for topics which space does not permit me to treat here.

Several lecturers at this Conference (Fröhlich, de Vega) dealt with r-matrices and Yang-Baxter equations, but did not go into the relations these concepts have with "quantum groups" or Lie-Hopf Algebras. I have therefore decided to include a very brief overview of this new and exciting topic.

Part I of the lecture deals with some aspects of groupoid theory, as applied to gauge and string physics. It contains a brief review of the necessary mathematics.

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Part II deals with the other topic: *Lie bigebras and "quantum groups"* and their uses in quantum field theory. This topic is again related to a subject about which I talked at an earlier of these conferences (Bonn 1973[B, see [M,71, 73]), where I made an attempt to advocate the use of W*-bigebras (Hopf–von Neumann) algebras, as a way of describing the (possibly broken) symmetries in algebraic quantum field theory. Since then Doplicher and Roberts [DR] have succeeded in partially completing the Doplicher-Haag-Roberts (DHR) program [DHR 69,71- -74], so that now it can be considered as a self-contained part of general quantum field theory (still missing is the incorporation of gauge theories). Although in the intervening time my interests have shifted in a more differential-geometric direction, I have often expressed my conviction that Hopf-algebraic aspects are bound to play an important role in quantum gauge theory (private conversations with, among others, Daniel Kastler, Rudolf Haag, Detlev Buchholz, and others). Unfortunately, the mathematical machinery seemed to have been lacking. At the International Congress of Mathematicians of 1986 P. Cartier presented in Berkeley the paper "Quantum Groups" of V. G. Drinfel'd (which appeared in Russian [D 86] and reached me in May, 1986), in which Lie-Hopf algebras (LHA) or Lie bigebras (Drinfel'd uses the longer name Lie- bialgebras) are introduced and their relations to solutions of the classical Yang-Baxter equations (CYBE) are discussed, and the notion of quantum group and their relation to the quantum Yang-Baxter equations (QYBE) are explored. Part 2 consists of a review some of Drinfel'd's definitions and an outline of the way I hope LHA-s will enter quantum field theory and conformal field theory in the near future.

The references to the two parts are given in different styles: those to Part 1 (groupoids) are in the form [Number], whereas those for Part 2 are in the form [Author, Year].

1. Groupoids, Gauge Theory, and Strings

1.1. INTRODUCTION

Although the "noncommutative differential geometry" of Alain Connes [1] has had a considerable impact on mathematics (see, e. g., [2]) few physicists paid attention to this very important development. The most notable exception was Daniel Kastler, who has heroically attempted to spread the new gospel in the mathematical physics community, cf. [3] as well as, more recently, Jean Bellissard [4], who (with collaborators) has pointed out that Connes' K-theory has remarkable applications in the theory of the quantum Hall effect as well as in other areas of condensed-matter physics. Under Daniel Kastler's influence, I have made some feeble attempts to call attention to this important development [5]. However, this situation has started to change: in a recent paper [6] (and a sequel), E. Witten has attempted to discuss the implications of Connes' noncommutative geometry for superstring theory [7], in particular, the interpretation of the BRS(T) charge operator Q_{BRS} [8] as a graded derivation. He proposed a "composition law" for open strings which strongly suggests an interpretation in terms of a groupoid — a concept which also plays a fundamental role in Connes' theory, and in the theory of foliated manifolds [10–12]. Among others, I have tried for several years to espouse the point of view that the BRS-charge and the Kugo-Ojima criterion for selecting the physical subspace in quantized gauge theories should be taken seriously, and that the group of gauge transformations and its Maurer-Cartan form (the "ghost") should be given consideration already at the classical level.

Recently Leonard Gross [13] has proposed an approach to gauge theories based on "lasso forms". Although his emphasis is on finding a measure on such forms, his results are closely related to a groupoid reformulation of gauge theory, as well as to early work by H. Loos [13a] as well as to my early attempts to quantize the holonomy of gauge fields [5a], and these relations need to be investigated further.

In 1986 I translated a paper of M. B. Mensky [14] which discusses a related approach to the theory of interacting strings, in terms of the group of paths on the path group of a manifold. Although Mensky's approach and aim is different, his paper (and presumably some of his other publications) makes some valuable technical points on the relations between the representations of the path groupoid, the fundamental group and the group of free paths on a homogeneous space. I will not discuss this paper here.

I will try to explain how one could possibly use the groupoid concept for the description of the composition of open strings and how it could be extended to closed strings. Since I am not an expert on superstring theory, these remarks should be interpreted as heuristic, and they are published in the hope that the people actively engaged in research will find them useful and will be able to exploit them more fully.

Since the notions of groupoid and algebroid are not very familiar to physicists. I start in Sec. 1.2 with a brief listing of the definitions and a few examples (for details see [9, 18]). Section 1.3 is devoted to a discussion of strings in terms of groupoids, thus giving some intuitive feeling for groupoids. Sec. 1.4 returns to the definition of Lie groupoids and algebroids and gives a brief account of the theory of connections and holonomy. Section 1.5 deals with a reformulation of gauge theory in terms of Lie groupoids and algebroids and their connection with principal bundle theory.

1.2. WHAT ARE GROUPOIDS?

A *groupoid* Γ can be thought of as a collection of transformations ("*arrows*") α, β, γ, of a set B of *objects* which can be identified with the set of *units* Γ^0 of Γ. The elements of B will denoted by $x, y, z \in B$, and their images in Γ by $\tilde{x}, \tilde{y}, \tilde{z} \in \Gamma$, respectively. The "object inclusion map" which identifies elements of the *base space* B with units in Γ^0 is denoted by $\epsilon : x \mapsto \tilde{x}$. If γ is an arrow from x to y then $x := s(\gamma)$ is called its *source* and $y := t(\gamma)$ is called its *target* (or source and target *projection*, respectively. The reversed arrow is called the *inverse* of γ and is denoted by γ^{-1}. The composition of arrows $\alpha\beta$ is defined only if the target of β is the source of α : $t(\beta) = s(\alpha)$ (mappings compose from right to left!). Composition is *associative* iff both terms $\alpha(\beta\gamma)$ and $(\alpha\beta)\gamma$ are defined. Obviously:

$$s(\gamma^{-1}) = t(\gamma),\ t(\gamma^{-1}) = s(\gamma), \quad \text{and} \quad \gamma^{-1}\gamma = \widetilde{s(\gamma)},\ \gamma\gamma^{-1} = \widetilde{t(\gamma)};$$

this is the reason why the arrows $\tilde{x}$ corresponding to the objects x are called *units* or *identities*. It is easy to see that they actually behave like units, i.e., if $s(\gamma) = x$ and $\gamma\eta = \eta$, then $\gamma = \tilde{x}$, etc. The set of composable pairs of arrows is denoted by Γ^2.

For two subsets A and C of Γ one uses the notation $\Gamma_A^C = t^{-1}(C) \cap s^{-1}(A)$ and $\Gamma\mid_A = \Gamma_A^A$, the restriction of the groupoid to A. If A consists of a single element $A = \{x\}$ then $\Gamma_x^x = G(x)$ is the *isotropy group* (or *vertex group*) of Γ at $x \in B$. There are two trivial extreme cases: *a)* when a groupoid consists of units only (no arrows) it is just a space without any structure; it can be identified with its base B; *b)* when the groupoid has only one object (unit), i.e., the arrows are always composable, and the groupoid reduces to a group.

It is sometimes convenient to identify the objects with the units as identities, i.e., omit the tilde and write, when defined: $x\gamma = \gamma$ and/or $\gamma x = x$. The definition of a *subgroupoid* is the obvious one; the set of units $\Gamma^0 = \tilde{B}$ is sometimes called the *base subgroupoid*.

A groupoid is called *transitive* if the map (s,t) from Γ to $B \times B = \epsilon^{-1}(\Gamma^0 \times \Gamma^0)$ (called the *anchor* by Mackenzie [18]) is surjective (onto) and it is called *principal* if it is bijective (one-to-one). A groupoid is called *connected* if for any two units x, y there exists an element α such that $x = s(\alpha)$ and $y = t(\alpha)$. It is convenient to think of a groupoid as "paths" (more precisely, as *homotopy classes or reparametrization classes of paths*, as will be explained below): the inverse path is defined, two paths can be composed if the end of one is the beginning of the next, associativity holds only if both compositions are defined, and left and right identities are defined as the addition of a zero path at the left or right end.

A *topological groupoid* is a groupoid where composition and taking of inverses (when they exist) are continuous. The properties of topological (smooth) groupoids are easily established locally in terms of germs. It turns out that topological groupoids are equivalent to Cartan principal bundles (principal bundles which are not necessarily locally trivial [18]). Similarly one can define *smooth groupoids* and (see below, Sec. 1.4) *Lie groupoids* and their associated *Lie algebroids*, structures which are closely related to principal bundles.

A *representation* of a topological groupoid Ω over a base space B is defined as a morphism of the groupoid into an *action* of a groupoid on a fibered space (M, p, B) over the same base space B (in particular, a vector bundle over B) defined as follows. Consider the "fibered product" $\Omega \star M = \{(\gamma, m) \in \Omega \times M | s(\gamma) = p(m)\}$. An *action* of Ω on (M, p, B) is a continuous map $\rho : \Omega \star M \to M, (\gamma, m) \mapsto \gamma m \in M$ with the properties:

$$(i) \qquad p(\gamma m) = t(\gamma), \qquad \forall\, (\gamma, m) \in \Omega \star M,$$

$$(ii) \qquad \beta(\gamma m) = (\beta\gamma)m, \quad \forall\, (\beta, \gamma) \in \Omega^2,\ (\gamma, m) \in \Omega \star M,$$

$$(iii) \qquad \widetilde{p(m)}m = m, \qquad \forall\, m \in M.$$

The cases of interest are when (M, p, B) is a topological group bundle and $m \mapsto \gamma m$ is a topological group isomorphism, or, when (M, p, B) is a *vector bundle* and $m \mapsto \gamma m$ is a linear map between fibers. In this case we can talk about a *linear representation of the groupoid* Ω. In particular, if the fibers are finite-dimensional vector spaces, not necessarily of the same dimensions, the linear maps among them (which represent the action groupoid) are $n \times m$ matrices, and the composition is defined only if the number of rows of the matrix on the left equals the number of columns of that on the right. An infinite-dimensional representation may have Hilbert bundles as objects and unitary transformations as elements of the groupoid.

As far as physics is concerned, the C^*-algebra of a groupoid may be the most useful concept, although it requires some technical concepts related to measure theory (this part is rather sketchy, and the reader should consult the literature for details). First of all one must define local compactness on a topological groupoid. Then one defines, as usual, the space of continuous functions with compact support on Γ, which will be denoted by $C^0(\Gamma)$. In Connes' theory the notion of measurable groupoid plays an important role. Connes defines a concept of transverse measure and Haar system for locally compact groupoids, concepts which extend the integration theory on groups. In terms of this noncommutative integration $C^0(\Gamma)$ can be turned into a C*-algebra by defining a convolution $f \star g$, and

an involution f^*. The representation theory of the groupoid is then defined in terms of a Hilbert bundle and its space of square-integrable sections. One of the most significant new concepts introduced by Connes is the construction of the holonomy groupoid of a foliated manifold. (The holonomy groupoid of a gauge-theory bundle should be familiar to physicists – its elements are the "Wilson parallel transporters" $\int e^A$ of lattice gauge theory along a path.)

Before continuing with the mathematical definitions of Lie groupoids, Lie algebroids, and connections, which are applicable to gauge theory I will briefly illustrate how groupoid-related ideas appear in the theory of open bosonic strings.

1.3. OPEN (BOSONIC) STRING INTERACTIONS AS GROUPOID OPERATIONS.

The elementary interactions of open and closed strings are described in the various reviews of John Schwarz [6]. Consider the most elementary interaction (of the Yang-Mills type) in which two strings break or join (Fig. 1). It is clear that since the words string and path are almost synonymous, these interactions can be interpreted directly in terms of groupoids. In other words, we have a "representation" of the path groupoid of $(D-1)$- dimensional space (the D-th dimension plays the role of time, and thus the string sweeps out a "world sheet" in D-space) by the open strings.

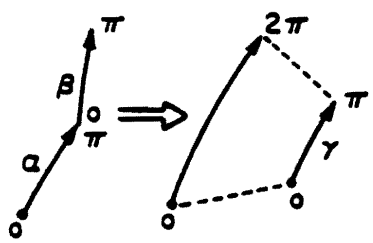

Fig. 1

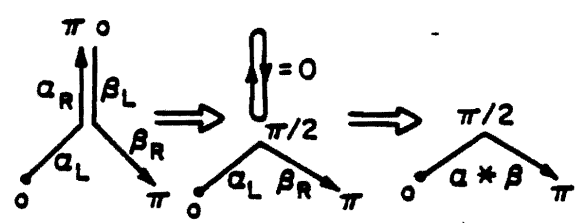

Fig. 2

On the other hand, in order to be able to accommodate the BRS-charge operator in string theory, Witten [6] has proposed a different composition law for open strings: each string is divided into a left half (parameter between 0 and $\pi/2$) and a right half (parameter running from $\pi/2$ to π). Two oriented open strings α and β can be composed if and only if the left half of the string α coincides with the right half of the second string β: these two halves are traversed in opposite directions, and are thus an "appendix" attached to the middle of the resulting string. The appendix is then "shed", resulting in a new string $\gamma = \alpha * \beta$ reparametrized with a parameter σ running again from 0 to π.

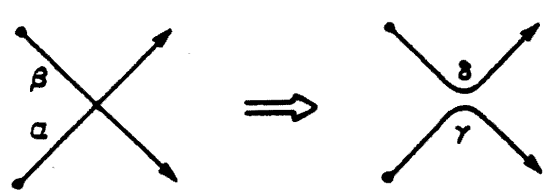

Fig. 3

Just like a chromosome, the new string resulting from the $*$-composition carries the "information of its two parents": the left half comes from string α, the right half, from string β (Fig. 2). Another type of interaction of oriented open strings is the exchange interaction: two strings cross in their midpoints and exchange right and left halves (Fig. 3). This is a new operation and can be interpreted as a groupoid operation only if we decompose the "crossing" into several steps as illustrated in Fig. 4: first the strings split at their midpoints and grow "virtual appendices" (parameters running respectively from 0 to $\pi/2$ and from $\pi/2$ to π); these appendices split, their halves are "flipped over", they recombine into new appendices ($\mu_L * \nu_R$ and $\nu_L * \mu_R$) which are then shed, resulting in the final configuration ($\alpha_L * \beta_R$ and $\beta_L * \alpha_R$). Reparametrizations are done "on the fly". This works in $(D-1)$ dimensions, but is hard to interpret in D dimensions, where the world sheets representing the strings and their appendices get hopelessly entangled.

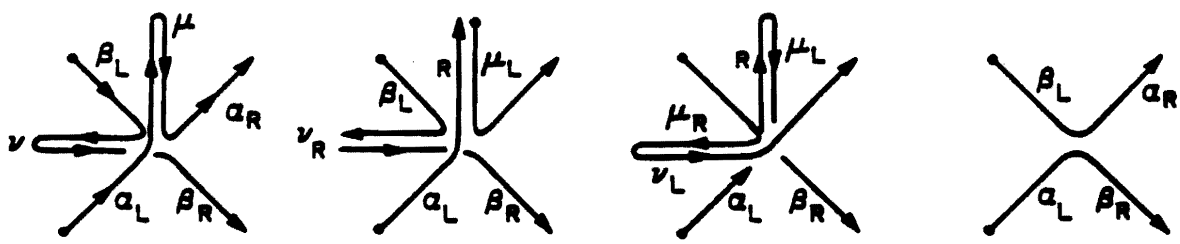

Fig. 4

It is clear that the composition $\alpha * \beta$, can be extended to three strings α, β, γ. However, these can be combined in two different ways: $\alpha * (\beta * \gamma)$ and $(\alpha * \beta) * \gamma$ (Fig. 5). As it stands this composition is not associative, since the two resulting parametrizations will be different. To get from one to the other we have to use the *homotopy* represented in Fig. 6.

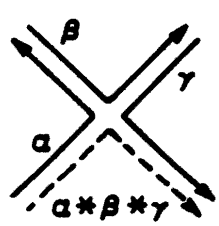

Fig. 5

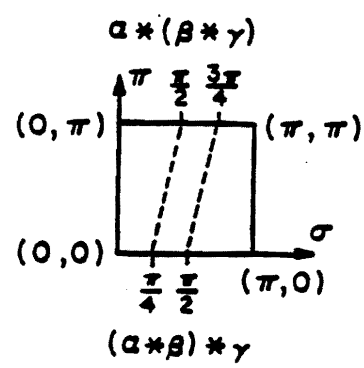

Fig. 6

This is why in place of paths (strings) one must consider *reparametrization classes* of paths (strings), in order to make the Witten $*$-product into a groupoid composition. It is also clear that this operation is noncommutative.

However, Witten's prescription cannot handle the interactions of closed strings (which is nonassociative) or the interaction of closed and open strings. In order to discuss these interactions one is forced to consider the 2-dimensional world sheets swept out by these strings in D-dimensional ambient space, as is done in the literature devoted to the field

theory of strings (cf [7]). See, however, Graeme Segal's talk [15] in this volume, where the interactions of closed strings are described in semigroup language.

It should also be noted that the Q_{BRS} operator, which appears in the "kinetic" term of the Witten action $\int(A * QA + \frac{2}{3}A * A * A)$ can be given a simple interpretation in the cohomology theory of Lie algebroids. I hope to return to this question in a more detailed publication soon.

1.4. LIE GROUPOIDS AND ALGEBROIDS. CONNECTIONS AND HOLONOMY

This section contains a very brief summary of facts about Lie groupoids, Lie algebroids and their relation to principal bundles, as well as the definition of connections.

The definition of a *Lie groupoid* Γ over a base manifold B is straightforward: one first makes Γ into a manifold, and requires that the projections s and t should be smooth (surjective submersions), and the multiplication (where defined) should be smooth. One then introduces a notion of *local triviality* and defines a *Lie groupoid as a locally trivial differentiable groupoid.* Equivalently, one can define a Lie groupoid as a differentiable groupoid for which the source map $s : \Gamma \to B$ is a *fibration* (surjective submersion). The same is also true for the target map $t : \Gamma \to B$. The *fibers* $\Gamma_x = s^{-1}(x)$ and $\Gamma^y = t^{-1}(y)$ are manifolds and any element $\gamma \in \Gamma$ such that $s(\gamma) = x, t(\gamma) = y$ defines a diffeomorphism between the fibers. A vector field ξ on Γ is defined to be right-invariant if it is vertical with respect to $s : s_*\xi = 0$ and $\gamma_*\xi = \xi$. On the other hand, already at the topological level, one can prove an equivalence between locally trivial groupoids and principal bundles. Much of the theory of reduction of the structure group of principal bundles also translates into groupoid language. There is obviously a close relation between Lie groupoids and smooth principal bundles. In particular, the theory of connections extends easily to Lie groupoids, if one starts out from the Atiyah sequence, which I recall briefly.

Atiyah defines a connection in a principal bundle $P(M, G, \pi)$ [19] (cf. also the forthcoming book [20]), as a splitting of the exact sequence of vector bundles $\mathcal{A}(P)$ (*Atiyah sequence*) defined as follows. Let $AdP = P \times_{Ad} \mathcal{G}$ be the bundle associated to P with fiber the Lie algebra $\mathcal{G}$ of the structure group G and the adjoint action of G on $\mathcal{G}$; let TP/G denote the quotient of the tangent bundle of P with respect to the right action of G, considered as a vector bundle over TM with projection the tangent map π_* of π, and let ι denote the inclusion map of $Ad(P)$ in TP/G (the identification of elements of $\mathcal{G}$ with fundamental vector fields. Then the Atiyah sequence $\mathcal{A}(P)$:

$$0 \longrightarrow Ad(P) \xrightarrow{\iota} TP/G \xrightarrow{\pi_*} TM \longrightarrow 0 \tag{1.1}$$

is exact, and a *connection is defined as a splitting of* (1.1), i.e, a vector bundle morphism $\chi : TM \longrightarrow TP/G$ such that $\pi_* \circ \chi = \mathrm{Id}_{TM}$. This induces a morphism $\omega : TP/G \longrightarrow Ad(P)$ such that $\iota \circ \omega = \mathrm{Id}_{TP/G}$ (to distinguish ω from the connection one-form sometimes denoted by the same letter, Mackenzie calls χ the connection and ω a *back-connection*. This splitting leads to the well-known direct-sum decomposition of the tangent spaces to P into horizontal and vertical parts, to the identification of ω with the $\mathcal{G}$-valued connection form defined as a section of $Ad(P)$, or to the definition of a connection in terms of a family of $\mathcal{G}$-valued one-forms over the open sets U_i of a trivializing atlas of M, with the known properties. Smoothness and equivariance under the right action of G completes the picture. The *curvature* is then defined in terms of a bracket for χ, or as a measure of failure of the connection form to be Maurer-Cartan, and the usual structure equations follow.

1.4.1. *Lie Algebroids and Lie-Algebra Bundles.* In the same way that the Lie algebra of a Lie group determines its local properties and, in most interesting cases, is sufficient for physical applications, the Lie algebroid of a Lie groupoid contains much of the local information about the Lie groupoid (and because of the existence of an exponential map, exhibits some similarities to the properties of the Lia algebra of a Lie group). Here is the definition:

A *Lie algebroid* on a manifold B is a vector bundle (A, p, B) together with a vector bundle map $q : A \longrightarrow TB$ (called the *anchor* of A by Mackenzie), and a bracket operation on sections, $[\ ,\ ] : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ ($\Gamma(A)$ is the vector space of smooth sections of A, not to be confused with the groupoid Γ) which is bilinear and skew-symmetric, satisfies the Jacobi identity and the two properties:

$$q([X,Y]) = [q(X), q(Y)], \qquad X, Y \in \Gamma(A); \tag{1.2a}$$

$$[X, uY] = u[X,Y] + q(X)(u)Y, \quad X, Y \in \Gamma(A), u \in C^\infty(B). \tag{1.2b}$$

(Mackenzie calls a Lie algebroid A *transitive* if q is a submersion, *regular* if q is of constant rank, and *totally intransitive* if $q = 0$.

A *Lie algebra bundle* (LAB) is a vector bundle (E, p, B) with a field of Lie brackets such that E admits an atlas $\{\psi_i : U_i \times \mathcal{G} \to L|_{U_i}\}$, ψ_i being Lie-algebra isomorphisms. Any LAB is a totally intransitive Lie algebroid, but the converse is not necessarily true. The most interesting example is the *Lie algebroid of covariant differential operators on a vector bundle* (see [18], Sec. 3.2).

It turns out that transitive Lie algebroids are the middle term of an Atiyah sequence of the type (1.1), and one is thus naturally led to the definition of a splitting, which defines an *infinitesimal connection*. Since the representation of the fundamental groupoid of the base manifold in a Lie groupoid defines a notion of *parallel transport*, and hence a *path connection*, it is necessary to distinguish these two notions.

1.4.2. *Infinitesimal Connections and Holonomy.* It is surprising that the relation between connections in a groupoid and connections in principal bundles, which was known for a number of years, has not caught the attention of many differential geometers and even less that of us physicists interested in gauge theories.

An *infinitesimal connection* in a Lie groupoid is defined as a splitting of the sequence $\mathcal{A} : L \longrightarrow A \longrightarrow TB$ associated to its Lie algebroid A (see [18], Ch. III), where L is the *adjoint Lie-algebra bundle* which makes the sequence exact. Thus a connection $\chi : TB \to A$ is a linear map such that $q \circ \chi = \mathrm{Id}_{TB}$. The theory of infinitesimal connections in Lie groupoids is practically identical to connection theory as outlined above. The curvature is defined either in terms of the Schouten-Nijenhuis bracket [20, L 83] for vector-bundle valued forms, or, simpler still, in terms of the brackets defined above, as a skew-symmetric vector bundle morphism $R_\chi : TB \oplus TB \to L$ defined by

$$\iota(R_\chi(X,Y)) = \chi[X,Y] - [\chi X, \chi Y], \quad X, Y \in \Gamma(TB). \tag{1.3}$$

The most promising aspect of the groupoid approach lies in the relation between infinitesimal connections and the action of its *holonomy groupoid*. The latter is defined in terms of a *path connection*, i.e., a mapping from the continuous paths in B (or reparametrization classes of paths) into piecewise smooth paths in Γ. Space does not permit me to go into all details, which can be found in [18]. The important result is that the holonomy

groupoid of a path-connection in a groupoid is a reduction of the groupoid (this is related to the Ambrose-Singer theorem for connections in principal bundles). In the Lie-groupoid-algebroid context the proof of the Ambrose-Singer theorem on the relation of curvature and holonomy is seen in a new light and becomes very simple. Since the monograph [18] treats this in detail, I will not dwell on this here. I have changed my terminology slightly from that used in the oral presentation of the talk, to bring it into conformity with the one used by Mackenzie. A brief outline of how gauge theory can be translated into this language (and what, if anything, can be learned from this) follows.

1. 5. A GROUPOID FORMULATION OF GAUGE THEORY.

A reformulation of gauge theories in terms of the holonomy groupoid is only a slight generalization of the formulation in terms of the holonomy group (Wilson loops – in lattice gauge theory). Thus, we replace the principal bundle $P(G, M, \pi)$ with structure group G and base space M by a Lie groupoid Γ and its Lie algebroid $A(\Gamma)$, and in place of the connection form ω and its curvature $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$ we use the connection χ in the algebroid and its curvature defined as in Eq. (1.3) (or in terms of the Schouten-Nijenhuis bracket).

The more global point of view, in terms of representations of the *holonomy groupoid* seems more promising, and the rest of this section is devoted to a quick survey of this. First, it is necessary to distinguish between the notions of a *curve* c_x^y between two points $x, y \in B$ considered as a one-dimensional submanifold of B with a *fixed parametrization*: $c_x^y : [0,1] \ni \sigma \mapsto c_x^y(\sigma) \in B, c_x^y(0) = x, c_x^y(1) = y$ and a *path* (or *path-class*) $\tilde{c}_x^y$ is an equivalence class of curves differing from each other by a reparametrization $\sigma \mapsto \sigma(\tau)$ or the attaching of an *appendix*, i.e., another curve attached to any point and traversed back and forth. An appendix may also be attached to the endpoint of a curve. However, attaching a *loop* at the end of a curve is to be considered a distinct operation since the "spoon"[13a] or "lasso"[13] formed in this way can pick up an element of curvature, and the composition of a path with a loop may produce a different kind of *parallel transport* from x to y than the path $\tilde{c}_x^y$. It is clear that the juxtaposition of paths (followed by a reparametrization of the resulting path) leads to a groupoid, where the inverse is the same path traversed in the opposite direction, and the source and target maps are the points x, y or appendices attached to them. I will denote this *path groupoid* by Π. The isotropy subgroup $G(x) = \Pi_x^x$ is identical to the loop group at the point x (reparametrization classes of loops based at this point), and is not to be confused with the *fundamental group*, which consists of homotopy classes of loops.

In order to describe a gauge theory, we consider a Lie groupoid Γ together with its Lie algebroid over B and associated families of vector bundles, and a representation of the path groupoid Π in Γ, as well as the induced action (*parallel transport*) in the associated vector bundles. This is implemented by means of the exponential map, and leads to a *product-integral (path-ordered exponential of the integral) of covariant differential operators* in these bundles. In other words, in the groupoid approach, we replace the connection one-form $A_\mu(x)dx^\mu$ by the action groupoid of parallel transport induced in associated vector bundles. The role of path-connection and holonomy becomes more prominent, and the theory is open to various generalizations, although at this level it is completely equivalent to conventional gauge theory in terms of connections and curvatures in principal bundles.

The group of gauge transformations, identified with the group of sections of the associated bundle $AdP = P \times_{AdG} G$ is seen in a new light in the groupoid interpretation, namely as an intertwining of the groupoid action with the adjoint action. One is naturally led to the BRS

cohomology as part of the cohomology theory of the corresponding Lie algebroid. Details of this will be found in my forthcoming book [20].

The new insight one gains on gauge theory by switching from principal bundles to Lie groupoids and algebroids is the renewed emphasis on *paths and parallel transport* of objects along paths in the base space. This becomes even more important in Kaluza-Klein type theories, where the dimensional reduction and compactification of parts of the fiber leads to a new fibered structure of the base space which becomes a locally a product of a space-time neighborhood and a compact homogeneous space. Thus a path on the base space might be thought of as a path in this fibered space, where the "appendix" which lies entirely in the homogeneous space is "macroscopically null-homotopic", and thus does not contribute to the holonomy, except at the scale of the compactified homogeneous manifold.

Another interesting aspect needing further study is the problem of higher-dimensional extended objects. We have already seen that open strings have a natural interpretation in terms of groupoids. The problem of defining a Stokes theorem [13, 14] for noncommutative differential forms leads to interesting extensions of groupoid theory to "ordered membranes" or higher-dimensional analogues.

2. Lie Bigebras and Quantum Groups

2.1. WHAT ARE BIGEBRAS ?

This section contains a brief review of the basic facts about algebras, cogebras, and bigebras (a.k.a. bialgebras or Hopf algebras). By *algebra* will be mean an associative algebra A over the field K ($K = \mathbf{R}$ or $\mathbf{C}$) with unit 1, and multiplication denoted $m : (x, y) \mapsto xy,\ x, y \in A$. The natural embedding of K in A (given by the unit) will be denoted by $u : K \hookrightarrow A$. The tensor product $\otimes_K$ will be denoted simply by $\otimes$.

Then multiplication can be viewed as a morphism $m : A \longrightarrow A \otimes A$ which makes the following diagram commutative (this simply expresses the associativity of the algebra A):

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "m\otimes Id"] \arrow[d, "Id\otimes m"] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"] & A \end{tikzcd} \tag{2.1}$$

The embedding of K in A can be described by the following commutative diagram:

$$\begin{tikzcd} K \otimes A \arrow[r, "u\otimes Id"] \arrow[d, equal] & A \otimes A \arrow[d, "m"] & A \otimes K \arrow[l, "Id\otimes u"'] \arrow[d, equal] \\ A \arrow[r, "Id"] & A & A \arrow[l, "Id"'] \end{tikzcd} \tag{2.2}$$

A *cogebra* (or coalgebra) is defined by simply turning all the arrows around in Eqs. (2.1) and (2.2). In other words, a vector space C over K is a *coassociative cogebra* with *comultiplication* $\Delta : C \longrightarrow C \otimes C$ (Δ stands for *diagonal map*) and *counit* ϵ (also called augmentation),

so that the following "dual" diagrams are commutative :

$$\begin{tikzcd} C\otimes C\otimes C & C\otimes C \arrow[l, "\Delta\otimes \mathrm{Id}"'] \\ C\otimes C \arrow[u, "\mathrm{Id}\otimes\Delta"] & C \arrow[l, "\Delta"'] \arrow[u, "\Delta"'] \end{tikzcd} \tag{2.3}$$

and

$$\begin{tikzcd} K\otimes C \arrow[d, equal] & C\otimes C \arrow[l, "\epsilon\otimes \mathrm{Id}"'] \arrow[r, "\mathrm{Id}\otimes\epsilon"] & C\otimes K \arrow[d, equal] \\ C & C \arrow[l, "\mathrm{Id}"'] \arrow[r, "\mathrm{Id}"] \arrow[u, "\Delta"'] & C \end{tikzcd} \tag{2.4}$$

A vector space which is both an algebra and a cogebra is called a *bigebra or bialgebra* (I prefer the shorter term "bigebra", used by Bourbaki, Dieudonné, and Cartier in the early papers on the subject). Some authors call such an object a *Hopf algebra*, but following Sweedler and others this term will be reserved for bigebras with an additional structure element, related to the group inverse, called an *antipode*.

The notion of antipode is motivated by considering the algebra $A = Fun(G)$ of functions on a group G (continuous functions for a locally compact group, smooth functions for a Lie group, etc.) with pointwise addition and multiplication.

We consider, for example, the algebra $F = L^\infty(G)$ with multiplication pointwise multiplication of functions on G, and the unit defined as the function $f = 1$. The algebra F is also a cogebra under the *comultiplication*

$$\Delta : F \longrightarrow F\otimes F : \Delta(f)(s,t) = f(st), \qquad f(st) \in L^\infty(G\times G) = F\otimes F, \quad f\in F, \tag{2.5}$$

with the counit naturally defined as evaluation at the group identity $\epsilon : f \mapsto f(e) \in \mathrm{R}$. This makes F into a *bigebra* or *bialgebra*, i.e., a vector space which is both an algebra and a cogebra. In our example the algebra is commutative, but the cogebra is not *cocommutative*, since $st \neq ts$.

The existence of the inverse g^{-1} in the group leads to the definition of the *antipode* (or *involution* j, whose properties can be derived from the example under consideration. Define the *involution* in F by $j(f)(g) = f(g^{-1})$. It is *involutive*: $j^2 = j\circ j = \mathrm{Id}$, and since $(gh)^{-1} = h^{-1}g^{-1}$, its interaction with Δ is given by $\sigma\circ\Delta\circ j = (j\otimes j)\circ\Delta$, where $\sigma : F\otimes F \to F\otimes F : f_1\otimes f_2 \mapsto f_2\otimes f_1$ permutes the two algebras.

The interaction of Δ, j, u, and m can be summarized in the following commutative diagram:

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\epsilon"'] & A\otimes A \arrow[r, shift left, "j\otimes\mathrm{Id}"] \arrow[r, shift right, "\mathrm{Id}\otimes j"'] & A\otimes A \arrow[r, "m"] & A \\ K & & & K \arrow[u, "u"'] \end{tikzcd} \tag{2.6}$$

where the mappings above and below the horizontal arrow correspond to the two orderings of factors in the tensor product, and the two copies of the field $K = \mathbf{R}$ or $\mathbf{C}$ at the bottom should be identified.

The set $(F, m, u, \Delta, \epsilon, j)$ is an example of a *commutative Hopf algebra* or *commutative bigebra*. The general definition follows by abstracting from this example, i.e., an *involutive bigebra* or *Hopf algebra* (not necessarily commutative or cocommutative) will be a vector space F endowed with a multiplication, unit, comultiplication, counit, and involution satisfying the diagrams (2.1)–(2.6). In the noncommutative case the definition of involution or antipode is not unique — one can introduce a second antipode, j', corresponding to the other order of multiplication but the same comultiplication. The two antipodes are "inverses" in the sense that $j'j = jj' = \mathrm{Id}$, but we no longer have $j^2 = \mathrm{Id}$ so that the term involution is no longer appropriate.

2.2. LIE BIGEBRAS AND THEIR APPLICATIONS

A *Lie bigebra* (or *Lie Hopf algebra* = LHA) is defined as a vector space $\mathcal{G}$ which has the structure of a Lie algebra and of a (Lie) cogebra, such that the map Δ —which in this context may be called a *cocommutator or cobracket* - - is a 1-cocycle (since $\mathcal{G}$ acts on $\mathcal{G} \otimes \mathcal{G}$ via the adjoint representation). Drinfel'd proved that the category of finite-dimensional Lie bigebras is equivalent to the category of connected simply connected Poisson-Lie groups.

A *Poisson-Lie group* (PLG) is a Lie group G with a Poisson manifold structure (cf., e.g., [L 83], [W 83]), on $Fun(G)$, so that the comultiplication $\Delta : Fun(G) \to Fun(G \times G) = Fun(G) \otimes Fun(G)$ is a Lie algebra homomorphism. A *quantization* of a Poisson algebra is defined as a *deformation* depending on a parameter $\hbar$ (cf. [L83], where additional references can be found).

Drinfel'd proves a number of other beautiful results on Poisson and co-Poisson algebras. He also establishes a remarkable relation with the r-matrices which are solutions of the classical Yang-Baxter equation, which was discussed extensively in the lectures by Fröhlich, de Vega, and others at this Conference. The motivation for this work came from the quantum inverse scattering approach to integrable dynamical systems.

Another, and even more exciting, suggestion made by Drinfel'd is the *quantization of universal enveloping algebras* — QUE-algebras — and the so-called QFSH (quantum formal series Hopf) algebras.

As far as I know at present no serious attempts have been made to relate Drinfel'd's work to the Doplicher-Roberts [DR 72, 84, 87] work, where Hopf algebras also play a prominent role. I hope to be able to return to this problem in the near future, and also explore how this machinery could be used in a quantum theory of gauge fields.

3. Outlook

It is safe to predict that the uses of Lie groupoids, their algebroids and connections, and their representations in C*- and W*-algebras will make some inroads into mathematical physics. The Hopf-algebra techniques which have appeared in two disjoint physics contexts — the Doplicher-Haag-Roberts approach to superselection sectors and symmetries in algebraic field theory, and the Drinfel'd-Belavin-Cherednik-Sklyanin papers, dealing with applications to Yang-Baxter equations, r-matrices, and inverse scattering —will also spread.

One can hope that these two directions will converge. One possible area of possible overlap is the circle of ideas discussed by Fröhlich in his talk, coming from two-dimensional field theories and leading to knot theory and braid groups.

Although it is not clear at present whether there is any direct link between the two seemingly disparate topics discussed in this talk, allow me to get somewhat speculative and express the hope that both Lie groupoids and Lie bigebras will find uses in a future theory which subsumes all the interesting aspects of gauge theories and string field theory. I will try to outline a vague "scenario" of how this could work. I also want to point out that these approaches should be tried out in another area where progress has been slow: quantum theory in gravitational fields and quantum theory of the gravitational field itself.

A future quantum theory of extended objects (be it strings, membranes, or theories which involve an underlying manifold which itself has a bundle structure with space-time as base and some "compactified Kaluza-Klein space" as fiber) has to incorporate two major ingredients: the differential geometry/topology of the underlying structures and the operator-algebraic aspects of fields, observables, states, and generalized superselection structure. Clearly the mathematical objects discussed in this talk will, in their present form or a future refined form, be necessary to describe these objects. Let me try to sketch a model of such a structure, in a very vague and qualitative way. Suppose that the quantum theory of such "extensons" (a temporary name for such a generic *extended but elementary* objects, such as strings, membranes Kaluza-Klein spaces) is described by some big algebra $\mathcal{F}$ — the *field algebra* and that there is a distinguished subalgebra $\mathcal{A}$ — the *observables* and an associated set of *states* $\mathcal{S}$, a subset of which can be given a manifold structure and identified with the space-time manifold. One of the most elementary notions which arises in this context is that of *holonomy*, i.e., the representation of reparametrization classes of paths in either the field algebra $\mathcal{F}$, or the space of states $\mathcal{S}$. In other words we are dealing with a *representation of the holonomy groupoid* of the "classically underlying structure" by the *groupoid of intertwiners* [DR 84, 86] of the appropriate algebras. On the other hand, these objects have a Hopf-algebra structure, since they also embody the duality of the internal symmetry (group) of the theory. A further study of these aspects might thus establish a link between the two seemingly disjoint structures discussed in the body of this talk.

Another aspect which has not been mentioned here at all (except that it is somehow implicit in superstring theories) is gravity, either as a quantum theory of gravity itself, or, more modestly, as a consistent theory of quantum fields in a gravitational background. This difficult subject has formed the object of much research, ranging from treatments which make little sense, dealing with such incomprehensible notions as "the *wave function of the Universe*" to some attempts at incorporating curved space into the algebraic approach to quantum field theory (see, e.g., [HNS 84], [FH 86], and remarks on using the tangent bundle I made at a previous Como gathering two years ago (last reference in [5])). Here too, we might want to see if progress could not be made by describing the effects of the curved background by its holonomy groupoid represented as intertwiners of the field algebras defined over different tangent spaces. This is a topic which deserves special attention, and I intend to return to it in the near future.

I hope these remarks, some of them perhaps too vague and cryptic, will have persuaded some of you that the subject deserves to be pursued further, and that both groupoids and algebroids and bigebras might play a role in future developments of unified theories of particle interactions.

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