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Quantization of WZNW Model in the Light-Cone Coordinate: Quadratic Algebra and Hierarchy

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ABSTRACT

We study the Hamiltonian formalism of the Wess-Zumino-Novikov-Witten (WZNW) model in the light-cone coordinate. In the quantization procedure we point out some subtleties. A quadratic algebra is derived from the fundamental Dirac brackets for the group-valued field $\tilde{g}$. Their associated Kac-Moody algebra as well as the corresponding Virasoro algebra, involutive conserved quantities, and hierarchy equations are also given.

I. Introduction

Abelian and nonabelian chiral models are interesting from various points of view.

Abelian chiral model is a 2-dimensional free boson system with constraints. The canonical quantization of this system on the light-cone coordinate is not trivial[1] due to constraints. This is not a mere toy model but a component of heterotic string theory.[2]

Nonabelian chiral model with the particular Wess-Zumino-Novikov-Witten term[3] (the WZNW model) are considered as a vacuum of string theory[4] thanks to their quantum conformal symmetry. For this conformal field theory (CFT) Green functions are calculable[5] by the methods of Belavin, Polyakov and Zamolodchikov.[6]

More importantly the WZNW model bears the essential resemblance of higher dimensional systems: the self-dual Yang Mills equation,[7] the super symmetric Yang Mills equations,[8] and the conformal super gravity equations.[9,10] If we want to extract as much information as possible for these higher dimensional theories, it is essential to understand the WZNW model clearly.

We choose to work in the light-cone coordinate. The light-cone coordinate is natural framework to study conformal symmetric system.[6] The linear equations constructed from integrable point of view for the many theories mentioned above[7–9] are the simplest form in this coordinate.[10] In addition, it is in the light-like coordinate that the linear systems exist for higher dimensional theories. Indeed this work turns out to be very helpful to the quantization of the four-dimensional self-dual Ynag-Mills system formulated in the J fields, see Ref.[11].

The aim of this paper is to make concise the canonical quantization procedure for the WZNW model. We point out some subtleties in the procedure of quantization. In particular, we emphasize the importance of the finite size effect to realize equation of motion at the Dirac bracket[12] level, and the importance of the first-class constraint in defining the Kac-Moody current.

This paper is organized as follows. In section 2, we apply the generalized Hamiltonian procedure[12] to the free boson system in the light-cone coordinate. We take care the existence of finite size effect on the Dirac bracket and realize the full equation of motion as a canonical equation. In section 3, we derive the quadratic algebra of Dirac brackets for the group-valued field g for the WZNW model through the same procedure as in section 2. The associated Kac-Moody algebra in the zero-mode modified field $\tilde{g}$ is than constructed. In section 4, from the Virasoro fields we construct local conserved quantities that have zero brackets to each other (i.e. involutive). They generate a hierarchy of nonlinear equations and their corresponding linear systems. Section 5, we give a summary and some discussion.

II. Abelian chiral model: An example

In this section, to illustrate the essential points of quantization in light-cone coordinate, we discuss the Dirac procedure[12] of quantizing a free boson system in 2-dimensional with finite size along the light-cone direction.

The action under study is

$$A = \int_{-\infty}^{\infty} dx^+ \int_0^L dx^- \ \partial_+\phi \, \partial_-\phi, \tag{2.1}$$

where x^+ and x^- are light-cone coordinates, $\partial_\pm \equiv \partial/\partial x^\pm$ and $\phi \equiv \phi(x^+, x^-)$ is a boson field in 2-dimensions. We can quantize the system considering either x^+ being the "time", or x^- being the "time". To set the convention, we call the former the "left" quantization, the latter the "right" quantization.

We first consider the "left" quantization with x^+ being the "time" and x^- the "space" with periodic boundary condition:

$$\phi(x^- + L) = \phi(x^-). \tag{2.2}$$

In generally, the periodicity is important since it is one of the basis for conserved transfer matrix.[13]

The definition of the canonical momenta and equal "time" x^+ Poisson brackets are

$$\pi(x^-) \equiv \frac{\partial A}{\partial(\partial_+\phi)} = \partial_-\phi(x^-), \tag{2.3}$$

$$\{\phi(x^-),\ \phi(y^-)\}_P = 0, \tag{2.4}$$

$$\{\pi(x^-),\ \phi(y^-)\}_P = \sum_{n=-\infty}^{\infty} \delta(x^- - y^- + nL), \tag{2.5}$$

$$\{\pi(x^-),\ \pi(y^-)\}_P = 0. \tag{2.6}$$

Eq. (2.3) is a constraint. We use $C(x^-)$ to express the constraint,

$$C(x^-) = \pi(x^-) - \partial_-\phi(x^-). \tag{2.7}$$

Note that since $C(x^-)$ is a function of x^- this equation means infinite number of constraints: one at each point of x^-. Now we can separate these constraints $C(x^-)$ to a first class constraint F and infinite number of second class constraints $S(x^-)$,

$$F \equiv \int_0^L dx^- \ C(x^-) \approx 0, \tag{2.8a}$$

$$S(x^-) \equiv C(x^-) - \frac{1}{L}\int_0^L dx^- \ C(x^-) = \pi(x^-) - \partial_-\phi(x^-) - \frac{1}{L}F = 0 \tag{2.8b}$$

Here F is a first class constraint because Poisson brackets between F and $S(x^-)$ are zero for all x^-.

Next we turn to the Hamiltonian. The canonical Hamiltonian is zero. Total Hamiltonian that has weakly zero Poisson bracket with $S(x^-)$ is

$$H_T = \lambda(x^+) \int_0^L dx^- \ C(x^-), \tag{2.9}$$

where $\lambda(x^+)$ is an arbitrary function of x^+. Needless to say, we can realize

equations of motion,

$$\partial_+\partial_-\phi = 0, \tag{2.10}$$

as canonical equations $\partial_+\phi = \{H_T, \phi\}$, using total Hamiltonian H_T and Poisson brackets eq.(2.4) to (2.6).

In the presence of those second class constraints eq.(2.8b) the Dirac brackets in this system are calculated:

$$\{\phi(x^-),\ \phi(y^-)\}_D = \frac{1}{4}\sum_{n=-\infty}^{\infty} \varepsilon(x^- - y^- + nL) - \frac{1}{2L}(x^- - y^-), \tag{2.11}$$

$$\{\pi(x^-),\ \phi(y^-)\}_D = \frac{1}{2}\sum_{n=-\infty}^{\infty} \delta(x^- - y^- + nL) + \frac{1}{2L}, \tag{2.12}$$

$$\{\pi(x^-),\ \pi(y^-)\}_D = -\frac{1}{2}\sum_{n=-\infty}^{\infty} \partial_-\delta(x^- - y^- + nL). \tag{2.13}$$

Here we discuss the feature of these Dirac brackets from two points of view: periodicity and realization of equation of motion.

The Dirac brackets eq.(2.11)~(2.13) are consistent with the periodic boundary condition. This is a new feature of our Dirac brackets in comparing with previous works.[1] We should remark that the second term in eq.(2.11) is a reflection of the finite system size.

In the Dirac procedure, the equations of motion can be derived from the total Hamitonian H_T and the Dirac brackets. The Canoncial equation of motions

$$\partial_+\phi(x^-) = \{H_T,\ \phi(x^-)\}_D = \lambda(x^+), \tag{2.14}$$

$$\partial_+\pi(x^-) = \{H_T,\ \pi(x^-)\}_D = 0, \tag{2.15}$$

and constraints

$$S(x^-) \equiv \pi(x^-) - \partial_-\phi(x^-) - \frac{1}{L}F = 0 \tag{2.16}$$

lead to Euler-Lagrange equation $\partial_+\partial_-\phi = 0$, either from eq.(2.14) or from eq.(2.15) and eq.(2.16). Here we see $\lambda(x^+)$ is essentially the x^+ dependent

part of the general solution of the Euler-Lagrange equation of motion,

$$\phi(x^+, x^-) = \phi_+(x^+) + \phi_-(x^-), \tag{2.17}$$

where ϕ_+, ϕ_- are arbitrary functions of x^+, x^- respectively. In the limit of L infinity, if we drop the last term of (2.16) then H_T becomes zero and we can not describe the x^+ depndent part $\phi_+(x^+)$ of the field $\phi(x^+, x^-)$. Therefore we should take care the existence of the first class constraint eq.(2.8a) and the L^{-1} terms in the Dirac brackets eqs.(2.11) and (2.12).

We can derive simpler expression of the Dirac bracket by introducing a modified field $\tilde{\phi}$ and its canonical momenta $\tilde{\pi}$:

$$\tilde{\phi} \equiv \phi(x^-) + \frac{F}{2L}x^-, \tag{2.18}$$

$$\tilde{\pi}(x^-) \equiv \partial_-\tilde{\phi} = \pi(x^-) - \frac{F}{2L}. \tag{2.19}$$

The added terms are precisely the zero modes. The brackets of $\tilde{\phi}$ and $\tilde{\pi}$ are simpler then those of ϕ and π:

$$\{\tilde{\phi}(x^-), \tilde{\phi}(y^-)\} = \frac{1}{4}\sum_{n=-\infty}^{\infty} \varepsilon(x^- - y^- + nL), \tag{2.20}$$

$$\{\tilde{\pi}(x^-), \tilde{\phi}(y^-)\} = \frac{1}{2}\sum_{n=-\infty}^{\infty} \delta(x^- - y^- + nL), \text{ and} \tag{2.21}$$

$$\{\tilde{\pi}(x^-), \tilde{\pi}(y^-)\} = -\frac{1}{2}\sum_{n=-\infty}^{\infty} \partial_-\delta(x^- - y^- + nL). \tag{2.22}$$

We have reproduced the previous results[1] in our periodical light-cone formalism. If we suppose that the field $\phi(x^-)$ takes a value on a loop then the first class constraint F can be recognized as a winding variable.

III. The WZNW Model

III.a. Quadratic Algebra of the Dirac Brackets

In this section, we study the Dirac procedure of quantizing the WZNW model. The fundamental Dirac bracket of this model is derived. We then show that the Euler-Lagrange equation of motion follow. We use some techniques developed in Ref. [14].

The following arguments are parallel to the case of free bosons in the section 2. We apply the periodic boundary condition and modify the field by a function of the first class constraint. The Kac-Moody algebra is realized with the modified field.

The action we consider is

$$Awz = \alpha \int_{-\infty}^{\infty} dx^+ \int_0^L dx^- \; tr(\partial_+ g \; \partial_- g^{-1})$$
$$- \alpha \int_{-\infty}^{\infty} dx^+ \int_0^L dx^- \int_0^1 d\rho \; tr\Big(\tilde{g}^{-1}\partial_\rho \tilde{g}[\tilde{g}^{-1}\partial_- \tilde{g}, \tilde{g}^{-1}\partial_+ \tilde{g}]\Big), \qquad (3.1)$$

where $\tilde{g} = g(x^+, x^-, \rho) \in \mathbf{GL(N)}$, $g = g(x^+, x^-, 1)$, and α is the coupling constant. The second term is called as Wess-Zumino term, Γ, that is represented in two dimensional form,

$$\Gamma = -\alpha \int_{-\infty}^{\infty} dx^+ \int_0^L dx^- \; tr(B(g, \partial_- g)\partial_+ g), \qquad (3.2)$$

where B is a function of g and $\partial_- g$.

We first consider the "left" quantization in which x^+ is the "time" and x^- is the "space" with periodic boundary condition, $g(x^+, x^-) = g(x^+, x^- + L)$.

The canonical momenta is

$$\pi_{\alpha\beta} \equiv \frac{\delta Awz}{\delta(\partial_+ g_{\alpha\beta})} = \alpha \; \partial_- g^{-1}_{\beta\alpha} - \alpha B_{\beta\alpha}, \qquad (3.3)$$

and equal "time" x^+ Piosson brackets are

$$\{g_{\alpha\beta}(x^-), g_{\gamma\delta}(y^-)\}_P = 0, \qquad (3.4)$$

$$\{\pi_{\alpha\beta}(x^-), g_{\gamma\delta}(y^-)\}_P = \delta_{\alpha\gamma}\delta_{\beta\delta} \sum_{n=-\infty}^{\infty} \delta(x^- - y^- + nL), \quad \text{and} \qquad (3.5)$$

$$\{\pi_{\alpha\beta}(x^-), \pi_{\gamma\delta}(y^-)\}_P = 0. \qquad (3.6)$$

Eq. (3.3) is a constraint $C_{\alpha\beta} = 0$, where

$$C_{\alpha\beta} \equiv \pi_{\alpha\beta} - \alpha\partial_- g^{-1}_{\beta\alpha} + \alpha B_{\beta\alpha}. \qquad (3.7)$$

More convenient form of this constraint is

$$e_{\rho\beta} = g_{\alpha\rho} C_{\alpha\beta}$$
$$= g_{\alpha\rho}\pi_{\alpha\beta} + \alpha(g^{-1}\partial_- g)_{\beta\rho} + \alpha(Bg)_{\beta\rho}. \qquad (3.8)$$

The Poisson brackets of the constraints $e_{\alpha\beta}(x^-) = 0$ are

$$\{e_{\alpha\epsilon}(x^-), e_{\delta\gamma}(y^-)\}_P$$
$$= -2\alpha \; \delta_{\gamma\alpha} \; \delta_{\epsilon\delta} \sum_{n=-\infty}^{\infty} \partial_x \delta(x^- - y^- + nL)$$
$$+ \Big[e^T_{\gamma\alpha}\delta_{\epsilon\delta} - \delta_{\gamma\alpha} e^T_{\epsilon\delta}\Big] \sum_{n=-\infty}^{\infty} \delta(x^- - y^- + nL). \qquad (3.9)$$

We separate the constraints $e_{\alpha\beta}(x^-)$ to the first class , $F^{\rho\alpha}$ and the second class $S^{\rho\sigma}(x^-)$:

$$F^{\rho\sigma} \equiv \int_0^L dx^- \; \lambda^{\rho\sigma}_{\alpha\gamma}(x^-) e_{\alpha\gamma}(x^-) \quad \text{and} \qquad (3.10)$$

$$S^{\rho\sigma}(x^-) \equiv \lambda^{\rho\sigma}_{\alpha\gamma}(x^-) e_{\alpha\gamma}(x^-) - \frac{1}{L} F^{\rho\sigma}, \qquad (3.11)$$

where $\lambda^{\rho\sigma}_{\alpha\gamma}(x^-)$ satisfies the boundary condition,

$$\lambda^{\rho\sigma}_{\alpha\gamma}(x^- = 0) = \delta^\rho_\alpha \; \delta^\sigma_\gamma, \quad \text{and} \qquad (3.12)$$

the definition of the first class constraint that has weakly zero bracket with the other constraints.

$$\{F^{\rho\sigma}, e_{\alpha\gamma}\} \approx 0, \qquad \int_0^L dx^- \lambda^{\rho\sigma}_{\alpha\gamma}(x^-) \sum_{n=-\infty}^{\infty} \partial_- \delta(x^- - y^- + nL) = 0. \quad (3.13)$$

The solution of eq. (3.13) is

$$\lambda^{\rho\sigma}_{\alpha\gamma}(x^-) = \lambda^{\rho\sigma}_{\alpha\gamma}(x^- = 0) = \delta^\rho_\alpha \delta^\sigma_\gamma, \quad (3.14)$$

i.e., constants.

Now we turn to the Hamiltonians. The canonical Hamiltonian is zero, so the total Hamiltonain of this model is solely from the constraint,

$$H_T = \Lambda_{\rho\sigma}(x^+) F^{\rho\sigma}. \quad (3.15)$$

One of the canonical equations of motion with Poisson bracket is

$$\partial_+ g_{\alpha\beta} = (g\Lambda)_{\alpha\beta}. \quad (3.16)$$

Equation (3.16) is consistent with Euler-Lagrange equations. Using it, we can derive the following equations

$$\partial_+ (g^{-1}\partial_- g)_{\alpha\beta} = -[\Lambda, g^{-1}\partial_- g]_{\alpha\beta}, \quad (3.17)$$

$$\partial_- \Lambda_{\alpha\beta} = 0. \quad (3.18)$$

These are nothing but Euler-Largrange equations $\partial_+(g\partial_- g^{-1}) = 0$ if we remember Λ is represented by g due to eq.(3.16),

$$\Lambda_{\alpha\beta} = (g^{-1}\partial_+ g)_{\alpha\beta}. \quad (3.19)$$

We notice that the first class constraint $F^{\rho\sigma}$ for the total hamiltonian is important to realize the Euler-Lagrange equations.

Now we are ready to study the Dirac bracket of this model. Here we use a tensor notation: the bracket of any tensors A and B is defined as[13]

$$\{A \overset{\otimes}{,} B\}_{\alpha\gamma,\beta\delta} \equiv \{A \otimes I,\ I \otimes B\}_{\alpha\gamma,\beta\delta} = \{A_{\alpha\beta}, B_{\gamma\delta}\}. \quad (3.20)$$

The fundamental Dirac bracket is calculated in the Appendix and they give a new quadratic algebra,

$$\{g(x^-) \overset{\otimes}{,} g(y^-)\}_D = g(x^-) \otimes g(y^-) M, \quad (3.21)$$

where

$$M \equiv -\frac{P}{2\alpha}\left(\frac{1}{2}\sum_n \varepsilon(x^- - y^- + nL) e^{(x^- - y^- + nL)\overset{\otimes}{L}} + \frac{1}{Q}\right), \quad (3.22)$$

$$Q \equiv \frac{1}{2\alpha}\left(F^T \otimes 1 - 1 \otimes F^T\right), \text{ and} \quad (3.23)$$

P is a permutation operator, $P(A \otimes B) = (B \otimes A)P$. The lesson of the free boson case lead us to introduce a modified field $\tilde{g}$:

$$\tilde{g}(x^-) \equiv g(x^-) e^{-F^T x^-/2\alpha L}. \quad (3.24)$$

The equal "time" bracket for $\tilde{g}(x^-)$ is

$$\{\tilde{g}(x^-) \overset{\otimes}{,} \tilde{g}(y^-)\}_D = \tilde{g}(x^-) \otimes \tilde{g}(y^-) \tilde{M}, \quad (3.25)$$

where

$$\tilde{M} = -\frac{P}{2\alpha}\left(\frac{1}{2}\sum_n \varepsilon(x^- - y^- + nL) e^{nQ} + \frac{1}{Q}\right). \quad (3.26)$$

A straightforward calculation proves that the quadratic algebra of the Dirac bracket, eq.(3.21), satisfies Jacobi identity. The first class constraint Q appears in the bracket of eq. (3.26) in an essential way unlike the free boson case, eq. (2.11), where the first-class constraint does not appear explicitly. It may suggest that the first class constraints become dynamical variables. Note that the algebra of g and $\tilde{g}$ are quadratic.

Similarly in the "right" quantization, we choose x^- as "time" and x^+ as "space" with periodic boundary conditions. We do the same quantization and obtain $\partial_-(g^{-1}\partial_+ g) = 0$ as equation of motion and eqs.(3.3) to eq.(3.26) with $g \leftrightarrow g^{-1}, x^+ \leftrightarrow x^-$.

III.b. Kac-Moody Algebra of the WZNW model

It is known that the current density satisfies the Kac-Moody algebra in WZNW model. We can show that the current density composed of the $\tilde{g}$ fields

$$j_- \equiv \tilde{g}\partial_-\tilde{g}^{-1}\,, \tag{3.27a}$$

satisfy neatly the Kac-Moody algebra:

$$\{\tilde{g}\partial_-\tilde{g}^{-1}(x^-)\overset{\otimes}{,}\tilde{g}\partial_-\tilde{g}^{-1}(y^-)\}$$
$$= \frac{P}{2\alpha}\left[\sum_n \delta'(x^- - y^- + nL) + (1\otimes\tilde{g}\partial_-\tilde{g}^{-1} - \tilde{g}\partial_-\tilde{g}^{-1}\otimes 1)\sum_n \delta(x^- - y^- + nL)\right]. \tag{3.27b}$$

On the other hand currents composed of g fields, $g\partial_- g^{-1}$, satisfy a more complicated algebra.

It is known that Kac-Moody algebra originates from a symmetry. The symmetry of the WZNW model is

$$g(x^+, x^-) \longrightarrow V(x^-)g(x^+, x^-)\bar{V}(x^+)\,. \tag{3.28}$$

and the Kac-Moody currents generate these symmetries.[3] In our formalizm, the current $j_-(x^-)$ generates the left symmetry $V(x^-)$ by a relation:

$$\left\{2\alpha\int_0^L dx^-\left[tr v(x^-)j_-(x^-)\right] \quad , \quad \tilde{g}(y^-)\right\}_D = v(y^-)\tilde{g}(y^-), \tag{3.29}$$

where $V(x^-) = 1 + v(x^-)$.

To generate the "right" symmetry $\bar{V}(x^+)$, we use the "right" Hamiltonian formalism where we choose the coordinate x^- as the "time", and x^+ as "space" with periodic condition. Then the "right" Kac-Moody current $j_+ \equiv \tilde{g}^{-1}\partial_+\tilde{g}$ generates the right symmetry $\bar{V}(x^+)$. The corresponding equations are simply eq.(3.27a) to eq.(3.29) with $\tilde{g} \leftrightarrow \tilde{g}^{-1}$ and $x^+ \leftrightarrow x^-$.

The field g can always be split into chiral parts, $g = L(x^-)R(x^+)$. To obtain the algebra for $L(x^-)$ or $R(x^+)$ from the g algebra, we need to consider the case of $\Lambda_{\alpha\beta} = 0$ in eq.(3.19). In the "left" Hamiltonian formalism where x^+ is the time, we then obtain $\partial_+\tilde{g} = 0$ and a "left" chiral field u can be defined,

$$u(x^-) \equiv \tilde{g}(x^-, x^+). \tag{3.30}$$

This $u(x^-)$ is $L(x^-)$ up to a constant (dynamical) factor on the right. Then eq.(3.25) reduces to "time" independent algebra:

$$\{u(x^-)\overset{\otimes}{,}u(y^-)\}_D = u(x^-)\otimes u(y^-)\tilde{M}. \tag{3.31}$$

The boundary condition is derived from eq.(3.24),

$$u(x^- + L) = u(x^-)e^{-F^T/2\alpha}. \tag{3.32}$$

Our quadratic algebra, eq.(3.31), is different from that of Ref. [15]. The difference may be from the choice of the group, *i.e.* $GL(N)$ or $SL(N)$.

Similarly the algebra for the "right" chiarl field can be obtained from $\tilde{g}$ in the "right" Hamiltonian forumlation in which x^- is the "time" and x^+ is the "space" with periodic condition.

IV. Virasoro Algebra and a Hierarchy of the WZNW Model

In this section, we discuss some bracket structures among conserved currents of the system. We then obtain a hierarchy of nonabelian chiral equations and linear systems through the Virasoro algebra and involutive local conserved quantities.

It is easy to see from eq.(3.17) and eq.(3.19) that the current defined in eq.(3.27a) is conserved,

$$\partial_+ j_- = \partial_+(\tilde{g}\partial_-\tilde{g}^{-1}) = 0\ . \tag{4.1}$$

Any function of j_- is also conserved. The Sugawara construction[16] gives the

Virasoro field,

$$U \equiv \alpha Tr(j_-)^2, \tag{4.2}$$

and $\partial_+ U = 0$ follows trivially from eq.(4.1). This density U generates coordinate transformation through the Dirac bracket,

$$\{U(x^-), \tilde{g}_{\alpha\beta}(y^-)\}_D = -\partial_- \tilde{g}_{\alpha\beta} \sum_n \delta(x^- - y^- + nL) \ , \tag{4.3a}$$

and satisfies Virasoro algebra,

$$\{U(x^-), U(y^-)\}_D = 2U(x^-) \sum_n \delta'(x^- - y^- + nL) + \partial_- U(x^-) \sum_n \delta(x^- - y^- + nL). \tag{4.3b}$$

In the following, we derive a hierarchy of WZNW equation. The linear systems of the hierarchy can also be written down explicitly. The derivation here makes use of only the Kac-Moody and Virasoro algebra. It is independent of the Hamiltonian formulation.

We know how to construct the infinite number of involutive local conserved quantities[17] from the Virasoro field. Since we don't have the central term in eq.(4.3), involutive local conserved quantities, $I_n[U]$, take the following simple forms:

$$I_n[U] \ = \ \frac{1}{n} \int_0^L dx^- \ U^n \quad , \qquad (n = 1, 2, \cdots) \ , \tag{4.4}$$

and they can be shown to be involutive among themselves, i.e.

$$\{ \ I_n[U], \ I_m[U]\}_D = \ 0. \tag{4.5}$$

In the following we show that a hierarchy of nonabelian chiral equations and linear systems can be derived from the conserved quantities, eq.(4.4). We introduce new time directions, t_n, through

$$\partial_{t_n}(\tilde{g}\partial_- \tilde{g}^{-1}) \ = \{\tilde{g}\partial_- \tilde{g}^{-1}, \ I_n[U] \ \}_D \ , \tag{4.6}$$

$$= \partial_-(\tilde{g}\partial_- \tilde{g}^{-1} \ U^{n-1} \ ) \ . \qquad (n = 1, 2, \cdots) \tag{4.7}$$

The nonlinear equations,eq.(4.7), can be identified as higher nonabelian chiral equations. We identify one of times, t_1, with x^-. It is easy to write the linear systems

$$\nabla_{t_n}\psi \equiv (\ \partial_{t_n} \ + \ \tilde{g}\partial_- \tilde{g}^{-1} \ U^{n-1} \ ) \ \psi(x^-, t_2, t_3, \cdots) = 0 \ , \qquad (n = 1, 2, \cdots). \tag{4.8}$$

The higher nonabelian chiral equations, eqs.(4.7) are produced as the curvatureless conditions:

$$[\nabla_{t_1}, \nabla_{t_n}] = 0. \tag{4.9}$$

Further from Eq. (4.7), we can show that

$$[\nabla_{t_n}, \nabla_{t_m}] = 0, \qquad \text{for} \qquad m, n = 2, 3, 4 \ldots . \tag{4.10}$$

are automatically satisfied. Thus the function ψ in the linear system exists and the linear system is exactly equivalent to the nonlinear equations eq. (4.7). Note that there is no spectral parameter in these linear systems. We can incorporate the conservation equation $\partial_+(\tilde{g}\partial_- \tilde{g}^{-1}) = 0$ into the hierarchy equations by adding one more equation to the linear system,

$$\partial_+ \psi(x^-, t_2, t_3, \ldots) = 0. \tag{4.11}$$

Similarly from the "right" Kac-Moody current $j_+ = \tilde{g}^{-1}\partial_+ \tilde{g}$ we can generate the "right" Virasoro field $\bar{U} \equiv \alpha Tr(j_+)^2$ and hierarchy. The corresponding equations are just eq.(4.1) to eq.(4.10) with $\tilde{g} \leftrightarrow \tilde{g}^{-1}$, $x^+ \leftrightarrow x^-$ and $U \rightarrow \bar{U}$.

V. Summary and Discussion

First, as a simple example to illustrate some subtleties involved in the quantization procedure, we treated the free boson system stressing the first class constraint. We showed that there is the size-dependent term in the fundamental Dirac brackets eqs.(2.11) and (2.12). The size dependent terms are necessary to give the full equation of motion in Dirac's formalism. In quantization, we needed to introduce a new quantity F such that $F \approx 0$ corresponds to the first class constraint, eq.(2.8a).

Then, WZNW model is treated in a similar fashion. We showed again that the finite size effect and the existence of the first class constraint are essential. The contribution of the constraint is used to define a new field $\tilde{g}(x)$. It is in the $\tilde{g}$ field that the Kac-Moody algebra emerges.

Lastly, we derived a hierarchy of nonlinear chiral equations eq.(4.7) through the existence of involutive infinite number of conserved quantities consisting of the Virasoro field. It is interesting that linear systems of the hierarchy exist.

Note that in this paper we treated x^+ coordinate as the "time" coordinate of our system. Another choice of x^- as "time" coordinate gives us another set of Kac-Moody-Virasoro algebra, conserved quantities and hierachy. The double structure is due to the existence of double (left and right) symmetries, eq.(3.28).

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Appendix

In this appendix, we note some points to derive the fundamental Dirac bracket. We have second class constraints:

$$S_{\alpha\beta}(x^-) \equiv S^{\alpha\beta}(x^-) = \int_0^L dy^- \left(\delta(x^- - y^-) - \frac{1}{L}\right) e_{\alpha\beta}(y^-). \tag{A.1}$$

The definition of the Dirac bracket is

$$\{A_{\alpha\beta}(x^-), B_{\gamma\delta}(y^-)\}_D \equiv \{A_{\alpha\beta}(x^-), B_{\gamma\delta}(y^-)\}_P$$
$$- \int_0^L dx^- \int_0^L dw^- \{A_{\alpha\beta}(x^-), S_{\mu\nu}(z^-)\}_P \{S_{\mu\nu}(z^-), S_{\rho\sigma}(w^-)\}_P^{-1} \{S_{\rho\sigma}(w^-), B_{\gamma\delta}(y^-)\}_P. \tag{A.2}$$

Here the inverse in the second term is defined as following.

$$\int_0^L dz^- \{S_{\zeta\xi}(x^-), S_{\mu\nu}(z^-)\}_P \{S_{\mu\nu}(z^-), S_{\rho\sigma}(w^-)\}_P^{-1} = \delta(x^- - w^-) - \frac{1}{L} \tag{A.3}$$

Usually the right hand side is just δ function. But $\{S_{\zeta\xi}, S_{\mu\nu}\}$ has no inverse in the usual sense because $S_{\alpha\beta}$ is projected on "the zero average function space", see eq.(A.1). Our alternative, eq.(A.3), is enough to define the Dirac bracket so that the constraint $S_{\alpha\beta}(x^-)$ has zero Dirac bracket with any fields.

We introduce a transpose T_c to use tensor notation. The definition of T_c is

$$\begin{aligned} \{S(x^-) \overset{\otimes}{,} S(y^-)\}^{T_c}_{\alpha\gamma,\beta\epsilon} &\equiv \{S(x^-) \overset{\otimes}{,} S(y^-)\}_{\alpha\beta,\gamma\epsilon} \\ &= \{S_{\alpha\gamma}(x^-), S_{\beta\epsilon}(y^-)\} \end{aligned} \tag{A.4}$$

The Dirac bracket, eq.(A.2) can written in the tensor notation:

$$\{A(x^-) \overset{\otimes}{,} B(y^-)\}_D \equiv \{A(x^-) \overset{\otimes}{,} B(y^-)\}_P$$
$$- \int_0^L dz^- \int_0^L dw^- \left[\{A(x^-) \overset{\otimes}{,} S(z^-)\}^{T_c}_P \left(\{S(z^-) \overset{\otimes}{,} S(w^-)\}^{T_c}_P\right)^{-1} \{S(w^-) \overset{\otimes}{,} B(y^-)\}^{T_c}_P\right]^{T_c}. \tag{A.5}$$

We used the transpose of S to obtain the bracket for the fundamental field $g(x^-)$:

$$\{g(x^-) \overset{\otimes}{,} g(y^-)\}_D = \{g(x^-) \overset{\otimes}{,} g(y^-)\}_P$$
$$- \int_0^L dz^- \int_0^L dw^- \left[\{g(x^-) \overset{\otimes}{,} S^T(z^-)\}^{T_c}_P (\{S^T(z^-) \overset{\otimes}{,} S^T(w^-)\}^{T_c}_P)^{-1}\right.$$
$$\left.\{S^T(w^-) \overset{\otimes}{,} g(y^-)\}^{T_c}_P\right]^{T_c} \tag{A.6}$$
$$= g(x^-) \otimes g(y^-) M, \tag{A.7}$$

where

$$M = -\frac{P}{2\alpha}\left(\frac{1}{2}\sum_n \varepsilon(x^- - y^- + nL) e^{(x^- - y^- + nL)\frac{Q}{L}} + \frac{1}{Q}\right), \text{ and} \tag{A.8}$$

$$Q = \frac{1}{2\alpha}\left(F^T \otimes 1 - 1 \otimes F^T\right). \tag{A.9}$$

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