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Time in 2+1 Dimensional Quantum Gravity*

S. Carlip

Department of Physics, University of California
Davis, CA 95616, U.S.A.

ABSTRACT

By examining the exact quantization of general relativity in 2+1 dimensions, we can investigate the nature of time in quantum gravity, while at the same time avoiding the difficult technical problems of 3+1 dimensions. It is shown that a manifestly gauge-invariant, time-independent quantization is possible, and is exactly equivalent — at least for simple spatial topologies — to a gauge-fixed quantization with an explicit choice of time. In particular, Hilbert space norms and inner products can be defined without any reference to time, and operators that commute with the super-Hamiltonian nevertheless permit a full dynamical description of 2+1 dimensional gravity. General relativity may thus need no fundamental revision in order to solve the "problem of time."

1. Introduction

Elsewhere in these proceedings, Unruh[1] has described some of the difficulties associated with the role of time in quantum gravity. The "problem of time" is a long-standing one, arising from the peculiar nature of time in general relativity.[2-3] Time is a coordinate, whose choice is largely arbitrary. In the classical theory, this arbitrariness is reflected in the fact that the solutions of the equations of motion do not depend on the choice of time-slicing; the diffeomorphisms which change the definition of time are generated by a constraint. In conventional canonical quanization, however, physical observables must commute with all constraints. For gravity, this means that observables must be time-independent, and it is not easy to see how such operators can describe dynamics. Moreover, time plays a special role in quantum theory — wave functions must be normalized so that total probability *at a given time* is unity — and it is difficult to even formulate such a condition in quantum gravity.

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Until now, attempts to investigate these problems have been stymied by the serious technical difficulties in formulating any quantum theory of gravity. General relativity is perturbatively nonrenormalizable, and conventional field theoretical methods fail, but we have few alternatives available. In the past few years, however, it has become apparent that much can be learned by working in 2+1 dimensions. In two spatial dimensions, general relativity becomes much simpler, and the constraint equations can be solved. The resulting physical phase space is finite dimensional, reducing the task of quantization to one of quantum mechanics rather than field theory. The problem of nonrenormalizability disappears, but many of the conceptual issues — including the problem of time — remain, now in a context in which they can be systematically explored.[4]

Two approaches to 2+1 dimensional quantum gravity have been proposed. The first, due to Witten,[5] is based on the first-order (Palatini) form of the Einstein action. It involves no explicit gauge-fixing; the theory is constructed from manifestly invariant holonomies of flat connections, with no choice of time-slicing. The procedure for quantization is clear and unambiguous, but the interpretation of the resulting gauge-invariant, time-independent operators and states is obscure.

The second approach, based on work of Moncrief[6] and Hosoya and Nakao,[7] uses a particular, rather arbitrary gauge-fixing procedure based on York's "extrinsic time,"[8] in which the trace of the extrinsic curvature is used as a time variable. Amplitudes are defined in only this gauge; the Hamiltonian is nonzero and complicated, but the resulting picture of time evolution is easy to interpret. Our goal is to try to understand time in quantum gravity by comparing this approach and Witten's.

2. ADM-York-Moncrief-Hosoya-Nakao Quantization

Let us begin by reviewing the gauge-fixed quantization of Moncrief, Hosoya and Nakao. We take spacetime to have the topology $M = \mathbb{R}\times\Sigma$, where Σ is a compact surface of genus h. In standard Arnowitt-Deser-Misner variables, the metric on M is

$$ds^2 = N^2dt^2 - g_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \quad . \tag{2.1}$$

The ordinary Einstein action, in Hamiltonian form, is then[9]

$$S = \int d^3x (-{}^{(3)}g)^{1/2}\, {}^{(3)}R = \int dt \int_\Sigma d^2x (\pi^{ij}\dot{g}_{ij} - N^i\mathcal{H}_i - N\mathcal{H}) \quad , \tag{2.2}$$

where the momentum conjugate to g_{ij} is $\pi^{ij} = \sqrt{g}\,(K^{ij} - g^{ij}K)$, with K^{ij} the extrinsic curvature of the surface $t = \text{const.}$, and

$$\mathcal{H}_i = -2\nabla_j \pi^j{}_i \quad , \qquad \mathcal{H} = \frac{1}{\sqrt{g}} g_{ij}g_{kl}(\pi^{ik}\pi^{jl} - \pi^{ij}\pi^{kl}) - \sqrt{g}R \tag{2.3}$$

are the supermomentum and super-Hamiltonian constraints.

Moncrief, Hosoya, and Nakao now partially gauge-fix the action by choosing

$$t = \tau = g^{-1/2} g_{ij} \pi^{ij} \quad . \tag{2.4}$$

τ is York's "extrinsic time," the mean curvature (i.e., the trace of the extrinsic curvature). To solve the Hamiltonian constraint $\mathcal{H} = 0$ in this gauge, we observe that any metric on a surface Σ is uniquely conformal to one of constant (intrinsic) curvature $k = 0$ or ± 1, where the value of k depends only on the topology of Σ. Let $\tilde{g}_{ij}$ be such a constant curvature metric, and write

$$g_{ij} = e^{2\lambda} \tilde{g}_{ij} \quad . \tag{2.5}$$

The constraint $\mathcal{H} = 0$ is then a differential equation for λ,

$$\Delta_{\tilde{g}} \lambda - \frac{1}{4} \tau^2 e^{2\lambda} + \frac{1}{2} \left[\tilde{g}^{-1} \tilde{g}_{ij} \tilde{g}_{kl} \tilde{\pi}^{ik} \tilde{\pi}^{jl} \right] e^{-2\lambda} - \frac{k}{2} = 0 \quad , \tag{2.6}$$

where $\Delta_{\tilde{g}}$ is the Laplacian for the metric $\tilde{g}_{ij}$ and

$$\tilde{\pi}^{ij} = \pi^{ij} - \frac{1}{2} g^{ij} g_{kl} \pi^{kl} \quad . \tag{2.7}$$

Moncrief has shown that this equation has a unique solution, determining the conformal factor λ as a function of $\tilde{g}$ and $\tilde{\pi}$.

The momentum constraints $\mathcal{H}_i = 0$, which generate diffeomorphisms of Σ, can also be simplified; in terms of $\tilde{g}$ and $\tilde{\pi}$, they become

$$\tilde{\nabla}_j \tilde{\pi}^{ij} = 0 \quad . \tag{2.8}$$

A point in the physical phase space is thus specified by a constant curvature metric $\tilde{g}$ and a transverse traceless tensor $\tilde{\pi}$ — in complex coordinates, a complex structure and a quadratic differential — modulo the diffeomorphisms generated by the constraints. This description gives the phase space a structure that has been studied extensively by mathematicians[10] and string theorists.[11] The constant curvature metrics parametrize the (Riemann) moduli space $\mathcal{M}$ of Σ, and the transverse traceless tensors are precisely the cotangent vectors to $\mathcal{M}$. The phase space is thus the cotangent bundle $T^*\mathcal{M}$, a space whose properties are well understood.

Although the super-Hamiltonian $\mathcal{H}$ vanishes on the physical phase space, the dynamics is not trivial. Moncrief has shown that there is an effective Hamiltonian

$$H = \int_\Sigma d^2x \, \sqrt{\tilde{g}} \, e^{2\lambda(m,p,\tau)} \tag{2.9}$$

with λ fixed by (2.6). The vanishing of the super-Hamiltonian reflects the invariance of the theory under redefinitions of time. But when we fix the time-slicing (2.4), we break this invariance; the Hamiltonian (2.9) describes the evolution *in this gauge.*

We now have a classical system with a finite dimensional phase space that has a cotangent bundle structure. In principle, such a system is easy to quantize. We take as the Hilbert space the space $L^2(\mathcal{M})$ of square integrable functions on moduli space; the basic operators are functions on $\mathcal{M}$, which act on the Hilbert space by multiplication, and cotangent vectors, which act by differentiation. The Hamiltonian (2.9) will in general be a complicated time-dependent function of these coordinates and momenta, and we may face difficult operator ordering problems, but we should be able to apply ordinary perturbation theory, for instance, to calculate amplitudes.

To make this construction less abstract, let us consider the simplest nontrivial topology for Σ, that of a torus T^2. Every metric g_{ij} on a torus is conformal to a flat one; the curvature k introduced above is zero. This flat metric is not unique, however, but rather depends on a complex number m, the modulus. A flat torus can be represented as a parallelogram with opposite sides identified (figure 1), and a conformal transformation can further normalize one side to have length 1. The modulus m then fixes the position of a vertex, uniquely determining the parallelogram.

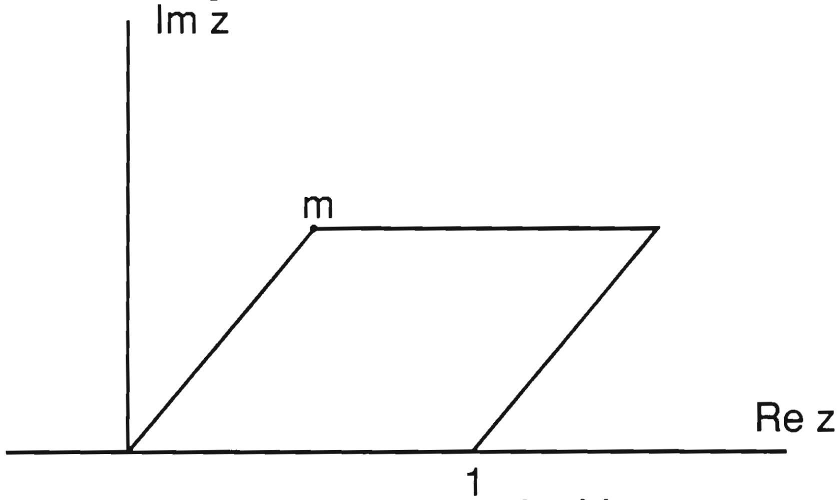

Figure 1. A flat torus of modulus m

In this coordinate system, the transverse traceless tensors $\tilde{\pi}$ are constant as well, so the Hamiltonian (2.9) becomes fairly simple (this is no longer the case for more complicated topologies). Hosoya and Nakao have shown that the trajectories of the modulus m determined by this Hamiltonian are semicircles in the upper half plane centered on the real axis, or, equivalently, geodesics for the Poincaré (constant negative curvature) metric on the upper half plane.*

There is one additional subtlety that must still be taken into account. Not every parallelogram gives a distinct torus; some are equivalent under elements of the "mapping class group," the group of diffeomorphisms that cannot be smoothly deformed

* This constant negative curvature metric is actually the natural (Weil-Petersson) metric for the mathematical description of the moduli space of a torus; see, for example, reference 11.

to the identity. A typical element of the mapping class group is a Dehn twist, a diffeomorphism performed by cutting a handle open into a cylinder, twisting one end by 360°, and gluing the ends back together (figure 2).

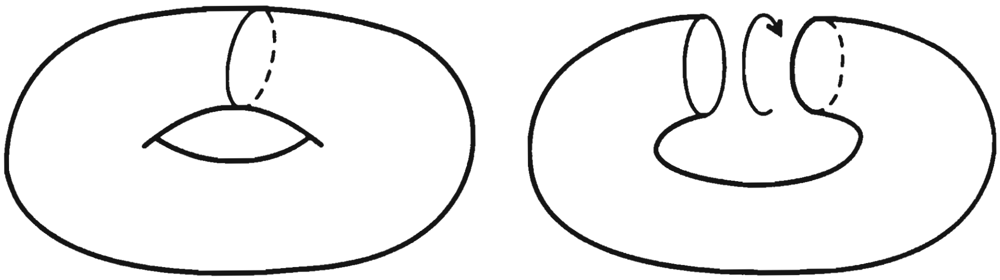

Figure 2. A Dehn twist of a torus

For the torus, there are two generators of the mapping class group, corresponding to the two generators of the fundamental group $\pi_1(T^2)$ or the two independent circumferences. Their action can be shown to be

$$S : m \to -\frac{1}{m} \ , \qquad T : m \to m+1 \ . \tag{2.10}$$

A fundamental region — a region that includes only one copy of each distinct torus — is given by $-\frac{1}{2} \leq m_1 \leq \frac{1}{2}$, $|m| \geq 1$.

The Hilbert space thus consists of functions of m invariant under (2.10). Wave functions must be square integrable, but the region of integration is not the entire half plane, but only a single fundamental region. The basic operators are (the real and imaginary parts of) m and $i\frac{d}{dm}$, and the Hamiltonian can be calculated to be

$$H = \frac{m_2}{\tau}\left[{p_1}^2 + {p_2}^2\right]^{1/2} \quad , \tag{2.11}$$

giving the circular motion described by Hosoya and Nakao in the classical theory.

3. Witten's Quantization

Witten takes a very different approach to the quantization of 2+1 dimensional gravity. He starts with the first-order form of the action (2.2), treating the local frame ${e^a}_\mu$ and the spin connection $\omega_{a\mu} = \frac{1}{2}\epsilon_{abc}{\omega_\mu}^{bc}$ as independent variables. The resulting structure is quite similar to Ashtekar's formulation of 3+1 dimensional general relativity.[12] The action in first-order form is

$$\begin{aligned} S &= \int d^3x\, \epsilon^{\rho\mu\nu} {e^a}_\rho \left(\partial_\mu \omega_{a\nu} - \partial_\nu \omega_{a\mu} + \epsilon_{abc} {\omega^b}_\mu {\omega^c}_\nu\right) \\ &= 2\int dt \int_\Sigma d^2x \left(-\epsilon^{ij} {e^a}_i \dot{\omega}_{aj} + {e^a}_0 \tilde{\Theta}_a + \omega_{a0}\Theta^a\right) \end{aligned} \tag{3.1}$$

with constraints

$$
\begin{aligned}
\Theta^a &= \frac{1}{2}\epsilon^{ij}\left(\partial_i e^a{}_j - \partial_j e^a{}_i + \epsilon^{abc}(\omega_{bi}e_{cj} - \omega_{ci}e_{bj})\right) \\
\tilde{\Theta}^a &= \frac{1}{2}\epsilon^{ij}\left(\partial_i \omega^a{}_j - \partial_j \omega^a{}_i + \epsilon^{abc}\omega_{bi}\omega_{cj}\right) \ .
\end{aligned}
\tag{3.2}
$$

Witten then observes that these constraints generate the Lie algebra ISO$(2,1)$, the 2+1 dimensional Poincaré group; moreover, the frame $e^a{}_i$ and the spin connection ω_{ai} together constitute an ISO$(2,1)$ connection on Σ. The conditions $\tilde{\Theta}^a = \Theta^a = 0$ restrict this connection to be flat, while at the same time the constraints generate local gauge transformations, requiring us to identify gauge-equivalent connections. The physical phase space is thus the space of flat ISO$(2,1)$ connections on Σ modulo gauge transformations.

To understand this phase space, let us recall some of the properties of flat connections. A connection describes parallel transport, and the curvature of a small region is characterized by the change of a vector parallel transported around that region. We might therefore expect a flat connection — one with curvature zero — to necessarily be trivial. This is indeed the case if space has the topology of a sphere or a plane. But if the topology is more complicated, there can be loops that do not surround regions of space (think of the circumferences of a torus), and parallel transport around these loops need not be trivial. In fact, a flat connection is completely characterized by its holonomies around these noncontractible loops.

In particular, a flat ISO$(2,1)$ connection is determined by a group homomorphism $\pi_1(\Sigma,*) \rightarrow \mathrm{ISO}(2,1)$, where $\pi_1(\Sigma,*)$ is the fundamental group of Σ ($*$ represents an arbitrary basepoint). A gauge transformation $\mathcal{G}(x)$ has the effect of conjugating all of the holonomies by $\mathcal{G}(*)\in\mathrm{ISO}(2,1)$. Our physical phase space can thus be characterized as the space of homomorphisms from $\pi_1(\Sigma,*)$ to ISO$(2,1)$ modulo conjugation. Like Moncrief's phase space, this space is a cotangent bundle, whose base space is now the space $\mathcal{N}$ of flat SO$(2,1)$ connections. As a space of flat connections, $\mathcal{N}$ can itself be characterized as a space of homomorphisms, now from $\pi_1(\Sigma,*)$ to SO$(2,1)$, modulo conjugation.

As we saw in the last section, a further subtlety arises from the existence of diffeomorphisms not deformable to the identity. Their effect can be understood as follows. To describe a homomorphism from $\pi_1(\Sigma,*)$ to ISO$(2,1)$, we must choose a set of $2h$ generators for the fundamental group of Σ (for the torus, two circumferences), and give their images in ISO$(2,1)$. Such a choice is not diffeomorphism-invariant, however: a Dehn twist will mix the generators (think of the two circumferences in figure 2), while leaving the group they generate unchanged. A Dehn twist thus acts as an automorphism of the fundamental group. In fact, the mapping class group $\mathcal{D}$ can be shown[13] to give precisely the outer automorphisms of $\pi_1(\Sigma,*)$. To obtain the true physical phase space, we must demand invariance under this action of $\mathcal{D}$.

As before, our classical phase space is a finite dimensional cotangent bundle, and we know how to write down the corresponding quantum theory. The Hilbert space is the space of $\mathcal{D}$-invariant L^2 functions on $\mathcal{N}$; the fundamental operators are a set of independent $\mathrm{SO}(2,1)$ holonomies, which act by multiplication, and their conjugate momenta, which act by differentiation. Note, however, that nowhere in this approach have we selected a time coordinate or gauge-fixed the action. States and operators are manifestly invariant under diffeomorphisms, including time translations; the effective Hamiltonian analogous to (2.9) is zero.

4. Classical Equivalence for the Torus

We now have two very different descriptions of 2+1 dimensional quantum gravity. In one, the meaning of time is reasonably clear — we know the classical trajectories, and the quantum Hamiltonian is nonzero — but our understanding has been gained at the expense of an arbitrary gauge choice and a complicated formalism. In the other, the theory is simple and manifestly invariant, but the meaning of time is obscure.

To compare these formulations, let us start by examining their classical relationship. Consider first the simplest case, for which the spatial topology is that of a torus T^2. Our first task is to understand the classical solutions in the Witten formalism.

The fundamental group of the torus is the Abelian group $\mathbb{Z} \oplus \mathbb{Z}$, so Witten's phase space is parametrized by two commuting elements of $\mathrm{ISO}(2,1)$ — i.e., a pair of commuting Poincaré transformations — up to conjugation. The space of all such pairs can be shown to have several distinct connected components; the relevant component for gravity is that for which the $\mathrm{SO}(2,1)$ piece of each transformation is a boost.* For the holonomies to commute, the two boosts must be parallel, and we can use our overall freedom of conjugation to put the transformations in the form

$$\begin{aligned} \Lambda_1 &: (t,x,y) \rightarrow (t\cosh\lambda + x\sinh\lambda,\ x\cosh\lambda + t\sinh\lambda,\ y+a) \\ \Lambda_2 &: (t,x,y) \rightarrow (t\cosh\mu + x\sinh\mu,\ x\cosh\mu + t\sinh\mu,\ y+b) \ . \end{aligned} \tag{4.1}$$

Now, Λ_1 and Λ_2 generate an isometry group $H \subset \mathrm{ISO}(2,1)$ of the Minkowski metric. A natural way to construct a spacetime from such a group is to look for a region $\mathcal{F}$ of Minkowski space on which H acts nicely (i.e., properly discontinuously), and to form the quotient $M = \mathcal{F}/H$. This amounts to using H to glue together a

$\star$ Another relevant component consists of a pair of pure translations, with no $\mathrm{SO}(2,1)$ elements. This component is only two dimensional, however, while the component we focus on here is four dimensional. There are also components involving rotations instead of boosts, but it is believed that the mapping class group does not act nicely on these; the quotient $\mathcal{N}/\mathcal{D}$ is probably not even Hausdorff, and there appear to be no nontrivial operators invariant under $\mathcal{D}$. See reference 4 for a further discussion of this issue.

flat patch of $\mathbb{R}^3$ to form a space with the topology $\mathbb{R}\times T^2$, taking elements of H as transition functions along the identified edges. It is not hard to show that this glued manifold inherits a metric and connection from Minkowski space with precisely the holonomies (4.1) used to define the gluing.

To see how this works in detail, define new Minkowski space coordinates

$$t = \frac{1}{\tau}\cosh u \quad , \qquad x = \frac{1}{\tau}\sinh u \quad . \tag{4.2}$$

The surfaces of constant τ can be shown to have mean curvature τ, so τ is the York time, and the transformations (4.1) become simply

$$\begin{aligned} \Lambda_1 &: (\tau, u, y) \rightarrow (\tau, u+\lambda, y+a) \\ \Lambda_2 &: (\tau, u, y) \rightarrow (\tau, u+\mu, y+b) \quad . \end{aligned} \tag{4.3}$$

Hence on each surface of constant τ, the fundamental region $\mathcal{F}$ is the torus $(u, y) \sim (u+\lambda, y+a) \sim (u+\mu, y+b)$.

We can make each of these tori look like figure 1 by defining new spatial coordinates (at fixed τ)

$$x' = \left(a^2 + \frac{\lambda^2}{\tau^2}\right)^{-1}\left(ay + \frac{\lambda}{\tau^2}u\right), \qquad y' = \left(a^2 + \frac{\lambda^2}{\tau^2}\right)^{-1}\left(\frac{\lambda y - au}{\tau}\right) \quad . \tag{4.4}$$

The spatial metric is then

$$d\sigma^2 = (a^2 + \frac{\lambda^2}{\tau^2})(dx'^2 + dy'^2) \quad , \tag{4.5}$$

which is periodic under the shifts

$$\begin{aligned} &(x', y') \rightarrow (x'+1,\, y') \\ &(x', y') \rightarrow (x' + \left(a^2 + \frac{\lambda^2}{\tau^2}\right)^{-1}\left(ab + \frac{\lambda\mu}{\tau^2}\right),\; y' + \left(a^2 + \frac{\lambda^2}{\tau^2}\right)^{-1}\frac{a\mu - \lambda b}{\tau}) \quad . \end{aligned} \tag{4.6}$$

From the definition of the modulus m (see figure 1), this periodicity implies that $d\sigma^2$ is conformal to the metric of a torus with

$$m = \left(a + \frac{i\lambda}{\tau}\right)^{-1}\left(b + \frac{i\mu}{\tau}\right) \quad . \tag{4.7}$$

We can also construct the momentum conjugate to m, using the Poisson brackets (derived from the action (3.1))

$$\{a,\mu\} = \{\lambda, b\} = \frac{1}{2} \quad . \tag{4.8}$$

We find that

$$p = -i\tau \left(a - \frac{i\lambda}{\tau}\right)^2 \quad . \tag{4.9}$$

These quantities have precisely the properties we need to identify them with the moduli and conjugate momenta in the Moncrief-Hosoya-Nakao description. In particular, it is easy to check that

$$\left(m_1 - \frac{1}{2}\left(\frac{\mu}{\lambda} + \frac{b}{a}\right)\right)^2 + {m_2}^2 = \frac{1}{4}\left(\frac{\mu}{\lambda} - \frac{b}{a}\right)^2 \quad , \tag{4.10}$$

so the τ evolution of m is a semicircle centered on the real axis. Further, under the action of the mapping class group on the holonomies,

$$\begin{aligned} S &: (a,\lambda) \rightarrow (b,\mu), \quad (b,\mu) \rightarrow (-a,-\lambda) \\ T &: (a,\lambda) \rightarrow (a,\lambda), \quad (b,\mu) \rightarrow (b+a,\mu+\lambda) \quad , \end{aligned} \tag{4.11}$$

m transforms according to (2.10). Witten's coordinates and momenta $\{a, b, \lambda, \mu\}$ thus parametrize the Moncrief-Hosoya-Nakao phase space. In fact, the two descriptions are equivalent under a (time-dependent) canonical transformation,

$$p_1 dm_1 + p_2 dm_2 = 2a d\mu - 2b d\lambda + H d\tau + dF \quad , \tag{4.12}$$

where

$$H = \frac{a\mu - \lambda b}{\tau} \tag{4.13}$$

is Moncrief's Hamiltonian (2.9) for the torus, and

$$F(m_1, m_2, \mu, \lambda) = -\frac{1}{m_2 \tau}\left[(\mu - m_1\lambda)^2 + {m_2}^2\lambda^2\right] \quad . \tag{4.14}$$

The passage from Moncrief's to Witten's variables is thus a standard procedure in classical mechanics, that of solving the equations of motion by finding an appropriate transformation to time-independent coordinates and momenta.

While we have considered only the torus, the same construction can be applied to arbitrary topologies. The holonomies of any flat $\mathrm{ISO}(2,1)$ connection define a group of isometries of the Minkowski metric, and this group can be used to glue together flat Minkowski patches to form an Einstein spacetime with the appropriate topology. This procedure is a particular case of a more general construction that Thurston[14–15] calls a "geometric structure." The general properties of such structures have been the subject of considerable work by mathematicians in the past few years. In particular, Mess[16] has shown that every maximal solution of the 2+1 dimensional Einstein equations corresponds uniquely to an $\mathrm{ISO}(2,1)$ isometry group and an associated flat connection, thus demonstrating the classical equivalence of our two approaches to 2+1 dimensional gravity.[17]

5. Quantum Equivalence for the Torus

We must still ask whether this relationship between classical formulations extends to the corresponding quantum theories. Two steps are necessary to demonstrate such an equivalence: the Hilbert spaces must be shown to be the same, and the (dynamical) operators of one approach must be identified with those of the other.

Most of the first step has already been carried out in the mathematical literature.[18–20] Although the space $\mathcal{N}$ of flat $\mathrm{SO}(2,1)$ connections on Σ seems quite different from the Riemann moduli space $\mathcal{M}$, the two are actually closely related. The detailed proof is fairly complicated, but the basic idea is simple, arising again from the concept of a geometric structure. Just as a flat $\mathrm{ISO}(2,1)$ connection determines an isometry group of the Minkowski metric on $\mathbb{R}^3$, so the holonomies of a flat $\mathrm{SO}(2,1)\simeq\mathrm{SL}(2,\mathbb{R})$ connection ω generate an isometry group of the Poincaré metric on the hyperbolic plane $\mathbb{H}^2$. This isometry group can be used to glue the edges of a region of $\mathbb{H}^2$ to form a constant negative curvature surface; this surface, in turn, represents a point in $\mathcal{M}$. The construction is closely analogous to the approach to $\mathbb{R}\times T^2$ discussed in the last section. In particular, the resulting surface may be viewed as the quotient of the hyperbolic plane by the holonomy group of ω. Conversely, the uniformization theorem of Riemann surface theory[10] guarantees that every constant negative curvature surface can be generated in this fashion. While there are subtleties related to the topologies of $\mathcal{M}$ and $\mathcal{N}$ and the action of the mapping class group, with a little care the equivalence of the Moncrief-Hosoya-Nakao and Witten Hilbert spaces can be rigorously demonstrated.

Showing the dynamical equivalence of the two quantum theories is a much harder task. For spaces of genus greater than one, the Moncrief-Hosoya-Nakao Hamiltonian is extremely complicated, and a detailed description of the dynamics has not yet been found. For the torus, however, the problem is much simpler, and an exact comparison is possible: we can explicitly construct operators on Witten's Hilbert space to represent the Moncrief-Hosoya-Nakao moduli.

Witten's Hilbert space for $M = \mathbb{R} \times T^2$ is characterized by four fundamental self-adjoint operators $\hat{a}$, $\hat{b}$, $\hat{\lambda}$, and $\hat{\mu}$, with commutators (see (4.8))

$$[\hat{a}, \hat{\mu}] = [\hat{\lambda}, \hat{b}] = \frac{i}{2} \quad . \tag{5.1}$$

To construct the fundamental operators in the Moncrief-Hosoya-Nakao picture, we start with the classical relations (4.7) and (4.9), and define a family of operators

$$\hat{m} = \left(\hat{a} + \frac{i\hat{\lambda}}{\tau}\right)^{-1} \left(\hat{b} + \frac{i\hat{\mu}}{\tau}\right), \qquad \hat{p} = -i\tau \left(\hat{a} - \frac{i\hat{\lambda}}{\tau}\right)^2 \quad , \tag{5.2}$$

where for now τ is simply an arbitrary parameter.* The operator ordering in (5.2) has been fixed by the requirement of modular invariance; that is, with this ordering the transformations (4.11) in Witten's quantization reproduce the transformations (2.10) of $\hat{m}$ and $\hat{p}$. Similarly, from (4.13) we can define a Hamiltonian operator

$$\hat{H} = \frac{\hat{a}\hat{\mu} - \hat{\lambda}\hat{b}}{\tau} \quad . \tag{5.3}$$

It is then not hard to show that

$$[\hat{m}_1, \hat{p}_1] = [\hat{m}_2, \hat{p}_2] = i \quad , \tag{5.4}$$

$$i\frac{d\hat{m}}{d\tau} = [\hat{m}, \hat{H}], \qquad i\frac{d\hat{p}}{d\tau} = [\hat{p}, \hat{H}] \quad . \tag{5.5}$$

The moduli $\hat{m}$ and their conjugate momenta $\hat{p}$ thus obey the correct Heisenberg equations of motion, have the proper commutators, and transform correctly under the mapping class group. In short, they satisfy exactly the requirements one would impose on the corresponding operators in the Moncrief-Hosoya-Nakao picture. Hence, for instance, if we simultaneously diagonalize $\hat{m}_1$ and $\hat{m}_2$, the resulting wave functions will obey the proper Schrödinger equation and have the right inner products. We can actually find these eigenfunctions; the state with eigenvalues $m = m_1 + im_2$ is

$$\psi(\mu, \lambda) = \text{const.}\,(\mu - m\lambda) \exp\left\{\frac{1}{im_2\tau}(\mu - m\lambda)(\mu - \bar{m}\lambda)\right\} \quad . \tag{5.6}$$

We have thus built the Moncrief-Hosoya-Nakao dynamics on the Witten Hilbert space. This construction has a rather odd feature, however. The York time variable τ

$\star$ $\hat{m}$ and $\hat{p}$ are not Hermitian, but they should not be, since m and p are not real; the true observables are the Hermitian and anti-Hermitian pieces $\hat{m}_1$, $\hat{m}_2$, $\hat{p}_1$, and $\hat{p}_2$.

has been introduced as an arbitrary parameter; it has seemingly appeared out of thin air in Witten's quantization. This should not be too surprising, however. To describe time evolution in general relativity, one must choose an arbitrary time-slicing and pick a parameter to label the slices. This choice in itself has no physical content. The interpretation of τ as a time variable comes from the geometry, or, in the quantum theory, from the correspondence principle — it is only after we know the geometric meaning of the Witten variables λ, μ, a, and b that we can interpret m and τ in terms of a constant mean curvature slicing. Once we understand the interpretation of the holonomies in terms of geometric structures, however, the meaning of the parameter τ is unambiguous. This point of view is quite close to that of Rovelli,[3] who has argued that any quantum theory can be defined in terms of variables that are constant on each classical trajectory, without any explicit reference to time.

There is another, equivalent way to understand the parameter τ. The choice of time-slicing in quantum gravity is a kind of gauge-fixing,[21] and correlation functions containing time-dependent operators are gauge-fixed quantities. But an object defined in a particular gauge can always be "invariantized" and viewed as a (usually nonlocal) gauge-invariant quantity.[22] In the Moncrief-Hosoya-Nakao quantization, the York time-slicing was presented as a gauge choice. But this slicing has an intrinsic geometric meaning. Given a classical spacetime M (with compact spatial topology) and a number τ, we can define the surface $\Sigma(\tau)$ to be the unique spatial slice of constant mean curvature τ. Similarly, we can define the Moncrief moduli $m(\tau)$ to be the moduli of the unique constant curvature surface conformal to $\Sigma(\tau)$, and the Moncrief Hamiltonian $H(\tau)$ to be the area of $\Sigma(\tau)$. We then have several one-parameter families of observables defined in a completely diffeomorphism-invariant fashion, which nevertheless depend on a parameter τ that plays the role of York time.

It can be shown[4] that these moduli obey the correct classical equations of motion, and that up to operator-ordering ambiguities, the corresponding operators satisfy the Heisenberg equations of motion (5.5). Since the holonomies of a flat ISO(2,1) connection completely determine the classical spacetime geometry, and therefore $m(\tau)$ and $H(\tau)$, it is not surprising that the corresponding quantum operators are closely related. If we had picked a different time-slicing, a similar parameter τ' could have been introduced. "Time"-dependent operators would again appear, but they would have a different τ' dependence and describe different geometric objects.

6. Remaining Questions

We have seen that the time-independent, manifestly invariant quantization of $2+1$ dimensional gravity suggested by Witten is equivalent to the gauge-fixed, time-dependent quantization of Moncrief, Hosoya, and Nakao. The Hilbert space structure, including the correct inner product and norm, can therefore be constructed without any explicit choice of time. Moreover, even though the Hamiltonian vanishes in

Witten's quantization, we can find "time"-dependent operators that answer all of the dynamical questions that can be asked in the explicitly time-dependent formulation.

Some important questions still remain. We have looked at one particular choice of time, York's constant mean curvature slicing. While it seems likely that other choices are equivalent, this has not been proven. As Unruh has emphasized, this problem can involve difficult questions of quantum measurement theory: one must ensure that changing the slicing does not change the temporal order of measurements.

We have also taken a particular approach to the quantization of constrained systems. In any such system, one may either first solve the constraints and then quantize — our procedure here — or first quantize and then impose the constraints as operators acting on the states. These two choices do not always give the same quantum theories,[23] and we do not know which is physically correct for gravity. For Witten's quantization, it can be shown that the choice does not matter, but this is a special feature, arising from the fact that the constraints are only first order in the momenta; in the Moncrief-Hosoya-Nakao variables, the two approaches may lead to different quantum theories.

Finally, of course, we must ask about the generalization to real gravity in $3+1$ dimensions. I have fairly little to say about this. Two necessary ingredients would be solving the constraints (perhaps in the new Capovilla-Dell-Jacobson formulation[24]) and developing an understanding of the physical meaning of diffeomorphism-invariant quantities (some generalization of the notion of geometric structure). But while the difficulties of $3+1$ dimensions are great, the $2+1$ dimensional model indicates that they are in some sense technical; the problem of time quantum gravity need not require a fundamental reformulation of general relativity or quantum theory.

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