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Decoherence of Multimode Thermal Squeezed Coherent States *

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Abstract

It is well known that any multimode positive definite quadratic Hamiltonian can be transformed into a Hamiltonian of uncoupled harmonic oscillators. Based on this theorem, the multimode thermal squeezed coherent states are constructed in terms of density operators. Decoherence of multimode thermal squeezed coherent states is investigated via the characteristic function and it is shown that the decohered (reduced) states are still thermal squeezed coherent states in general.

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Disclaimer

1 Introduction

In the past three decades, decoherence (reduction) has become more and more widely recognized, being studied in many fields of physics from non–equilibrium statistical mechanics [1] to quantum measurement and quantum cosmology [2, 3]. The purpose of this paper is to investigate decoherence of multimode thermal Squeezed Coherent States (SqCS's).

In the literature there are many equivalent definitions for one–mode, two–mode [4, 5, 6, 7] and multimode SqCS's [8]. However, thermal SqCS's are usually defined for one–mode [9]. (Two–mode thermal SqCS in thermo–field formalism is effectively one–mode.) Therefore, in this paper we first introduce a general definition of the multimode thermal SqCS's in terms of density operators. Then we will discuss two related representations—the Wigner function and the characteristic function—and will show that the latter is the better representation for decoherence problems. Finally we will use the characteristic function to study the decoherence of multimode thermal SqCS's.

This paper is organized as follows: In Sec. 2 notations, conventions and a lemma on matrix are introduced for the mathematics used in this paper. In Sec. 3 a unified definition of multimode SqCS's with the aid of a special kind of Hamiltonian is presented. In Sec. 4 the multimode thermal SqCS is constructed by thermalizing the multimode SqCS defined in Sec. 3. In Sec. 5 we calculate the Wigner functions and the characteristic functions of some multimode thermal SqCS's. In Sec. 6 the decoherence of multimode thermal SqCS is effectuated and it is shown that the decohered state is still a thermal SqCS.

2 Mathematical Preliminaries

Throughout this paper, $\hbar$ is set equal to 1; "$\dagger$" denotes hermitian conjugate and "t" denotes the transpose of a matrix. The physical system under consideration is of n degrees of freedom, hence the dummy indices run from 1 to n unless otherwise specified.

We use $\vec{x} =< x_1, x_2, \ldots, x_n >$ and $\vec{k} =< k_1, k_2, \ldots, k_n >$ for the n–dimensional canonical coordinate and momentum respectively. Thus $< \vec{x}; \vec{k} >$ is a vector in

$2n$–dimensional phase space. $\vec{q}$ and $\vec{p}$ denote the n-dimensional position and momentum operators corresponding to the canonical variables $\vec{x}$ and $\vec{k}$. The Canonical Commutation Relations (CCR's) are:

$$[\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0, \qquad [\hat{q}_i, \hat{p}_j] = i\delta_{ij}. \tag{1}$$

$|0\rangle$ denotes the n–mode Fock vacuum state, i.e., the ground state of an n–dimensional harmonic oscillator with unit mass and frequency:

$$\langle \vec{x} | 0 \rangle = \pi^{-\frac{n}{4}} \exp[-\frac{1}{2}(\vec{x})^2]. \tag{2}$$

The number operators are defined in the ordinary way:

$$\hat{N}_i = \frac{1}{2}(\hat{p}_i^2 + \hat{q}_i^2 - 1). \tag{3}$$

The (phase space) displacement operator $\hat{D}(< \vec{x}; \vec{k} >)$ is defined as:

$$\hat{D}(< \vec{x}; \vec{k} >) = \exp[i(\vec{k} \cdot \vec{q} - \vec{x} \cdot \vec{p})]. \tag{4}$$

$\hat{D}(< \vec{x}; \vec{k} >)$ is unitary and has the following properties:

$$\hat{D}^\dagger(< \vec{x}; \vec{k} >) = \hat{D}^{-1}(< \vec{x}; \vec{k} >) = \hat{D}(- < \vec{x}; \vec{k} >), \tag{5}$$

$$\hat{D}(< \vec{x}; \vec{k} >) < \vec{q}; \vec{p} > \hat{D}^{-1}(< \vec{x}; \vec{k} >) =< (\vec{q} - \vec{x}); (\vec{p} - \vec{k}) > . \tag{6}$$

The coherent state is defined as [4]:

$$|\vec{x}, \vec{k}\rangle = \hat{D}(< \vec{x}, \vec{k} >)|0\rangle. \tag{7}$$

Another kind of unitary operator we will use in this paper are the elements of metaplectic group **Mp(2n,R)**—the quantum analogue of symplectic group **Sp(2n,R)**. **Mp(2n,R)** is an $n(2n+1)$–dimensional Lie group with its algebra spanned by $\{\hat{q}_i\hat{q}_j, \hat{p}_i\hat{p}_j, \hat{q}_i\hat{p}_j + \hat{p}_j\hat{q}_i\}$. The elements of the Lie algebra of **Mp(2n,R)** can be organized as anti–hermitian operators in the following form:

$$\begin{aligned} \hat{\Phi}(m) &= \frac{i}{2}\sum_{i,j=1}^{n}[\alpha_{ij}\hat{q}_i\hat{q}_j + \beta_{ij}\hat{p}_i\hat{p}_j + \gamma_{ij}(\hat{q}_i\hat{p}_j + \hat{p}_j\hat{q}_i)] \\ &= \frac{i}{2} < \vec{q}; \vec{p} > \begin{pmatrix} \alpha & \gamma \\ \gamma^t & \beta \end{pmatrix} < \vec{q}; \vec{p} >^t \\ &= \frac{i}{2} < \vec{q}; \vec{p} > Jm < \vec{q}; \vec{p} >^t, \end{aligned} \tag{8}$$

where $\alpha_{ij} = \alpha_{ji}$, $\beta_{ij} = \beta_{ji}$ and

$$m = \begin{pmatrix} -\gamma^t & -\beta \\ \alpha & \gamma \end{pmatrix} \in \mathbf{sp(2n,r)} \tag{9}$$

is a $2n \times 2n$ real matrix [10], while

$$J = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{-1} & \mathbf{0} \end{pmatrix}, \quad \mathbf{1} = n \times n \text{ unit matrix.} \tag{10}$$

From CCR's, we have:

$$[\hat{\Phi}(m), <\vec{q};\vec{p}>^t] = \begin{pmatrix} \gamma^t & \beta \\ -\alpha & -\gamma \end{pmatrix} <\vec{q};\vec{p}>^t = -m <\vec{q};\vec{p}>^t, \tag{11}$$

and

$$[\hat{\Phi}(m_1), \hat{\Phi}(m_2)] = \hat{\Phi}([m_1, m_2]). \tag{12}$$

Therefore the Lie algebra of $\mathbf{Mp(2n,R)}$ is isomorphic to $\mathbf{sp(2n,r)}$—the Lie algebra of $\mathbf{Sp(2n,R)}$

The action of $\exp[\hat{\Phi}(m)] \in \mathbf{Mp(2n,R)}$ on $<\vec{q};\vec{p}>$ can be defined and calculated from (11):

$$\exp[\hat{\Phi}(m)] <\vec{q};\vec{p}>^t \exp[-\hat{\Phi}(m)] = \exp(-m) <\vec{q};\vec{p}>^t, \tag{13}$$

where $\exp(-m) \in \mathbf{Sp(2n,R)}$.

Now we replace $\exp(-m)$ in (13) by a general element $S \in \mathbf{Sp(2n,R)}$ and try to find a unitary operator $\hat{U}(S) \in \mathbf{Mp(2n,R)}$ such that

$$\hat{U}(S) <\vec{q};\vec{p}>^t \hat{U}(S)^{-1} = S <\vec{q};\vec{p}>^t. \tag{14}$$

From linear algebra and group theory, we know that there is a unique polar decomposition $S = RP$ for any element S in $\mathbf{Sp(2n,R)}$, where R is orthogonal, P is symmetric and positive definite, and both R and P are in $\mathbf{Sp(2n,R)}$. Therefore we can always put $S = \exp(m_R)\exp(m_P)$, where $R = \exp(m_R)$, $P = \exp(m_P)$, and both m_R and m_S are elements in $\mathbf{sp(2n,r)}$ (m_P is symmetric and unique, while m_R is anti-symmetric and not unique) [11]. The element $\hat{U}(S) \in \mathbf{Mp(2n,R)}$ which is unitary and satisfies (14) can be constructed as follows:

$$\hat{U}(S) = \exp[\hat{\Phi}(-m_P)] \exp[\hat{\Phi}(-m_R)], \tag{15}$$

where $\exp[\hat{\Phi}(-m_P)]$ corresponds to a generalized squeezing and $\exp[\hat{\Phi}(-m_R)]$ to a rotation in $2n$-dimensional phase space. This decomposition is crucial in the construction of multimode SqCS since the Fock vacuum is an eigenstate of $\exp[\hat{\Phi}(-m_R)]$, hence only the degrees of freedom in $\exp[\hat{\Phi}(-m_P)]$ are effective in the SqCS constructed as $\hat{U}(S)|0\rangle$ [8].

Lemma [12]

If M is a symmetric and positive definite $2n \times 2n$ matrix, then there exists a matrix $S \in \mathbf{Sp(2n,R)}$ (but not unique), such that

$$M = S^t \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} S, \tag{16}$$

where $\omega = \mathrm{diag}(\omega_1, \omega_2, \ldots, \omega_n)$, $\omega_j > 0$ for all j.

Remarks:

(1) $S \in \mathbf{Sp(2n,R)}$ if and only if $S^t J S = J$ by definition.

(2) ω_j is not an eigenvalue of M in general.

(3) The eigenvalues of JM are $\pm i\omega_j$'s, hence we can calculate ω_j's from JM as an ordinary eigenvalue problem.

(4) If the matrix C_j corresponds to a 2-dimensional rotation on the $<x_j, k_j>$ plane, then

$$C_j^t \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} C_j = C_j^t C_j \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} = \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}. \tag{17}$$

Therefore S in (16) can be replaced by $C_j S$ and hence is not unique.

3 Unified Definition of Multimode Squeezed Coherent States

The Hamiltonian we will use in this section is of n-mode, inhomogeneously quadratic, time-independent, and with its quadratic part positive definite:

$$\hat{H} = \frac{1}{2} <\vec{q};\vec{p}> M <\vec{q};\vec{p}>^t + V <\vec{q};\vec{p}>^t, \tag{18}$$

where M is a $2n \times 2n$, symmetric and positive definite (hence invertible) real matrix and V is a $1\times 2n$ real vector. This kind of Hamiltonian can be transformed into the "standard form":

$$\hat{H} = \hat{D}_0\hat{U}_0\hat{H}_0\hat{U}_0^{-1}\hat{D}_0^{-1} + \text{constant}, \tag{19}$$

where $\hat{D}_0$ is a displacement operator, $\hat{U}_0$ is an operator in $\mathbf{Mp(2n,R)}$ and

$$\hat{H}_0 = \sum_{i=1}^{n}\omega_i\hat{N}_i, \qquad \omega_i > 0. \tag{20}$$

Using the formulas discussed in last section, the derivation of (19) is straightforward:

$$\begin{aligned}
\hat{H} &= \frac{1}{2} < \vec{q};\vec{p} > M < \vec{q};\vec{p} >^t + V < \vec{q};\vec{p} >^t \\
&= \frac{1}{2}[< \vec{q};\vec{p} > + VM^{-1}]M[< \vec{q};\vec{p} >^t + M^{-1}V^t] + \text{constant} \\
&= \frac{1}{2} < (\vec{q}-\vec{x}_0);(\vec{p}-\vec{k}_0) > M < (\vec{q}-\vec{x}_0);(\vec{p}-\vec{k}_0) >^t + \text{constant} \\
&= \frac{1}{2}\hat{D}_0 < \vec{q};\vec{p} > \hat{D}_0^{-1}M\hat{D}_0 < \vec{q};\vec{p} >^t \hat{D}_0^{-1} + \text{constant} \\
&= \hat{D}_0[\frac{1}{2} < \vec{q};\vec{p} > M < \vec{q};\vec{p} >^t]\hat{D}_0^{-1} + \text{constant} \\
&= \hat{D}_0[\frac{1}{2} < \vec{q};\vec{p} > S^t \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} S < \vec{q};\vec{p} >^t]\hat{D}_0^{-1} + \text{constant} \\
&= \hat{D}_0[\frac{1}{2}\hat{U}_0 < \vec{q};\vec{p} > \hat{U}_0^{-1} \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} \hat{U}_0 < \vec{q};\vec{p} >^t \hat{U}_0^{-1}]\hat{D}_0^{-1} + \text{constant} \\
&= \hat{D}_0\hat{U}_0[\frac{1}{2} < \vec{q};\vec{p} > \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} < \vec{q};\vec{p} >^t]\hat{U}_0^{-1}\hat{D}_0^{-1} + \text{constant} \\
&= \hat{D}_0\hat{U}_0[\frac{1}{2}\sum_{i=1}^{n}\omega_i(\hat{q}_i^2+\hat{p}_i^2)]\hat{U}_0^{-1}\hat{D}_0^{-1} + \text{constant} \\
&= \hat{D}_0\hat{U}_0[\sum_{i=1}^{n}\omega_i\hat{N}_i]\hat{U}_0^{-1}\hat{D}_0^{-1} + \text{constant} \\
&= \hat{D}_0\hat{U}_0\hat{H}_0\hat{U}_0^{-1}\hat{D}_0^{-1} + \text{constant},
\end{aligned} \tag{21}$$

where $VM^{-1} = - < \vec{x}_0;\vec{k}_0 >$, $\hat{D}_0 = \hat{D}(< \vec{x}_0;\vec{k}_0 >)$ and $\hat{U}_0 = \hat{U}(S)$.

Without loss of generality, we can always drop the constant term and consider

$$\hat{H} = \hat{D}_0\hat{U}_0\hat{H}_0\hat{U}_0^{-1}\hat{D}_0^{-1}. \tag{22}$$

It is easy to see that the normalized ground state of this Hamiltonian is:

$$\hat{D}_0\hat{U}_0|0\rangle \propto \hat{D}_0\exp[\hat{\Phi}(-m_P)]|0\rangle, \tag{23}$$

which is a SqCS in general, it contains the coherent state ($U_0 = 1$, $D_0 \neq 1$) and the squeezed state ($U_0 \neq 1$, $D_0 = 1$) as two special cases.

Therefore we can take (23) as a unified definition of the multimode SqCS. However, since those ω_i's in $\hat{H}_0$ do not appear in (23), the correspondence between (22) and (23) is many–to–one. The non–uniqueness of S, hence $\hat{U}_0$, will not cause any trouble, because we have shown that S is unique up to some 2–dimensional rotations in phase space, and rotations correspond to $\exp[\hat{\Phi}(-m_R)]$ in $\hat{U}(S)$ which will not appear in (23).

4 Multimode Thermal Squeezed Coherent States

Consider immersing a physical system described by the Hamiltonian (22) in a heat bath of temperature T. This constitutes a canonical ensemble and the density operator of this system is:

$$\begin{aligned}
\hat{\rho} &= Z^{-1}\exp(-\beta\hat{H}) \\
&= Z^{-1}\exp[-\beta(\hat{D}_0\hat{U}_0\hat{H}_0\hat{U}_0^{-1}\hat{D}_0^{-1})] \\
&= Z^{-1}\hat{D}_0\hat{U}_0\exp(-\beta\hat{H}_0)\hat{U}_0^{-1}\hat{D}_0^{-1},
\end{aligned} \tag{24}$$

where

$$\beta = \frac{1}{kT}, \qquad Z = Tr[\exp(-\beta\hat{H})] = Tr[\exp(-\beta\hat{H}_0)]. \tag{25}$$

This density operator $\hat{\rho}$ describes a mixed state unless $T = 0$. In the limit as $T \to 0$, since

$$\lim_{\beta\to\infty}\exp(-\beta\hat{H}_0) = |0\rangle\langle 0|, \tag{26}$$

we have

$$\hat{\rho} = \hat{D}_0\hat{U}_0|0\rangle\langle 0|\hat{U}_0^{-1}\hat{D}_0^{-1}, \tag{27}$$

which corresponds to the pure SqCS (23), hence (23) is a special case of (24) and (24) is a "thermalized state" of (23). Therefore we can take (24) as the definition of multimode thermal SqCS.

5 Representations of Multimode Thermal Squeezed Coherent States

There are many equivalent representations of the density operator $\hat{\rho}$, e.g., the coordinate representation, P–representation, Q–representation, Fock space representation, Wigner function and characteristic function, etc. In this paper we will discuss the last two representations.

5.1 Wigner Function

The Wigner function of a density operator $\hat{\rho}$ is defined as [13, 14]:

$$W(\vec{x};\vec{k}) = \pi^{-n}\int_{-\infty}^{\infty} d\vec{y}\exp(2i\vec{k}\cdot\vec{y})\rho(\vec{x}-\vec{y},\vec{x}+\vec{y}), \tag{28}$$

where $\rho(\vec{x},\vec{x}')$ is the coordinate representation of the density operator $\hat{\rho}$.

The Wigner function can also be put into the following form [15]:

$$W(\vec{x};\vec{k}) = Tr[\hat{\rho}\hat{\Delta}_W(<\vec{x};\vec{k}>)], \tag{29}$$

where the "Wigner operator" $\hat{\Delta}_W(<\vec{x};\vec{k}>) = \pi^{-n}\hat{D}(2<\vec{x};\vec{k}>)\exp(i\pi\sum_{i=1}^{n}\hat{N}_i)$ is a well–defined hermitian operator with $<\vec{x};\vec{k}>$ as its parameters.

The Wigner function is normalized by definition:

$$\int_{-\infty}^{\infty} d\vec{x}d\vec{k}W(\vec{x};\vec{k}) = 1, \tag{30}$$

and it is real because the Wigner operator is hermitian. However, the Wigner function is not always positive–definite and it is thus called the quasi–probability distribution function over the "phase space" $(\vec{x};\vec{k})$.

In the following, we will calculate the Wigner functions for some thermal SqCS's. First let us consider the simplest one–mode case, i.e.,

$$\hat{H} = \frac{1}{2}\omega(\hat{p}^2+\hat{q}^2-1) = \omega\hat{N}, \tag{31}$$

the density operator is:

$$\hat{\rho} = Z^{-1}\exp(-\beta\omega\hat{N}), \tag{32}$$

and the Wigner function takes the form [14]:

$$W(x,k) = Tr[\hat{\rho}\hat{\Delta}_W(x,k)] = \frac{1}{\pi}\tanh(\frac{\beta\omega}{2})\exp[-\tanh(\frac{\beta\omega}{2})(x^2+k^2)]. \tag{33}$$

In the limit as $T\to 0$, (33) becomes

$$W(x,k) = \frac{1}{\pi}\exp[-(x^2+k^2)], \tag{34}$$

which is exactly the Wigner function of the vacuum state [14].

Noticing that the Wigner function (33) is a Gaussian distribution function in (x,k), we can use the exponent of (33) to define the "Wigner ellipse" in the phase space (x,k) as:

$$\tanh(\frac{\beta\omega}{2})(x^2+k^2) = 1. \tag{35}$$

The area of the Wigner ellipse represents the range of uncertainty of the corresponding state. In this simplest case, the Wigner ellipse is a circle with radius $\sqrt{\coth(\frac{1}{2}\beta\omega)} \geq 1$ and with its center at the origin.

Next we consider the general one–mode Hamiltonian:

$$\hat{H} = \omega\hat{D}_0\hat{U}_0\hat{N}\hat{U}_0^{-1}\hat{D}_0^{-1}, \tag{36}$$

the density operator is:

$$\hat{\rho} = Z^{-1}\hat{D}_0\hat{U}_0\exp(-\beta\omega\hat{N})\hat{U}_0^{-1}\hat{D}_0^{-1}. \tag{37}$$

The Wigner function takes the form [16]:

$$\begin{aligned} W(x,k) &= Z^{-1}Tr[\hat{D}_0\hat{U}_0\exp(-\beta\omega\hat{N})\hat{U}_0^{-1}\hat{D}_0^{-1}\hat{\Delta}_W(x,k)] \\ &= Z^{-1}Tr[\exp(-\beta\omega\hat{N})\hat{U}_0^{-1}\hat{D}_0^{-1}\hat{\Delta}_W(x,k)\hat{D}_0\hat{U}_0] \\ &= Z^{-1}Tr[\exp(-\beta\omega\hat{N})\hat{\Delta}_W(x',k')] \\ &= \frac{1}{\pi}\tanh(\frac{\beta\omega}{2})\exp[-\tanh(\frac{\beta\omega}{2})(x'^2+k'^2)], \end{aligned} \tag{38}$$

where

$$\begin{pmatrix} x' \\ k' \end{pmatrix} = S\begin{pmatrix} x-x_0 \\ k-k_0 \end{pmatrix}, \tag{39}$$

and the Wigner ellipse is:

$$\tanh(\frac{\beta\omega}{2})(x - x_0\ ,\ k - k_0)S^t S \begin{pmatrix} x - x_0 \\ k - k_0 \end{pmatrix} = 1. \tag{40}$$

For the n–mode cases we first consider the uncoupled Hamiltonian, i.e., $\hat{H} = \hat{H}_0$. The Wigner function in this case is a product of each individual one–mode Wigner function:

$$W(\vec{x};\vec{k}) = \pi^{-n}[\prod_{i=1}^{n} \tanh(\frac{\beta\omega_i}{2})] \exp[-\sum_{i=1}^{n} \tanh(\frac{\beta\omega_i}{2})(x_i^2 + k_i^2)]. \tag{41}$$

In this multimode case, we can define the "Wigner ellipsoid" in $2n$–dimensional phase space as:

$$\sum_{i=1}^{n} \tanh(\frac{\beta\omega_i}{2})(x_i^2 + k_i^2) = 1, \tag{42}$$

or equivalently,

$$(\vec{x};\vec{k}) \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} (\vec{x};\vec{k})^t = 1, \tag{43}$$

where $T = \mathrm{diag}(\tanh(\frac{1}{2}\beta\omega_1), \tanh(\frac{1}{2}\beta\omega_2), \ldots, \tanh(\frac{1}{2}\beta\omega_n))$. Analogously, the $2n$–dimensional volume of the Wigner ellipsoid represents the range of uncertainty.

For the most general Hamiltonian $\hat{H} = \hat{D}_0\hat{U}_0\hat{H}_0\hat{U}_0^{-1}\hat{D}_0^{-1}$, analogue to (38), the Wigner function is:

$$\begin{aligned} W(\vec{x};\vec{k}) &= \pi^{-n}[\prod_{i=1}^{n} \tanh(\frac{\beta\omega_i}{2})] \\ &\quad \exp[-(\vec{x} - \vec{x}_0; \vec{k} - \vec{k})S^t \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} S(\vec{x} - \vec{x}_0; \vec{k} - \vec{k}_0)^t], \end{aligned} \tag{44}$$

and the Wigner ellipsoid becomes:

$$(\vec{x} - \vec{x}_0; \vec{k} - \vec{k})S^t \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} S(\vec{x} - \vec{x}_0; \vec{k} - \vec{k}_0)^t = 1. \tag{45}$$

5.2 Characteristic Function

The characteristic function of a density operator $\hat{\rho}$ is defined as:

$$X(\vec{x};\vec{k}) = Tr[\hat{\rho}\hat{D}(-\vec{x};-\vec{k})], \tag{46}$$

From the symplectic Fourier transformation of the Wigner operator $\hat{\Delta}_W(\vec{x};\vec{k})$:

$$\begin{aligned} \mathbf{F}[\hat{\Delta}_W(\vec{x};\vec{k})] &= \int_{-\infty}^{\infty} d\vec{x}' d\vec{k}' \hat{\Delta}_W(\vec{x}';\vec{k}') \exp[-i(\vec{x}' \cdot \vec{k} - \vec{k}' \cdot \vec{x})] \\ &= \hat{D}(-\vec{x};-\vec{k}), \end{aligned} \tag{47}$$

we can see that the characteristic function is the symplectic Fourier transformation of the Wigner function:

$$\mathbf{F}[W(\vec{x};\vec{k})] = X(\vec{x};\vec{k}). \tag{48}$$

The normalization condition of the Wigner function corresponds to $X(\vec{0};\vec{0}) = 1$ in the characteristic function. Since the operator $\hat{D}(-\vec{x};-\vec{k})$ is unitary instead of hermitian, $X(\vec{x};\vec{k})$ is complex in general.

The characteristic function of the general n–mode thermal SqCS, which corresponds to the Wigner function (44), is:

$$X(\vec{x};\vec{k}) = \exp[-\frac{1}{4}(\vec{x};\vec{k})S^t \begin{pmatrix} T^{-1} & 0 \\ 0 & T^{-1} \end{pmatrix} S(\vec{x};\vec{k})^t + i(\vec{x} \cdot \vec{k}_0 - \vec{k} \cdot \vec{x}_0)]. \tag{49}$$

5.3 Covariance Matrix

For an n–mode (mixed) state with density operator $\hat{\rho}$, the covariance matrix is a $2n \times 2n$ matrix of the form:

$$\begin{pmatrix} U & Q \\ Q^t & V \end{pmatrix}, \tag{50}$$

$$U_{ij} \equiv \langle(\hat{q}_i - \langle\hat{q}_i\rangle)(\hat{q}_j - \langle\hat{q}_j\rangle)\rangle = \langle\hat{q}_i\hat{q}_j\rangle - \langle\hat{q}_i\rangle\langle\hat{q}_j\rangle, \tag{51}$$

$$V_{ij} \equiv \langle(\hat{p}_i - \langle\hat{p}_i\rangle)(\hat{p}_j - \langle\hat{p}_j\rangle)\rangle = \langle\hat{p}_i\hat{p}_j\rangle - \langle\hat{p}_i\rangle\langle\hat{p}_j\rangle, \tag{52}$$

$$\begin{aligned} Q_{ij} &\equiv \frac{1}{2}\langle(\hat{q}_i - \langle\hat{q}_i\rangle)(\hat{p}_j - \langle\hat{p}_j\rangle) + (\hat{p}_j - \langle\hat{p}_j\rangle)(\hat{q}_i - \langle\hat{q}_i\rangle)\rangle \\ &= \langle\frac{1}{2}(\hat{q}_i\hat{p}_j + \hat{p}_j\hat{q}_i)\rangle - \langle\hat{q}_i\rangle\langle\hat{p}_j\rangle, \end{aligned} \tag{53}$$

where $\langle\hat{q}_i\rangle \equiv Tr(\hat{\rho}\hat{q}_i)$, etc.

For the thermal SqCS which corresponds to the Wigner function (44) or the characteristic function (49), it can be proved that the covariance matrix is

$$\frac{1}{2}S^{-1} \begin{pmatrix} T^{-1} & 0 \\ 0 & T^{-1} \end{pmatrix} (S^{-1})^t. \tag{54}$$

6 Decoherence Problems

6.1 General Theory of Decoherence

Consider a quantum system which contains two subsystems (A) and (B) with the density operator $\hat{\rho}_{AB}$. Any (monomial) operator $\hat{O}$ which corresponds to a measurement on the system can be decomposed into $\hat{O}_A \otimes \hat{O}_B$, where $\hat{O}_A$ corresponds to a measurement on and only on (A) and $\hat{O}_B$ correspondingly on (B). If we decohere this system by ignoring (B), i.e., not making any measurement on (B), then the operator $\hat{O}$ will be reduced to $\hat{O}_A \otimes \hat{\mathbf{1}}$ and the expectation value of $\hat{O}_A$ will become:

$$\begin{aligned}\langle \hat{O}_A \rangle &= Tr[\hat{\rho}_{AB}(\hat{O}_A \otimes \hat{\mathbf{1}})] \\ &= Tr_{(A)} Tr_{(B)}[\hat{\rho}_{AB}(\hat{O}_A \otimes \hat{\mathbf{1}})] \\ &= Tr_{(A)}[(Tr_{(B)}(\hat{\rho}_{AB}))\hat{O}_A] \\ &= Tr[\hat{\rho}_A \hat{O}_A], \end{aligned} \tag{55}$$

where $Tr_{(A)}/\ Tr_{(B)}$ represents the "partial trace" which only takes trace with respect to the degrees of freedom of (A)/(B), and $\hat{\rho}_A = Tr_{(B)}(\hat{\rho}_{AB})$ is a well-defined reduced density operator.

If the Wigner function $W(\vec{x}_A, \vec{x}_B; \vec{k}_A, \vec{k}_B)$ corresponds to the original density operator $\hat{\rho}_{AB}$, then the reduced Wigner function corresponding to $\hat{\rho}_A$ is [14]:

$$W_A(\vec{x}_A; \vec{k}_A) = \int_{-\infty}^{\infty} d\vec{x}_B d\vec{k}_B W(\vec{x}_A, \vec{x}_B; \vec{k}_A, \vec{k}_B). \tag{56}$$

As for the characteristic function, if $X(\vec{x}_A, \vec{x}_B; \vec{k}_A, \vec{k}_B)$ corresponds to $\hat{\rho}_{AB}$, the reduced characteristic function corresponding to $\hat{\rho}_A$ will take the form:

$$X_A(\vec{x}_A; \vec{k}_A) = X(\vec{x}_A, \vec{0}; \vec{k}_A, \vec{0}), \tag{57}$$

which is a restriction of the original $X(\vec{x}_A; \vec{x}_B, \vec{k}_A; \vec{k}_B)$ to a subspace in the $2n$–dimensional phase space. From the mathematical point of view, it is easier to use the characteristic function to study decoherence problems.

6.2 Decoherence of a Thermal Squeezed Coherent State From n–Mode to m–Mode

For a given characteristic function of an n–mode thermal SqCS:

$$\begin{aligned} & X(x_1, x_2, \ldots, x_m, \ldots, x_n; k_1, k_2, \ldots, k_m, \ldots, k_n) \\ = \ & \exp\{-\frac{1}{4}(x_1, x_2, \ldots, x_m, \ldots, x_n; k_1, k_2, \ldots, k_m, \ldots, k_n) \\ & S^t \begin{pmatrix} T^{-1} & 0 \\ 0 & T^{-1} \end{pmatrix} S(x_1, x_2, \ldots, x_m, \ldots, x_n; k_1, k_2, \ldots, k_m, \ldots, k_n)^t \\ & + i\sum_{j=1}^{n}(x_j k_{0j} - k_j x_{0j})\}. \end{aligned} \tag{58}$$

The reduced characteristic function is:

$$\begin{aligned} & X(x_1, x_2, \ldots, x_m, \vec{0}; k_1, k_2, \ldots, k_m, \vec{0}) \\ = \ & \exp\{-\frac{1}{4}(x_1, x_2, \ldots, x_m, \vec{0}; k_1, k_2, \ldots, k_m, \vec{0}) \\ & S^t \begin{pmatrix} T^{-1} & 0 \\ 0 & T^{-1} \end{pmatrix} S(x_1, x_2, \ldots, x_m, \vec{0}; k_1, k_2, \ldots, k_m, \vec{0})^t \\ & + i\sum_{j=1}^{m}(x_j k_{0j} - k_j x_{0j})\} \\ = \ & \exp\{-\frac{1}{4}(x_1, x_2, \ldots, x_m; k_1, k_2, \ldots, k_m,)K(x_1, x_2, \ldots, x_m; k_1, k_2, \ldots, k_m)^t \\ & + i\sum_{i=1}^{m}(x_i k_{0i} - k_i x_{0i})\}, \end{aligned} \tag{59}$$

where the matrix K is $2m \times 2m$ and still symmetric and positive–definite, its elements are a subset of the the elements of $S^t \begin{pmatrix} T^{-1} & 0 \\ 0 & T^{-1} \end{pmatrix} S$:

$$K_{i,j} = [S^t \begin{pmatrix} T^{-1} & 0 \\ 0 & T^{-1} \end{pmatrix} S]_{i,j}, \tag{60}$$

$$K_{i+m,j} = [S^t \begin{pmatrix} T^{-1} & 0 \\ 0 & T^{-1} \end{pmatrix} S]_{i+n,j}, \tag{61}$$

$$K_{i,j+m} = [S^t \begin{pmatrix} T^{-1} & 0 \\ 0 & T^{-1} \end{pmatrix} S]_{i,j+n}, \tag{62}$$

$$K_{i+m,j+m} = [S^t \begin{pmatrix} T^{-1} & 0 \\ 0 & T^{-1} \end{pmatrix} S]_{i+n,j+n}, \tag{63}$$

where $1 \leq i, j, \leq m$.

From the Lemma in Sec. 1, we can find a $2m \times 2m$ symplectic matrix σ such that:

$$K = \sigma^t \begin{pmatrix} \tau & 0 \\ 0 & \tau \end{pmatrix} \sigma, \tag{64}$$

where $\tau = \mathrm{diag}(\tau_1, \tau_2, \ldots, \tau_m)$, $\tau_i > 0$, for all $i = 1, 2, \ldots, m$.

We can make a further restriction on τ_i from the following physical consideration: Since the reduced density operator $\hat{\rho}_A = Tr_{(B)}(\hat{\rho}_{AB})$ is well-defined, it will never correspond to any non-physical state. Noticing that (59) is of the same form as (49), comparison with (54) shows the covariance matrix of this decohered state to be:

$$K = \frac{1}{2}\sigma^{-1} \begin{pmatrix} \tau & 0 \\ 0 & \tau \end{pmatrix} (\sigma^{-1})^t. \tag{65}$$

Since σ^{-1} corresponds to a symplectic (hence canonical) transformation on the canonical coordinates,

$$\frac{1}{2} \begin{pmatrix} \tau & 0 \\ 0 & \tau \end{pmatrix} \tag{66}$$

is also a covariance matrix for the same state in another canonical coordinates. This guarantees that $\tau_i \geq 1$ for all $i = 1, 2, \ldots, m$, otherwise (59) will give a state that violates the uncertainty principle. Therefore we conclude that the reduced characteristic function (59) corresponds to an m-mode thermal SqCS.

7 Conclusion

The results of this paper are threefold (1) A unified construction of multimode (thermal) SqCS's. (2) Proof of the statement: The decohered multimode thermal SqCS is still a (multimode) thermal SqCS. (3) Introduction of the decohering technique via characteristic function, which is very efficient and can be applied to many related problems.

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