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Canonical Analysis and Symmetries for a Non-Relativistic System of Anyons

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Abstract

A thorought analysis of the canonical formalism for a non-relativistic ideal gas of anyons is performed. The generators of the rigid and gauge continuous symmetries are explicitely given and the interplay between both types of symmetries within the group algebra is exhibited. Reduced and Dirac quantizations are introduced and it is argued that they may not be equivalent.

Resum

Hom realitza una anàlisi del formalisme canònic per a un gas ideal no-relativista d'anyons. Es construeixen els generadors tant de les simetries rígides com de les gauge i se'n exhibeix una interessant interrelació. Es presenten també les quantitzacions de Dirac i reduïda del model i se'n discuteix llur possible no-equivalència.

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1 Introduction

It is currently well known [1] that physics in two-space dimensions has room for an infinite family –parametrized by a phase– of different statistics for identical particles which, after Wilzeck, are named anyons. Only when this phase takes the ± 1 values we recover the usual statistics of bosons and fermions. This striking difference of 2+1 space-time physics, with respect to higher dimensions, can be traced to the fact that the configuration space of a system of identical particles in two space dimensions is –if an exclusion principle holds– multiply connected. The "exchange" of particles can then be implemented by the braid group, which has a family of inequivalent one-dimensional representations that can be labeled by a phase. This is just the phase associated with the anyon statistics. Instead, higher dimensions only allow for the permutation group, whose two one-dimensional representations account for the statistical behauvior of the particles we see in the nature: bosons and fermions.

Since our real world is not 2+1, we only expect to realize anyons in nature as collective states of bosons and fermions and for physical systems that admit a description in terms of two space dimensions. This is the case of some Condensed Matter Systems and thereby the way is open to the application of anyon physics to the fractional quantum Hall effect –where its relevance has already been shown– and to high T_c superconductivity. Dynamics in the presence of infinite cosmic strings may also bring anyons into the play [3] .

Anyon statistics may be implemented in bosons or fermions by attatching a fictitious charge-flux tube to each particle. An elegant way of getting such an artifact is to couple the matter sector with a gauge field governed by an abelian Chern Simons term. It turns out that that this field has not degrees of freedom by its own, its role being that of introducing a new interaction among the particles themselves that should give rise to the anyon statistics. Except for the possible application to cosmic strings scenarios, the anyon physics is non-relativistic. In this framework, the classical Lagrangian setting of an ideal gas of anyons is the following:

$$L = \sum_{\alpha=1}^{N} \frac{m}{2}\dot{\mathbf{x}}_\alpha^2 - q\int d^2x\, a_\mu j^\mu + \frac{\mu}{2}\int d^2x\, \epsilon^{\rho\sigma\tau} a_\rho \partial_\sigma a_\tau \tag{1.1}$$

where j^μ is the conserved current corresponding to the particles:

$$j^\mu = (\rho, j^i), \qquad \rho(\mathbf{x}) = \sum_{\alpha=1}^{N} \delta^2(\mathbf{x}-\mathbf{x}_\alpha), \qquad j^i(\mathbf{x}) = \sum_{\alpha=1}^{N} \dot{x}_\alpha^i \delta^2(\mathbf{x}-\mathbf{x}_\alpha).$$

Our 2+1 metric is $\eta^{ij} = (+,-,-)$ [1] and the Levi-Civitta tensor has $\epsilon_{012} = +1$.

The reduced Hamiltonian formalism associated to this Lagrangian has been worked out in the literature. Thus in ref [2] a completelly gauge-fixed Hamiltonian is obtained in a rather direct way, from which a quantization (a "reduce first" quantization) program can be performed. The same Hamiltonian is obtained in [3] using a gauge fixing that contains the one introduced in [2] (i.e.the Coulomb gauge) plus the Weyl (or temporal) gauge. A more careful analysis shows that both parts of the gauge fixing are incompatible. This fact renders the approach of [3] somewhat insatisfactory.

As of now, a thorought analysis of the full constraint structure and the gauge and rigid symmetries in phase space of (1.1) is still lacking. The purpose of this paper is to provide with such a study. The reduced Hamiltonian of [2] will then show up when the appropiate gauge fixing is introduced, but the whole knowledge of the system prior to the gauge fixing procedure is worth because: a) It allows for the study of other compatible gauge fixings which could eventually be of interest, b) It leaves the system ready to perform a Dirac quantization, that is to say, a "first quantize and then reduce" procedure. There is no way that could guarantee the equivalence between the Dirac and the reduced quantisation. In fact it has been pointed out that some ambiguities arise in the quantisation of non-abelian Chern Simons topological theories [4]. Many other results and examples on the equivalence or non-equivalence of both quantisation schemes can be found in the literature [5]. And c) This study is worth for another reason too: It provides us with an explicit realisation of the interplay between rigid and gauge symmetries within the full algebra of continuous canonical symmetries of the theory. The role of Dirac's quantisation is then emphasized as a natural way to restore the usual algebra or rigid symmetries in the quantum formalism.

In section 2 the constraint analysis in canonical and Lagrangian formalism is performed. The algebra of the generators of rigid and gauge symmetries is

[1] Although the system is non-relativistic, we keep relativistic notation to raise and lower indices, etc

displayed in section 3 and the interplay between both types of symmetries is exhibited. Section 4 is devoted to quantization and section 5 to conclusions.

2 Constraint analysis

2.1 Canonical formalism

Now we are going to perform the canonical analysis of the system described by (1.1). The Lagrangian analysis will come easily as a bonus from the canonical one.

An integration by parts on space components allows to rewrite L as:

$$\begin{aligned} L &= \sum_{\alpha=1}^{N} \frac{m}{2}\dot{\mathbf{x}}_\alpha^2 + \sum_{\alpha=1}^{N} q\, \dot{\mathbf{x}}_\alpha \cdot \mathbf{a}(\mathbf{x}_\alpha) \\ &+ \int d^2x\, a_0(-q\rho + \mu\epsilon^{ij}\partial_i a_j) - \frac{\mu}{2}\int d^2x\, \epsilon^{ij} a_i \dot{a}_j \end{aligned} \tag{2.1}$$

This is going to be our starting point to get the canonical formulation. The usual definition of momenta gives:

$$p_\alpha^i = -m\dot{x}_\alpha^i - qa^i(\mathbf{x}_\alpha) \tag{2.2.a}$$

$$\pi^i = \frac{\mu}{2}\epsilon^{ij} a_j \tag{2.2.b}$$

$$\pi^0 = 0 \tag{2.2.c}$$

where $\mathbf{p}_\alpha$, π, π^0 are the conjugate momenta for $\mathbf{x}_\alpha, \mathbf{a}, a^0$, respectively. Equations (2.2) define the Legendre map between tangent and cotangent spaces.

From (2.2) we can readily identify the primary constraints of the Hamiltonian formulation [2]:

$$\pi^0 \simeq 0$$

$$\phi^i := \pi^i - \frac{\mu}{2}\epsilon^{ij} a_j \simeq 0$$

[2] Dirac's weak equality $\simeq$ means equality on the constraint surface.

The surface defined by these constraints in cotangent space is nothing but the image of the tangent space under the Legendre mapping. Notice that the constraints ϕ^1,ϕ^2 constitute a couple of second class constraints:

$$\{\phi^i(\mathbf{y}), \phi^j(\mathbf{x})\} = -\mu\epsilon^{ij}\delta^2(\mathbf{x}-\mathbf{y}). \tag{2.3}$$

Observe that the Poisson bracket structure has now a particle-like component, $\{x_\alpha^i, p_\beta^j\} = \delta_{\alpha\beta}\eta^{ij}$, and a field-like component, $\{a^\mu(\mathbf{y}), \pi^\nu(\mathbf{x})\} = \eta^{\mu\nu}\delta^2(\mathbf{x}-\mathbf{y})$.

The Lagrangian energy function is obtained by substituting the definitions of the momenta into

$$E_L := \sum_{\alpha=1}^{N} p_\alpha^i \dot{x}_\alpha^i + \int d^2y\, \pi_\mu \dot{a}^\mu - L$$

thus obtaining

$$E_L = \sum_{\alpha=1}^{N} \frac{m}{2}\dot{\mathbf{x}}_\alpha^2 - \int d^2x\, a_0(-q\rho + \mu\epsilon^{ij}\partial_i a_j) \tag{2.4}$$

The Hamiltonian is defined as the function of canonical coordinates (i.e., function of cotangent space) the pullback of which (i.e., the substitution of the momenta variables by the definitions (2.2)) is just the Lagrangian energy. The existence of primary constraints (or rephrased: the fact that the Legendre map is not surjective) makes the definition of the Hamiltonian as a function of the cotangent space ambiguous. Indeed, it is only uniquely defined on the primary constraint surface. The most general Hamiltonian –the Dirac Hamiltonian H_D– can then be written as

$$H_D = H_C + \int d^2x\, \xi_0\pi^0 + \int d^2x\, \lambda_i\phi^i \tag{2.5}$$

where H_C can be taken to be the "naïve" canonical Hamiltonian that one can get directly, using (2.2), from equation (2.4):

$$H_C = \frac{1}{2m}\sum_{\alpha=1}^{N}(\mathbf{p}_\alpha + q\mathbf{a}(\mathbf{x}_\alpha))^2 - \int d^2x\, a_0(-q\rho + \mu\epsilon^{ij}\partial_i a_j). \tag{2.6}$$

ξ_0 and λ_i are, at this moment, arbitrary function of space-time variables (they can also be thought as the Lagrange multipliers for the primary constraints, but

here we are not going to think of them as independent variables). For the time being, there is no preferred choice for these functions and we will try to apply consistency requirements in order to –perhaps– determine them. If one sticks from the scratch with the "simplest" choice of putting these functions to zero, then one is taking the wrong way, as we will see right now from two different poins of view.

First we can notice that these functions ξ_0 and λ_i become known functions of the tangent space.[3] In fact, according to the Hamilton equations for the Dirac Hamiltonian:

$$\begin{aligned} \dot{a}^i &= \{a^i, H_D\} = -\lambda_i \\ \dot{a}^0 &= \{a^0, H_D\} = \xi^0 \end{aligned} \tag{2.7}$$

We will make use later of these results.

On the other hand, there are consistency requirements that may give this functions a canonical determination –sometimes. This is indeed the case for the λ_j due to the fact that ϕ^i are second class constraints. Since we need to fulfill the obvious consistency requirement that the primary constraints have to be preserved under the dynamical evolution, we have:

$$0 \simeq \dot{\phi}^i(\mathbf{y}) = \{\phi^i(\mathbf{y}), H_D\} \simeq \{\phi^i(\mathbf{y}), H_C\} + \int d^2x\ \lambda_j \{\phi^i(\mathbf{y}), \phi^j(\mathbf{x})\}$$

or, using (2.3)

$$0 = \frac{q}{m} \sum_{\alpha=1}^{N} (\mathbf{p}_\alpha + q\mathbf{a}(\mathbf{x}_\alpha))\delta^2(\mathbf{y} - \mathbf{x}_\alpha) - \mu\epsilon^{ki}\partial_k a_0(\mathbf{y}) - \mu\epsilon^{ij}\lambda_j(\mathbf{y})$$

The multipliers λ_j are therefore determined (up to primary constraints):

$$\lambda_j(\mathbf{y}) = \frac{q}{\mu m} \sum_{\alpha=1}^{N} (p^i_\alpha + qa^i(\mathbf{x}_\alpha))\delta^2(\mathbf{y} - \mathbf{x}_\alpha)\epsilon^{ij} + \partial_j a_0(\mathbf{y}) \tag{2.8}$$

and the Dirac Hamiltonian acquires the new form:

$$H_D = H_{FC} + \int d^2x\ \xi_0 \pi^0 \tag{2.9}$$

[3] This is an standard fact when dealing with constrained systems, as can be seen in references [6]. In reference [7] this is extended to higher order formalisms.

where [4]

$$\begin{aligned} H_{FC} &= \frac{1}{2m} \sum_{\alpha=1}^{N} (\mathbf{p}_\alpha + q\mathbf{a}(\mathbf{x}_\alpha))^2 + \frac{q}{\mu m} \sum_{\alpha=1}^{N} (\mathbf{p}_\alpha + q\mathbf{a}(\mathbf{x}_\alpha))\phi^k(\mathbf{x}_\alpha)\epsilon^{ik} \\ &\quad - \int d^2x\ a_0(-q\rho + \mu\epsilon^{ij}\partial_i a_j + \partial_k \phi^k) \end{aligned} \tag{2.10}$$

Preservation in time of the constraint π^0 do not lead to any determination of ξ_0 but it introduces a new (secondary) constraint:

$$\psi(\mathbf{y}) := -q\rho(\mathbf{y}) + \mu\epsilon^{ij}\partial_i a_j(\mathbf{y}) + \partial_k \phi^k(\mathbf{y}) \simeq 0 \tag{2.11}$$

The dynamical consistency of our system requires again that this secondary constraint be preserved in time (we are following the steps of the stabilization algorithm first devised by Dirac). Considering

$$\{\rho(\mathbf{y}), H_D\} = \{\rho(\mathbf{y}), H_C\} = \frac{1}{m} \sum_{\alpha=1}^{N} (\mathbf{p}_\alpha + q\mathbf{a}(\mathbf{x}_\alpha) + \frac{q}{\mu}\epsilon^{ik}\phi^k(\mathbf{x}))\frac{\partial}{\partial y^i}\delta^2(\mathbf{y} - \mathbf{x}_\alpha)$$

and

$$\{a_j(\mathbf{y}), H_D\} = \{a_j(\mathbf{y}), H_{FC}\} = \lambda_j(\mathbf{y}),$$

one can check that

$$\{\psi(\mathbf{y}), H_D\} = \{\psi(\mathbf{y}), H_{FC}\} = 0 \tag{2.12}$$

exactly, which shows that no tertiary constraints arise. The stabilization algorithm has been completed. One can also easily verify that the constraint ψ of (2.11) is first class:

$$\begin{aligned} \{\psi(\mathbf{y}), \phi^k(\mathbf{x})\} &= 0 \\ \{\psi(\mathbf{y}), \pi^0(\mathbf{x})\} &= 0 \\ \{\psi(\mathbf{y}), \psi^k(\mathbf{x})\} &= 0 \end{aligned} \tag{2.13}$$

Summing up, we have two second class constraints in the theory, ϕ^1 and ϕ^2, and two first class constraints π^0 and ψ, which appear at the primary and secondary level respectively, and a first class Hamiltonian H_{FC} from which the Dirac Hamiltonian is obtained by adding the piece $\int d^2x\ \xi_0\pi^0$.

[4] FC stands for "first class".

2.2 Lagrangian formalism

Once the Hamiltonian analysis of (1.1) has been made, its Lagrangian counterpart in tangent space is easily performed by just relying on the former one. It can be shown [6] that all the Lagrangian constraints appear from either two ways: a) as the pullback of the Hamiltonian constraints (i.e., the substitution of the momenta by its Lagrangian definitons (2.2)) –the primary ones excluded because its pullback vanishes–, or b) as the relation that equals the Lagrangian determination of the Lagrange multipliers with the pullback of its canonical determination –when this determination exists. In our case the mechanism of a) gives the constraint (Gauss law):

$$\mu\epsilon^{ij}\partial_i a_j \simeq q\rho \tag{2.14}$$

whereas b) gives, according to (2.7) and (2.8):

$$\mu(\partial_0 a_i - \partial_i a_0) \simeq -\frac{q}{\mu} j^i \tag{2.15}$$

These are the only Lagrangian constraints of the theory, and all them show up in the first step of the stabilisation algorithm. Using the standard definitions of the electric and magnetic field, we can write (2.14) and (2.15) as

$$B \simeq \frac{q}{\mu}\rho, \qquad E^k \simeq \frac{q}{\mu}\epsilon^{ki} j^i. \tag{2.16}$$

We see, therefore, that the physical configurations of the electric and the magnetic fields are completely determined by the current of the charged particles.

3 Symmetries

Now we go back to the phase space formalism. The generators throught the Poisson Bracket of continuous symmetries (either rigid or gauge) in constrained systems must satisfy certain conditions. If the first class Hamiltonian H_{FC} and the primary first class constraints (which we will call (pfc)) of the theory are given, then the necessary and sufficent conditions for a given function $G(q,p;t)$ to generate an infinitesimal Dynamical Symmetry Transformation (DST) (i.e., that maps solutions of the equations of motion into other solutions) are the following [8] [5]:

$$G \text{ is a first class function,} \tag{3.1.a}$$

$$\{(pfc), G\} \equiv (pfc), \tag{3.1.b}$$

$$\{G, H_{FC}\} + \frac{\partial G}{\partial t} \equiv (pfc). \tag{3.1.c}$$

(The first is an obvious condition since we need the motions generated by G be tangent to the constraint surface.) With these conditions in mind, we can proceed to the analysis of the gauge transformations and the rigid transformations for our system.

3.1 The gauge generator

The first class Hamiltonian and the two first class constraints satisfy:

$$\{\pi^0, H_{FC}\} = \psi, \qquad \{\psi, H_{FC}\} = 0, \qquad \{\psi, \pi^0\} = 0.$$

Therefore the function

$$G[\Lambda] = \int d^2x\, (\dot{\Lambda}(\mathbf{x},t)\pi^0(\mathbf{x}) - \Lambda(\mathbf{x},t)\psi(\mathbf{x})) \tag{3.2}$$

with Λ being an infinitesimal arbitrary function of space-time variables, satisfies the required conditions (3.1) to generate –through Poisson bracket– a gauge transformation. Indeed in our case:

$$\{(pfc), G\} = 0, \qquad \{G, H_{FC}\} + \frac{\partial G}{\partial t} = (pfc)$$

The gauge transformation is defined by $\delta f = \{f, G\}$, and we have:

$$\begin{aligned} \delta x^i_\alpha &= 0, & \delta p_{\alpha i} &= -q\partial_i\Lambda(\mathbf{x}_\alpha, t), \\ \delta a_0(\mathbf{x},t) &= \partial_t\Lambda(\mathbf{x},t), & \delta a_i(\mathbf{x},t) &= \partial_i\Lambda(\mathbf{x},t), \\ \delta\pi^0(\mathbf{x},t) &= 0, & \delta\pi^i(\mathbf{x},t) &= -\tfrac{\mu}{2}\epsilon^{ki}\partial_i\Lambda(\mathbf{x},t). \end{aligned} \tag{3.3}$$

Since there is only one primary first class constraint, the generator (3.2) exhausts all the gauge freedom available to our system.

[5] We use hereafter Dirac's strong equalities $\equiv$ to express standard equalities up to quadratic terms in the constraints

Notice that the structure of this gauge transformation is already present in the Dirac Hamiltonian (2.9) if one takes into account the equation of motion (2.7). Then:

$$H_D = \frac{1}{2m}\sum_{\alpha=1}^{N}(\mathbf{p}_\alpha + q\mathbf{a}(\mathbf{x}_\alpha))^2 + \frac{q}{\mu m}\sum_{\alpha=1}^{N}(\mathbf{p}_\alpha + q\mathbf{a}(\mathbf{x}_\alpha))\phi^k(\mathbf{x}_\alpha)\epsilon^{ik} + G[a_0]. \quad (3.4)$$

the arbitrary function of the dynamics, a_0, just being the arbitrary function associated to the gauge transformation.

3.2 Observables and rigid symmetries

The physical observables are the gauge invariant quantities (to be more precise, here we mean gauge invariance "on shell", i.e., up to constraints; in the language of ref. [9] this is called "conditional invariance"). In our case we can check that all the constraints are gauge invariant quantities –as it should be because the gauge generator is first class–. But there are other observables with much more physical interest; from (3.3) we see that

$$\delta(p^i_\alpha + qa^i(\mathbf{x}_\alpha)) = \{p^i_\alpha + qa^i(\mathbf{x}_\alpha), G\} = 0$$

From this relation and (3.3) we see that x^i_α and $\mathcal{R}^i_\alpha := p^i_\alpha + qa^i(\mathbf{x}_\alpha)$ are gauge invariant quantities. From these we can form some N-particle observables [6]:

$$\begin{array}{rclrcl} \mathcal{X}^i & = & \sum_{\alpha=1}^N x^i_\alpha, & \mathcal{R}^i & = & \sum_{\alpha=1}^N \mathcal{R}^i_\alpha, \\ \mathcal{L} & = & \sum_{\alpha=1}^N \epsilon^{ij} x^i_\alpha \mathcal{R}^j_\alpha, & \mathcal{C} & = & \frac{1}{2}\sum_{\alpha=1}^N (\mathbf{x}_\alpha)^2, \\ \mathcal{E} & = & \frac{1}{2m}\sum_{\alpha=1}^N (\mathcal{R}_\alpha)^2, & \mathcal{A} & = & \sum_{\alpha=1}^N x^i_\alpha \mathcal{R}^i_\alpha. \end{array} \quad (3.5)$$

From these observables we can construct the following constants of motion:

$$\mathcal{R}^i, \quad \mathcal{X}^i + \frac{1}{m}\mathcal{R}^i t, \quad \mathcal{L}, \quad \frac{1}{2}\mathcal{A} + \mathcal{E}t, \quad \mathcal{C} + \frac{1}{m}\mathcal{A}t + \frac{1}{m}\mathcal{E}t^2. \quad (3.6)$$

We can easily check that all these quantities weakly commute with the first class Hamiltonian H_{FC}.

In a non-constrained canonical theory, the constants of motion become automatically generators –through Poisson Bracket (PB)– of continuous symmetries (DST) of the system. *This is no longer true for constrained systems.* In this case, the conditions for a given function (of canonical variables and time) to be a symmetry generator can be read off from (3.1). But it is obvious that the requirements of (3.1) are much stronger than merely the fulfillment of the conditions of being a constant of motion [7]. Since the constraints are special cases of constants of motion, we will take advantage of them to modify (3.6) in order to get DST generators. A careful analysis of the commutation relations of the constants of motion (3.6) with the Hamiltonian H_{FC} shows that the substitution

$$\mathcal{R}^i_\alpha \longrightarrow \mathcal{P}^i_\alpha = \mathcal{R}^i_\alpha + \frac{q}{\mu}\epsilon^{ik}\phi^k(\mathbf{x}_\alpha) + \frac{q}{\mu m}\epsilon^{ik}\pi_0(\mathbf{x}_\alpha)\mathcal{R}^k_\alpha. \quad (3.7)$$

makes all these quantities generators of DST. We have arrived, therefore, to the following generators:

$$\begin{array}{rclrcl} \mathcal{P}^i & := & \sum_{i=1}^N \mathcal{P}^i_\alpha & \mathcal{L} & := & \sum_\alpha^N \epsilon^{ij} x^i_\alpha \mathcal{P}^j_\alpha \\ \mathcal{B} & := & \mathcal{X}^i + \frac{1}{m}\mathcal{P}^i t & \mathcal{D} & := & \frac{1}{2}\sum_{\alpha=1}^N x^i_\alpha \mathcal{P}^i_\alpha + \frac{1}{2m}\sum_{\alpha=1}^N (\mathcal{P}_\alpha)^2 t \end{array}$$

$$\mathcal{K} := \frac{1}{2}\sum_{\alpha=1}^{N}(\mathbf{x}_\alpha)^2 + \frac{1}{m}\sum_{\alpha=1}^{N} x^i_\alpha \mathcal{P}^i_\alpha t + \frac{1}{2m^2}\sum_{\alpha=1}^{N}(\mathcal{P}_\alpha)^2 t^2 \quad (3.8)$$

Since the generators of DST close under PB, we could expect our rigid symmetries generators close among themselves. However, if we perform for instance the PB of the generators $\mathcal{P}^i$,we obtain

$$\{\mathcal{P}^i, \mathcal{P}^j\} \equiv \frac{q}{m}\epsilon^{ij} G[\rho(\mathbf{x})]$$

with G the gauge generator defined in (3.2) and with the understanding that now $\dot{\rho} = \{\rho, H_{FC}\} + \partial\rho/\partial t$. This result is not that surprising because the whole set of DST contains both rigid and gauge symmetries, but it shows an interesting interplay between these two types of symmetries. Moreover, notice that $\mathcal{P}^i$ are just the generators of space translations of the system –if we switch off the interaction we obviously get the standard generators of translations for a system of free particles– and they are expected to commute. In fact *they indeed commute on the physical states* (either in the classical or the quantum version of the system, which will be considered later) *if it is taken into account that*

[6] there are obvious generalisations of these observables when the masses of the particles are not necessarily the same.

[7] It is not dificult to show that constants of motion are always observables

the gauge transformations do not modify the physical states and therefore the gauge generator is implemented as the null operator on them. In fact, all the commutations of the generators (3.8) are plagued with the presence of gauge generators in the r.h.s.. For instance:

$$\begin{aligned}
\{\mathcal{P}^i,\mathcal{L}\} &\equiv \epsilon^{ij}\mathcal{P}^j + \frac{q}{\mu}G[\sum_{\alpha=1}^{N} x^i_\alpha \delta^2(\mathbf{x}-\mathbf{x}_\alpha)] \\
\{\mathcal{B}^i,\mathcal{P}^i\} &\equiv N\eta^{ij} + \frac{q}{\mu m}\epsilon^{ij}G[t\rho(\mathbf{x})] \\
\{\mathcal{B}^i,\mathcal{L}\} &\equiv \epsilon^{ij}\mathcal{B}^j + \frac{q}{\mu m}G[t\sum_{\alpha=1}^{N} x^i_\alpha \delta^2(\mathbf{x}-\mathbf{x}_\alpha)]
\end{aligned}$$

and so on. One could think that the use of the Dirac Bracket (DB) instead of the PB would make this gauge generators terms disappear, but in fact all the commutations we can make out of the generators (3.8) are exactly the same for both parenthesis. This kind of commutation relations is an example of the general fact that rigid and gauge symmetry generators obbey an algebra as:

$$\{\text{gauge},\text{gauge}\} = \text{gauge} \tag{3.9}$$

$$\{\text{gauge},\text{rigid}\} = \text{gauge} \tag{3.10}$$

$$\{\text{rigid},\text{rigid}\} = \text{rigid}+\text{gauge} \tag{3.11}$$

as it is easily verified if one considers that the gauge generators appearing in the left hand side depend on arbitrary functions that can be set to zero.

Going back to our system, the "physical" algebra for the generators (3.8) is the one given by the PB *modulo gauge generators* (m.g.g.):

$$\begin{array}{lll@{\qquad}lll}
\{\mathcal{P}^i,\mathcal{P}^j\} & \equiv & 0 \ \text{(m.g.g.)} & \{\mathcal{P}^i,\mathcal{L}\} & \equiv & \epsilon^{ij}\mathcal{P}^j \ \text{(m.g.g.)} \\
\{\mathcal{B}^i,\mathcal{L}\} & \equiv & \epsilon^{ij}\mathcal{B}^j \ \text{(m.g.g.)} & \{\mathcal{P}^i,\mathcal{E}\} & \equiv & 0 \ \text{(m.g.g.)} \\
\{\mathcal{B}^i,\mathcal{E}\} & \equiv & -\frac{1}{m}\mathcal{P}^i \ \text{(m.g.g.)} & \{\mathcal{D},\mathcal{E}\} & \equiv & -2\mathcal{E} \ \text{(m.g.g.)} \\
\{\mathcal{K},\mathcal{E}\} & \equiv & -\frac{1}{m}\mathcal{D} \ \text{(m.g.g.)} & \{\mathcal{P}^i,\mathcal{B}^j\} & \equiv & N\delta^{ij} \ \text{(m.g.g.)} \\
\{\mathcal{P}^i,\mathcal{D}\} & \equiv & \mathcal{P}^i \ \text{(m.g.g.)} & \{\mathcal{P}^i,\mathcal{K}\} & \equiv & \mathcal{B}^i \ \text{(m.g.g.)} \\
\{\mathcal{L},\mathcal{D}\} & \equiv & 0 \ \text{(m.g.g.)} & \{\mathcal{L},\mathcal{K}\} & \equiv & 0 \ \text{(m.g.g.)} \\
\{\mathcal{B}^i,\mathcal{D}\} & \equiv & -\mathcal{B}^i \ \text{(m.g.g.)} & \{\mathcal{B}^i,\mathcal{K}\} & \equiv & 0 \ \text{(m.g.g.)} \\
\{\mathcal{D},\mathcal{K}\} & \equiv & \mathcal{K} \ \text{(m.g.g.)} & & &
\end{array} \tag{3.12}$$

This has been done at the classical level. At the quantum level we apply the correspondence principle to get the commutators of the operators from the classical DB –to get rid of second class constraints. The result is that, on the quantum physical states, either in Dirac or reduced quantisation -more on this below-, the algebra of the generators of rigid DST is formally the same as (3.12). In (3.12) we recognize the symmetry group first exhibited by Jackiw [3] which is an extended (with central charge) $2+1$ Galilei group plus time dilations (generated by $\mathcal{D}$) and time special conformal transformations (generated by $\mathcal{K}$). Here we have realized this symmetry in the phase space of the system with generators implemented as canonical transformations.

4 Quantization

4.1 The reduced quantization

The reduced quantization method eliminates the gauge degrees of freedom at the classical level, i.e., before quantization. This is usually done by introducing new constraints –the gauge fixing constraints– which render the system of constraints a second class one. Then the Dirac bracket [8] is the starting structure –in the constraint surface– to apply the correspondence principle to get the quantum commutators. This program applies in our case as follows:

We introduce as a gauge fixing the Coulomb gauge:

$$\nabla\mathbf{a} = 0 \tag{4.1}$$

Its stability under the evolution generated by (2.9) gives the new constraints

$$\partial_k\lambda_k(\mathbf{y}) = 0 \tag{4.2}$$

with λ defined in (2.8).

This last constraint determines a_0 :

$$\triangle a_0(\mathbf{y}) = -\frac{q}{\mu m}\sum_{\alpha=1}^{N}(p^i_\alpha + qa^i(\mathbf{x}_\alpha))\partial_j\delta^2(\mathbf{y}-\mathbf{x}_\alpha)\epsilon^{ij}, \tag{4.3}$$

which shows explicitely that the Weyl gauge was not available.

[8] The DB is nothing but the parenthesis associated to the pullback to the constraint surface of the original simplectic structure in phase space

In turn, the stability of $\partial_k \lambda_k(\mathbf{y}) = 0$ under H_D simply determines the Lagrange multiplier ξ_0. No more constraints appear.

So we have ended up with six second class constraints:

$$\phi^1 \simeq 0, \quad \phi^2 \simeq 0, \quad \psi \simeq 0, \quad \nabla \mathbf{a} \simeq 0, \quad \pi_0 \simeq 0, \quad \partial_k \lambda_k \simeq 0. \tag{4.4}$$

To get the physical degrees of freedom we first eliminate π^i from the first pair of constraints $\pi^i = \frac{\mu}{2}\epsilon^{ij} a_j$. Next, $\mathbf{a}$ is obtained from the second pair by solving the equations $\mu\epsilon^{ij}\partial_i a_j = q\rho$ and $\partial_i a_i = 0$. The last pair is used to eliminate the variables a_0 and π_0. The reduced (or physical) space is described by the variables $\mathbf{x}_\alpha$ and $\mathbf{p}_\alpha$. It is not dificult to verify that these variables maintain its canonical character under the DB:

$$\{x^i_\alpha, p^j_\beta\}^* = \delta_{\alpha\beta}\eta^{ij} \tag{4.5}$$

The dynamics in the constrained surface is given by H_D (with the determination just obtained for the multiplier ξ_0) through the D.B.. But under the D.B. we can put the second class constraints (i.e.: all the constraints) to zero and thus we get the reduced Hamiltonian H_R:

$$H_R = \frac{1}{2m}\sum_{\alpha=1}^{N}(\mathbf{p}_\alpha + q\mathbf{a}(\mathbf{x}_\alpha))^2 \tag{4.6}$$

with $\mathbf{a}$ given by

$$a^i(\mathbf{x}) = \frac{q}{2\pi\mu}\int d^2y\, \frac{\epsilon^{ij}(\mathbf{x}-\mathbf{y})_j}{|\mathbf{x}-\mathbf{y}|^2}\rho(\mathbf{y}) \tag{4.7}$$

which depends –through ρ– on the positions of all the particles.

So far we have concluded the setting for the reduced quantization formalism: the bracket (4.1) and the Hamiltonian (4.6) are ready to proceed, through the correspondence principle, to the standard quantization. This is just the result obtained in [2]. Our analysis emphasizes the role of the Dirac Bracket to get a consistent elimination of the unphysical degrees of freedom.

4.2 The Dirac quantization

The Dirac quantization of this system with first and second class constraints goes in two steps. We begin by eliminating the second class constraints through the Dirac Bracket, then the first class constraints are implemented as operators in our Hilbert space, thus defining the physical states as those anihilated by them.

The second class constraints ϕ^1 and ϕ^2, which are zero in the DB, allow for the elimination of π^2 and a_2 in terms of a_1 and π^1, which form a couple of canonical variables under the DB. In the Schrodinger representation [10] we associate to π^1 the operator $-i\delta/\delta a$ where a stands for a_1. To the constraint π^0 we associate the operator $-i\delta/\delta a_0$ which has to anihilate our physical states. This means that our wave functions do not depend on a_0.

The remaining constraint is implemented as an operator acting on the Schrodinger wave functional:

$$\left\{q\rho(\mathbf{x}) - \mu\frac{\partial}{\partial x^2}a(\mathbf{x}) + 2i\frac{\partial}{\partial x^1}\frac{\delta}{\delta a(\mathbf{x})}\right\}\psi[\mathbf{x}_\alpha, a(\mathbf{x})] = 0 \tag{4.8}$$

As of now, it is not clear to us whether the Dirac quantization, which has equation (4.8) as starting point, is equivalent or not to the reduced formulation already introduced in the literature. Indeed, althought the number of field degrees of freedom –one– for $a(\mathbf{x})$ matches with the number of equations –one– in (4.8), it is not obvious that this degree of freedom can be completely eliminated in terms of the particle degrees of freedom.

5 Conclusions

The complete canonical analysis of a non-relativistic ideal gas of anyons and its continuous symmetry transformations shows an interesting interplay between its rigid and gauge symmetries. This is the main result of the present paper, and gives an example of a typical structure of the algebra of the full group of continuous symmetries which is not currently displayed. This special feature of our system illuminates a general point of the Dirac quantization programme (we restrict ourselves to a case with first class constraints only): Since the physical states must be invariant under the gauge transformations, they have to be anihilated by the gauge generators of the theory. But since the gauge generators are constructed as combinations of the first class constraints, we finally arrive to the usual requirement that these constraints are implemented as operators anihilating the physical states. The result –the Dirac quantization–

is the same but the conceptual point has shifted from simply implementing the constraints to the more physical requirement as to the null action of the gauge generators on the physical states. Once this is done, there remains the quocient group of rigid symmetries as the candidate for the symmetry group of the quantum system.

Besides the main result mentioned above, we sketch the reduced and the Dirac quantization for our system. the reduced quantization is introduced once the gauge fixing constraints are given. Then, the Dirac Bracket structure for the full set of constraints makes the system ready for quantization through the correspondence principle. In the Dirac quantization, with no gauge fixing at all, there is only one constraint equation, which is (4.8). At this point, the equivalence between both quantization procedures is not clear and deserves further investigation.

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