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STATISTICS, AHARONOV-BOHM AND HOLES

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Abstract

It is shown that the existence of non-bosonic quantum statistics does not require a topologically non-trivial configuration space. This is also so for the Aharonov-Bohm effect. A hole in configuration space is only necessary if the wavefunction is multiple-valued and there is no point where the action corresponding to paths going through this point is either undefined or divergent or where discontinuities appear in momentum space when the path sweeps over the point. The role of boundary conditions at these special points and their meaning as contact or zero-range interactions compatible with self-adjointness are explained.

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I. Introduction

In defining indistinguishable particles in the Feynman path integral formalism for nonrelativistic quantum mechanics Laidlaw and DeWitt [1] explicitely exclude points of coincidence of two or more particles. This allows for a homotopical classification of closed paths in configuration space. Weighing paths belonging to different homotopy classes with different phases, which correspond to one-dimensional unitary representations of the fundamental group of the multiply connected configuration space, leads then to the non-bosonic quantum statistics; in two space-dimensions to anyons and fermions [2]. As the authors remark, the question of whether two point particles can simultaneously occupy the same point in space is one they do not address, they just say that excluding points of coincidence leads straightforwardly to physically meaningful results. No doubt these remarks reflect some uneasiness about the assumption which, since then, is usually made in defining indistinguishable particles in the path integral formalism, i.e. excluding points of coincidence, and they leave open the question on whether this assumption is necessary or disposable.

Leinaas and Myrheim mention, in their work on identical particles [2], that the points of coincidence in the configuration space of identical particles are singular and that when they are retained the space is simply connected. Then they proceed excluding these points because the multiple connectedness of the resulting space suggests a relation to the different statistics.

This attitude has been taken over by many of the authors studying quantum statistics [3]- [6], because the condition which allows contributions corresponding to different paths to carry a different phase is that these paths cannot be deformed continuously into one another and it is generally believed that this requires a multiply connected space. As we will see, multiple connectedness is a sufficient but not a necessary condition. It is also often stated that if identical particles were allowed to occupy the same point in configuration space they would then be necessarily bosonic [3], [4], [5]. Again

some uneasiness is reflected by the need some authors feel in justifying this discarding of trajectories for which particles meet by invoking a hard core repulsion [3], or the centrifugal barrier which always exists for non-bosonic identical particles in two dimensions [6].

Throwing away some configurations with only a teleological justification, i.e. as a means for introducing quantum statistics, seems a too ad hoc foundation for quantum statistics. Furthermore, even assuming that coinciding configurations do not contribute to the path integral, one could always just keep them; would that rule out statistics other than the bosonic? Finally, translating this exclusion to one-dimensional quantum mechanics would forbid exchanging identical particles to start with!

It is our aim here to point out that the essential characteristic of touching paths for allowing for non-bosonic statistics is not their exclusion but that the point of contact is a singular point where configuration space is not differentiable and which is such that if excluded (but it is not) configuration space becomes homotopically non-trivial. Paths can be deformed continuously in configuration space from one homotopy class to a contiguous one, but in doing so they jump discontinuously in phase space , and thus allows for discontinuities in the action when the path sweeps over the singular point. The discontinuity is $\hbar$ times the exponent of the phase characterizing non-bosonic statistics.

There is thus no need to exclude coinciding configurations, and we agree in this with Bourdeau and Sorkin [7], who very recently took up this issue in two space-dimensions. They conclude that discarding coinciding configurations leads to a loss of physical information, because there is (for bosons and anyons) a one-parameter family of possible self-adjoint extensions of the hamiltonian. They also propose a plausible limiting procedure which selects one unique extension, which turns out to be the one for which the wavefunction remains finite at the point of overlap. What seems to have been overlooked is that the need for self-adjointness is independent of the inclusion or exclusion of the point of overlap; the quantum mechanics on $\Re^+$ and on cones have

been thoroughly and beautifully worked out by Farhi and Gutmann [8] and by Kay and Studer [9], and it is evident that whether the origin is included or not, or whether the tip of the cone is included or not is irrelevant: the hamiltonian better be self-adjoint anyhow. These authors show furthermore that the behaviour of the wavefunction at the origin or tip depends only on whether contact interactions are present or not (see also [10] and [11]). The same has been found in the context of statistics in two space-dimensions, and anyons in particular [12], [13]: wavefunctions are singular at the point of coincidence if and only if nontrivial contact interactions are present, they are otherwise vanishing for anyons and finite for bosons. All this bring us to yet another issue: is there a relation between excluding points of coincidence and excluding contact interactions? It is clear from references [8], [9], [12] and [13] that there is none, but given the confusion we will also come shortly back to this. Let us finally insist that one can, if one wishes, disregard touching path contributions, but it is neither necessary nor physically justified, it is at most practical because one comprehends quicker the introduction of phases in a space with a hole than in a space with a singular point.

There are several other closely related issues which have caused confusion which we take up here and, hopefully, clarify. They have to do with the definition of spin in two space-dimensions, with the relation between the Aharonov-Bohm effect and multiple connectedness, with the conditions which allow wavefunctions to be multiple-valued and with contact or zero range interactions, boundary conditions and self-adjoint extensions.

We will work in two space-dimensions throughout. Section II reviews the quantum mechanics of one particle when a point in $\Re^2$ is somehow singled out. After this we deal with two identical particles and conclude in section III.

II. One particle and one special point

There are three quantum mechanical problems on a plane in which a point is singled out. Two of them, the Aharonov-Bohm (AB) [14] and the Dirac delta, contact or zero range interaction (ZR) [10], [11] problems are defined on $\Re^2$, the third, the hole (H) [1] problem is defined on $\Re^2 - \{\vec{0}\}$, which is multiply connected.

The AB problem is given by a vector gauge field

$$\vec{A} = \frac{\Phi}{2\pi}\vec{\nabla}\varphi \equiv \frac{\Phi}{2\pi}\frac{\hat{u}_\varphi}{r}, \qquad \varphi = \arctan\frac{y}{x}, \qquad r = \sqrt{x^2+y^2} \tag{1}$$

which is singular at the origin and a pure gauge elsewhere, and which leads to a magnetic field

$$B \equiv \epsilon^{ij}\partial_i A^j = \Phi\delta(\vec{r}) \tag{2}$$

so that Φ is the magnetic flux. If the particle has electric charge q the propagator is given by

$$G_{AB}(\vec{r_2}, t_2; \vec{r_1}, t_1) = \int_\Gamma [d\vec{r}] exp\left[\frac{i}{\hbar}\int_{t_1}^{t_2} dt\left(\frac{M}{2}\dot{\vec{r}}^2 + \frac{q}{c}\dot{\vec{r}}\cdot\vec{A}\right)\right] \tag{3}$$

with $\Gamma \equiv \{\vec{r}(t)\}$ the set of all paths which go from $\vec{r_1} = \vec{r}(t_1)$ to $\vec{r_2} = \vec{r}(t_2)$. With $\Gamma \equiv \overset{\circ}{\Gamma} \bigcup \Gamma_{\{\vec{0}\}}$ a given path $\gamma \in \Gamma_{\{\vec{0}\}}$ if it crosses the origin; if not $\gamma \in \overset{\circ}{\Gamma}$. As $\Re^2$ is translationally invariant the measure $[d\vec{r}]$ is homogeneous.

The wavefunction, for which

$$\psi(\vec{r_2}, t_2) = \int d\vec{r_1} G(\vec{r_2}, t_2; \vec{r_1}, t_1)\psi(\vec{r_1}, t_1) \tag{4}$$

satisfies the Schrödinger equation for the hamiltonian

$$H_{AB} = \frac{1}{2M}\left[-i\hbar\vec{\nabla} - \frac{q}{c}\vec{A}\right]^2 = -\frac{\hbar^2}{2M}\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial}{\partial\varphi} - i\frac{q}{2\pi}\frac{\Phi}{\hbar c}\right)^2\right] \tag{5}$$

and is single-valued,

$$exp\left[-i2\pi\frac{L}{\hbar}\right]\psi(\vec{r}, t) = \psi(R^{-1}(2\pi)\vec{r}, t) = \psi(\vec{r}, t) \tag{6}$$

here $L \equiv -i\hbar\frac{\partial}{\partial\varphi}$ is the angular momentum operator. Since it is sometimes stated that the AB effect leads to a multiple-valued wavefunction let us dwell on this a bit more. We follow here an approach different from reference [15], where this issue was put straight.

One can write (3) as

$$G(\vec{r_2}, t_2; \vec{r_1}, t_1) \equiv \overset{\circ}{G}(\vec{r_2}, t_2; \vec{r_1}, t_1) + G_{\{\vec{0}\}}(\vec{r_2}, t_2; \vec{r_1}, t_1) \tag{7}$$

in a straightforward notation. Using polar coordinates, and for $\vec{r_1} \neq \vec{0}, \vec{r_2} \neq \vec{0}$,

$$\overset{\circ}{G}(\vec{r_2}, t_2; \vec{r_1}, t_1) = \sum_{n\in Z}\overset{\circ}{G}(r_2, \varphi_2 + 2\pi n, t_2; r_1, \varphi_1, t_1) \tag{8}$$

with $0 \leq \varphi_i < 2\pi$, and, from (1) and (3)

$$\overset{\circ}{G}_{AB}(r_2, \varphi_2 + 2\pi, t_2; r_1, \varphi_1, t_1) = exp\left[\frac{iq\Phi}{2\pi\hbar c}(\varphi_2 + 2\pi n - \varphi_1)\right]\cdot$$

$$\cdot\overset{\circ}{G}_o(r_2, \varphi_2 + 2\pi n, t_2; r_1, \varphi_1, t_1) \tag{9}$$

where the subscript o means $\vec{A} = 0$. Under a full rotation

$$exp\left[-i2\pi\frac{L}{\hbar}\right]\overset{\circ}{G}(r_2, \varphi_2 + 2\pi n, t_2; r_1, \varphi_1, t_1) = \overset{\circ}{G}(r_2, \varphi_2 + 2\pi(n-1), t_2; r_1, \varphi_1, t_1) \tag{10}$$

so that $\overset{\circ}{G}(\vec{r_2}, t_2; \vec{r_1}, t_1)$ given by (8) remains invariant.

Let us now write

$$G_{\{\vec{0}\}}(\vec{r_2}, t_2; \vec{r_1}, t_1) \equiv \int_{t_1}^{t_2} dt\mathcal{G}_{\{\vec{0}\}}(\vec{r_2}, t_2; t; \vec{r_1}, t_1) \tag{11}$$

where t is the time of the last crossing of the origin. Then

$$\mathcal{G}_{\{\vec{0}\}}(\vec{r_2}, t_2; t; \vec{r_1}, t_1) = \int d\vec{r'}\,\overset{\circ}{G}(\vec{r_2}, t_2; \vec{r'}, t')\mathcal{G}_{\{\vec{0}\}}(\vec{r'}, t'; t; \vec{r_1}, t_1) \tag{12}$$

with $t < t' < t_2$. It now follows immediately that as $\overset{\circ}{G}$ is invariant under a full rotation so is $\mathcal{G}_{\{\vec{0}\}}$ and thus finally $G_{\{\vec{0}\}}$. The invariance of G under a full rotation is thus proven and (6) follows immediately. One should not

forget that the idealized solenoid (2) is a limit of regular solenoids for which certainly wavefunctions have to be single-valued.

There are however solutions of the Schrödinger equation corresponding to (5), [16]

$$\overline{\psi}(\vec{r_2}, t_2) = exp\left[\frac{iq}{\hbar c}\int_{t_1}^{t_2} \dot{\vec{r}} \cdot \vec{A} dt\right] \psi_0(\vec{r_1}, t_1) \tag{13}$$

where the integral is performed along a path $\gamma \in \overset{\circ}{\Gamma}$ and where ψ_0 is a solution of the free Schrödinger equation, and which is obviously not single-valued:

$$exp\left[-i2\pi\frac{L}{\hbar}\right]\overline{\psi}(\vec{r}, t) = exp\left[-i\frac{q\Phi}{\hbar c}\right]\overline{\psi}(\vec{r}, t) \tag{14}$$

This function $\overline{\psi}$ is thus not the correct wavefunction. It is only correct locally, but not in any finite region of space.

Since the wavefunction is single-valued angular momentum is quantized by integers and separation of variables

$$\psi(\vec{r}) = e^{im\varphi}\phi_\nu(r), \qquad m \in Z, \qquad \nu \equiv \mid m - \frac{q\Phi}{2\pi\hbar c} \mid \tag{15}$$

leads to the radial hamiltonian

$$H_{AB,r} = -\frac{\hbar^2}{2M}\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{\nu^2}{r^2}\right] \tag{16}$$

whose regular solution satisfies

$$\phi_\nu(0) = 0, \qquad \nu \neq 0; \qquad 0 < \phi_0(0) < \infty \tag{17}$$

and whose singular solution is irrelevant for the pure AB problem but will soon play an important role. These formulae solve completely the AB problem in the operator formalism.

In the path integral formalism there is a hitch, however, because contributions coming from paths which cross the origin are intrinsically undefined, as so is the change in φ when the path goes through the origin. We do not

know what $G_{AB,\{\vec{0}\}}$ is. Recall that a propagator can be written as

$$G(\vec{r_2}, t_2; \vec{r_1}, t_1) = \sum\!\!\!\!\!\!\int \psi_E^*(\vec{r_1}, t_1)\psi_E(\vec{r_2}, t_2) \tag{18}$$

where ψ_E are eigenfunctions of the hamiltonian and one sums and/or integrates over the whole spectrum. Issues of normalizability and degeneracy can easily be avoided. The only arbitrariness in (18) lies in the boundary condition which the eigenfunctions have to satisfy at the origin, and which are given by (17) for the pure AB problem . Thus $G_{AB,\{\vec{0}\}}$ should be fixed by (17). This is indeed the case. Recall

$$G(\vec{r_2}, t_2; \vec{r_1}, t_1) = \int d\vec{r} G(\vec{r_2}, t_2; \vec{r}, t) G(\vec{r}, t; \vec{r_1}, t_1), \qquad t_1 < t < t_2 \tag{19}$$

which is independent ot t. Then the contributions from paths which at time t pass at a distance $R \ll min(r_1, r_2)$ is

$$dG_{\{\vec{0},R\}}(\vec{r_2}, t_2; t; \vec{r_1}, t_1) \simeq v(t) R dt G(\vec{r_2}, t_2; \vec{0}, t) G(\vec{0}, t; \vec{r_1}, t_1) \tag{20}$$

where $v(t)$ is the mean velocity of paths passing near the origin at time t and where continuity of the r.h.s. of (20) has been assumed. Integrating over time and taking $R \to 0$ leads to

$$G_{\{\vec{0}\}}(\vec{r_2}, t_2; \vec{r_1}, t_1) \simeq \lim_{R\to 0} R \int_{t_1}^{t_2} dt v(t) G(\vec{r_2}, t_2; \vec{0}, t) G(\vec{0}, t; \vec{r_1}, t_1) \tag{21}$$

which vanishes unless the assumption of continuity fails, more specifically unless the propagator, when the initial or the final point is at the origin, diverges. This is however excluded by (17) so that one concludes $G_{AB,\{\vec{0}\}} = 0$. This, incidentally, shows that there is no need of screening the solenoid in the AB problem, as it is usually done [14], [17], [18], as paths going through it do not contribute. One might still think that choosing (17) is equivalent to screening, but we will soon see that this is not so: choosing (17) means not having a ZR interaction, that is, having a pure AB problem.

The second problem is the Dirac delta, contact interaction or zero range problem [10], [11]

$$V(\vec{r}) = g\delta(\vec{r}) \tag{22}$$

which, as it stands, is either trivial (no physical interaction) when $g > 0$, or collapses (ground state of infinite negative energy) for $g < 0$. The proper treatment of (22) requires either a regularization or, equivalently, a boundary condition at the origin.

The first approach [10] is based on substituting (22) by

$$V_R(r) = \frac{\hbar^2}{MR^2\left(\ln\frac{R}{R_o} + \gamma\right)}\theta(R-r), \qquad R_o > 0, \qquad R > 0 \tag{23}$$

where R is the regulating length which is taken to zero at the end, γ is Euler's constant, R_o is an arbitrary lenght scale and θ is the step function. The potential (23) is, when $R \to 0$, of vanishing range and leads to a finite bound state energy and to a finite s-wave phase shift. There exists no other circular well potential with these features.

Equivalently (22) can be substituted by a boundary condition for $\phi_m(r)$ [11]

$$\lim_{r\to 0}\frac{\phi_0(r)}{\ln\frac{r}{R_o}} = \frac{1}{\gamma}\lim_{r\to 0}\left(\phi_0(r) - \lim_{r'\to 0}\frac{\phi_0(r')}{\ln\frac{r'}{R_o}}\ln\frac{r}{R_o}\right), \qquad \phi_{m\neq 0}(0) = 0 \tag{24}$$

which leads to exactly the same physics as (23). This boundary condition is the most general which ensures that the hamiltonian, in fact its extensions, as there is an arbitrary parameter in (24), is self-adjoint. When $R_o \to \infty$ the interaction disappears and the s-wave wavefunction becomes regular at the origin: non-trivial contact interactions are unambiguously characterized by a singular (but certainly normalizable!) wavefunction, which is what (24) implies for $0 < R_o < \infty$.

Notice that exactly as the AB problem, the ZR problem is defined on $\Re^2$, origin included; in the regularized approach eq. (23) leads to an absolutely

standard problem of quantum mechanics on $\Re^2$, only that at the end the limit $R \to 0$ is taken. The propagator is given by

$$G_{ZR}(\vec{r_2}, t_2; \vec{r_1}, t_1) = \lim_{R\to 0}\int_\Gamma [d\vec{r}] exp\left[\frac{i}{\hbar}\int_{t_1}^{t_2} dt\left(\frac{M}{2}\dot{\vec{r}}^2 - V_R(\vec{r})\right)\right]$$

$$\equiv \lim_{R\to 0}\int_\Gamma [d\vec{r}]_R exp\left[\frac{i}{\hbar}\int_{t_1}^{t_2} dt\left(\frac{M}{2}\dot{\vec{r}}^2\right)\right] \tag{25}$$

This proves that a boundary condition as (24) is equivalent to a redefinition of the measure around the origin. This is, for $\Re^+$, very carefully explained in references [8], [19]. Writing

$$G_{ZR}(\vec{r_2}, t_2; \vec{r_1}, t_1) = \lim_{R\to 0}\overset{o}{G}_R(\vec{r_2}, t_2; \vec{r_1}, t_1) + \lim_{R\to 0} G_{\{R\}}(\vec{r_2}, t_2; \vec{r_1}, t_1) \tag{26}$$

where we distinguish paths which pass at a distance less than R from the origin from the others, one finds, from (25)

$$G_{ZR}(\vec{r_2}, t_2; \vec{r_1}, t_1) = G_0(\vec{r_2}, t_2; \vec{r_1}, t_1) + G_{ZR,\{0\}}(\vec{r_2}, t_2; \vec{r_1}, t_1) \tag{27}$$

so that $G_{ZR,\{\vec{0}\}} \neq 0$. Thus paths going through the origin contribute if and only if there is a nontrivial ZR there.

The third problem is a free particle on $\Re^2 - \{\vec{0}\}$, i.e. on a plane with a hole. The propagator is given by [1]

$$G_\theta(\vec{r_2}, t_2; \vec{r_1}, t_1) = \sum_{n\in Z} e^{in\theta}\overset{o}{G}_o(r_2, \varphi_2 + 2\pi n, t_2; r_1, \varphi_1, t_1) \tag{28}$$

with $0 \leq \theta < 2\pi$. From this, (4) and (10) one finds that the wavefunction is multiple-valued

$$exp\left[-i2\pi\frac{L}{\hbar}\right]\psi_\theta(\vec{r}, t) = e^{i\theta}\psi_\theta(\vec{r}, t) \tag{29}$$

One can make the wavefunction single-valued through the unitary transformation

$$\psi(\vec{r}, t) = e^{\frac{i\theta\varphi}{2\pi}}\psi_\theta(\vec{r}, t) = \psi(R^{-1}(2\pi)\vec{r}, t) \tag{30}$$

so that ψ satisfies (4) with

$$G_H(\vec{r_2}, t_2; \vec{r_1}, t_1) \equiv e^{\frac{i\theta\varphi_2}{2\pi}} G_\theta(\vec{r_2}, t_2; \vec{r_1}, t_1) e^{\frac{-i\theta\varphi_1}{2\pi}} \tag{31}$$

which from (8) and (9) leads to

$$G_H(\vec{r_2}, t_2; \vec{r_1}, t_1) = \overset{\circ}{G}_{AB}(\vec{r_2}, t_2; \vec{r_1}, t_1) \tag{32}$$

with $\theta \equiv \frac{q\Phi}{\hbar c}$. Since $G_{AB,\{\vec{0}\}} = 0$ the H problem is equivalent to the AB problem, $G_{AB} = G_H$. This might seem surprising, as in one case one has a missing point and in the other a singular magnetic field. But there is nothing which allows to distinguish formally these two problems. In both cases, all what the origin does it to allow for a discontinuity in the action when the path, which is continuously deformed, sweeps over the origin. In the H problem the discontinuities, i.e. the θ in (28), are allowed because the hole does not actually allow the path to sweep over the origin, and in the AB problem they are allowed because, as mentioned before, paths which cross the origin lead to undefined actions.

One might still think that a hole requires making sure that probability is not lost at the hole, i.e. that wavefunctions are such that the hamiltonian is self-adjoint. This has been done behind the scene in (28), as (30) satisfies a Schrödinger equation corresponding to the hamiltonian (16) and (28) corresponds to (17) or, more properly for $\Re^2 - \{\vec{0}\}$, to

$$\lim_{r\to 0} \phi_\nu(r) = 0, \qquad \nu \neq 0; \qquad 0 < \lim_{r\to 0} \phi_0(r) < \infty \tag{33}$$

The AB problem and the H problem are two versions of the same mathematics. The multiple-valued unitary transformation trades a singular magnetic field for a hole. The combined H-AB problem, if one chooses not to transform one into the other, has been studied in reference [20].

Let us now combine the AB/H problem with the ZR problem. This can be done starting from any of them. Consider, say, the AB/H problem [21]. We have choosen (33) as boundary conditions, because they correspond to the absence of a physical ZR. The general boundary conditions which lead to all the self-adjoint extensions are generalizations of (24) which include θ.

The relevant formulae are, for the regular wavefunction $u_\nu(r) \equiv r^\nu \phi_\nu(r)$, [12]

$$\lim_{r\to 0}\left(u_\nu(r) \pm (R_o)^{2\nu}\frac{du_\nu(r)}{d(r^{2\nu})}\right) = 0, \qquad 0 < \nu < 1$$
$$\lim_{r\to 0} u_\nu(r) = 0, \qquad \nu \geq 1 \tag{34}$$

with $R_o > 0$ an arbitrary lenght scale. The case $\nu = 0$ is given by (24). The limit $R_o \to 0$ in the lower sign expression of (34) corresponds to the regular solution, which vanishes at the origin, (33), for which no ZR is present. All other values of R_o correspond to singular but, for $0 < \nu < 1$, normalizable solutions, reflecting the presence of a physical ZR. Again, and paralleling (25), this corresponds to a modification of the measure around the origin which is natural in the sense that the origin is singled out anyhow. The equivalent of (26) leads to

$$G_C(\vec{r_2}, t_2; \vec{r_1}, t_1) = G_{AB}(\vec{r_2}, t_2; \vec{r_1}, t_1) + G_{C,\{\vec{0}\}}(\vec{r_2}, t_2; \vec{r_1}, t_1) \tag{35}$$

where the subscript C means complete or combined, and $G_{C,\{\vec{0}\}}$ corresponds to paths which pass at vanishing distance from the origin, and which is nonvanishing in presence of a nontrivial ZR.

It is amusing to notice that one can as well start from the ZR problem, in its regularized version (23). A standard path which goes through the origin will get a contribution to the action from (23)

$$\int_{t_1}^{t_2} dt V_R(r) \propto \frac{R}{v} V_R(0) \tag{36}$$

where v is the average velocity and which blows up when the $R \to 0$. The action being divergent when going through the origin allows a discontinuity of it when the path sweeps the origin: this is precisely what the hole does, (28). Of course, the final outcome is the same combined problem as before.

This is an instance where multivaluedness appears without multiconnectedness. We will see in the next section that the phases associated to quantum statistics are another example of multivaluedness in a simply connected configuration space.

One can add to whatever has been done up to now a regular, rotationally invariant potential in a straightforward manner, whithout changing any of our results.

One last, but important point concerning one particle. The rotation group on $\Re^2$ is $SO(2)$, which is abelian. Its unitary irreducible representations are either one-dimensional or infinite-dimensional. We are limiting ourselves here to the former, i.e. wavefunctions do not have components. The total angular momentum would be $J = L + s$, $s \in \Re$ being the spin. Then, under a 2π rotation one would have

$$exp\left[-i2\pi\frac{J}{\hbar}\right]\psi(r,\varphi) = exp\left[-i2\pi\frac{s}{\hbar}\right]\psi(r,\varphi-2\pi) = exp\left[-i2\pi\frac{s}{\hbar}\right]\psi(r,\varphi) \tag{37}$$

One should notice the difference between this and (29): no point is singled out in (37) which holds for rotations around any point, while (29) only holds for rotations around the origin, which is a special point. There seems to be some confusion related to this too [5].

III. Two identical particles. Conclusions

Consider two indistinguishable particles. Neglecting the c.o.m. coordinates and identifying configurations which are indistinguishable, $\vec{r} \equiv -\vec{r}$, $\vec{r}$ being the relative coordinates, one obtains a configuration space $\overline{\Re^+} \times \Re P^1$. Here $\overline{\Re^+}$ are the non-negative real numbers and $\Re P^1$ is the one-dimensional real projective space, which is equivalent to the one-dimensional circle S^1. This configuration space can be represented as a cone [2]. There is one singular point in this space, different from all the others, which is $\vec{r} = 0$, the apex of the cone, due to which the space is not a manifold . A two-dimensional space with one distinguishable point can allow the introduction of phases even if the point is included in the space, unlike what happens with $\Re^2$, which has no special point. This is so because when one continuously deforms a path in configuration space until it goes through the apex, $\dot{\vec{r}}$ has a discontinuity

there so that the deformation is not continuous in momentum space. Thus continuity arguments do not allow to relate the actions of nearby paths which are on different sides of the apex, because these paths are nearby in configuration space but not in phase space. There are continuous deformations in phase space which connect these paths, but of course these deformations are not small and thus the corresponding actions can differ in finite amounts. This is then where a phase comes in, paralleling (28), substituting $2\pi n$ by πn as a full rotation around the apex corresponds to $\varphi \to \varphi + \pi$. Since a full rotation exchanges the identical particles once, the equivalent of (29) leads to non-bosonic statistics for $\theta \neq 0$ [22]. Fermions correspond to $\theta = \pi$, bosons to $\theta = 0$, and anyons to all others values of θ.

One can now proceed to make the wavefunction single-valued as in (30), only that recalling that we have now two identical particles the introduction of the charge would introduce an unwanted Coulomb interaction, so that one should take $q \to 0$ with $\theta \equiv \frac{q\Phi}{\hbar c}$ fixed. Also, although this transformation introduces a gauge field paralleling (1) it does not introduce a magnetic field in the sense of (2), as derivatives are not defined at the apex [2]. This only helps, as the need of taking $\Phi \to \infty$ would make the magnetic field doubly singular. In short, it is physically more convenient not to use the electromagnetic analogy and just state that multivaluedness characterizing non-bosonic statistics can be traded for an interaction corresponding to the potential $V_\theta \equiv \frac{\hbar}{\pi}\theta\dot{\varphi}$. Notice the quantum character of the potential, as corresponds to the fact that quantum statistics has no classical counterpart.

The analogy with the hole problem also holds with respect to the ZR problem: exactly as at the hole, one has to make sure that no probability is lost at the apex. The issue of self-adjoint extensions on the cone has been elegantly solved (for $\theta = 0$) in reference [9]. For $\theta \neq 0$ the solution [12] parallels the study of the combined problem of the previous section, without modification, except the obvious one that $\nu \equiv \mid m - \frac{\theta}{\pi} \mid$, $\frac{m}{2} \in Z$.

In conclusion: Neither non-bosonic statistics nor the Aharonov-Bohm effect require multiply connected configuration spaces implemented by holes.

The discontinuity of velocities when the path goes through the point of coincidence of two identical particles (the apex of the cone) shows that paths which are nearby in configuration space and lead to nearby actuons are far apart in phase space and this is enough to allow for fermions and anyons. It is only when one requires that configuration space be a manifold that holes appear [23]. The idealized solenoid of the Aharonov-Bohm effect does not require screening; its single-valued wavefunction can be transformed into a multiple-valued one by singular multiple-valued gauge transformation, which gauges away the gauge field and drills a hole at the location of the magnetic field. The hole is only a consequence of a very special transformation. Holes are only required by multivaluedness if there are no points which lead to undefined or divergent actions, or which lead to discontinuities in momentum space.

Special points of any of these types always require to make sure through boundary conditions that the hamiltonian is self-adjoint. One generally has a choice of self-adjoint extensions: of these all but one imply a priori unrelated attractive contact interaction, always characterized by a singular (but normalizable) wavefunction. One can do things in the reverse order, with the same result: a contact interaction allows, as it leads to a divergent action, to introduce multivaluedness without a hole.

That excluding coinciding configurations is arbitrary and unnecessary has been put forward also in a new approach to quantum mechanics based on diffeomorphism groups [24].

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