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Perturbative Renormalization in Quantum Mechanics

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Abstract

Some quantum mechanical potentials, singular at short distances, lead to ultraviolet divergences when used in perturbation theory. Exactly as in quantum field theory, but much simpler, regularization and renormalization lead to finite physical results, which compare correctly to the exact ones. The Dirac delta potential, because of its relevance to triviality, and the Aharonov-Bohm potential, because of its relevance to anyons, are used as examples here.

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An important instrument of present days physics, quantum field theory, is permeated by short distance singularities, which are thoroughly understood in the framework of regularized and renormalized perturbation theory. Our non-perturbative understanding, mainly lattice-bound, is not so firm, and exact solutions for physically relevant theories are basically absent. Quantum mechanics does not usually have short distance singularities, but they show up if the potential is singular enough (but not too much: the Hamiltonian should be bounded from below and self-adjoint). Then regularization and renormalization consistently cure the short distance singularities and lead to physical results independent of the precise regulator and independent of the precise renormalization scheme. Furthermore, renormalized perturbation theory reproduces the exact solutions for physical magnitudes.

The two problems we have chosen to study are best considered in two dimensions. First, the Dirac delta, zero-range or contact interaction, because already its exact solution is most conveniently obtained by regulating and renormalizing, because one can perform perturbation theory to all orders, and because of its relevance to triviality [1]. Second, the Aharonov-Bohm potential, because it perturbatively induces a new interaction absent in the exact setting and because of being at the foundation of anyon physics [2], [3].

Recall that the Schrödinger equation (throughout this letter we will use $2M = \hbar = 1$) for positive energies $E = k^2$ is equivalent to the Lippman-Schwinger equation

$$\psi(\vec{r}) = \psi_0(\vec{r}) - \int d^2r' G_{k,+}(\vec{r}-\vec{r'}) V(\vec{r'}) \psi(\vec{r'}),$$

where ψ_0 is a solution of the free Schrödinger equation, and $G_{k,+}$ is the free propagator. In the two dimensional case that we are interested in

$$G_{k,+}(\vec{r}-\vec{r'}) = \frac{i}{4} H_0^{(1)}\left(k|\vec{r}-\vec{r'}|\right),$$

$H_0^{(1)}$ being the first Hankel function of zero order, which in the asymptotic limit behaves as an outgoing wave and has the short distance behavior

$$G_{k,+}(\vec{\epsilon}) \underset{\epsilon\to 0}{\sim} \frac{i}{4} - \frac{1}{2\pi}\left(\ln(\frac{k\epsilon}{2}) + \gamma\right), \qquad \vec{\epsilon} \equiv \vec{r} - \vec{r'}, \tag{1}$$

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where γ is Euler's constant. One particular way to find the solution of the Schrödinger equation consists in solving iteratively the Lippman-Schwinger equation, so that

$$\begin{aligned}\psi(\vec{r}) &= \psi_0(\vec{r}) - \int d^2\vec{r'} G_{k,+}(\vec{r}-\vec{r'}) V(\vec{r'}) \psi_0(\vec{r'}) \\ &+ \int d^2\vec{r'} d^2\vec{r''} G_{k,+}(\vec{r}-\vec{r'}) V(\vec{r'}) G_{k,+}(\vec{r'}-\vec{r''}) V(\vec{r''}) \psi_0(\vec{r''}) + \dots \quad (2)\end{aligned}$$

In the scattering problem, the above approach together with the asymptotic condition

$$\psi(\vec{r}) \underset{r\to\infty}{\sim} e^{i\vec{k}\cdot\vec{r}} + \frac{e^{i\left(kr+\frac{\pi}{4}\right)}}{\sqrt{r}} f(k,\theta), \quad (3)$$

allows to define the Born series to compute the scattering amplitude f in perturbation theory. In this context, it is easy to understand that for a potential $V(\vec{r})$ which is singular enough when $r \to 0$, the Born series will contain divergent coefficients, as the propagator exhibits the short distance logarithmic behavior (1). We will illustrate this fact by studying the perturbative approach of two different potentials which can be solved exactly but have that peculiarity.

Let us focus first our attention on the perturbative approach to the Dirac delta potential, zero-range or contact interaction. Let us recall that contact interactions have been studied exactly in the literature [4]-[10]. Delta function potentials in two and three dimensions are a nice and simple example where the concepts of regularization and renormalization, which are common in quantum field theory but hardly used in quantum mechanics, are very useful for obtaining non-trivial results. One correct treatment of the problem requires first regularization. This can be achieved by substituting the delta function potential $g_0\delta^{(2)}(\vec{r})$, $g_0 < 0$, into the Schrödinger equation by, say,

$$V_R(r) = \frac{g_0}{2\pi R}\delta(r-R); \qquad R > 0, \quad (4)$$

which in the $R \to 0$ limit reproduces the original potential. The Schrödinger equation for negative energies can then easily be solved by well-known methods, finding for small R one binding energy $E_0(R) = -\frac{4}{R^2}e^{-2\gamma}e^{4\pi/g_0}$. For $R \to 0$ the bound state energy goes to $-\infty$. However, one can choose a

coupling constant depending on the regulator in such a way that when the regulator is removed the binding energy stays finite. This happens when

$$\frac{1}{g_0(R)} = \frac{1}{2\pi}\ln\frac{R}{R_0}, \qquad R < R_0,$$

where R_0 is a length which measures the strength of the interaction. Because one is used to describe interactions with dimensionless couplings one introduces the renormalized coupling constant $g_r(\mu)$, which depends on an arbitrary momentum scale μ,

$$\frac{1}{g_r(\mu)} \equiv \frac{1}{g_0(R)} - \frac{1}{2\pi}\left(\ln\frac{\mu R}{2} + \gamma\right), \quad (5)$$

in terms of which the binding energy reads $E_0 = -\mu^2 e^{4\pi/g_r(\mu)}$. Notice that E_0 depends on $g_r(\mu)$, but not on μ, that is, the explicit and implicit (through g_r) dependence on μ of E_0 cancel. Physics is determined by the value of $g_r(\mu)$ at an arbitrary value of μ.

Similar features are found in the scattering problem, $E = k^2 > 0$. Only when the coupling constant depends on R as above is it possible to find the finite and renormalized scattering amplitude

$$f(k,\theta) = \left(\frac{\pi}{2k}\right)^{1/2} \frac{1}{\ln\frac{k}{\mu} - \frac{i\pi}{2} - \frac{2\pi}{g_r(\mu)}}. \quad (6)$$

Contact interactions have also been studied in the literature with the use of some other regularizations, such as a circular well potential [6], [9], or in momentum space [4], [5], [7], arriving to the conclusion that regularization and renormalization is a powerful approach for obtaining non-trivial results which are furthermore independent of the particular regulator used. It is also important to stress that there is a large amount of arbitrariness in defining $g_r(\mu)$: a specific definition, as (5), defines a renormalization scheme, but one could have added a constant to the r.h.s. of (5) which would modify the dependence of E_0 and of $f(k,\theta)$ in $g_r(\mu)$ without actually changing physics. This is seen immediately recalling that the dependence of $f(k,\theta)$ on $g_r(\mu)$ and μ can both be traded for a dependence on E_0, which is fixed by a renormalization condition [6].

From references [4]-[10] we learn that the delta function potential in two space dimensions: i) is most appealingly treated by regularization and renormalization of the coupling constant; ii) exhibits the typical quantum scale anomaly associated to the process of renormalization, since the problem was originally scale invariant but the final answer is not; iii) is only noticed by s-waves, since for higher waves the centrifugal barrier dominates over the delta function potential, which therefore is completely screened; iv) can be substituted by a boundary condition which ensures the self-adjoint character of the radial free Hamiltonian; v) is the formal non-relativistic limits of a $g\phi^4$ quantum field theory.

After recalling the basic features of contact interactions, we will address here its perturbative approximation. One expects that perturbative renormalization should be used to deal with this problem, and in fact this is so. If one substitutes directly $g_0\delta^{(2)}(\vec{r})$ in (2), the second and higher order results are clearly logarithmically divergent. As we already mentioned, this is due to the short distance behavior of the propagator (1). To compute explicitly the divergences, we will introduce the regulated expression for the delta function potential given in (4). The first term in the Born series of the scattering amplitude, if $\vec{k}_i$ and $\vec{k}_f$ are the momenta of the incident and scattered particles respectively, is

$$f_B^{(1)} = -\frac{1}{2\sqrt{2\pi k}}\int d^2r e^{-i\vec{k}_f\cdot\vec{r}}V_R(\vec{r})e^{i\vec{k}_i\cdot\vec{r}} = \frac{-g_P}{2\sqrt{2\pi k}} + O(R),$$

where we call the coupling in which we perturb g_P. Although the first order approximation is finite when the regulator is removed, the second order one is logarithmically divergent

$$\begin{aligned} f_B^{(2)} &= \frac{1}{2\sqrt{2\pi k}}\int d^2r'd^2r'' e^{-i\vec{k}_f\cdot\vec{r}'}V_R(\vec{r}')G_{k,+}(\vec{r}'-\vec{r}'')V_R(\vec{r}'')e^{i\vec{k}_i\cdot\vec{r}''} \\ &= \frac{1}{2\sqrt{2\pi k}}g_P^2\left(\frac{i}{4}-\frac{1}{2\pi}\ln\frac{kRe^\gamma}{2}\right) + O(R). \end{aligned} \tag{7}$$

In order to get rid of this divergence in the scattering amplitude we will choose another potential given by

$$V_R(r) = V_R^{(1)}(r) + V_R^{(2)}(r) = \frac{g_P}{2\pi R}\left(1-\frac{g_P}{2\pi}\ln\frac{R\Lambda e^\gamma}{2}\right)\delta(r-R), \tag{8}$$

where Λ is an arbitrary momentum. The computation up to second order in g_P using the above potential leads to a finite answer, since the second order in g_P in (8) exactly cancels the logarithmic divergence in (7), and then

$$f_B = \frac{-g_P}{2\sqrt{2\pi k}}\left(1-g_P\left(\frac{i}{4}-\frac{1}{2\pi}\ln\frac{k}{\Lambda}\right)\right) + O(g_P^3).$$

Notice that the modification introduced in (8) is only aimed at making the physical magnitudes finite when $R\to 0$, not at introducing a new scale Λ. Thus g_P has to depend on Λ in such a way as to cancel the Λ dependence of f_B.

Perturbation theory up to third order in g_P using potential (8) would yield divergent quantities once again

$$f_B = \frac{-g_P}{2\sqrt{2\pi k}}\left(1-g_P\left(\frac{i}{4}-\frac{1}{2\pi}\ln\frac{k}{\Lambda}\right)+g_P^2\left(\frac{i}{4}-\frac{1}{2\pi}\ln\frac{kRe^\gamma}{2}\right)^2\right) + O(g_P^4).$$

The potential that leads to finite results up to g_P^3 is given by

$$V_R(r) = V_R^{(1)}+V_R^{(2)}+V_R^{(3)} = \frac{g_P}{2\pi R}\left(1-\frac{g_P}{2\pi}\ln\frac{R\Lambda e^\gamma}{2}+\frac{g_P^2}{4\pi^2}\ln^2\frac{R\Lambda e^\gamma}{2}\right)\delta(r-R),$$

so that one obtains

$$f_B = \frac{-g_P}{2\sqrt{2\pi k}}\left(1-g_P\left(\frac{i}{4}-\frac{1}{2\pi}\ln\frac{k}{\Lambda}\right)+g_P^2\left(\frac{i}{4}-\frac{1}{2\pi}\ln\frac{k}{\Lambda}\right)^2\right) + O(g_P^4).$$

One can easily compute to all orders in perturbation theory, and the series obtained can be summed up giving

$$V_R(r) = \frac{1}{2\pi R}\frac{g_P}{1+\frac{g_P}{2\pi}\left(\ln\frac{R\Lambda}{2}+\gamma\right)}\delta(r-R).$$

and

$$f_B = \left(\frac{\pi}{2k}\right)^{1/2}\frac{-g_P/2\pi}{1+g_P\left(\frac{i}{4}-\frac{1}{2\pi}\ln\frac{k}{\Lambda}\right)}. \tag{9}$$

The above result agrees with the exact one (6) after imposing $\Lambda e^{2\pi/g_P} = \mu e^{2\pi/g_r}$, which implies $g_P(\Lambda = \mu) \equiv g_r(\mu)$. The bare coupling constant is defined from $V_R(r)$ as

$$g_0(R) \equiv \frac{g_P(\Lambda)}{1 + \frac{g_P(\Lambda)}{2\pi}\left(\ln\frac{R\Lambda}{2} + \gamma\right)},$$

which coincides with (5). This identification should be understood as a renormalization condition, that in principle, one should impose at each order of the perturbative expansion.

The use of the specific regulator we chose allows to easily compute to all orders the Born scattering amplitude and sum the perturbative results. In general, with the use of some other regulator (e.g. a circular well potential), renormalized perturbation theory does not reproduce both the exact regularized potential and the exact scattering amplitude. This latter is easily reproduced, which is all what physics demands, but the summed regularized potential is not the same as the exact one.

This analysis explains how the perturbative approach of contact interactions in quantum mechanics has to be performed, which is close both in spirit and technically to standard perturbative regularization and renormalization of quantum field theory. We have seen that in order to obtain finite results up to a certain order in perturbation theory one has to add some "counterterms" to the potential which cancel the logarithmic divergences produced in the coefficients of the Born series. Then these divergences are reabsorbed into the bare parameters of the theory, while physical finite magnitudes can be defined. This is exactly the same process that is followed when renormalizing a quantum field theory perturbatively.

We will address now the perturbative approach to the Aharonov-Bohm problem, that also requires the techniques of renormalization. The Aharonov-Bohm gauge potential is $\vec{A} = \frac{\Phi}{2\pi r}\hat{u}_\varphi$ where Φ is the magnetic flux, and $\hat{u}_\varphi$ is the unitary vector in the azimuthal direction and r is the radial distance. We define $\alpha \equiv -\frac{e\Phi}{2\pi}$, e being the electric charge of the particle. Restricting the study of the problem to $0 < \alpha < 1$, the exact scattering amplitude of this problem has been computed by several authors [11], [12], [13], [14] and for an

incident wave with momentum $\vec{k}_i = (-k, 0)$ is, in the non-forward direction [1],

$$f(k,\theta) = -\frac{\sin\pi\alpha}{\sqrt{2\pi k}}\left(1 - i\tan\frac{\theta}{2}\right); \qquad -\pi < \theta < \pi.$$

The limit $\alpha \to 0$ can be taken in the above expression, obtaining

$$f(k,\theta) = -\frac{\pi\alpha}{\sqrt{2\pi k}}\left(1 - i\tan\frac{\theta}{2}\right) + O(\alpha^3). \tag{10}$$

Notice that there is no α^2 contribution in (10). This is the result one would expect to recover using perturbation theory up to second order. As remarked by several authors [15], [16], [17], the Born series fails to give the correct results. We will show here how one should proceed to obtain the correct answers.

The perturbation in this problem is

$$V(\vec{r}) = -2i\frac{\alpha}{r^2}\frac{\partial}{\partial\varphi} + \frac{\alpha^2}{r^2} \equiv V_\alpha^{(1)}(\vec{r}) + V_{\alpha^2}^{(2)}(r).$$

The Born approximation in first order in α gives

$$f_B^{(1)} = \frac{i\pi\alpha}{\sqrt{2\pi k}}\tan\frac{\theta}{2}.$$

This does not agree with the result obtained in (10) in first order in α since there is one term which is missing. As noticed by Corinaldesi and Rafeli [15], in the first order approximation the s-wave contribution to the scattering amplitude is absent.

The correction in α^2 comes from two terms. The first one, from the iteration of $V_\alpha^{(1)}$, is finite, and can be computed best by going to momentum

[1] As noticed in the literature [12], [13], [14], if the exact Aharonov-Bohm scattering wavefunction obeys the asymptotic condition (3), then the scattering amplitude exhibits a singular contribution in the forward direction. This singularity can be removed by modifying that asymptotic condition as it was done in [11]. Here we will restrict our study to the non-foward direction, noticing that also the perturbative approach yields singularities in the forward direction.

space and computing the integral

$$f_B^{(2),1} = \frac{4\alpha^2}{2\sqrt{2\pi k}} \int \frac{pdpd\varphi}{p^2 - k^2 - i\epsilon} \frac{kp\sin\varphi}{(k^2 + p^2 + 2kp\cos\varphi)} \frac{kp\sin(\theta - \varphi)}{(k^2 + p^2 - 2kp\cos(\varphi - \theta))}.$$

The angular integral can be trivially evaluated by expanding each fraction in Gegenbauer polynomials. After some algebra, one arrives at

$$f_B^{(2),1} = -\frac{2\pi\alpha^2}{2\sqrt{2\pi k}} \left(\ln(2\cos\theta/2) + \frac{i\pi}{2}\right).$$

The second contribution of order α^2 to the Born scattering amplitude comes from $V_{\alpha^2}^{(2)}$ and it yields a logarithmically divergent integral

$$f_B^{(2),2} = -\frac{\alpha^2}{2\sqrt{2\pi k}} \int d^2r \frac{e^{i(\vec{k}_i - \vec{k}_f)\cdot\vec{r}}}{r^2} = -\frac{2\pi\alpha^2}{2\sqrt{2\pi k}} \int_0^\infty dr \frac{J_0(\kappa r)}{r},$$

where J_0 is the Bessel function of zero order, and $\kappa = |\vec{k}_i - \vec{k}_f| = 2k\cos\theta/2$. Introducing a short distance cut-off, we can compute the explicit form of the divergence

$$f_B^{(2),2} = -\frac{2\pi\alpha^2}{2\sqrt{2\pi k}} \left(-\gamma - \ln\frac{\kappa R}{2} + O(R)\right).$$

Adding all contributions up to second order in α^2, the Born scattering amplitude is

$$f_B = \frac{\pi\alpha}{\sqrt{2\pi k}} \left(i\tan\frac{\theta}{2} + \alpha\left(-i\frac{\pi}{2} + \gamma + \ln\frac{kR}{2}\right)\right) + O(\alpha^3). \qquad (11)$$

This does not reproduce the correct result in first order in α and yields a divergent quantity in second order, even though the correct result does not have contribution in α^2. Notice, as well, the kind of divergence in (11): it is exactly of the same form as the one we found in the second order perturbation of the contact interaction, eq. (7). In fact, introducing a new interaction in the perturbed Hamiltonian, a zero-range or contact interaction $V(\vec{r}) = 2\pi\alpha\delta^{(2)}(\vec{r})$, the wrong first order result is corrected, while the whole second order result, including the logarithmic divergence, is cancelled, so that one arrives at precisely the scattering amplitude (10).

We should remark here that the crossed terms in the scattering amplitude in α^2 corresponding to the the delta function potential times $V_\alpha^{(1)}$ trivially vanish, the reason being that they act in orthogonal Hilbert subspaces: while the contact interaction only affects the s-wave sector, $V_\alpha^{(1)}$ vanishes in that subspace.

The contact interaction introduced here is a repulsive one. This very likely reflects its origin as an auxiliary interaction only seen in perturbation theory, but which is not present in the exact treatment (recall that repulsive contact interactions are trivial). It contributes to implement a feature of the exact wavefunction, i.e. that it vanishes at the origin, which perturbations in the Aharonov-Bohm potential alone are not able to implement, leading to the short distance divergence of $f_B^{(2),2}$. Notice also that the coupling constant of this interaction does not become renormalized, since its unique job is to make finite the perturbative theory of the Aharonov-Bohm problem. Obviously, one could add a different contact interaction to the Hamiltonian and proceed with its perturbative study. This was studied exactly in ref. [18].

We should mention here that similar results were first found studying the non relativistic quantum field theory model corresponding to the Aharonov-Bohm effect [19]. In refs. [20], [21], [22] it was shown that in order that the theory is finite a ϕ^4 interaction is needed. We find parallel results in quantum mechanics: the introduction of a new interaction in the perturbed Hamiltonian, a delta function potential, which is the quantum mechanical counterpart of ϕ^4, is needed to ensure the finiteness of the perturbative approach.

These results should also be applied to the study of the perturbative theory of anyonic quantum mechanics in the bosonic end, where one also finds logarithmic divergent quantities. This problem has been considered recently by several authors [23], [24], [25], but we think that the solution we offer here, not unrelated to some of the ones suggested by these authors, is simple, systematic and straightforward.

In conclusion: quantum mechanics for singular enough potentials leads to a perturbation theory anchored on regularization and renormalization. It shows in a very simple setting many features of renormalized perturbation

theory in quantum field theory, but allows to go far beyond by, i.e. comparing to the known exact results.

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