

QUANTUM INTEGRABILITY AND CLASSICAL INTEGRABILITY

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Abstract: Quantum integrability and classical integrability are compared by studying their commuting objects, the constants of motion. In particular we discuss the problem of constructing a quantum integrable system when a classically integrable system is given.

1 INTRODUCTION

Integrable dynamical systems have been studied quite actively in recent years, most often from the classical point of view. Here we will discuss the differences of classical and quantum integrability, especially the existence of classically or quantumly commuting operators. We will only consider systems with a finite number of degrees of freedom. (The material in Sections 1 to 4 have been discussed in more detail in Ref. [1])

In the literature there are many definitions for integrability so let us start with the definition that will be used here:

Definition A Hamiltonian system of N degrees of freedom is called integrable if there are N globally defined, independent functions  $I = \{I_1, \dots, I_N\}$ , i.e. mappings from the phase space to the reals, which commute with each other with respect to the Poisson bracket:

$$\{I_i, I_j\}_{PB} = 0, \quad i, j = 1, \dots, N,$$

The  $I_i$ 's of the above definition are called constants of motion.

The above definition was for classical mechanics. What about quantum mechanics? Of course classical and quantum mechanics differ in many ways and some concepts cannot be used in both. In the algebraic sense they are quite close to each other and we can transport the above definition to quantum mechanics:

Definition: A quantum mechanical Hamiltonian system is called quantum integrable if there are N independent, globally defined operators  $I = \{I_1, \dots, I_N\}$ , commuting with each other i.e.

$$\{I_i, I_j\} = 0, \quad i, j = 1, \dots, N$$

In the above definitions we used the term "independent" whose meaning we still have to discuss. In classical mechanics the vague idea of functional independency, which was meant above, is defined by the more rigorous concept of linear independency of the one-forms  $dl_i$  [2]. How should independency be defined in quantum mechanics? There does not seem to exist an equally convenient definition. In the following we will encounter only differential or integral operators for which the definition is not so problematic, and analogues to classical mechanics can be used. Nevertheless, this term needs

a rigorous study.

Suppose then that we are given a classically integrable system, i.e. a set of functions  $I_i$  whose Poisson brackets vanish. If we next make the usual operator substitutions for the momenta we get operators  $I_i$  (we could have ordering ambiguities!). Do we now have  $\{I_i, I_j\} = 0$ ? Not always, although this has been claimed. In the following we shall discuss the various problems that we are faced with when we try to construct a quantum integrable system from a classically integrable one.

2 QUANTUM MECHANICS WITH C-NUMBERS

In the following we will mostly be doing quantum mechanics with c-number functions rather than with operators. There are two reasons for doing it here: 1) Since we are comparing classical and quantum mechanics it is useful to have similar objects in both, 2) for computer algebra systems, where many of the necessary computations were done, it is more convenient to have commuting objects.

For the present purposes we need only replacements for the objects (operators) and for the only algebraic operation between them (commutator). There are many isomorphisms that could be used, but we will only consider linear correspondence rules. They are characterized by a function  $f(x, y)$  (see [3]) with  $f$  given we translate c-number function  $A(p, q)$  into an operator  $\hat{A}(p, \hat{q})$  using the (formal) integral

$$\hat{A}(p, \hat{q}) = \int d^N p' d^N q' \delta^N(x - 2\pi i \hat{p}' - y + y'(q' - q)) / (2\pi i)^N.$$

We require the transformation properties  $f(p) \rightarrow f(\hat{p})$ ,  $g(q) \rightarrow g(\hat{q})$ , which imply  $f(0, y) = f(x, 0) = 1$ . In the Weyl rule  $f(x, y) = 1$  and for the standard ordering rule ( $\hat{p}$ 's to the right)  $f(x, y) = \exp(-ixy/(2\pi i))$ . In the following we will only use the Weyl-Wigner rule.

The major reason for using the Weyl rule is that then the replacement for the commutator is defined by the Moyal bracket which is quite convenient. The Moyal bracket is defined by

$$\{A, B\}_{MB} = A(p, q) \frac{1}{2} \sin \left[ \frac{1}{2} (\vec{\sigma}_q \cdot \vec{\sigma}_p - \vec{\sigma}_p \cdot \vec{\sigma}_q) \right] B(p, q). \quad (2)$$

where the arrows indicate the direction to which the derivative operates. Note that (2) reduces to the Poisson bracket when  $\hbar \rightarrow 0$ . Another property of considerable importance is that both brackets respect time reversal parity ( $p \rightarrow -p$ ). If the Hamiltonian is even in momenta this parity conservation implies that both the classical and quantum c-number invariants can be assumed to have a definite parity, i.e. they will be either even or odd in  $p$ .

3 AUTOMATIC EXTENSION FROM CM TO QM

The Moyal and Poisson brackets give different results only if the higher order terms in (2) contribute. If the Hamiltonian is of the type

$$H = (p_x^2 + p_y^2)^{1/2} + V(x, y). \quad (3)$$

then in the higher order terms all q-derivatives must operate on V and the p-derivatives on I. In particular if the invariant is at most second order

In moments the higher order terms in the Moyal bracket do not contribute at all, and in that case classical integrability does imply quantum integrability.

The classical invariant can be used as the c-number quantum invariant also in some other cases. For example in systems whose classical integrability follows from the existence of a Lax pair it turns out that often the commuting invariants can be chosen so that the variables appear in combinations which also yield quantum integrability. For further details and many examples see [4].

For many systems it is therefore true that classical integrability implies quantum integrability without any problems. However, this is not always the case.

#### 4 QUANTUM CORRECTIONS TO THE SECOND INVARIANT

When the invariant is sufficiently complicated the higher order terms in the Moyal bracket do contribute. In such cases the natural first thing to try is an expansion in  $\hbar^2$  for the invariant. As the zeroth order term we take the classical invariant  $I_{cl}$ , the higher order terms are taken as unknowns. The resulting equations that follow from  $\{H, I\}_{MB} = 0$  are then solved order by order in  $\hbar$ . If this equation has a solution  $I_{qu}$  it can be regarded as a "deformation" of the classical invariant  $I_{cl}$ .  $I_{qu}$  is the c-number representative of a quantum operator which commutes with the quantum Hamiltonian. In many cases just one additional term is enough. Several examples of this procedure are given in [1].

Although the quantum operator remains unchanged it is possible to change the appearance of the c-number representative by changing the  $\hbar^2$  deformation rule [i.e. the function  $f$  in (1)]. If in some ordering rule the  $\hbar^2$  deformations vanish one can say that that ordering gives a method for an integrability preserving quantization of the system. In [1] it was shown how in some specific cases such an ordering rule can be found. Unfortunately there is no one ordering rule or deformation that works for all classically integrable models.

#### 5 DEFORMATIONS IN THE HAMILTONIAN

Even these corrections to the second invariant are not always sufficient to make a classically integrable system quantum integrable. As the by now standard example let us take the Holt Hamiltonian

$$H_{cl} = (p_x^2 + p_y^2)/2 + 3/4x^{4/3} + x^{-2/3}y^2. \tag{4}$$

which is classically integrable with the second invariant [5]

$$I_{cl} = p_y^3 + 3/2p_x p_x^2 + [-9/2x^{4/3} + 3x^{-2/3}y^2]p_y + 9x^{1/3}p_x^2. \tag{5}$$

If we now try a second invariant  $I_{qu}$  with the same leading part as in  $I_{cl}$  it turns out that the new system of equations has no solution.

However, now we can solve the equations if we allow an  $\hbar^2$  correction term in both the invariant and the Hamiltonian. We have found that

$$H_{qu} = H_{cl} - 5/72\hbar^2 x^{-2}, \text{ and} \tag{6}$$

$$I_{qu} = I_{cl} - 5/72\hbar^2 x^{-2} p_y \tag{7}$$

indeed have a vanishing Moyal bracket. Calculations similar to the above can be carried out with the other integrable Holt potentials whose invariants are of order 4 and 6 in p [6]. Also in these cases we have found that the correction term to the Hamiltonian is the same,  $-5/72\hbar^2 x^{-2}$ .

Another example of this type is the Fokas-Lagerstrom potential

$$H_{cl} = (p_x^2 + p_y^2)/2 + (xy)^{-4/3}, \tag{8}$$

which is classically integrable [7]. For quantum integrability the Hamiltonian must be deformed to [8]

$$H_{qu} = H_{cl} - 5/72\hbar^2 [x^{-2} + y^{-2}], \tag{9}$$

and then the second invariant is

$$I_{qu} = p_x p_y (xp_y - yp_x) + 2(xy)^{-2/3} (xp_x - yp_y) - 5/36\hbar^2 [xy^{-2} p_x^2 - yx^{-2} p_y^2]. \tag{10}$$

It is quite surprising that the deformations (6) and (9) are of the same type, including numerical factors. Also the correction seems to be associated with the variable that appears with a fractional power in the rest of the potential. Note also that the correction term cannot be eliminated by a canonical point transformation [1], or a change in the ordering rule. [Of course one can define an ad hoc correspondence rule by stating that  $H_{cl}$  transforms to the operator whose c-number representative is  $H_{qu}$ , but such a rule does not satisfy the requirements of a linear correspondence rule discussed in Sec. 2]

Recently we have been able to solve this intriguing question by applying a canonical transformation together with a change in the time variable [9]. We have found that starting with a Rosen-Bellias type potential

$$V = 2/3[mx^3/3 + xy^2] - 3\hbar/2 \tag{11}$$

there is a sequence of transformations which turn this potential into Holt type

$$V = 3/4mx^{4/3} + x^{-2/3}y^2 - \hbar x^{-2/3}. \tag{12}$$

Both potentials are integrable for  $m = 1, 6, 16$ . The step that produces the  $\hbar q$  type correction terms is the point transformation  $q = f(Q)$  followed by multiplication of the Hamiltonian by  $f'(Q)^2$ . The correction that must be made in the quantum version is in general given by a "Scharzian derivative" [9], which for  $f(Q) = cQ$  reduces to

$$dV = -5/72\hbar^2 Q^{-2}. \tag{13}$$

There is another sequence of transformations that takes one from the radial quartic potential

$$V = \hbar(x^2 + y^2)^{-2}/24 \tag{14}$$

into the Fokas-Lagerstrom potential

$$V = -2g(xy)^{-2/3}. \tag{15}$$

