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AN ATTEMPT TO CALCULATE THE EFFECTIVE LAGRANGIAN FOR LOW LYING PSEUDOSCALAR MESONS FROM QCD STRONG COUPLING EXPANSION[1]

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ABSTRACT

An attempt to derive the effective Lagrangian for low lying pseudoscalar mesons is given in QCD strong coupling expansion based on the idea of the effective field theory with a physical cut-off Λ.This theory provides the information about chiral symmetry breaking, and the quark condensates are calculable. The obtained effective Lagrangian contains the exact Wess-Zumino-Witten term and the complete Gasser-Leutwyler chiral Lagrangian with all the coefficients $F_0,B_0,L_1,L_2,\cdots,L_{10},H_1$ and H_2 given analytically as functions of the two fundamental parameters Λ and g_s (effective coupling constant in the cut-off QCD theory). Λ and g_s are then determined by taking the data of m_π and m_k as inputs. Up to order-$1/g_s^2$ contributions, the calculated m_η,F_π,F_k,F_η, quark condensates, pion- pion scattering lengths,and pseudoscalar-meson form factors are all in reasonable agreement with experiments.

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I. Introduction

There has been a lot of interest in the study and application of low energy effective Lagrangians in recent years. Effective Lagrangians provide powerful and convenient means for studying physics of low energy hadrons[1]-[3], heavy flavor decays[4], precise tests of electroweak theory including possible contributions from new physics[5][6], and strongly interacting electroweak symmetry breaking mechanisms[7]. Therefore it concerns most of the currently interesting problems in particle physics ranging from 10^2 MeV to TeV.

The general form of the effective Lagrangian for low lying pseudoscalar mesons was given by Gasser and Leutwyler[1] based on a symmetry argument.Up to p^4 the non-anomalous chiral Lagrangian consists of fourteen terms and the coefficients $F_0,B_0,L_1,L_2,\cdots,L_{10},H_1$ and H_2 are unknown parameters which should be determined by imposing certain number of low energy hadronic data as inputs[1]. Although this chiral Lagrangian has been widely used, it is certainly interesting to investigate further the relation between the chiral Lagrangian and the underlying theory QCD. This is important not only for theoretical reasons but also for practical needs since once the relation is known, the number of independent unknown parameters can be significantly reduced, and this will increase the predictability of the theory. Moreover, once we can handle the nonperturbative dynamics of the underlying theory governing this relation,we can further generalize the study to the more interesting up to date problems such as the strongly interacting electroweak symmetry breaking mechanisms which are needed in the experimental study of electroweak symmetry breaking mechanisms at SSC and LHC.

There have been various attempts to derive the chiral Lagrangian from some underlying theories, and most of them are cut-off theories looking like effective field theories [8][5]. In a series of papers, it is shown that the chiral Lagrangian can be obtained from the Nambu-Jona-Lasinio (NJL) type four-fermion theory [9][10]. In this

kind of approach, there is a calculable quark condensate breaking the chiral symmetry and Goldstone bonson fields are the auxiliary fields for the quark composite operators $\bar{q}\gamma_5 t_i q$, where t_i is the generator of the flavor group. Both the linear and the nonlinear realizations can be adopted in this approach and the low lying vector and axial-vector mesons can also be included. The obtained phenomenological results are encouraging[9][10]. However, more work is needed for understanding further the precise relation between the NJL type Lagrangian and QCD from the point of view of the effective field theory [8][5]. Another approach[11] starts from the cut-off QCD Lagrangian with u,d,and s quarks and taking the large N_c (number of colors) limit. The cut-off Λ is defined as the characteristic parameter in an exponential type regularization in which non-minimal chiral anomaly exists in the real part of the effective action.The Goldstone boson field $U = exp(-i\lambda^a\pi^a/F_\pi)$ is formally introduced as a collective mode and the Lagrangian of U with external sources is determined from the requirement that the U-field theory gives the same non-minimal anomaly as that in the original theory. All the coefficients F_0,B_0,L_1,L_2,$\cdots$,L_{10},H_1 and H_2 are then expressed in terms of N_c, Λ,and the gluon condensate. Unfortunately, the information about chiral symmetry breaking is not included in this approach.The obtained pion decay constant F_π is proportional to Λ rather than proportional to the vacuum expectation value breaking the chiral symmetry as in the conventional understanding. To fit the experimental value of F_π, the cut-off Λ should take a rather low value $\Lambda = 320MeV$ [11] which is even lower than the masses of kaons and η (the pseudo-Goldstone- bonsons contained in U). In order to incorporate with chiral symmetry breaking, a phenomenological interaction $M_Q(\bar{q}_R U q_L + \bar{q}_L U^\dagger q_R)$ is added to the QCD Lagrangian in this approach,where M_Q is a free parameter triggering the chiral symmetry breaking[11]. Therefore the quark condensates are not calculable in this approach. There is also a different approach based on the consideration of the instanton induced chiral symmetry breaking[12]. However, in the obtained meson spectrum, the lightest meson apart from the pseudo-Goldstone-bosons is the

tensor meson. i.e. there is no ρ-meson in this theory. Owing to the above situation, further improved investigations on the derivation of chiral Lagrangian from QCD is really desired.

In this paper, we will give an attempt to derive the chiral Lagrangian based on the idea of the effective field theory from cut-off QCD with three light flavors in the approach of the strong coupling expansion. The physical cut-off is expected to be higher than m_η to include all the eight low lying pseudoscalar mesons in the chiral Lagrangian, and it should be still in the several hundred MeV region to have a large enough effective QCD coupling constant g_s so that the strong coupling expansion works. Note that this finite g_s should not be simply identified with the conventional scheme-dependent renormalized coupling constant in the full-range renormalizable QCD theory[5]. In an effective field theory[8][5], all particles with masses heavier than Λ and the high momentum part of the light particles are integrated out which contribute a tower of extra terms $\mathcal{L}_H^{eff}$ in the total cut-off QCD Lagrangian. In $\mathcal{L}_H^{eff}$, terms with higher dimensional operators (nonrenormalizable terms) are suppressed in low energy processes[5]. Therefore we keep only dimension$\leq$4 operators in $\mathcal{L}_H^{eff}$.With a certain consideration of gauge invariance of the final theory,we obtain a NJL type four- fermion interaction of the light quark fields after integrating over the gluon fields in the leading order part in $1/g_s^2$ expansion. Results similar to those in Ref.[9]-[10] can then be obtained but with fewer free parameters.Furthermore, we also take into account, in this paper, the order-$1/g_s^2$ corrections which reflect the gluon self-interactions, so that our total results are different from those in Ref.[9]-[10]. Apart from the pseudoscalar mesons, the NJL type four-fermion theory includes also the scalar,vector,and axial-vector mesons. Vacuum stability condition determines the nonvanishing vacuum expectational value (VEV) v of the scalar field which breaks the chiral symmetry. Taking the nonlinear realization[13] and integrating out the scalar,vector,and axial-vector fields in the stationary phase approximation (semiclassical approximation), we get the effective Lagrangian for the

pseudoscalar mesons alone which contains the exact Wess-Zumino-Witten (WZW) term[14] and the complete Gasser-Leutwyler chiral Lagrangian[1] with the coefficients $F_0, B_0, L_1, L_2, \cdots, L_{10}, H_1$ and H_2 expressed in terms of N_c, N_f (number of flavors), Λ, and g_s. Fixing $N_c = N_f = 3$, there are only two fundamental parameters Λ and g_s in this theory. They can be determined by taking the data of m_π and m_k as inputs. The determined g_s is really large and the value of Λ is $\Lambda = 830 MeV$ which is really reasonable, therefore the present approach is self-consistent. The theory can then predict $m_\eta, F_\pi, F_k, F_\eta$, the quark condensates, the $\pi - \pi$ scattering lengths, and the pseudoscalar-meson form factors, etc, and they are all in reasonable agreement with experiments.

This paper is organized as follows. In Sec.II, we give the generating functional in the cut-off QCD theory and derive the effective meson theory in the linear realization. Sec.III deals with the nonlinear realization, and the derivation of the WZW term and the general form of the effective Lagrangian including scalar, pseudoscalar, vector and axial-vector mesons at low energies. The gap equation is also derived there from vacuum stability. Sec.IV is devoted to the derivation of the low energy effective Lagrangian for pseudoscalar mesons (chiral Lagrangian) from integrating out the scalar, vector, and axial-vector meson- fields [15]. The Gasser-Leutwyler coefficients are calculated up to next-to-the-leading order in $1/g_s^2$ expansion. In Sec.V, we present the determination of Λ and g_s, and the phenomenological predictions of the obtained chiral Lagrangian for the low energy physics of the pseudoscalar mesons. Sec.VI contains a brief summary and some discussions on the present approximation and further investigations. Some of the technical details in the derivation of the effective Lagrangian are given in the Appendices.

II. The Generating Functional and the Linear Realization of Meson Fields

1. The Generating Functional

Consider the system of gluons B_μ^a and N_f flavors of light quarks q. The QCD Lagrangian formally including the θ-vacuum effect can be written as[16]

$$\mathcal{L}_{QCD} = \mathcal{L}_G(B) + \bar{q}(i\partial\!\!\!/ - B\!\!\!/ - m)q \,, \tag{1}$$

where m is the current quark mass matrix, $B_\mu \equiv (\lambda_a/2) B_\mu^a$, and

$$\mathcal{L}_G(B) \equiv -\frac{1}{2g_s^2} tr_c(B_{\mu\nu}B^{\mu\nu}) - \frac{\theta}{16\pi^2} tr_c(B_{\mu\nu}\tilde{B}^{\mu\nu}) \,, \tag{2}$$

in which tr_c is the trace in the color space, $B_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu + i[B_\mu, B_\nu]$ and $\tilde{B}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\lambda\kappa}B^{\lambda\kappa}$. In (1) and (2), we have rescaled the gluon field $g_s B_\mu^a \to B_\mu^a$ for convenience. We then introduce a physical momentum cut-off Λ and consider the effective field theory for the light particles below Λ. Actually we shall take $N_f = 3$, therefore Λ should be at least higher than twice of the constituent s-quark mass, roughly speaking $700 MeV$, to include all the eight pseudoscalar mesons in the effective theory. A natural value of Λ is about the c-quark mass. However, in practice we shall evolve the theory down to a lower value of Λ in the several hundred MeV region so that the effective coupling constant g_s is large and the first term in (2) can be treated as a perturbation (strong coupling expansion). The specific value of Λ will be determined later. According to the idea of the effective field theory, heavy quark fields (with masses greater than Λ) and the high momentum part of the light particles should be integrated out, and this will affect the physics below Λ, so that total Lagrangian $\mathcal{L}_{QCD}^{(\Lambda)}$ in the effective field theory is not merely (1)-(2). The effect of integrating out the physics above Λ may contribute two kinds of terms in $\mathcal{L}_{QCD}^{(\Lambda)}$. The first kind contains terms of the same form as those in (1)-(2), and this can be simply absorbed into (1)-(2) by redefining the fields and the coupling

constant g_s. The second kind contains terms different from those in (1)-(2), and we shall call this $\mathcal{L}_H^{eff}$. Therefore

$$\mathcal{L}_{QCD}^{(\Lambda)} = \mathcal{L}_{QCD} + \mathcal{L}_H^{eff} \ . \tag{3}$$

It is $\mathcal{L}_{QCD}^{(\Lambda)}$ that describes the complete physics in the cut-off theory and thus the finite effective coupling constant g_s in (3),which is directly physical, is not equivalent to the renormalized coupling constant in any of the conventional renormalization schemes in the full range renormalizable QCD theory[5]. In principle, g_s can be determined from the "matching" condition at the scale Λ [5]. However, the present problem is nonperturbative so that the "matching" condition is hard to realize. Therefore we take g_s as an unknown parameter and determine it from the low energy data of hadrons.We expect the determined g_s to be large, otherwise the present approach will not be self-consistent. Terms in $\mathcal{L}_H^{eff}$ may contain various kinds of field operators. From simple dimensional analysis, we infer that terms containing operators of dimension greater than 4 must have coefficients proportional to negative powers of Λ (or the heavy quark mass), so that these terms are suppressed in the study of low energy processes. As an approximation, we neglect these terms in this paper. There can also be terms containing lower dimensional operators in $\mathcal{L}_H^{eff}$, and these should be kept in (3). It is known that introducing a momentum cut-off Λ will affect the manifest local gauge invariance of the Lagrangian [5][17]. Indeed, if we integrated out the light quark field in $\mathcal{L}_{QCD}$ with a momentum cut-off Λ, a term $-\frac{1}{2}M^2 B_\mu^a B^{a\mu}$ with

$$M^2 = \frac{N_f \Lambda^2}{16\pi^2} \tag{4}$$

will emerge in the effective Lagrangian for gluons [17], which violates the gauge invariance. The above integration also gives rise to dimension-4 operators (and higher dimensional operators) of gluon fields. If we keep only the terms with leading powers of Λ in the coefficients of these operators, these dimension-4 operators do not violate gauge invariance. So that, in this approximation, the only gauge-

noninvariant contribution which we have to consider is the $-\frac{1}{2}M^2 B_\mu^a B^{a\mu}$ term. On the other hand, the integration over the high momentum part of the light-quark-field, belonging to $\mathcal{L}_H^{eff}$, contributes a corresponding term $\frac{1}{2}M^2 B_\mu^a B^{a\mu}$. Taking into account this term, the two gauge-noninvariant terms just cancel each other which just reflects the gauge invariance of the original full-range QCD. In the effective field theory, the high momentum part of the light-quark-field is always integrated out, i.e. the term $\frac{1}{2}M^2 B_\mu^a B^{a\mu}$ is always there in $\mathcal{L}_H^{eff}$ no matter how we treat the low momentum part of the light-quark-field in $\mathcal{L}_{QCD}$. Therefore the original gauge invariance is not guaranteed to be kept exactly at any step in the calculation if the low momentum part and the high momentum part are not treated on equal footing. In this paper, we only require that after integrating over the low momentum part of the light-quark-field in $\mathcal{L}_{QCD}$, the final effective Lagrangian for gluons contributed from the total $\mathcal{L}_{QCD}^{(\Lambda)}$ takes the gauge invariant form. This is really the spirit of the original gauge invariance in the full-range theory, and thus we may regard this requirement as a kind of "matching" condition which "matches" the gauge invariance in the two theories. Therefore the only dimension$\leq$4 operator that should be taken into account in $\mathcal{L}_H^{eff}$ is $\frac{1}{2}M^2 B_\mu^a B^{a\mu}$, and our $\mathcal{L}_{QCD}^{(\Lambda)}$ is

$$\mathcal{L}_{QCD}^{(\Lambda)} = \mathcal{L}_G(B) + \frac{1}{2}M^2 B_\mu^a B^{a\mu} + \bar{q}(i\partial\!\!\!/ - B\!\!\!/ - m)q \ . \tag{5}$$

The inclusion of the $\frac{1}{2}M^2 B_\mu^a B^{a\mu}$ term is formally similar to the discussion on QED in Ref.[17], but the physical interpretation is different. In (5), the momentum cut-off is understood to be expressed in terms of a cut-off function, say $\theta(\Lambda^2 - p^2)$, in the momentum representation of $\mathcal{L}_{QCD}^{(\Lambda)}$.

Next we consider the generating functional for this system. Let $I^{a\mu}$ be the external source for the gluon field B_μ^a, and $J_S, J_P, J_V^\mu, J_A^\mu$ be the external sources for the composite quark operators $-\bar{q}q, i\bar{q}\gamma_5 q, \bar{q}\gamma^\mu q$, and $\bar{q}\gamma^\mu\gamma_5 q$,respectively. $J_S, J_P, J_V^\mu, J_A^\mu$ are colorless $N_f \times N_f$ matrices. For the convenience of deriving the WZW term, we shall take the Euclidean space-time formulation with the metric $g^{\mu\nu} = -\delta^{\mu\nu}$ in the

following calculations. The generating functional $Z[I,J]$ can then be written as

$$Z[I,J] = \int \mathcal{D}q\mathcal{D}\bar{q}\mathcal{D}B^a_\mu \, exp\int d^4x \, \{ \, \mathcal{L}_G(B) + \frac{1}{2}M^2 B^a_\mu B^{a\mu} + I^a_\mu B^{a\mu} + \\ + \bar{q}(i\partial\!\!\!/ - B\!\!\!/ - J_S + iJ_P\gamma_5 + V\!\!\!/ + A\!\!\!/\gamma_5)q\} \ , \tag{6}$$

where

$$\mathcal{L}_G(B) = -\frac{1}{2g_s^2}tr_c(B_{\mu\nu}B^{\mu\nu}) - i\frac{\theta}{16\pi^2}tr_c(B_{\mu\nu}\tilde{B}^{\mu\nu}) \ . \tag{7}$$

Here we have put the current quark mass matrix m into the definition of J_S, i.e. the physical case corresponds to $J_S = m$ rather than $J_S = 0$.

The θ-term violates CP, so that it plays a special role in $\mathcal{L}^{(\Lambda)}_{QCD}$. We shall keep its complete contribution in our calculation. For this purpose, we make the following local chiral transformation

$$B^a_\mu(x) \rightarrow B^a_\mu(x) \quad , \quad q(x) \rightarrow [g(x)P_R + g^\dagger(x)P_L]q(x) \ , \tag{8}$$

where $P_{\substack{R\\L}} \equiv \frac{1}{2}(1 \pm \gamma_5)$ and $g(x)$ is an $N_f \times N_f$ unitary matrix.We know that the functional measure $\mathcal{D}q\mathcal{D}\bar{q}$ is not invariant under the chiral transformation (8)[18] and we have (See Appendix A)

$$\mathcal{D}q\mathcal{D}\bar{q} \rightarrow \mathcal{D}q\mathcal{D}\bar{q} \, exp\{i\frac{N_c}{240\pi^2}\int_{D_5} tr'(u^{-1}du)^5 + i\int_{S^4}\mathcal{A}(u,J_L,J_R) \\ -\int d^4x[lndet(u)]\frac{1}{16\pi^2}tr_c(B_{\mu\nu}\tilde{B}^{\mu\nu}) \ \} \ , \tag{9}$$

where, tr' is the trace in the flavor space and the Lorentz spinor space, D_5 is a five-dimensional disc bounded by S^4,

$$u(x) \equiv g(x)g(x), \qquad J^\mu_{\substack{R\\L}} \equiv J^\mu_V \pm J^\mu_A, \qquad J_{\substack{R\\L}} \equiv J^\mu_{\substack{R\\L}} dx_\mu, \tag{10}$$

and

$$\mathcal{A}(u,J_L,J_R) \equiv \frac{N_c}{48\pi^2}tr'\{ \, [J^u_L(J_R dJ_R + dJ_R J_R + J^3_R - R^2 J_R) \\ -u^{-1}J_L u(-J_R R J_R + R dJ_R) + \frac{1}{2}J_L L J_L L] \\ -(J_L \leftrightarrow J_R, u \leftrightarrow u^{-1}) + \frac{1}{2}(J_L u J_R u^{-1})^2\} - (u=1) \ , \tag{11}$$

in which

$$R \equiv u^{-1}du, \ L \equiv duu^{-1}, \quad J^u_L \equiv u^{-1}J_L u + R, \quad J^u_R \equiv uJ_R u^{-1} - L. \tag{12}$$

The first term in (9) is just the WZW term. The second term in (9) is an anomaly term related to the external sources J^μ_R and J^μ_L [19] and it vanishes when $J^\mu_R = J^\mu_L = 0$, but it affects the topological baryon current [10]. We then take u to satisfy

$$lndet(u) = -i\theta \quad , \tag{13}$$

so that (6) becomes

$$Z[I,J] = \int \mathcal{D}q\mathcal{D}\bar{q}\mathcal{D}B^a_\mu \, exp\{\int d^4x \, [\, \mathcal{L}_K(B) + \frac{1}{2}M^2 B^a_\mu B^{a\mu} + I^a_\mu B^{a\mu} + \\ \bar{q}(i\partial\!\!\!/ - B\!\!\!/ - J'_S + iJ'_P\gamma_5 + V\!\!\!/' + A\!\!\!/'\gamma_5)q] + \\ +i\frac{N_c}{240\pi^2}\int_{D_5} tr'(u^{-1}du)^5 + i\int_{S^4}\mathcal{A}(u,J_L,J_R)\} \ , \tag{14}$$

where

$$\begin{aligned} -J'_S + iJ'_P\gamma_5 &= (gP_R + g^\dagger P_L)\,(-J_S + iJ_P\gamma_5)\,(gP_R + g^\dagger P_L), \\ V\!\!\!/' + A\!\!\!/' &= g^\dagger(i\partial\!\!\!/\, g) + g^\dagger(V\!\!\!/ + A\!\!\!/)g, \\ V\!\!\!/' - A\!\!\!/' &= g(i\partial\!\!\!/\, g^\dagger) + g(V\!\!\!/ - A\!\!\!/)g^\dagger, \\ \mathcal{L}_K(B) &\equiv \mathcal{L}_G(B)\,|_{\theta=0} = -\tfrac{1}{2}g_s^{-2}tr_c(B_{\mu\nu}B^{\mu\nu}). \end{aligned} \tag{15}$$

Treating $\mathcal{L}_K(B)$ as a perturbation, the generating functional $Z[J] \equiv Z[0,J]$ can be further written as

$$Z[J] = exp\{+i\frac{N_c}{240\pi^2}\int_{D_5} tr'(u^{-1}du)^5 + i\int_{S^4}\mathcal{A}(u,J_L,J_R)\} \times \\ \times exp(S_K[\frac{\delta}{\delta I}]) \int \mathcal{D}q\mathcal{D}\bar{q}\mathcal{D}B^a_\mu \, exp\int d^4x \, \{ \, \frac{1}{2}M^2 B^a_\mu B^{a\mu} + I^a_\mu B^{a\mu} + \\ +\bar{q}(i\partial\!\!\!/ - B\!\!\!/ - J'_S + iJ'_P\gamma_5 + V\!\!\!/' + A\!\!\!/'\gamma_5)q\} \,|_{I=0}, \tag{16}$$

where

$$S_K[B] \equiv \int d^4x\mathcal{L}_K(B). \tag{17}$$

The integrand under $\int \mathcal{D}B^a_\mu$ in (16) is simple. The gluon field is quadratic in the exponential. Therefore the $\int \mathcal{D}B^a_\mu$ integration can be carried out [20] and we obtain

$$\begin{aligned} Z[J] &= exp\{i\frac{N_c}{240\pi^2}\int_{D_5} tr'(u^{-1}du)^5 + i\int_{S^4}\mathcal{A}(u,J_L,J_R)\}\times \\ &\times exp(S_K[\frac{\delta}{\delta I}])\int \mathcal{D}q\mathcal{D}\bar{q}\, exp\int d^4x\,\{\,\bar{q}(i\partial\!\!\!/ - J'_S + iJ'_P\gamma_5 + V\!\!\!/' + A\!\!\!/'\gamma_5)q - \\ &-\frac{1}{2M^2}(I^{a\mu} - \bar{q}\frac{\lambda^a}{2}\gamma^\mu q)\,(I^a_\mu - \bar{q}\frac{\lambda^a}{2}\gamma_\mu q)\}\,|_{I=0}\quad . \end{aligned} \tag{18}$$

Here we arrive at a four-fermion interaction. Next we make an inverse chiral transformation $g^\dagger(x)$ to transform back to the original quark field, i.e.

$$q(x)\rightarrow[g^\dagger(x)P_R + g(x)P_L]q(x). \tag{19}$$

Now

$$\begin{aligned} \mathcal{D}q\mathcal{D}\bar{q} \;\rightarrow\; & \mathcal{D}q\mathcal{D}\bar{q}\, exp\{-i\frac{N_c}{240\pi^2}\int_{D_5} tr'(u^{-1}du)^5 - i\int_{S^4}\mathcal{A}(u,J_L,J_R) \\ & +\int d^4x[lndet(u)]\frac{1}{16\pi^2}tr_c(F_{I\mu\nu}\tilde{F}^{\mu\nu}_I)\}, \end{aligned} \tag{20}$$

where

$$F_{I\mu\nu} \equiv \frac{\lambda^a}{2}F^a_{I\mu\nu} \equiv \partial_\mu\frac{I_\nu}{M^2} - \partial_\nu\frac{I_\mu}{M^2} - i[\frac{I_\mu}{M^2},\frac{I_\nu}{M^2}], \tag{21}$$

and $\tilde{F}_{I\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\lambda\kappa}F^{\lambda\kappa}_I$, in which $I_\mu \equiv \frac{\lambda^a}{2}I^a_\mu$. With (13) the generating functional can be written as

$$\begin{aligned} Z[J] &= exp(S_K[\frac{\delta}{\delta I}])\int \mathcal{D}q\mathcal{D}\bar{q}\, exp\int d^4x\,\{\,\bar{q}(i\partial\!\!\!/ - J_S + iJ_P\gamma_5 + V\!\!\!/ + A\!\!\!/\gamma_5 + \frac{I\!\!\!/}{M^2})q \\ &-\frac{1}{2M^2}(\bar{q}\frac{\lambda^a}{2}\gamma^\mu q)\,(\bar{q}\frac{\lambda^a}{2}\gamma_\mu q) - \frac{1}{2M^2}I^a_\mu I^{a\mu} - i\frac{\theta}{16\pi^2}tr_c(F_{I\mu\nu}\tilde{F}^{\mu\nu}_I)\,\}\,|_{I=0}\quad . \end{aligned} \tag{22}$$

In (22) the θ-term is completely expressed in terms of $tr_c(F_{I\mu\nu}\tilde{F}^{\mu\nu}_I)$ after the gluon-field integration.

The four-fermion interaction in (22) is composed of two colored currents. It can be expressed in terms of color singlet operators through Fierz reordering and the

result is

$$\begin{aligned} Z[J] &= exp(S_K[\frac{\delta}{\delta I}])\int \mathcal{D}q\mathcal{D}\bar{q}\, exp\int d^4x\,\{\,\bar{q}(i\partial\!\!\!/ - J_S + iJ_P\gamma_5 + V\!\!\!/ + A\!\!\!/\gamma_5 + \frac{I\!\!\!/}{M^2})q + \\ &+\frac{1}{4M^2}[(\bar{q}t_jq)^2 + (\bar{q}t_ji\gamma_5q)^2 - \frac{1}{2}(\bar{q}t_j\gamma_\mu q)^2 - \frac{1}{2}(\bar{q}t_j\gamma_\mu\gamma_5 q)^2 + \frac{1}{N_c}(\bar{q}\gamma_\mu q)^2] \\ &-\frac{1}{2M^2}I^a_\mu I^{a\mu} - i\frac{\theta}{16\pi^2}tr_c(F_{I\mu\nu}\tilde{F}^{\mu\nu}_I)\,\}\,|_{I=0}\ , \end{aligned} \tag{23}$$

where $t_j, j = 0,1,2,\cdots,N_f$-1, is the generator of the $U(N_f)$ flavor group, normalized as

$$t_j = \begin{cases} \frac{1}{\sqrt{N_f}}\ , & j=0 \\ \frac{\lambda_j}{\sqrt{2}}\ , & j\neq 0\,. \end{cases} \tag{24}$$

Eq.(23) is to be compared with the NJL type four-fermion interactions considered in Ref.[9]-[10]. We see that there is only one parameter Λ in the four-fermion interactions in (23). This specific form comes from the color octet current-current interaction in (22) which is the characteristic of QCD. Furthermore, our $Z[J]$ includes also the corrections in $1/g_s^2$ expansion through $exp(S_K[\frac{\delta}{\delta I}])$ which includes the self- interactions of gluons. Therefore (23) is different from the theory considered in Ref.[9]-[10].

The coefficients of the four-fermion terms in (23) are of $O(M^{-2})\sim O(\Lambda^{-2})$. One may question about our approximation in which the four-fermion terms in $\mathcal{L}^{eff}_H$ are not considered. The fact is that the four-fermion terms in $\mathcal{L}^{eff}_H$ can only come from integrating out the high momentum part of the gluon fields.Actually this part of integration has already been included in the way of doing the integration over B^a_μ in (16)[20] which leads to the four-fermion interaction in (18). Therefore, by definition, there is no $O(\Lambda^{-2})$ four-fermion term left in $\mathcal{L}^{eff}_H$. Since there is still a cut-off function associated with $\mathcal{L}_K(B)$, there can be $O(g_s^{-2}\Lambda^{-2})$ four- fermion terms in $\mathcal{L}^{eff}_H$. However, as we shall see in Sec.V that g_s^{-2} is very small, so that we can neglect such terms in $\mathcal{L}^{eff}_H$. In this paper, we only take account of $O(g_s^{-2})$ terms and neglect all the $O(g_s^{-2}\Lambda^{-2})$ terms through out the calculations.

Next we introduce auxiliary fields $\hat{S}_j$, $\hat{P}_j$, $\hat{V}_j^\mu$, and $\hat{A}_j^\mu$ for the composite operators $-\frac{1}{2}M^{-2}\bar{q}t_jq$, $\frac{i}{2}M^{-2}\bar{q}t_j\gamma_5 q$, $-\frac{1}{4}M^{-2}\bar{q}t_j\gamma^\mu q$, and $-\frac{1}{4}M^{-2}\bar{q}t_j\gamma^\mu\gamma_5 q$, respectively,in the standard way. They are regarded as the meson fields. Eq.(23) can then be converted into

$$Z[J] = exp(S_K[\frac{\delta}{\delta I}]) \int DqD\bar{q}\, D\hat{S}D\hat{P}\, D\hat{V}D\hat{A}\, exp\int d^4x\, \{ -\frac{1}{2M^2}I_\mu^a I^{a\mu} -$$

$$-i\frac{\theta}{16\pi^2}\, tr_c(F_{I\mu\nu}\tilde{F}_I^{\mu\nu}) + \bar{q}[i\partial\!\!\!/ + \frac{I\!\!\!/}{M^2} - (\hat{S}+J_S) + i(\hat{P}+J_P)\gamma_5 + (\hat{V}\!\!\!/ + V\!\!\!/_V) + (\hat{A}\!\!\!/ + A\!\!\!/_A)\gamma_5]q +$$

$$+ M^2[-\hat{S}^2 - \hat{P}^2 + 2\hat{V}^2 + 2\hat{A}^2] + 2M^2(\frac{1}{1-\frac{2N_f}{N_c}} - 1)\, \hat{V}_0^2\} \,|_{I=0} \ , \qquad (25)$$

in which $\hat{S}\equiv t_j\hat{S}_j$,$\hat{S}^2\equiv\hat{S}_j\hat{S}_j$, $\hat{P}\equiv t_j\hat{P}_j$, $\hat{P}^2\equiv\hat{P}_j\hat{P}_j$, $\cdots$, etc. For convinience, we introduce the new variables $S\equiv\hat{S}+J_S$, $P\equiv\hat{P}+J_S$, $V^\mu\equiv\hat{V}^\mu+J_V^\mu$, $A^\mu\equiv\hat{A}^\mu+J_A^\mu$. Carrying out the quark-field integration, we get

$$Z[J] = exp(S_K[\frac{\delta}{\delta I}]) \int DSDPDVDA\, exp\{lnDet(i\partial\!\!\!/ - S + iP\gamma_5 + V\!\!\!/_I + A\!\!\!/\gamma_5) +$$

$$+\int d^4x[-\frac{1}{2M^2}I_\mu^a I^{a\mu} - i\frac{\theta}{16\pi^2}tr_c(F_{I\mu\nu}\tilde{F}_I^{\mu\nu}) + 2M^2(\frac{1}{1-\frac{2N_f}{N_c}} - 1)\,(V_0^\mu - J_{V_0}^\mu)^2$$

$$+ M^2(-(S-J_S)^2 - (P-J_P)^2 + 2(V^\mu - J_V^\mu)^2 + 2(A^\mu - J_A^\mu)^2)]\} \,|_{I=0} \ , \qquad (26)$$

where Det is the functional determinant,and

$$V_I^\mu \equiv V^\mu + \frac{I^\mu}{M^2} \ . \qquad (27)$$

Eq.(26) is our generating fuctional for the meson fields in the linear realization.

2. The Local $U(N_f)_L \times U(N_f)_R$ Symmetry

Let $g_L(x)$ and $g_R(x)$ be the group element of $U(N_f)_L$ and $U(N_f)_R$,respectively.Consider the following local $U(N_f)_L \times U(N_f)_R$ transformation

$$\begin{aligned}
S'(x) + iP'(x) &= g_R(x)[S(x) + iP(x)]g_L^\dagger(x) \\
S'(x) - iP'(x) &= g_L(x)[S(x) - iP(x)]g_R^\dagger(x) \\
V'_\mu(x) + A'_\mu(x) &= g_R(x)[V_\mu(x) + A_\mu(x)]g_R^\dagger(x) + ig_R(x)\partial_\mu g_R^\dagger(x) \\
V'_\mu(x) - A'_\mu(x) &= g_L(x)[V_\mu(x) - A_\mu(x)]g_L^\dagger(x) + ig_L(x)\partial_\mu g_L^\dagger(x) \\
J'_S(x) + iJ'_P(x) &= g_R(x)[J_S(x) + iJ_P(x)]g_L^\dagger(x) \\
J'_S(x) - iJ'_P(x) &= g_L(x)[J_S(x) - iJ_P(x)]g_R^\dagger(x) \\
J'_{V\mu}(x) + J'_{A\mu}(x) &= g_R(x)[J_{V\mu}(x) + J_{A\mu}(x)]g_R^\dagger(x) + ig_R(x)\partial_\mu g_R^\dagger(x) \\
J'_{V\mu}(x) - J'_{A\mu}(x) &= g_L(x)[J_{V\mu}(x) - J_{A\mu}(x)]g_L^\dagger(x) + ig_L(x)\partial_\mu g_L^\dagger(x) \\
I_\mu^{a\prime}(x) &= I_\mu^a(x)
\end{aligned} \qquad (28)$$

All terms in (26), except the fermion determinant term $lnDet(i\partial\!\!\!/ - S + iP\gamma_5 + V\!\!\!/_I + A\!\!\!/\gamma_5)$, are manifestly invariant under the local chiral transformation (28). The transformation property of $lnDet(i\partial\!\!\!/ - S + iP\gamma_5 + V\!\!\!/_I + A\!\!\!/\gamma_5)$ can be examined as follows. The transformation of the covariant derivative

$$iD(x,y) \equiv (i\partial\!\!\!/ - S + iP\gamma_5 + V\!\!\!/_I + A\!\!\!/\gamma_5)\delta^4(x-y) \qquad (29)$$

is

$$D'(x,y) = [g_R(x)P_R + g_L(x)P_L]D(x,y)\,[g_L^\dagger(y)P_R + g_R^\dagger(y)P_L] \ , \qquad (30)$$

hence

$$lnDet(iD') = lnDet(iD) + lnDet(g_Rg_L^\dagger P_R + g_Lg_R^\dagger P_L) \ . \qquad (31)$$

Now

$$\begin{aligned}
RelnDet(g_Rg_L^\dagger P_R + g_Lg_R^\dagger P_L) &= \frac{1}{2}Trln\,(g_Rg_L^\dagger P_R + g_Lg_R^\dagger P_L) + \frac{1}{2}Trln\,(g_Lg_R^\dagger P_R + g_Rg_L^\dagger P_L) \\
&= \frac{1}{2}Trln(P_L + P_R) = 0 \ ,
\end{aligned} \qquad (32)$$

i.e. $Det(g_Rg_L^\dagger P_R + g_Lg_R^\dagger P_L)$ is formally a unimodular phase factor. Therefore *the real part of* $lnDet(iD)$ *is invariant under the transformation* (28)[21]. Moreover, when

$g_R(x) = g_L(x) = g(x)$, we have

$$Trln(g_R g_L^\dagger P_R + g_L g_R^\dagger P_L) \mid_{g_R=g_L=g} = Trln(P_L + P_R) = 0 \ .$$

Therefore *the imaginary part of* $lnDet(i\mathcal{D})$ *is invariant under the diagonal transformation* $g_L(x) = g_R(x) = g(x)$ *but is not invariant under the chiral transformation* (28).

Thus, except for $ImlnDet(i\mathcal{D})$, all terms in (26) are invariant under the local chiral transformation (28). This symmetry has been considered in Ref.[1] as the basis of determining the form of the chiral Lagrangian.

III. The Nonlinear Realization and the General Form of the Low Energy Effective Lagrangian for Mesons

1.Nonlinear Realization

Instead of S and P, we introduce a polar parametrization (nonlinear realization) for $S + iP$ [10]

$$S(x) + iP(x) = \Omega(x)[\Sigma(x) + v]\Omega(x) \ , \tag{33}$$

where $\Omega(x)$ is a unitary matrix whose phase can be parametrized by the Goldstone boson fields, and $\Sigma(x) + v$ is an hermitian matrix. Here we have formally separated out a possible VEV v of the Σ-field, i.e. $\Sigma - J_S$ is the quantum fluctuation around v. Since $S_0(x)$ is the auxiliary field for $-\frac{1}{2M^2}(\bar{q}t_0 q)$, we see from (33) that $< \bar{q}q >$ is related to v by the simple relation

$$< \bar{q}q >= -2\sqrt{N_f}M^2 < S_0 >= -2N_f M^2 v \ . \tag{34}$$

The value of v is to be determined by the minimum of the effective potential, so that $< \bar{q}q >$ is calculable in this theory. If $v \neq 0$, chiral symmetry is broken dynamically by $< \bar{q}q >$.

It is also useful to introduce

$$U(x) \equiv \Omega(x)\Omega(x) \tag{35}$$

which is also unitary. Under the transformation (28), the fields Σ(x) , Ω(x) , and $U(x)$ transform as

$$\begin{aligned} \Sigma'(x) &= h(x)\Sigma(x)h^\dagger(x) \ , \\ \Omega'(x) &= h(x)\Omega(x)g_L^\dagger(x) = g_R(x)\Omega(x)h^\dagger(x) \ , \\ U'(x) &= g_R(x)U(x)g_L^\dagger(x) \ , \end{aligned} \tag{36}$$

where the unitary matrix $h(x)$ is the element of an extra hidden local $U(N_f)_h$ group [13] coming from the freedom of introducing the nonlinear realization.

2.The WZW Term

Now we evaluate $ImlnDet(i\mathcal{D})$. For a given $\mathcal{D}$, we make a chiral transformation with $g_R(x) = \Omega(x)$, $g_L(x) = \Omega^\dagger(x)$. From (30) we have

$$i\mathcal{D} \rightarrow i\mathcal{D}_\Omega = [i\partial\!\!\!/ - (\Sigma(x) + v) - V\!\!\!/_{I\Omega}(x) - A\!\!\!/_\Omega(x)\gamma_5]\delta^4(x-y) \ , \tag{37}$$

where $V\!\!\!/_{I\Omega}$ and $A\!\!\!/_\Omega$ are given by (28).Consider a change of Ω ,$\Omega \rightarrow \Omega + \delta\Omega$. Then

$$\begin{aligned} \delta(lnDetI\mathcal{D}_\Omega) &= Tr(\mathcal{D}_\Omega^{-1}\delta\mathcal{D}_\Omega) = \\ &= \tfrac{1}{2}Tr[\mathcal{D}_\Omega^{-1}\ (\delta\Omega\Omega^\dagger - \Omega^\dagger\delta\Omega)\ \mathcal{D}_\Omega - \mathcal{D}_\Omega^{-1}\mathcal{D}_\Omega\ (\delta\Omega\Omega^\dagger - \Omega^\dagger\delta\Omega) + \\ &\quad + \mathcal{D}_\Omega^{-1}\ (\delta\Omega\Omega^\dagger + \Omega^\dagger\delta\Omega)\ \gamma_5\mathcal{D}_\Omega\ + \mathcal{D}_\Omega^{-1}\mathcal{D}_\Omega\ (\delta\Omega\Omega^\dagger + \Omega^\dagger\delta\Omega)\ \gamma_5] \\ &= Tr[(\delta\Omega\Omega^\dagger\ + \Omega^\dagger\delta\Omega)\gamma_5] \ . \end{aligned} \tag{38}$$

This functional trace is evaluated in Appendix A, and we get

$$\delta(lnDetI\mathcal{D}_\Omega) = Tr[\ (\delta\Omega\Omega^\dagger + \Omega^\dagger\delta\Omega)\ G(I, V_\Omega, A_\Omega)] \ , \tag{39}$$

where

$$G(I,V,A) = \frac{N_c}{\pi^2}\epsilon^{\mu\nu\rho\sigma}\ [\frac{1}{3}A_\mu A_\nu A_\rho A_\sigma\ + \frac{1}{32}(V_{\mu\nu}V_{\rho\sigma}\ + \frac{1}{3}A_{\mu\nu}A_{\rho\sigma}) -$$

$$-\frac{1}{12}(A_\mu A_\nu V_{\rho\sigma} + V_{\mu\nu}A_\rho A_\sigma + A_\mu V_{\nu\rho}A_\sigma)] + \frac{1}{16\pi^2}tr_c(F_{I\mu\nu}\tilde{F}_I^{\mu\nu}) , \qquad (40)$$

with $V_{\mu\nu} \equiv \partial_\mu V_\nu - \partial_\nu V_\mu - i[V_\mu, V_\nu] - i[A_\mu, A_\nu]$, $A_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - i[V_\mu, A_\nu] - i[A_\mu, V_\nu]$. The difference between our $G(I, V, A)$ and the function $G(V, A)$ in Ref.[10] is only the term $\frac{1}{16\pi^2}tr_c(F_{I\mu\nu}\tilde{F}_I^{\mu\nu})$ in (40), which is independent of Ω . Therefore we can simply quote the result of integrating over $\delta\Omega$ in Ref.[10] and get (see Appendix A)

$$\begin{aligned} &lnDet(i\mathcal{D}_\Omega) - lnDet(i\mathcal{D}) = \\ &= i\frac{N_c}{240\pi^2}\int_{D_5} tr'(U^{-1}dU)^5 + i\int_{S^4}\mathcal{A}(U, A_L, A_R) + \int d^4x(lndetU)\,\frac{1}{16\pi^2}tr_c(F_{I\mu\nu}\tilde{F}_I^{\mu\nu}) , \end{aligned} \qquad (41)$$

where $A_R^\mu = V^\mu + A^\mu$, $A_L^\mu = V^\mu - A^\mu$, and $\mathcal{A}(U, A_L, A_R)$ is given in (11) with J_V , J_A replaced by V , A , respectively. Since $ReInDet(i\mathcal{D})$ is invariant under the $\Omega(x)$ transformation, the L.H.S. of (41) is simply $iImlnDet(i\mathcal{D}_\Omega) - iImlnDet(i\mathcal{D})$. Thus we have

$$\begin{aligned} iImlnDet(i\mathcal{D}) &= -i\frac{N_c}{240\pi^2}\int_{D_5} tr'(U^{-1}dU)^5 - i\int_{S^4}\mathcal{A}(U, A_L, A_R) - \\ &\quad -\int d^4x(lndetU)\,\frac{1}{16\pi^2}tr_c(F_{I\mu\nu}\tilde{F}_I^{\mu\nu}) + iImlnDet(i\mathcal{D}_\Omega) . \end{aligned} \qquad (42)$$

From (37) and (28) we see that the integration constant $iImlnDet(i\mathcal{D}_\Omega)$ does not contain pure U terms. The first term on R.H.S. of (42) is just the WZW term. Therefore our $iImlnDet(i\mathcal{D})$ contains the exact WZW term together with other anomaly terms.

3.Effective Lagrangian for Mesons

In terms of Σ and U , our generating functional (26) can be written as

$$Z[J] = exp[P(J)] \int \mathcal{D}\mu(U,\Sigma)\mathcal{D}V\mathcal{D}A\; exp\; S_{eff}[U, \Sigma, V, A, J, \theta] , \qquad (43)$$

in which $\mathcal{D}\mu(U,\Sigma)$ is the functional measure for the transformed variables in (33) ,

$$P(J) \equiv M^2\int d^4x\{-J_S^2 - J_P^2 + 2J_V^2 + 2J_A^2 + 2(\frac{1}{1-\frac{2N_f}{N_c}} - 1)J_{V_0}^2\} , \qquad (44)$$

$$\begin{aligned} S_{eff}[U, \Sigma, V, A, J, \theta] &\equiv \int d^4x\mathcal{L}_{eff}(U, \Sigma, V, A, J, \theta) \\ &= S_{eff}^{(MJ)}[U, \Sigma, V, A, J] + S_{eff}^{(M1)}[U, \Sigma, V, A] + S_{eff}^{(M2)}[U, \Sigma, V, A, \theta] , \end{aligned} \qquad (45)$$

where $P(J)$ is a pure external source term, and the three parts in $S_{eff}[U, \Sigma, V, A, J, \theta]$ are

$$\begin{aligned} S_{eff}^{(MJ)}[U, \Sigma, V, A, J] &\equiv \int d^4x\mathcal{L}_{eff}^{(MJ)}(U, \Sigma, V, A, J) \\ &= M^2\int d^4x\{J_S[\Omega\Sigma\Omega + \Omega^\dagger\Sigma\Omega^\dagger + (U + U^\dagger)v] \\ &\quad -iJ_P[\Omega\Sigma\Omega - \Omega^\dagger\Sigma\Omega^\dagger + (U - U^\dagger)v] - \\ &\quad -4J_V^\mu V_\mu - 4J_A^\mu A_\mu + 4(1 - \frac{1}{1-\frac{2N_f}{N_c}})J_{V_0}V_0\} , \end{aligned} \qquad (46)$$

$$\begin{aligned} S_{eff}^{(M1)}[U, \Sigma, V, A] &\equiv \int d^4x\mathcal{L}_{eff}^{(M1)}(U, \Sigma, V, A) \\ &= M^2\int d^4x\{-(\Sigma + v)^2 + 2V^2 + 2A^2 + 2(\frac{1}{1-\frac{2N_f}{N_c}} - 1)V_0^2\} \\ &\quad -i\frac{N_c}{240\pi^2}\int_{D_5} tr'(U^{-1}dU)^5 - i\int_{S^4}\mathcal{A}(U, A_L, A_R) , \end{aligned} \qquad (47)$$

$$\begin{aligned} S_{eff}^{(M2)}[U, \Sigma, V, A, \theta] &\equiv \int d^4x\mathcal{L}_{eff}^{(M2)}(U, \Sigma, V, A, \theta) \\ &= ln\,\{\,exp(S_K[\frac{\delta}{\delta I}])\; exp[\,Re\,Tr\,ln(i\mathcal{D}) + iIm\,Tr\,ln(i\mathcal{D}_\Omega) + \\ &\quad +\int d^4x\,[-\frac{(I^a)^2}{2M^2} - \frac{1}{16\pi^2}(i\theta + lndetU)tr_c(F_{I\mu\nu}\tilde{F}_I^{\mu\nu})]\}\,|_{I=0} , \end{aligned} \qquad (48)$$

which are, respectively, the meson-source interaction part, the meson-field self-interaction part independent of $S_K[\frac{\delta}{\delta I}]$, and the pure meson part containing $S_K[\frac{\delta}{\delta I}]$ (non-leading orders of $1/g_s^2$ expansion). We see that the WZW term and the anomaly term $\int_{S^4}\mathcal{A}(U, A_L, A_R)$ are all in $S_{eff}^{(M1)}$, i.e. they do not receive any corrections in the $1/g_s^2$ expansion.

The symmetry of the effective action can be examined as follows. The term $lndet\ U$ in $S_{eff}^{(M2)}$ is not invariant under the chiral $U(N_f)_L \times U(N_f)_R$ transforma-

tion.According to (35), $detU$ transforms as

$$detU' = (detg_R)(detg_L)^{-1}(detU) \ . \tag{49}$$

For the $SU(N_f)_L$ and $SU(N_f)_R$ subgroups, we have $detg_L = 1$, $detg_R = 1$; for the diagonal $U_V(1)$ subgroup, we have $detg_R = detg_L$. Therefore $lndetU$ is invariant under the transformation of the subgroup $SU(N_f)_L \times SU(N_f)_R \times U_V(1)$. In other words, $lndetU$ explicitly breaks the $U_A(1)$ subgroup of $U(N_f)_L \times U(N_f)_R$. However, $lndetU$ appears together with $i\theta$ in the combination

$$i\bar{\theta} \equiv i\theta + lndetU \tag{50}$$

in $S_{eff}^{(M2)}$. If we take the transformation of θ to be

$$\theta' = \theta + ilndetg_R - ilndetg_L \ , \tag{51}$$

the combination $i\bar{\theta}$ can be made invariant under the complete $U(N_f)_L \times U(N_f)_R$ transformation. This is just what was mentionned in Ref.[1].

There is an extra hidden symmetry $U(N_f)_h$ (cf.(36)) in the nonlinear realization due to the freedom of defining Σ and Ω . The present theory also respects this extra symmetry.

Further evaluation of $S_{eff}^{(M2)}$ includes: (a) the evaluation of $ReTrln(i\not{D})$, and (b) the operation of $exp(S_K[\frac{\delta}{\delta J}])$ $(1/g_s^2$ expansion). At low energies, we can expand $ReTrln(i\not{D})$ in power series of covariant derivatives of the meson fields. Formally we write it as

$$ReTrln(i\not{D}) = \sum_n R_n \ , \tag{52}$$

where R_n is the sum of terms containing dimension-n field operators. From dimensional analysis,we infer that the coefficients in R_n for $n > 4$ must be proportional to negative powers of the fundamental scale parameter Λ, so that at low energies these terms are suppressed [22]. As an approximation, we shall keep only $n \leq 4$ terms

in (52). The expansion can be carried out by using the technique in Ref.[23] , and the details are given in Appendix B. The results of R_0 , R_1 , $\cdots$, R_4 are

$$R_0 = \frac{1}{2} Tr\ ln\ (v^2 + \not{D}_I \not{D}_I\ ) \ ,$$

$$R_1 = v\int d^4x \int \frac{d^4k}{(2\pi)^4}\, tr(\Delta_k \Sigma) \ ,$$

$$R_2 = \int d^4x \int \frac{d^4k}{(2\pi)^4}\, tr\{\ (\frac{1}{2}\Delta_k - v^2\Delta_k^2)\Sigma^2 + \frac{v^2}{16}(\Delta_k\gamma_\nu\Delta_k\gamma^\nu)(D^\mu U^\dagger D^\mu U) +$$
$$+[\Delta_k g^{\mu\nu} - 2\Delta_k(iD_I^\mu + k^\mu)\Delta_k(iD_I^\nu + k^\nu)][(V_\mu - J_{V\mu})(V^\mu - J_V^\mu) + (A_\mu - J_{A\mu})(A^\mu - J_A^\mu)]\}\ ,$$

$$R_3 = -\int d^4x \int \frac{d^4k}{(2\pi)^4}\, tr\{\ v\Delta_k^2\Sigma^3 + \frac{v^3}{4}(\Delta_k\gamma_\nu\Delta_k\gamma^\nu\Delta_k)(D^\mu U^\dagger D_\mu U)\Omega^\dagger\Sigma\Omega +$$
$$+\frac{v}{16}(\Delta_k\gamma_\nu\Delta_k\gamma^\nu)[(D^\mu D_\mu U^\dagger)\Omega\Sigma\Omega + (D^\mu D_\mu U)\Omega^\dagger\Sigma\Omega^\dagger]\}\ ,$$

$$R_4 = \int d^4x \int \frac{d^4k}{(2\pi)^4}\, tr\{\ -(\frac{1}{4}\Delta_k^2 + \frac{v^4}{2}\Delta_k^4)\Sigma^4 + \frac{v^2}{48}(\Delta_k^2\gamma_\rho\Delta_k\gamma^\rho)(D^\mu D^\nu U^\dagger)(D_\mu D_\nu U) +$$
$$+v^4[(\frac{1}{288}\Delta_k\gamma_\rho\Delta_k\gamma_\sigma\Delta_k\gamma^\rho\Delta_k\gamma^\sigma - \frac{1}{72}\Delta_k\gamma_\rho\Delta_k\gamma^\rho\Delta_k\gamma_\sigma\Delta_k\gamma^\sigma)(D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U) +$$
$$+(\frac{1}{288}\Delta_k\gamma_\rho\Delta_k\gamma^\rho\Delta_k\gamma_\sigma\Delta_k\gamma^\sigma - \frac{5}{576}\Delta_k\gamma_\rho\Delta_k\gamma_\sigma\Delta_k\gamma^\rho\Delta_k\gamma^\sigma)(D_\mu U^\dagger D_\nu U D^\mu U^\dagger D^\nu U)] +$$
$$+(-\frac{5}{24}\Delta_k^2 - \frac{1}{192}\Delta_k[\gamma_\rho,\gamma_\sigma]\Delta_k\gamma^\rho\gamma^\sigma + \frac{1}{24}\Delta_k(iD_I^\rho + k^\rho)\Delta_k(iD_I^\sigma + k^\sigma)\Delta_k[\gamma_\rho,\gamma_\sigma] -$$
$$-\frac{1}{6}\Delta_k(iD_I^\rho + k^\rho)\Delta_k(iD_{I\rho} + k_\rho)\Delta_k(iD_I^\sigma + k^\sigma)\Delta_k(iD_{I\sigma} + k_\sigma))(V_{\mu\nu}V^{\mu\nu} + A_{\mu\nu}A^{\mu\nu}) -$$
$$-(\frac{v^2}{8}\Delta_k\gamma_\nu\Delta_k\gamma^\nu\Delta_k - \frac{v^4}{2}\Delta_k^3\gamma_\nu\Delta_k\gamma^\nu + \frac{v^2}{3}\Delta_k^3)(D^\mu U^\dagger D_\mu U)\Omega^\dagger\Sigma^2\Omega - \frac{v^2}{3}\Delta_k^3[D_\mu(\Omega^\dagger\Sigma\Omega^\dagger)][D^\mu(\Omega\Sigma\Omega)] -$$
$$-v^2(\frac{1}{4}\Delta_k\gamma_\nu\Delta_k\gamma^\nu\Delta_k + \frac{1}{3}\Delta_k^3)([D^\mu(\Omega^\dagger\Sigma\Omega^\dagger)](D_\mu U)\Omega^\dagger\Sigma\Omega + [D^\mu(\Omega^\dagger\Sigma\Omega^\dagger)]\Omega\Sigma\Omega^\dagger(D_\mu U)) +$$
$$+\frac{v^4}{4}(\Delta_k^2\gamma_\nu\Delta_k^2\gamma^\nu)(D^\mu U^\dagger)\Omega\Sigma\Omega^\dagger(D_\mu U)\Omega^\dagger\Sigma\Omega -$$
$$-iv^2[-\frac{1}{96}\Delta_k[\gamma_\rho,\gamma_\sigma]\Delta_k\gamma^\rho\Delta_k\gamma^\sigma + \frac{1}{12}\Delta_k(iD_{I\lambda} + k_\lambda)\Delta_k(iD_{I\kappa} + k_\kappa)\Delta_k\gamma_\tau\Delta_k(g^{\kappa\tau}\gamma^\lambda - g^{\lambda\tau}\gamma^\kappa)] \times$$
$$\times[(V^{\mu\nu} - A^{\mu\nu})D_\mu U^\dagger D_\nu U + (V^{\mu\nu} + A^{\mu\nu})D_\mu U D_\nu U^\dagger]\}\ , \tag{53}$$

where

$$
\begin{aligned}
D_I^\mu &\equiv \partial^\mu\,|_{only\ on\ I} \ -i\frac{I^\mu}{M^2}\ , \\
D^\mu U &\equiv \partial^\mu U - i(V^\mu + A^\mu)U + iU(V^\mu - A^\mu)\ , \\
D^\mu U^\dagger &\equiv \partial^\mu U^\dagger - i(V^\mu - A^\mu)U^\dagger + iU^\dagger(V^\mu + A^\mu)\ , \\
D^\mu(\Omega\Sigma\Omega) &\equiv \partial^\mu(\Omega\Sigma\Omega) - i(V^\mu + A^\mu)\Omega\Sigma\Omega + i\Omega\Sigma\Omega(V^\mu - A^\mu)\ , \\
D^\mu(\Omega^\dagger\Sigma\Omega^\dagger) &\equiv \partial^\mu(\Omega^\dagger\Sigma\Omega^\dagger) - i(V^\mu - A^\mu)\Omega^\dagger\Sigma\Omega^\dagger + i\Omega^\dagger\Sigma\Omega^\dagger(V^\mu + A^\mu)\ , \\
\Delta_k &\equiv (v^2 + k^2 + \not{D}_I\ \not{D}_I\ - 2ik_\mu D_I^\mu)^{-1}\ .
\end{aligned}
\tag{54}
$$

We have checked that (52) and (53) are consistent with the formula obtained from using the Seeley-Dewitt expansion [24] in Ref.[11].

Next we introduce the operator

$$
\mathcal{O}[\frac{\delta}{\delta I}] \equiv exp(S_K[\frac{\delta}{\delta I}]) \cdot exp\{\frac{1}{2}Trln(v^2 + \not{D}_I\ \not{D}_I\ ) - \int d^4x\frac{(I^a)^2}{2M^2}\ \}\ , \tag{55}
$$

and define

$$
a_0 \equiv \mathcal{O}[\frac{\delta}{\delta I}]\ \cdot\ 1\ . \tag{56}
$$

It can be shown that up to the next-to-the-leading order in $1/g_s^2$ expansion, the anomalous part of $S_{eff}^{(M2)}$ (imaginary part in the curly bracket in (48)) does not contribute. In this paper we only present the contribution of the normal part in $S_{eff}^{(M2)}$. Up to $n \leq 4$ it is

$$
\begin{aligned}
&S_{eff}^{(M2)}[U,\Sigma,V,A,\theta]_{norm} \equiv \int d^4x\mathcal{L}_{eff}^{(M2)}(U,\Sigma,V,A,\theta)_{norm} = [lna_0 + \\
&a_0^{-1}\mathcal{O}[\frac{\delta}{\delta I}](R_1 + R_2 + \frac{1}{2}R_1^2 + R_3 + R_1R_2 + \frac{1}{6}R_1^3 + R_4 + R_1R_3 + \frac{1}{2}R_2^2 + \frac{1}{2}R_1^2R_2 + \frac{1}{24}R_1^4) - \\
&-\{a_0^{-1}\mathcal{O}[\frac{\delta}{\delta I}]R_1\}_{I=0}\{a_0^{-1}\mathcal{O}[\frac{\delta}{\delta I}](R_3 + R_1R_2 + \frac{1}{6}R_1^3)\} - \frac{1}{2}\{a_0^{-1}\mathcal{O}[\frac{\delta}{\delta I}](R_2 + \frac{1}{2}R_1^2)\}^2 + \\
&+\{a_0^{-1}\mathcal{O}[\frac{\delta}{\delta I}]R_1\}_{I=0}^2\{a_0^{-1}\mathcal{O}[\frac{\delta}{\delta I}](R_2 + \frac{1}{2}R_1^2)\} - \frac{1}{4}\{a_0^{-1}\mathcal{O}[\frac{\delta}{\delta I}]R_1\}^4]_{I=0}\ .
\end{aligned}
\tag{57}
$$

Substituting (53) into (57) , we get

$$
\begin{aligned}
&S_{eff}^{(M2)}[U,\Sigma,V,A,\theta]_{norm} \equiv \int d^4x\mathcal{L}_{eff}^{(M2)}(U,\Sigma,V,A,\theta)_{norm} = \\
&= ln\ a_0 + \int d^4x[tr_f\{va_1\Sigma + (\frac{1}{2}a_1 - v^2a_2)\Sigma^2 - va_2\Sigma^3 - (\frac{1}{4}a_2 + \frac{v^4}{2}a_9)\Sigma^4 + \frac{v^2}{16}a_3(D_\mu U^\dagger D^\mu U)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\Lambda^2}{32\pi^2}[(V_\mu - J_{V\mu})(V^\mu - J_V^\mu) + (A_\mu - J_{A\mu})(A^\mu - J_A^\mu)] + \frac{v^2}{48}a_6(D^\mu D^\nu U^\dagger)(D_\mu D_\nu U) + \\
&+v^4(\frac{a_{13}}{288} - \frac{a_{12}}{72})(D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U) + v^4(\frac{a_{12}}{288} - \frac{5a_{13}}{576})(D_\mu U^\dagger D_\nu U D^\mu U^\dagger D^\nu U) + \\
&+(\frac{5}{24}a_1 + \frac{1}{192}a_4 - \frac{1}{24}a_8 - \frac{1}{6}a_{14})(V_{\mu\nu}V^{\mu\nu} + A_{\mu\nu}A^{\mu\nu}) - \frac{v^3}{4}a_6(D_\mu U^\dagger D^\mu U)\Omega^\dagger\Sigma\Omega - \\
&-(\frac{v^2}{8}a_6 - \frac{v^4}{2}a_{10} + \frac{v^2}{3}a_5)(D_\mu U^\dagger D^\mu U)\Omega^\dagger\Sigma^2\Omega - \frac{v^2}{3}a_5[D_\mu(\Omega^\dagger\Sigma\Omega^\dagger)][D^\mu(\Omega\Sigma\Omega)] - \\
&-\frac{v}{16}a_3[(D_\mu D^\mu U^\dagger)\Omega\Sigma\Omega + (D_\mu D^\mu U)\Omega^\dagger\Sigma\Omega^\dagger] + \frac{v^4}{4}a_{11}(D_\mu U^\dagger)\Omega\Sigma\Omega^\dagger(D^\mu U)\Omega^\dagger\Sigma\Omega - \\
&-v^2(\frac{a_6}{4} + \frac{a_5}{3})[(D_\mu(\Omega^\dagger\Sigma\Omega^\dagger))(D^\mu U)\Omega^\dagger\Sigma\Omega + (D_\mu(\Omega^\dagger\Sigma\Omega^\dagger))\Omega\Sigma\Omega^\dagger(D^\mu U)] + \\
&+iv^2(\frac{a_7}{96} - \frac{a_{15}}{12})[(V_{\mu\nu} - A_{\mu\nu})(D^\mu U^\dagger)(D^\nu U) + (V_{\mu\nu} + A_{\mu\nu})(D^\mu U)(D^\nu U^\dagger)]\} + \\
&+\frac{v^2}{2}a_{16}(tr_f\Sigma)^2 - \frac{v^2}{48}a_{17}tr_f(\partial_\mu\Sigma\partial^\mu\Sigma)(tr_f\Sigma) - \frac{a_{18}}{384}[(tr_fV_{\mu\nu})(tr_fV^{\mu\nu}) + (tr_fA_{\mu\nu})(tr_fA^{\mu\nu})] + \\
&+(\frac{v}{2}a_{16} - v^3a_{19})(tr_f\Sigma)(tr_f\Sigma^2) + (\frac{1}{8}a_{16} - \frac{v^2}{2}a_{19} + \frac{v^4}{2}a_{21})(tr_f\Sigma^2)^2 - \\
&-\frac{v^2}{16}(\frac{1}{2}a_{20} - v^2a_{22})(tr_f\Sigma^2)tr_f(D_\mu U^\dagger D^\mu U) + \frac{v^3}{16}a_{20}(tr_f\Sigma)tr_f(D_\mu U^\dagger D^\mu U) + \\
&+\frac{v^2}{8}a_{20}(tr_f\Sigma)tr_f[(D_\mu U^\dagger)(D^\mu(\Omega\Sigma\Omega))] - v^2a_{19}(tr_f\Sigma)(tr_f\Sigma^3) - \frac{v^4}{4}a_{23}(tr_f\Sigma)tr_f[(D_\mu U^\dagger)(D^\mu U) \\
&+\frac{v^4}{1152}(5a_{24} - 2a_{25})tr_f(D_\mu U^\dagger D^\mu U)tr_f(D_\nu U^\dagger D^\nu U) + \frac{v^4}{576}(4a_{25} - a_{24})tr_f(D_\mu U^\dagger D_\nu U)tr_f(D^\mu U \\
&+\frac{v^3}{6}a_{26}(tr_f\Sigma)^3 + \frac{v^4}{32}a_{28}(tr_f\Sigma)^2tr_f(D_\mu U^\dagger D^\mu U) + (\frac{v^2}{8}a_{26} - \frac{v^4}{2}a_{27})(tr_f\Sigma)^2(tr_f\Sigma^2) + \frac{v^4}{24}a_{29}(tr_f\Sigma
\end{aligned}
\tag{58}
$$

where tr_f is the trace in the flavor space,and

$$
a_1 \equiv a_0^{-1}\mathcal{O}[\frac{\delta}{\delta I}]\int \frac{d^4k}{(2\pi)^4}tr''\Delta_k(x,x)\ |_{I=0}\ ,
$$

$$
a_2 \equiv a_0^{-1}\mathcal{O}[\frac{\delta}{\delta I}]\int \frac{d^4k}{(2\pi)^4}tr''\Delta_k^2(x,x)\ |_{I=0}\ ,
$$

$$
a_3 \equiv a_0^{-1}\mathcal{O}[\frac{\delta}{\delta I}]\int \frac{d^4k}{(2\pi)^4}tr''(\Delta_k\gamma_\mu\Delta_k\gamma^\mu)(x,x)\ |_{I=0}\ ,
$$

$$
a_4 \equiv a_0^{-1}\mathcal{O}[\frac{\delta}{\delta I}]\int \frac{d^4k}{(2\pi)^4}tr''(\Delta_k[\gamma_\mu,\gamma_\nu]\Delta_k\gamma^\mu\gamma^\nu)(x,x)\ |_{I=0}\ ,
$$

$$a_5 \equiv a_0^{-1}\mathcal{O}[\frac{\delta}{\delta I}]\int \frac{d^4k}{(2\pi)^4} tr''\Delta_k^3(x,x)\,|_{I=0}\,,$$

$$a_6 \equiv a_0^{-1}\mathcal{O}[\frac{\delta}{\delta I}]\int \frac{d^4k}{(2\pi)^4} tr''(\Delta_k^2\gamma_\mu\Delta_k\gamma^\mu)(x,x)\,|_{I=0}\,,$$

$$a_7 \equiv a_0^{-1}\mathcal{O}[\frac{\delta}{\delta I}]\int \frac{d^4k}{(2\pi)^4} tr''(\Delta_k[\gamma_\mu,\gamma_\nu]\Delta_k\gamma^\mu\Delta_k\gamma^\nu)(x,x)\,|_{I=0}\,,$$

$$a_8 \equiv a_0^{-1}\mathcal{O}[\frac{\delta}{\delta I}]\int \frac{d^4k}{(2\pi)^4} tr''(\Delta_k(iD_{I\mu}+k_\mu)\Delta_k(iD_{I\nu}+k_\nu)\Delta_k[\gamma^\mu,\gamma^\nu])(x,x)\,|_{I=0}\,,$$

$$a_9 \equiv a_0^{-1}\mathcal{O}[\frac{\delta}{\delta I}]\int \frac{d^4k}{(2\pi)^4} tr''\Delta_k^4(x,x)\,|_{I=0}\,,$$

$$a_{10} \equiv a_0^{-1}\mathcal{O}[\frac{\delta}{\delta I}]\int \frac{d^4k}{(2\pi)^4} tr''(\Delta_k^3\gamma_\mu\Delta_k\gamma^\mu)(x,x)\,|_{I=0}\,,$$

$$a_{11} \equiv a_0^{-1}\mathcal{O}[\frac{\delta}{\delta I}]\int \frac{d^4k}{(2\pi)^4} tr''(\Delta_k^2\gamma_\mu\Delta_k^2\gamma^\mu)(x,x)\,|_{I=0}\,,$$

$$a_{12} \equiv a_0^{-1}\mathcal{O}[\frac{\delta}{\delta I}]\int \frac{d^4k}{(2\pi)^4} tr''(\Delta_k\gamma_\mu\Delta_k\gamma^\mu\Delta_k\gamma_\nu\Delta_k\gamma^\nu)(x,x)\,|_{I=0}\,,$$

$$a_{13} \equiv a_0^{-1}\mathcal{O}[\frac{\delta}{\delta I}]\int \frac{d^4k}{(2\pi)^4} tr''(\Delta_k\gamma_\mu\Delta_k\gamma_\nu\Delta_k\gamma^\mu\Delta_k\gamma^\nu)(x,x)\,|_{I=0}\,,$$

$$a_{14} \equiv a_0^{-1}\mathcal{O}[\frac{\delta}{\delta I}]\int \frac{d^4k}{(2\pi)^4} tr''(\Delta_k(iD_{I\mu}+k_\mu)\Delta_k(iD_I^\mu+k^\mu)\Delta_k(iD_{I\nu}+k_\nu)\Delta_k(iD_I^\nu+k^\nu))(x,x)\,|_{I=0}\,,$$

$$a_{15} \equiv a_0^{-1}\mathcal{O}[\frac{\delta}{\delta I}]\int \frac{d^4k}{(2\pi)^4} tr''(\Delta_k(iD_{I\mu}+k_\mu)\Delta_k(iD_{I\nu}+k_\nu)(\Delta_k\gamma^\nu\Delta_k\gamma^\mu-\Delta_k\gamma^\mu\Delta_k\gamma^\nu))(x,x)\,|_{I=0}\,,$$

$$a_{16} \equiv a_0^{-1}\mathcal{O}[\frac{\delta}{\delta I}]\int d^4y\int \frac{d^4k}{(2\pi)^4} tr''\Delta_k(x,x)\int \frac{d^4q}{(2\pi)^4} tr''\Delta_q(y,y)\,|_{I=0,conn}\,,$$

$$a_{17} \equiv a_0^{-1}\mathcal{O}[\frac{\delta}{\delta I}]\int d^4y(x-y)_\mu(x-y)^\mu\int \frac{d^4k}{(2\pi)^4} tr''\Delta_k(x,x)\int \frac{d^4q}{(2\pi)^4} tr''\Delta_q(y,y)\,|_{I=0,conn}\,,$$

$$a_{18} \equiv a_0^{-1}\mathcal{O}[\frac{\delta}{\delta I}]\int d^4y\int \frac{d^4k}{(2\pi)^4} tr''(\Delta_k(x,x)[\gamma_\mu,\gamma_\nu])\int \frac{d^4q}{(2\pi)^4} tr''(\Delta_q(y,y)\gamma^\mu\gamma^\nu)\,|_{I=0,conn}\,,$$

$$a_{19} \equiv a_0^{-1}\mathcal{O}[\frac{\delta}{\delta I}]\int d^4y\int \frac{d^4k}{(2\pi)^4} tr''\Delta_k(x,x)\int \frac{d^4q}{(2\pi)^4} tr''\Delta_q^2(y,y)\,|_{I=0,conn}\,,$$

$$a_{20} \equiv a_0^{-1}\mathcal{O}[\frac{\delta}{\delta I}]\int d^4y\int \frac{d^4k}{(2\pi)^4} tr''\Delta_k(x,x)\int \frac{d^4q}{(2\pi)^4} tr''(\Delta_q\gamma_\mu\Delta_q\gamma^\mu)(y,y)\,|_{I=0,conn}\,,$$

$$a_{21} \equiv a_0^{-1}\mathcal{O}[\frac{\delta}{\delta I}]\int d^4y\int \frac{d^4k}{(2\pi)^4} tr''\Delta_k^2(x,x)\int \frac{d^4q}{(2\pi)^4} tr''\Delta_q^2(y,y)\,|_{I=0,conn}\,,$$

$$a_{22} \equiv a_0^{-1}\mathcal{O}[\frac{\delta}{\delta I}]\int d^4y\int \frac{d^4k}{(2\pi)^4} tr''\Delta_k^2(x,x)\int \frac{d^4q}{(2\pi)^4} tr''(\Delta_q\gamma_\mu\Delta_q\gamma^\mu)(y,y)\,|_{I=0,conn}\,,$$

$$a_{23} \equiv a_0^{-1}\mathcal{O}[\frac{\delta}{\delta I}]\int d^4y\int \frac{d^4k}{(2\pi)^4} tr''\Delta_k(x,x)\int \frac{d^4q}{(2\pi)^4} tr''(\Delta_q^2\gamma_\mu\Delta_q\gamma^\mu)(y,y)\,|_{I=0,conn}\,,$$

$$a_{24} \equiv a_0^{-1}\mathcal{O}[\frac{\delta}{\delta I}]\int d^4y\int \frac{d^4k}{(2\pi)^4} tr''(\Delta_k\gamma_\mu\Delta_k\gamma^\mu)(x,x)\int \frac{d^4q}{(2\pi)^4} tr''(\Delta_q\gamma_\nu\Delta_q\gamma^\nu)(y,y)\,|_{I=0,conn}\,,$$

$$a_{25} \equiv a_0^{-1}\mathcal{O}[\frac{\delta}{\delta I}]\int d^4y\int \frac{d^4k}{(2\pi)^4} tr''(\Delta_k\gamma_\mu\Delta_k\gamma_\nu)(x,x)\int \frac{d^4q}{(2\pi)^4} tr''(\Delta_q\gamma^\mu\Delta_q\gamma^\nu)(y,y)\,|_{I=0,conn}\,,$$

$$a_{26} \equiv a_0^{-1}\mathcal{O}[\frac{\delta}{\delta I}]\int d^4y\int d^4z\int \frac{d^4k}{(2\pi)^4} tr''\Delta_k(x,x)\int \frac{d^4q}{(2\pi)^4} tr''\Delta_q(y,y)\int \frac{d^4p}{(2\pi)^4} tr''\Delta_p(z,z)\,|_{I=0,conn}\,,$$

$$a_{27} \equiv a_0^{-1}\mathcal{O}[\frac{\delta}{\delta I}]\int d^4y\int d^4z\int \frac{d^4k}{(2\pi)^4} tr''\Delta_k(x,x)\int \frac{d^4q}{(2\pi)^4} tr''\Delta_q(y,y)\int \frac{d^4p}{(2\pi)^4} tr''\Delta_p^2(z,z)\,|_{I=0,conn}\,,$$

$$a_{28} \equiv a_0^{-1}\mathcal{O}[\frac{\delta}{\delta I}]\int d^4y\int d^4z\int \frac{d^4k}{(2\pi)^4} tr''\Delta_k(x,x)\int \frac{d^4q}{(2\pi)^4} tr''\Delta_q(y,y)\times$$
$$\times\int \frac{d^4p}{(2\pi)^4} tr''(\Delta_p\gamma_\mu\Delta_p\gamma^\mu)(z,z)\,|_{I=0,conn}\,,$$

$$a_{29} \equiv a_0^{-1}\mathcal{O}[\frac{\delta}{\delta I}]\int d^4y\int d^4z\int d^4u\int \frac{d^4k}{(2\pi)^4} tr''\Delta_k(x,x)\int \frac{d^4q}{(2\pi)^4} tr''\Delta_q(y,y)\times$$
$$\times\int \frac{d^4p}{(2\pi)^4} tr''\Delta_p(z,z)\int \frac{d^4r}{(2\pi)^4} tr''\Delta_r(u,u)\,|_{I=0,conn}\,, \tag{59}$$

in which tr'' is the trace in the color and Lorentz spinor spaces (not including flavors) , and the subscript *conn* means taking only the connected pieces in the terms with two or more $\frac{\delta}{\delta I}$ operations. We see that the non-Abelian self-interactions of the gluon fields in the underlying QCD theory reside now in the coefficients a_0 , a_1 ,$\cdots$, a_{29} through the $\mathcal{O}[\frac{\delta}{\delta I}]$ operation.

Substituting (46),(47), and (58) into (45), we get a lengthy expression for the low energy effective Lagrangian

$$\mathcal{L}_{eff}[U,\Sigma,V,A,J,\theta] == \mathcal{L}_{eff}^{(MJ)}[U,\Sigma,V,A,J]+\mathcal{L}_{eff}^{(M1)}[U,\Sigma,V,A]+\mathcal{L}_{eff}^{(M2)}[U,\Sigma,V,A,\theta]\,, \tag{60}$$

which contain 44 terms. The constant term lna_0 is only relevant in the study of the physical vacuum. If we simply regard the coefficients a_0 , a_1 ,$\cdots$, a_{29} as free parameters, this effective Lagrangian is just a generalization of the Gasser-Leutwyler effective Lagrangian to the case including pseudoscalar, scalar, vector, and axial-vector mesons. This general form is independent of the $1/g_s^2$ expansion.

The advantage of the present approach is that all the coefficients a_0 , a_1 ,$\cdots$, a_{29} in $\mathcal{L}_{eff}^{(M2)}(U,\Sigma,V,A,J,\theta)$ (cf.(59)) are calculable order by order in$1/g_s^2$ expansion as functions of the two fundamental parameters Λ and g_s (The VEV v can be calculated from the vacuum stability condition (cf.(68)).). In the following, we list the calculated a_1 ,$\cdots$, a_{29} up to order-$1/g_s^2$ in the $1/g_s^2$ expansion.

$$a_1 = \frac{N_c}{4\pi^2}\hat{a}_1+\frac{1}{g_s^2}\frac{(N_c^2-1)}{(16\pi^2M^2)^2}\{72b_0\hat{a}_2-24b_1\hat{a}_1+6v^2\hat{a}_1\hat{a}_2-16x\hat{a}_1\hat{a}_3-24b_0v^2ln(1+\frac{\Lambda^2}{v^2})+$$
$$+\frac{24N_c\hat{a}_1}{(16\pi^2M^2)}[-\hat{a}_1(4b_0+v^2\hat{a}_1)+v^2(8b_0+v^2\hat{a}_1)ln(1+\frac{\Lambda^2}{v^2})]+\frac{24N_c^2}{(16\pi^2M^2)^2}v^2\hat{a}_1^3\hat{a}_3\}\ ,$$

$$a_2 = \frac{N_c}{4\pi^2}ln(1+\frac{\Lambda^2}{v^2})-\frac{1}{g_s^2}\frac{(N_c^2-1)}{(16\pi^2M^2)^2}\{+12\hat{a}_1\hat{a}_3+24\hat{a}_2^2-[144b_0-48b_1+12v^2\hat{a}_1+32x(\hat{a}_1+\hat{a}_2)]\times$$
$$\times ln(1+\frac{\Lambda^2}{v^2})-\frac{16N_c\hat{a}_1}{(16\pi^2M^2)}[9\hat{a}_2^2-(3v^2(\hat{a}_1+3\hat{a}_2)+60b_0)ln(1+\frac{\Lambda^2}{v^2})]-$$
$$-\frac{24N_c^2\hat{a}_1^2}{(16\pi^2M^2)^2}[4v^2\hat{a}_3-\hat{a}_2^2+v^2\hat{a}_1ln(1+\frac{\Lambda^2}{v^2})]\}\ ,$$

$$a_3 = \frac{N_c}{\pi^2}ln(1+\frac{\Lambda^2}{v^2})-\frac{1}{g_s^2}\frac{(N_c^2-1)}{(16\pi^2M^2)^2}\{48\hat{a}_1\hat{a}_3+24\hat{a}_2^2-[576b_0+48v^2\hat{a}_1+32x(\hat{a}_2+4\hat{a}_1)]ln(1+\frac{\Lambda^2}{v^2})-$$
$$-\frac{16N_c\hat{a}_1}{(16\pi^2M^2)}[9\hat{a}_2^2-(12v^2(\hat{a}_1+\hat{a}_2)+96b_0]ln(1+\frac{\Lambda^2}{v^2})-\frac{6N_c^2\hat{a}_1^2}{(16\pi^2M^2)^2}[16v^2\hat{a}_3-\hat{a}_2^2+16v^2\hat{a}_1ln(1+\frac{\Lambda^2}{v^2})]\}\ ,$$

$$a_4 = -\frac{6N_c}{\pi^2}ln(1+\frac{\Lambda^2}{v^2})-\frac{1}{g_s^2}\frac{(N_c^2-1)}{(16\pi^2M^2)^2}\{288\hat{a}_1\hat{a}_3-96[36b_0+4b_1+3v^2\hat{a}_1-8x\hat{a}_1]ln(1+\frac{\Lambda^2}{v^2})-$$
$$-\frac{384N_c\hat{a}_1}{(16\pi^2M^2)}[12b_0+v^2(3\hat{a}_1+\hat{a}_2)]ln(1+\frac{\Lambda^2}{v^2})+\frac{576N_c^2}{(16\pi^2M^2)^2}v^2\hat{a}_1^3ln(1+\frac{\Lambda^2}{v^2})\}\ ,$$

$$a_5 = \frac{N_c}{8\pi^2v^2}+\frac{1}{g_s^2}\frac{(N_c^2-1)}{(16\pi^2M^2)^2}\{\frac{1}{v^2}(60b_0-36b_1)-\frac{2x}{v^2}(3\hat{a}_2+\hat{a}_1)-[18\hat{a}_1+54\hat{a}_2+18\hat{a}_3+12xln(1+\frac{\Lambda^2}{v^2})]\times$$
$$\times ln(1+\frac{\Lambda^2}{v^2})+\frac{8N_c}{(16\pi^2M^2)}[-3\hat{a}_1^2-\frac{6b_0}{v^2}(\hat{a}_1+8\hat{a}_2)+((54\hat{a}_1+9\hat{a}_2)\hat{a}_2-$$

$$-(36v^2\hat{a}_1+192b_0)ln(1+\frac{\Lambda^2}{v^2}))ln(1+\frac{\Lambda^2}{v^2})]+\frac{4N_c^2\hat{a}_1}{(16\pi^2M^2)^2}[14\hat{a}_1^2-18(\hat{a}_1\hat{a}_2+\hat{a}_2\hat{a}_3)ln(1+\frac{\Lambda^2}{v^2})]\}\ ,$$

$$a_6 = \frac{N_c}{2\pi^2v^2}+\frac{1}{g_s^2}\frac{(N_c^2-1)}{(16\pi^2M^2)^2}\{\frac{1}{v^2}(60b_0-12b_1)-\frac{32x}{v^2}(\hat{a}_1+3\hat{a}_2)-[18\hat{a}_1+36\hat{a}_2+6\hat{a}_3+192xln(1+\frac{\Lambda}{v^2}$$
$$\times ln(1+\frac{\Lambda^2}{v^2})+\frac{16N_c}{(16\pi^2M^2)}[-6\hat{a}_1^2-\frac{12}{v^2}b_0(\hat{a}_1+2\hat{a}_2)+(\hat{a}_2(54\hat{a}_1+\frac{9}{2}\hat{a}_2)-$$
$$-(48v^2\hat{a}_1+240b_0)ln(1+\frac{\Lambda^2}{v^2}))ln(1+\frac{\Lambda^2}{v^2})]+\frac{N_c^2\hat{a}_1}{\pi m^2}[80\hat{a}_1^2-(36\hat{a}_2\hat{a}_3+54\hat{a}_2^2)ln(1+\frac{\Lambda^2}{v^2})]\}\ ,$$

$$a_7 = -\frac{3N_c}{\pi^2v^2}+\frac{1}{g_s^2}\frac{(N_c^2-1)}{(16\pi^2M^2)^2}\{\frac{48}{v^2}[30b_0+2b_1+x(4\hat{a}_1+\hat{a}_2)]+48[9\hat{a}_1+6\hat{a}_2-\hat{a}_3+2xln(1+\frac{\Lambda^2}{v^2})]\times$$
$$\times ln(1+\frac{\Lambda^2}{v^2})+\frac{192N_c}{(16\pi^2M^2)}[3\hat{a}_1^2+\frac{6b_0}{v^2}(\hat{a}_1+2\hat{a}_2)-(9\hat{a}_1\hat{a}_2-(8v^2\hat{a}_1+24b_0)ln(1+\frac{\Lambda^2}{v^2}))ln(1+\frac{\Lambda^2}{v^2})]+$$
$$+\frac{36N_c^2}{(16\pi^2M^2)^2}\hat{a}_1\hat{a}_2(2\hat{a}_1+\hat{a}_2)ln(1+\frac{\Lambda^2}{v^2})\}\ ,$$

$$a_8 = \frac{1}{g_s^2}\frac{(N_c^2-1)}{(16\pi^2M^2)^2}\{-36\hat{a}_2^2-(96b_1+48x\hat{a}_2)ln(1+\frac{\Lambda^2}{v^2})+\frac{N_c}{(16\pi^2M^2)}[-15\hat{a}_1\hat{a}_2^2-24\hat{a}_1^2\hat{a}_2+12\hat{a}_1\hat{a}_2\hat{a}$$
$$+(-72b_1\hat{a}_2+204b_0\hat{a}_1+18v^2\hat{a}_1(\hat{a}_1-\hat{a}_2))ln(1+\frac{\Lambda^2}{v^2})]+$$
$$+\frac{N_c^2\hat{a}_1}{(16\pi^2M^2)^2}[-9\hat{a}_2^2\hat{a}_3-36v^2\hat{a}_1(\hat{a}_1-2\hat{a}_2)ln(1+\frac{\Lambda^2}{v^2})]\}\ ,$$

$$a_9 = \frac{N_c}{24\pi^2v^4}+\frac{1}{g_s^2}\frac{(N_c^2-1)}{(16\pi^2M^2)^2}\{\frac{8}{v^2}(\hat{a}_1-\frac{14}{3}\hat{a}_2-3\hat{a}_3)+\frac{16}{v^4}b_0-\frac{32}{3v^2}x\frac{\Lambda^2}{v^2}-[\frac{128}{v^2}x+192ln(1+\frac{\Lambda^2}{v^2})]\times$$
$$\times ln(1+\frac{\Lambda^2}{v^2})+\frac{8N_c}{(16\pi^2M^2)}[\frac{1}{v^2}(21\hat{a}_1\hat{a}_2-\hat{a}_1^2+6\hat{a}_2^2-192b_0)-\frac{16}{v^4}b_0\hat{a}_2-12(3\hat{a}_1-(3\hat{a}_1+2\hat{a}_2)ln(1+\frac{\Lambda^2}{v^2}$$
$$\times ln(1+\frac{\Lambda^2}{v^2})]-\frac{N_c^2}{(16\pi^2M^2)^2}[\frac{192}{v^2}\hat{a}_1\hat{a}_2^2+48(\hat{a}_1^2+8\hat{a}_1\hat{a}_3-16\hat{a}_2^2)ln^2(1+\frac{\Lambda^2}{v^2})]\}\ ,$$

$$a_{10} = \frac{N_c}{6\pi^2v^4}+\frac{1}{g_s^2}\frac{(N_c^2-1)}{(16\pi^2M^2)^2}\{\frac{16}{v^2}(2\hat{a}_1-\frac{19}{3}\hat{a}_2-\frac{9}{2}\hat{a}_3)+\frac{64}{v^4}b_0-\frac{20}{3v^2}x\frac{\Lambda^2}{v^2}-(\frac{80}{v^2}x+528ln(1+\frac{\Lambda^2}{v^2}))\times$$
$$\times ln(1+\frac{\Lambda^2}{v^2})-\frac{8N_c}{(16\pi^2M^2)}[\frac{1}{v^2}(4\hat{a}_1^2-56\hat{a}_1\hat{a}_2-\frac{21}{2}\hat{a}_2^2+528b_0)+\frac{16}{v^4}b_0\hat{a}_2+$$
$$+(104\hat{a}_1-(90\hat{a}_1+42\hat{a}_2)ln(1+\frac{\Lambda^2}{v^2}))ln(1+\frac{\Lambda^2}{v^2})]+\frac{3N_c^2}{(16\pi^2M^2)^2}[\frac{1}{v^2}(4\hat{a}_1^2\hat{a}_2-52\hat{a}_2^2\hat{a}_1)-$$
$$-(34\hat{a}_1^2+176\hat{a}_1\hat{a}_2-256\hat{a}_2^2-224v^2\hat{a}_1ln(1+\frac{\Lambda^2}{v^2}))ln^2(1+\frac{\Lambda^2}{v^2})]\}\ ,$$

$$a_{11} = \frac{N_c}{6\pi^2 v^4} + \frac{1}{g_s^2}\frac{(N_c^2-1)}{(16\pi^2 M^2)^2}\{\frac{16}{v^2}(2\hat{a}_1 - \frac{19}{3}\hat{a}_2 - 3\hat{a}_3) + \frac{64}{v^4}b_0 - \frac{20}{3v^2}x\frac{\Lambda^2}{v^2} - 32[\frac{x}{v^2} + 12ln(1+\frac{\Lambda^2}{v^2})]\times$$

$$\times ln(1+\frac{\Lambda^2}{v^2}) - \frac{32N_c}{(16\pi^2 M^2)}[\frac{1}{v^2}(\hat{a}_1^2 - 14\hat{a}_1\hat{a}_2 - \frac{3}{2}\hat{a}_2^2 + 96b_0) + \frac{4}{v^4}b_0\hat{a}_2 +$$

$$+(20\hat{a}_1 - (9\hat{a}_1 + 6\hat{a}_2)ln(1+\frac{\Lambda^2}{v^2}))ln(1+\frac{\Lambda^2}{v^2})] + \frac{4N_c^2}{(16\pi^2 M^2)^2}[\frac{2}{v^2}(\hat{a}_1^2\hat{a}_2 - 24\hat{a}_2^2\hat{a}_1) -$$

$$-(3\hat{a}_1^2 + 24\hat{a}_1\hat{a}_2 + 8\hat{a}_2^2 - 96v^2\hat{a}_1 ln(1+\frac{\Lambda^2}{v^2}))ln^2(1+\frac{\Lambda^2}{v^2})]\} ,$$

$$a_{12} = \frac{2N_c}{3\pi^2 v^4} + \frac{1}{g_s^2}\frac{(N_c^2-1)}{(16\pi^2 M^2)^2}\{\frac{16}{v^2}(2\hat{a}_1 - \frac{10}{3}\hat{a}_2 - \frac{9}{2}\hat{a}_3) + \frac{64}{v^4}b_0 - \frac{144}{3v^2}x\frac{\Lambda^2}{v^2} - [\frac{1440}{v^2}x + 432ln(1+\frac{\Lambda^2}{v^2})]\times$$

$$\times ln(1+\frac{\Lambda^2}{v^2}) - \frac{16N_c}{(16\pi^2 M^2)}[\frac{1}{v^2}(8\hat{a}_1^2 - 40\hat{a}_1\hat{a}_2 - \frac{15}{2}\hat{a}_2^2 + 672b_0) + \frac{32}{v^4}b_0\hat{a}_2 +$$

$$+(144\hat{a}_1 - (180\hat{a}_1 + 30\hat{a}_2)ln(1+\frac{\Lambda^2}{v^2}))ln(1+\frac{\Lambda^2}{v^2})] + \frac{96N_c^2}{(16\pi^2 M^2)^2}[\frac{1}{v^2}(\hat{a}_1^2\hat{a}_2 - 2\hat{a}_2^2\hat{a}_1) -$$

$$-(5\hat{a}_1^2 + 7\hat{a}_1\hat{a}_2 - 8\hat{a}_2^2 - 13v^2\hat{a}_1 ln(1+\frac{\Lambda^2}{v^2}))ln(1+\frac{\Lambda^2}{v^2})]\} ,$$

$$a_{13} = -\frac{N_c}{3\pi^2 v^4} - \frac{1}{g_s^2}\frac{(N_c^2-1)}{(16\pi^2 M^2)^2}\{\frac{8}{v^2}(2\hat{a}_1 - \frac{10}{3}\hat{a}_2 - \frac{3}{2}\hat{a}_3) + \frac{32}{v^4}b_0 - \frac{64}{3v^2}x\frac{\Lambda^2}{v^2} + 8(\frac{64}{v^2}x - 9ln(1+\frac{\Lambda^2}{v^2}))\times$$

$$\times ln(1+\frac{\Lambda^2}{v^2}) - \frac{32N_c}{(16\pi^2 M^2)}[\frac{1}{v^2}(2\hat{a}_1^2 - 10\hat{a}_1\hat{a}_2 + \frac{3}{2}\hat{a}_2^2 - 48b_0) + \frac{8}{v^4}b_0\hat{a}_2 + 6(6\hat{a}_1 + \hat{a}_2)ln^2(1+\frac{\Lambda^2}{v^2})] +$$

$$+\frac{48N_c^2}{(16\pi^2 M^2)^2}[\frac{1}{v^2}(\hat{a}_1^2\hat{a}_2 - 2\hat{a}_2^2\hat{a}_1) + 4(\hat{a}_1^2 + 2\hat{a}_1\hat{a}_3 - 4\hat{a}_2^2)ln^2(1+\frac{\Lambda^2}{v^2})]\} ,$$

$$a_{14} = \frac{N_c}{4\pi^2}ln(1+\frac{\Lambda^2}{v^2}) + \frac{1}{g_s^2}\frac{(N_c^2-1)}{(16\pi^2 M^2)^2}\{6\hat{a}_2\hat{a}_4 - \frac{1}{2}\hat{a}_2\hat{a}_3 - 24\hat{a}_1\hat{a}_5 - 6\hat{a}_2^2 - 8\hat{a}_3^2 + 2[84b_0 + 3b_1 - 4b_2 + 12v^2\hat{a}_1$$

$$-4x(6\hat{a}_1 - 3\hat{a}_2 + 2\hat{a}_3)]ln(1+\frac{\Lambda^2}{v^2}) + \frac{N_c}{(16\pi^2 M^2)}[\frac{29}{4}\hat{a}_2^3 + \frac{3}{2}\hat{a}_1\hat{a}_2^2 - 12\hat{a}_2\hat{a}_1^2 + 14\hat{a}_1\hat{a}_3^2 + 17\hat{a}_2^2\hat{a}_3 + 24\hat{a}_1^2\hat{a}_4 -$$

$$-34\hat{a}_1\hat{a}_2\hat{a}_3 - 12\hat{a}_1\hat{a}_2\hat{a}_4 + (426b_0\hat{a}_1 + \frac{491}{4}b_0\hat{a}_2 + 63b_1\hat{a}_1 + 2b_1\hat{a}_2 - 74v^2\hat{a}_1^2 +$$

$$+15v^2\hat{a}_1\hat{a}_2 - 6v^2\hat{a}_1\hat{a}_3)ln(1+\frac{\Lambda^2}{v^2})] + \frac{N_c^2}{(16\pi^2 M^2)^2}[6\hat{a}_1^4 - 15\hat{a}_1^3\hat{a}_2 - 3\hat{a}_1^2\hat{a}_2^2 - \frac{11}{4}\hat{a}_1\hat{a}_2^3 + \frac{5}{8}\hat{a}_2^4 -$$

$$-30\hat{a}_1^2\hat{a}_3^2 + \frac{39}{2}\hat{a}_1^2\hat{a}_2\hat{a}_3 + \frac{45}{4}\hat{a}_1\hat{a}_2^2\hat{a}_4 + 6\hat{a}_1^2\hat{a}_2\hat{a}_5 + 6(12\hat{a}_1^3 - 9\hat{a}_1^2\hat{a}_2 + 20\hat{a}_1^2\hat{a}_3)v^2 ln(1+\frac{\Lambda^2}{v^2})]\} ,$$

$$a_{15} = \frac{1}{g_s^2}\frac{(N_c^2-1)}{(16\pi^2 M^2)^2}\{\frac{24}{v^2}(2b_1 + x\hat{a}_2) - 24xln^2(1+\frac{\Lambda^2}{v^2}) - \frac{N_c}{(16\pi^2 M^2)}[\frac{4}{v^2}(25b_0\hat{a}_1 - 114b_0\hat{a}_2 + 9b_1\hat{a}_2) -$$

$$-(72\hat{a}_1^2 - 15\hat{a}_1\hat{a}_2 + 15\hat{a}_2^2 + 6\hat{a}_1\hat{a}_3 + \frac{11}{2}\hat{a}_2\hat{a}_3 + 6(v^2\hat{a}_1 + 20b_0 + 16b_1)ln(1+\frac{\Lambda^2}{v^2}))ln(1+\frac{\Lambda^2}{v^2})] +$$

$$+\frac{N_c^2}{(16\pi^2 M^2)^2}[18\hat{a}_1^3 - (18\hat{a}_1^2\hat{a}_2 - \frac{63}{2}\hat{a}_1\hat{a}_2^2 - 27v^2\hat{a}_1\hat{a}_2 ln(1+\frac{\Lambda^2}{v^2}))ln(1+\frac{\Lambda^2}{v^2})]\} ,$$

$$a_{16} = -\frac{1}{g_s^2}\frac{96(N_c^2-1)N_c}{(16\pi^2 M^2)^4}\hat{a}_1^2\hat{a}_3^2 ,$$

$$a_{17} = -\frac{1}{g_s^2}\frac{768(N_c^2-1)N_c}{(16\pi^2 M^2)^4}\hat{a}_1\hat{a}_3 ln(1+\frac{\Lambda^2}{v^2}) ,$$

$$a_{18} = 0 ,$$

$$a_{19} = -\frac{1}{g_s^2}\frac{192(N_c^2-1)N_c}{(16\pi^2 M^2)^4}\hat{a}_1\hat{a}_3(\hat{a}_1 + \hat{a}_2)ln(1+\frac{\Lambda^2}{v^2}) ,$$

$$a_{20} = -\frac{1}{g_s^2}\frac{192(N_c^2-1)N_c}{(16\pi^2 M^2)^4}\hat{a}_1\hat{a}_3(4\hat{a}_1 + \hat{a}_2)ln(1+\frac{\Lambda^2}{v^2}) ,$$

$$a_{21} = -\frac{1}{g_s^2}\frac{384(N_c^2-1)N_c}{(16\pi^2 M^2)^4}(\hat{a}_1 + \hat{a}_2)^2 ln^2(1+\frac{\Lambda^2}{v^2}) ,$$

$$a_{22} = -\frac{1}{g_s^2}\frac{384(N_c^2-1)N_c}{(16\pi^2 M^2)^4}(\hat{a}_1 + \hat{a}_2)(4\hat{a}_1 + \hat{a}_2)ln^2(1+\frac{\Lambda^2}{v^2}) ,$$

$$a_{23} = -\frac{1}{g_s^2}\frac{48(N_c^2-1)N_c}{(16\pi^2 M^2)^4}\hat{a}_1\hat{a}_3[\frac{1}{v^2}(5\hat{a}_1 + 4\hat{a}_3 + \Lambda^2) + 12ln^2(1+\frac{\Lambda^2}{v^2})] ,$$

$$a_{24} = -\frac{1}{g_s^2}\frac{384(N_c^2-1)N_c}{(16\pi^2 M^2)^4}(16\hat{a}_1^2 + 8\hat{a}_1\hat{a}_2 + \hat{a}_2^2)ln^2(1+\frac{\Lambda^2}{v^2}) ,$$

$$a_{25} = -\frac{1}{g_s^2}\frac{192(N_c^2-1)N_c}{(16\pi^2 M^2)^4}(8\hat{a}_1^2 + 4\hat{a}_1\hat{a}_2 - \hat{a}_2^2)ln^2(1+\frac{\Lambda^2}{v^2}) ,$$

$$a_{26} = a_{27} = a_{28} = a_{29} = 0 , \tag{61}$$

in which

$$\begin{aligned}
\hat{a}_n &\equiv \Lambda^2 - nv^2 ln(1+\frac{\Lambda^2}{v^2}) , \qquad n = 1, 2, \cdots \\
b_0 &\equiv -\frac{1}{8}\Lambda^4 + \frac{v^2}{4}\Lambda^2 - \frac{v^4}{4}ln(1+\frac{\Lambda^2}{v^2}) , \\
b_1 &\equiv \frac{1}{8}\Lambda^4 - \frac{v^2}{2}\Lambda^2 + \frac{3}{4}v^4 ln(1+\frac{\Lambda^2}{v^2}) , \\
b_2 &\equiv \frac{1}{8}\Lambda^4 - \frac{3v^2}{4}\Lambda^2 + \frac{3v^4}{2}ln(1+\frac{\Lambda^2}{v^2}) , \\
x &\equiv \frac{3N_c}{(16\pi^2 M^2)}(\frac{1}{4}\Lambda^4 + \frac{v^2\hat{a}_1^2 N_f}{(16\pi^2 M^2)}) .
\end{aligned} \tag{62}$$

In the calculation of the Λ-dependent terms in the coefficients, we have kept only the parts which are divergent as $\Lambda \to \infty$. In this approximation, the local $U(N_f)_L \times U(N_f)_R$ symmetry (28) is not violated, (for the logarithm function,we have kept the entire $ln(1+\frac{\Lambda^2}{v^2})$ without expanding it in powers of $\frac{v^2}{\Lambda^2}$ because the property of the vacuum is sensitive to this function (cf (69)) and we want to determine the physical vacuum as accurate as possible.). For convergent integrations we simply take $\Lambda \to \infty$ to preserve the local $U(N_f)_L \times U(N_f)_R$ symmetry (28) (This is equivalent to taking a finit Λ together with the necessary terms in $\mathcal{L}_H^{eff}$ to preserve the chiral symmetry [21]).

4.The Physical Vacuum

It is well-known that the physical vacuum is the state minimizing the effective potential, and the minimization concerns only the scalar field. Neglecting the meson-loop contributions (U-field loop corrections taken into account in Sec.V), the extremum condition for the effective potential is simply

$$\frac{\partial S_{eff}(U,\Sigma,V,A,J,\theta)}{\partial v}\,|_{vac.} \;=\; 0 \;, \tag{63}$$

where $S_{eff}(U,\Sigma,V,A,J,\theta)$ is given in (45). Since $U=1$ and Σ,V,A all vanish in the vacuum,eq.(63) simply reads (cf. (46),(47),(48), and (58))

$$\frac{\partial}{\partial v}\;[\,lna_0 + \int d^4x\,(2M^2 J_S v - M^2 v^2)\,]\,|_{vac.} \;=0\;. \tag{64}$$

From (55),(56), and (59) we see that

$$\frac{\partial}{\partial v}\,lna_0\;|_{vac.} \;=\int d^4x\,a_1 v\;|_{vac.}\;. \tag{65}$$

Therefore (64) reduces to

$$[\,2M^2\,J_S + v(a_1 - 2M^2)\,]\,|_{vac.} \;=\; 0\;. \tag{66}$$

What we are interested in is the dynamical breaking of the chiral symmetry. So that, we will first turn off the current quark mass m and study chiral symmety

breaking from (66), and then turn on m and treat it as a perturbation (chiral perturbation). In this prescription, we should take $J_S\,|_{vac.}=0$ instead of $J_S\,|_{vac.}=m$ in (66) so that, in the chiral limit, (66) is

$$v(a_1 - 2M^2) \;=\; 0\;. \tag{67}$$

The coefficient a_1 is calculable order by order in $1/g_s^2$ expansion (cf.(59)). If $a_1<0$, there is only a trivial solution $v=0$ of (67), i.e. chiral symmetry is unbroken. If $a_1>0$,there is also a nontrivial solution $v\neq 0$ provided

$$a_1 \;=\; 2M^2\;. \tag{68}$$

It is easy to see that the effective potential $V_{eff} = -S_{eff}(1,v,0,0,0,0)$ increases with v when v is very large. If $a_1\,|_{v=0} > 2M^2$, the solution $v=0$ is a maximum of V_{eff}, so that the physical vacuum is given by the nontrivial solution with v determined by (68). Eq.(68) is just the gap equation in the present approach and it looks very simple.

As an example, let us look at the specific form of (68) to leading order in $1/g_s^2$ expansion. a_1 is given in (61) and (62) which is

$$a_1 = \frac{N_c}{4\pi^2}\hat{a}_1 = \frac{N_c}{4\pi^2}[\Lambda^2 - v^2 ln(1+\frac{\Lambda^2}{v^2})]\;,$$

therefore (68) reads

$$\frac{v^2}{\Lambda^2}ln(1+\frac{\Lambda^2}{v^2}) = 1 - \frac{N_f}{2N_c}\;. \tag{69}$$

For $N_c = N_f = 3$, $a_1\,|_{v=0}$ is greater than $2M^2$ and (69) does have a $v\neq 0$ solution which gives the chiral symmetry breaking in this theory. Note that (69) cannot be simply applied to the case of $N_f>3$ in QCD since when heavy quarks are included,the current quark mass matrix m cannot be treated as a perturbation and thus the chiral limit form of vacuum stability condition (68) no longer makes sense.

Taking into account of the order-$1/g_s^2$ contributions with a_1 given by (61) and (62), we can see from (68) that chiral symmetry breaking takes place only if $\alpha_s \equiv \frac{g_s^2}{4\pi}$ is greater than a critical value

$$\alpha_c = \frac{24\pi(N_c^2-1)}{N_f^2(2N_c-N_f)} \ . \tag{70}$$

For $N_c = N_f = 3$, this critical value is α_c=22.3 . Therefore in the broken chiral symmetry phase α_s is large, so that the strong coupling expansion really makes sense.

IV. Low Energy Effective Lagrangian for Pseudoscalar Mesons

The low energy effective Lagrangian for U can be obtained by carrying out the integration with respect to Σ,V,and A in (43). The U-field contains the pion, the kaon, the η and the η' mesons. Our present study does not include the physics related to the $\eta-\eta'$ mixing, so that we cannot describe the $\eta-\eta'$ system properly. In the following, we derive the effective Lagrangian for the low lying flavor octet pseudoscalar mesons by simply turning off the flavor singlet field, and we regard the η-meson as a member of the flavor octet. The integration over Σ,V,and A cannot be done exactly. We consider the following two commonly used approximations.

1.The VEV Approximation (VEVA)

This is the approximation taken in Ref.[9]-[10]. In this approximation,instead of really doing the integration, the auxiliary fields are taken to be their VEV's, i.e. $\Sigma = \frac{1}{2}[\Omega(J_S - iJ_P)\Omega + \Omega^\dagger(J_S+iJ_P)\Omega^\dagger]$, $V^\mu = J_V^\mu$, $A^\mu = J_A^\mu$. Then the normal part of the effective Lagrangian for U is

$$\begin{aligned}\mathcal{L}_{eff}(U,J) &= lna_0 + tr_f\{M^2v[((J_S-iJ_P)U+(J_S+iJ_P)U^\dagger]+\\ &+\frac{1}{4}(2M^2-v^2a_2)[(J_S+iJ_P)U^\dagger(J_S+iJ_P)U^\dagger+(J_S-iJ_P)U(J_S-iJ_P)U]-v^2\}-\\ &-\frac{1}{2}v^2a_2(J_S-iJ_P)(J_S+iJ_P)+tr_f\{\frac{v^2}{16}a_3(\nabla_\mu U^\dagger\nabla^\mu U)+\frac{v^2}{48}a_6(\nabla_\mu\nabla_\nu U^\dagger)(\nabla^\mu\nabla^\nu U)-\\ &-\frac{v^4}{288}(4a_{12}-a_{13})(\nabla_\mu U^\dagger\nabla^\mu U\nabla_\nu U^\dagger\nabla^\nu U)+\frac{v^4}{576}(2a_{12}-5a_{13})(\nabla_\mu U^\dagger\nabla_\nu U\nabla^\mu U^\dagger\nabla^\nu U)+\\ &+\frac{v}{16}(a_3-2v^2a_6)(\nabla_\mu U^\dagger\nabla^\mu U)[U^\dagger(J_S+iJ_P)+(J_S-iJ_P)U]+\\ &+\frac{1}{192}(40a_2+a_4-8a_8-32a_{14})(F_{V\mu\nu}F_V^{\mu\nu}+F_{A\mu\nu}F_A^{\mu\nu})+\\ &+i\frac{v^2}{96}(a_7-8a_{15})(F_R^{\mu\nu}\nabla_\mu U\nabla_\nu U^\dagger+F_L^{\mu\nu}\nabla_\mu U^\dagger\nabla_\nu U)\ \}+\\ &+\frac{v^4}{1152}(5a_{24}-2a_{25})tr_f(\nabla_\mu U^\dagger\nabla^\mu U)tr_f(\nabla_\nu U^\dagger\nabla^\nu U)+\\ &+\frac{v^4}{576}(4a_{25}-a_{24})tr_f(\nabla_\mu U^\dagger\nabla_\nu U)tr_f(\nabla^\mu U^\dagger\nabla^\nu U)+\\ &+\frac{v^2}{8}a_{16}tr_f[(J_S+iJ_P)U^\dagger+(J_S-iJ_P)U]tr_f[(J_S+iJ_P)U^\dagger+(J_S-iJ_P)U]+\\ &+\frac{v^3}{32}a_{20}tr_f(\nabla_\mu U^\dagger\nabla^\mu U)tr_f[(J_S+iJ_P)U^\dagger+(J_S-iJ_P)U]-\\ &-\frac{a_{18}}{384}[(tr_fF_{V\mu\nu})(tr_fF_V^{\mu\nu})+(tr_fF_{A\mu\nu})(tr_fF_A^{\mu\nu})]\ ,\end{aligned} \tag{71}$$

where

$$\begin{aligned}\nabla_\mu &\equiv D_\mu\ |_{V^\mu=J_V^\mu\,,\,A^\mu=J_A^\mu}\ ,\\ F_V^{\mu\nu} &\equiv \partial^\mu J_V^\nu-\partial^\nu J_V^\mu+i[J_V^\mu,J_V^\nu]+i[J_A^\mu,J_A^\nu]\ ,\\ F_A^{\mu\nu} &\equiv \partial^\mu J_A^\nu-\partial^\nu J_A^\mu+i[J_V^\mu,J_A^\nu]+i[J_A^\mu,J_V^\nu]\ ,\\ F_{\substack{R\\L}}^{\mu\nu} &\equiv F_V^{\mu\nu}\pm F_A^{\mu\nu}\ .\end{aligned} \tag{72}$$

As has been mentioned in Ref.[1] that in the low energy expansion, we should regard U as $O(p^0)$, J_V^μ and J_A^μ as $O(p^1)$,J_S and J_P as $O(p^2)$. Eq.(71) is just a low energy expansion up to $O(p^4)$. The $O(p^4)$ term with $tr_f[(\nabla_\mu\nabla_\nu U^\dagger)(\nabla^\mu\nabla^\nu U)]$ can be further simplified.Differentiating by parts and dropping the total derivatives, we get

$$\begin{aligned}tr_f[(\nabla_\mu\nabla_\nu U^\dagger)(\nabla^\mu\nabla^\nu U)] &= tr_f[(\nabla_\mu\nabla^\mu U^\dagger)(\nabla_\nu\nabla^\nu U)-F_{L\mu\nu}U^\dagger F_R^{\mu\nu}U+\\ &+(F_{V\mu\nu}F_V^{\mu\nu}+F_{A\mu\nu}F_A^{\mu\nu})+iF_{R\mu\nu}\nabla^\mu U\nabla^\nu U^\dagger+iF_{L\mu\nu}\nabla^\mu U^\dagger\nabla^\nu U]\ .\end{aligned} \tag{73}$$

This $O(p^4)$ term can be further simplified by using the $O(p^2)$ Euler-Lagrange equation of U which is

$$\nabla_\mu\nabla^\mu U = -U(\nabla_\mu U^\dagger\nabla^\mu U) - \frac{8M^2}{va_3}[U(J_S-iJ_P)U-(J_S+iJ_P)]$$

$$\nabla_\mu\nabla^\mu U^\dagger = -(\nabla_\mu U^\dagger\nabla^\mu U)U^\dagger - \frac{8M^2}{va_3}[U^\dagger(J_S+iJ_P)U^\dagger - (J_S-iJ_P)] \ . \qquad (74)$$

From (73) and (74),we have

$$\begin{aligned} tr_f[(\nabla_\mu\nabla_\nu U^\dagger)(\nabla^\mu\nabla^\nu U)] &= tr_f[\nabla_\mu U^\dagger\nabla^\mu U\nabla_\nu U^\dagger\nabla^\nu U + \\ &+(\frac{8M^2}{va_3})^2(2J_S^2+2J_P^2-(J_S-iJ_P)U(J_S-iJ_P)U-(J_S+iJ_P)U^\dagger(J_S+iJ_P)U^\dagger)+ \\ &+(F_{V\mu\nu}F_V^{\mu\nu}+F_{A\mu\nu}F_A^{\mu\nu})-F_{L\mu\nu}U^\dagger F_R^{\mu\nu}U+iF_{R\mu\nu}\nabla^\mu U\nabla^\nu U^\dagger+iF_{L\mu\nu}\nabla^\mu U^\dagger\nabla^\nu U] \ . \qquad (75) \end{aligned}$$

With (75) and the $SU(3)$ identity [1]

$$\begin{aligned} tr_f(\nabla_\mu U^\dagger\nabla_\nu U\nabla^\mu U^\dagger\nabla^\nu U) &= -2tr_f(\nabla_\mu U^\dagger\nabla^\mu U\nabla_\nu U^\dagger\nabla^\nu U)+\frac{1}{2}[tr_f(\nabla_\mu U^\dagger\nabla^\mu U)]^2+ \\ &+tr_f(\nabla_\mu U^\dagger\nabla_\nu U)tr_f(\nabla^\mu U^\dagger\nabla^\nu U) \ , \qquad (76) \end{aligned}$$

eq.(71) can be written as

$$\begin{aligned} \mathcal{L}_{eff}(U,J) &= \frac{1}{4}F_0^2\{tr_f(\nabla_\mu U^\dagger\nabla^\mu U)+2B_0tr_f[((J_S-iJ_P)U+(J_S+iJ_P)U^\dagger]\}+ \\ &+L_1[tr_f(\nabla_\mu U^\dagger\nabla^\mu U)]^2+L_2tr_f(\nabla_\mu U^\dagger\nabla_\nu U)tr_f(\nabla^\mu U^\dagger\nabla^\nu U)+ \\ &+L_3tr_f(\nabla_\mu U^\dagger\nabla^\mu U\nabla_\nu U^\dagger\nabla^\nu U)+ \\ &+2B_0L_4tr_f(\nabla_\mu U^\dagger\nabla^\mu U)tr_f[(J_S+iJ_P)U^\dagger+(J_S-iJ_P)U]+ \\ &+2B_0L_5tr_f[\nabla_\mu U^\dagger\nabla^\mu U((J_S+iJ_P)U^\dagger+(J_S-iJ_P)U)]+ \\ &+4B_0^2L_6[tr_f((J_S-iJ_P)U+(J_S+iJ_P)U^\dagger)]^2+ \\ &+4B_0^2L_7[tr_f((J_S-iJ_P)U-(J_S+iJ_P)U^\dagger)]^2+ \\ &+4B_0^2L_8tr_f[(J_S-iJ_P)U(J_S-iJ_P)U+(J_S+iJ_P)U^\dagger(J_S+iJ_P)U^\dagger]- \\ &-iL_9tr_f(F_{R\mu\nu}\nabla^\mu U\nabla^\nu U^\dagger+F_{L\mu\nu}\nabla^\mu U^\dagger\nabla^\nu U) \\ &+L_{10}tr_f(U^\dagger F_{R\mu\nu}UF_L^{\mu\nu})+H_1tr_f(F_{R\mu\nu}F_R^{\mu\nu}+F_{L\mu\nu}F_L^{\mu\nu}) \\ &+4B_0^2H_2tr_f[(J_S-iJ_P)(J_S+iJ_P)]- \\ &-\frac{a_{18}}{384}[(tr_fF_{V\mu\nu})(tr_fF_V^{\mu\nu})+(tr_fF_{A\mu\nu})(tr_fF_A^{\mu\nu})] \ , \qquad (77) \end{aligned}$$

in which

$$F_0^2=\frac{v^2}{4}a_3, \qquad B_0=\frac{8M^2}{va_3}, \qquad L_1=\frac{v^4}{1152}(2a_{12}-5a_{13}+5a_{24}-a_{25}),$$

$$\begin{aligned} &L_2=\frac{v^4}{576}(2a_{12}-5a_{13}-a_{24}+4a_{25}), \qquad L_3=\frac{v^2}{48}[a_6-v^2(a_{12}-a_{13})], \\ &L_4=\frac{v^2a_3a_{20}}{512}\frac{v^2}{M^2}, \quad L_5=\frac{v^2a_3}{256M^2}(a_3-2v^2a_6), \quad L_6=\frac{a_3^2a_{16}}{2048}\frac{v^4}{M^4}, \quad L_7=0, \\ &L_8=\frac{a_3^2}{512}\frac{v^2}{M^2}(1-\frac{1}{2}\frac{v^2}{M^2}a_2)-\frac{v^2}{192}a_6, \ L_9=-\frac{v^2}{96}(2a_6+a_7-8a_{15}), \ L_{10}=-\frac{v^2}{48}a_6, \\ &H_1=\frac{1}{384}(40a_2+a_4+8v^2a_6-8a_8-32a_{14}), \qquad H_2=\frac{1}{512}(\frac{16}{3}v^2a_6-\frac{v^4}{M^4}a_2a_3^2) \ . \end{aligned} \qquad (78)$$

Eq.(77) contains the complete Gasser-Leutwyler chiral Lagangian [1] up to $O(p^4)$ with the coefficients given by (78). Fixing $N_c=N_f=3$,the VEV v can be solved from (68) for given Λ and g_s. Therefore all the Gasser-Leutwyler coefficients are expressed in terms of the two fundamental parameters Λ and g_s through the calculable a_n's (cf.(59)) whose up to $O(1/g_s^2)$ formulae are given in (61) and (62). There is an extra term (the last term) in (77) containing only the $U_V(1)$ and $U_A(1)$ external sources.From (61) we see that a_{18} is at most of $O(g_s^{-4})$. If we gauge the flavor degree of freedom and regard the sources J_V^μ and J_A^μ as the corresponding gauge fields, the effect of this extra term is that they make the $U_V(1)$ and $U_A(1)$ effective coupling constants different from the $SU(N_f)$ coupling constant.

The explicit form of (78) to leading order in the $1/g_s^2$ expansion can be easily read out from (61) and (62). With the relation (69), we have

$$\begin{aligned} &F_0^2=\frac{N_c}{4\pi^2}v^2ln(1+\frac{\Lambda^2}{v^2})=\frac{\Lambda^2}{4\pi^2}(N_c-\frac{N_f}{2}), \qquad B_0=\frac{N_fv}{2N_c-N_f}, \qquad L_1=\frac{N_c}{384\pi^2}, \\ &L_2=2L_1, \qquad L_3=-4L_1, \qquad L_4=0, \\ &L_5=\frac{N_c}{16\pi^2}\frac{N_c-\frac{N_f}{2}}{N_f}ln(1+\frac{\Lambda^2}{v^2}), \qquad L_6=0, \qquad L_7=0, \qquad (7 \\ &L_8=L_1[24\frac{N_c}{N_f}(1-\frac{N_c}{N_f})(1-\frac{N_f}{2N_c})ln(1+\frac{\Lambda^2}{v^2})-1], \qquad L_9=8L_1 \qquad L_{10}=-4L_1, \\ &H_1=L_3ln(1+\frac{\Lambda^2}{v^2}), \qquad H_2=L_2[1-24\frac{(N_c-\frac{N_f}{2})^2}{N_f^2}ln(1+\frac{\Lambda^2}{v^2} \end{aligned}$$

It is interesting to see that, in the present approximation, L_1,L_2,L_3,L_4,L_6,L_7,L_9, and L_{10}, all coinside with the corresponding large-N_c and chiral limit results in Ref.[11].

In the $N_c \to \infty$ limit our F_0^2 also coinsides with the corresponding chiral limit result in Ref.[11]. However, the physical origin of our F_0^2 is different. Our F_0 is scaled by v rather than directly by Λ (Note that F_0 vanishes if $v = 0$.). This is consistent with the conventional understanding of F_0 which leads to the Goldberger-Treiman relation. Different from the result in Ref.[11], chiral symmetry breaking (cf.(69)) makes our F_0^2 proportional to $N_c - \frac{N_f}{2}$ rather than simply to N_c. This difference is significant for $N_c = N_f = 3$ since it makes our cut-off Λ larger than that in Ref.[11] by a factor of $\sqrt{2}$, ie. Λ=453MeV, if one requires F_0^2 to fit the experiment. However, this value of Λ is still lower than m_K and m_η. For $N_c = N_f = 3$ the coefficients L_1,L_2,L_3,L_9,and L_{10} in (78) are in reasonable agreement with experiments [11], but L_8 in (78) has a wrong sign as compared with the experiment [1]. In the following,we consider a better approximation.

2.The Stationary Phase Approximation (SPA)

In this approximation, the functional integration with respect to Σ,V and A is approximately represented by the contribution of the classical orbit (the solution of the Euler-Lagrange equations for Σ, V^μ and A^μ). This is the leading term in the semiclassical expansion, and thus further quantum corrections can be calculated in the standard way. The SPA reflects more about the path-integration than the simple VEVA does.

The Euler-Lagrange equations for Σ,V^μ,and A^μ are

$$\frac{\delta S_{eff}[U,\Sigma,V,A,J,\theta]}{\delta\Sigma} = 0 \quad , \quad \frac{\delta S_{eff}[U,\Sigma,V,A,J,\theta]}{\delta V_\mu} = 0 \quad , \quad \frac{\delta S_{eff}[U,\Sigma,V,A,J,\theta]}{\delta A_\mu} = 0. \tag{80}$$

The classical solutions needed for calculating the effective Lagrangian up to $O(p^4)$ are

$$\Sigma = \frac{M^2}{2v^2a_2}[\Omega(J_S - iJ_P)\Omega + \Omega^\dagger(J_S + iJ_P)\Omega^\dagger] + \frac{c_1^2c_2^2}{16va_2}(a_3 - 2v^2a_6)\Omega(\nabla_\mu U^\dagger\nabla^\mu U)\Omega^\dagger +$$

$$+(2a_2 - N_f a_{16})^{-1}\{\frac{M^2a_{16}}{2v^2a_2}tr_f[U(J_S - iJ_P) + U^\dagger(J_S + iJ_P)] +$$

$$+ \frac{c_1^2c_2^2}{16va_2}[a_{16}(a_3 - 2v^2a_6) + v^2a_2]tr_f(\nabla_\mu U^\dagger\nabla^\mu U)\} \ , \tag{81}$$

$$V_c^\mu = J_V^\mu + i\frac{v^2}{8}a_3c_2[U(\nabla^\mu U^\dagger) - (\nabla^\mu U^\dagger)U] \ ,$$

$$A_c^\mu = J_A^\mu + i\frac{v^2}{8}a_3c_2[U(\nabla^\mu U^\dagger) + (\nabla^\mu U^\dagger)U] \ ,$$

where

$$c_1 \equiv \frac{(2N_f - N_c)\Lambda^2}{8\pi^2} \quad , \quad c_2 \equiv [\frac{(2N_f - N_c)\Lambda^2}{8\pi^2} + \frac{1}{2}v^2a_3]^{-1} \ . \tag{82}$$

Then the normal part of the effective Lagrangian for U is

$$\mathcal{L}_{eff}(U,J) = lna_0 + tr_f\{\frac{v^2}{16}a_3c_1c_2(\nabla_\mu U^\dagger\nabla^\mu U) + M^2v[(J_S - iJ_P)U + (J_S + iJ_P)U^\dagger]\} +$$

$$+tr_f\{\frac{v^2}{48}a_6c_1^2c_2^2(\nabla_\mu\nabla^\mu U^\dagger)(\nabla_\nu\nabla^\nu U) - \frac{v^4}{96}[a_3c_2^2(1 - \frac{v^2}{4}a_3c_2)\{c_1^2c_2(a_6 + a_7 - 4a_{15}) +$$

$$+\frac{1}{16}a_3(1 - \frac{v^2}{4}a_3c_2)(40a_2 + a_4 - 8a_8 - 32a_{14})\} + \frac{1}{6}c_1^4c_2^4(2a_{12} - 5a_{13})](\nabla_\mu U^\dagger\nabla_\nu U\nabla^\mu U^\dagger\nabla^\nu U) -$$

$$-\frac{V^2}{192}[a_3c_2^2(1 - \frac{v^2}{4}a_3c_2)\{c_1^2c_2(2a_6 + a_7 - 8a_{15}) - \frac{1}{8}a_3(1 - \frac{v^2}{4}a_3c_2)(8a_2 + a_4 - 8a_8 - 32a_{14})\} +$$

$$+\frac{4}{3}c_1^4c_2^4(\frac{32}{9}v^4a_2a_{12} - \frac{8}{9}v^4a_2a_{13} - 4v^4a_6^2 + 4v^2a_3a_6 - a_3^2)](\nabla_\mu U^\dagger\nabla^\mu U)^2 +$$

$$+\frac{M^2c_1^2c_2^2}{16a_2}(a_3 - 2v^2a_6)(\nabla_\mu U^\dagger\nabla^\mu U)[(J_S - iJ_P)U + U^\dagger(J_S + iJ_P)] +$$

$$+\frac{M^2}{4v^2a_2}[(J_S - iJ_P)U(J_S - iJ_P)U + (J_S + iJ_P)U^\dagger(J_S + iJ_P)U^\dagger] +$$

$$+i\frac{v^2}{384}[4c_1^2c_2^2(2a_6 + a_7 - 8a_{15}) + a_3c_2(1 - \frac{v^2}{4}a_3c_2)(40a_2 + a_4 - 8a_8 - 32a_{14})] \times$$

$$\times[(F_V^{\mu\nu} + F_A^{\mu\nu})(\nabla_\mu U\nabla_\nu U^\dagger) + (F_V^{\mu\nu} - F_A^{\mu\nu})(\nabla_\mu U^\dagger\nabla_\nu U)] -$$

$$-\frac{v^2}{384}[8a_6c_1^2c_2^2 - a_3c_2(1 - \frac{v^2}{4}a_3c_2)(40a_2 + a_4 - 8a_8 - 32a_{14})](F_{V\mu\nu} + F_{A\mu\nu})U(F_V^{\mu\nu} - F_A^{\mu\nu})U^\dagger +$$

$$+\frac{1}{192}[4v^2a_6c_1^2c_2^2 + (1 - \frac{v^2}{2}a_3c_2 + \frac{v^4}{8}a_3^2c_2^2)(40a_2 + a_4 - 8a_8 - 32a_{14})](F_{V\mu\nu}F_V^{\mu\nu} + F_{A\mu\nu}F_A^{\mu\nu}) -$$

$$-M^2(1 - \frac{M^2}{2v^2a_2})(J_S - iJ_P)(J_S + iJ_P)\} +$$

$$+c_1^4c_2^4[\frac{v^4}{1152}(5a_{24}-2a_{25})+\{\frac{v^2}{128}a_{20}(a_3-2v^2a_6)+\frac{N_fv^4}{512}a_{20}^2+\frac{a_{16}}{32a_2}(a_3-2v^2a_6)^2\}c_0][tr_f(\nabla_\mu U^\dagger\nabla^\mu U)]^2$$

$$+\frac{v^4}{576}c_1^4c_2^4(4a_{25}-a_{24})tr_f(\nabla_\mu U^\dagger\nabla_\nu U)tr_f(\nabla^\mu U^\dagger\nabla^\nu U)+$$

$$+\frac{M^2c_0c_1^2c_2^2}{16}[va_{20}+\frac{a_3a_{16}}{va_2}-\frac{2va_6a_{15}}{a_2}]tr_f(\nabla_\mu U^\dagger\nabla^\mu U)tr_f[(J_S-iJ_P)U+(J_S+iJ_P)U^\dagger]+$$

$$+\frac{M^4c_0a_{16}}{4v^2a_2}\{tr_f[(J_S-iJ_P)U+(J_S+iJ_P)U^\dagger]\}^2-\frac{a_{18}}{384}(tr_fF_{V\mu\nu})(tr_fF_V^{\mu\nu})-$$

$$-\frac{a_{18}}{384}(1-\frac{v^2}{2}a_3c_2)^2(tr_fF_{A\mu\nu})(tr_fF_A^{\mu\nu}) \quad , \tag{83}$$

where

$$c_0\equiv(2a_2-N_fa_{16})^{-1} \quad . \tag{84}$$

After using the $O(p^2)$ Euler-Lagrange equation (74) and the $SU(3)$ identity (76),eq.(83) can also be reduced to the standard form (77) with

$$F_0^2=\frac{v^2}{4}a_3c_1c_2 \quad , \quad B_0=\frac{8M^2}{va_3c_1c_2} \quad ,$$

$$L_1=\frac{v^4}{1152}\{c_1^4c_2^4(2a_{12}-5a_{13}+5a_{24}-2a_{25})-3a_3c_2^2(1-\frac{v^2}{4}a_3c_2)[c_1^2c_2(2a_6+a_7-8a_{15})+$$

$$+\frac{1}{8}a_3(1-\frac{v^2}{4}a_3c_2)(40a_2+a_4-8a_8-32a_{14})]\}+$$

$$+\frac{c_0c_1^4c_2^4}{512}\{2\frac{a_{16}}{a_2}(a_3-2v^2a_6)^2+N_fv^4a_{20}^2+4v^2a_{20}(a_3-2v^2a_6)\} \quad ,$$

$$L_2=\frac{v^4}{576}\{c_1^4c_2^4(2a_{12}-5a_{13}-a_{24}+4a_{25})-3a_3c_2^2(1-\frac{v^2}{4}a_3c_2)[c_1^2c_2(2a_6+a_7-8a_{15})+$$

$$+\frac{1}{8}a_3(1-\frac{v^2}{4}a_3c_2)(40a_2+a_4-8a_8-32a_{14})]\} \quad ,$$

$$L_3=\frac{c_1^4c_2^4}{256a_2}[(a_3-2v^2a_6)^2-\frac{16}{3}(a_{12}-a_{13})]+\frac{v^4}{96}a_3c_2^2(1-\frac{v^2}{4}a_3c_2)[c_1^2c_2(a_6+\frac{3}{2}a_7-12a_{15})-$$

$$-\frac{3}{16}a_3(1-\frac{v^2}{4}a_3c_2)(-40a_2-a_4+8a_8+32a_{14})]+\frac{v^2}{48}a_6c_1^2c_1^2 \quad ,$$

$$L_4=\frac{a_3c_0c_1^3c_2^3}{256a_2}[(a_3-2v^2a_6)a_{16}+v^2a_2a_{20}] \quad , \quad L_5=\frac{a_3c_1^3c_2^3}{256a_2}(a_3-2v^2a_6) \quad ,$$

$$L_6=\frac{a_3^2a_{16}c_0c_1^2c_2^2}{1024a_2} \quad , \quad L_7=0 \quad , \quad L_8=c_1^2c_2^2(\frac{a_3^2}{1024a_2}-\frac{v^2}{192}a_6) \quad , \tag{85}$$

$$L_9=-\frac{v^2}{384}[4c_1^2c_2^2(2a_6+a_7-8a_{15})+a_3c_2(1-\frac{v^2}{4}a_3c_2)(40a_2+a_4-8a_8-32a_{14})] \quad ,$$

$$L_{10}=-\frac{v^2}{384}[8a_6c_1^2c_2^2-a_3c_2(1-\frac{v^2}{4}a_3c_2)(40a_2+a_4-8a_8-32a_{14})] \quad ,$$

$$H_1=\frac{v^2}{96}a_6c_1^2c_2^2+\frac{1}{384}(1-\frac{v^2}{2}a_3c_2+\frac{v^4}{8}a_3^2c_2^2)(40a_2+a_4-8a_8-32a_{14}) \quad ,$$

$$H_2=\frac{c_1^2c_2^2}{512a_2}[a_3^2(1-\frac{2v^2}{M^2}a_2)+\frac{16}{3}v^2a_2a_6] \quad .$$

Thus we have also obtained the Gasser-Leutwyler chiral Lagrangian in the SPA, but the coefficients given in (85) are different from those in the VEVA (cf.(78)). To leading order in the strong coupling expansion, we see from (61) and (62) that the explicit from of (85) is

$$F_0^2=\frac{(2N_f-N_c)}{12\pi^2}v^2ln(1+\frac{\Lambda^2}{v^2})=\frac{\Lambda^2}{12\pi^2}(2N_f-N_c)(1-\frac{N_f}{2N_c}), \quad B_0=\frac{3N_fv}{2(2N_f-N_c)(1-\frac{N_f}{2N_c})},$$

$$L_1=\frac{N_c}{31104\pi^2}\{32(1-\frac{N_f}{2N_c})^2(1+\frac{N_f}{N_c})^2ln(1+\frac{\Lambda^2}{v^2})+16(1-\frac{N_f}{2N_c})(1+\frac{N_f}{N_c})(\frac{2N_f}{N_c}-1)^2+(\frac{2N_f}{N_c}-1)^4\},$$

$$L_2=2L_1,$$

$$L_3=\frac{N_c}{5184\pi^2}\{[-32(1-\frac{N_f}{2N_c})^2(1+\frac{N_f}{N_c})^2+(\frac{2N_f}{N_c}-1)^4]ln(1+\frac{\Lambda^2}{v^2})-24(1-\frac{N_f}{2N_c})(1+\frac{N_f}{N_c})(\frac{2N_f}{N_c}-1)^2+$$

$$+8(1-\frac{N_f}{2N_c})(1-\frac{2N_f}{N_c})^2-\frac{16}{3}(1-\frac{N_f}{2N_c})^2(1-\frac{2N_f}{N_c})^2+6(1-\frac{2N_f}{N_c})^2\} \, ,$$

$$L_4=0,$$

$$L_5=\frac{N_c}{1728\pi^2}(\frac{2N_f}{N_c}-1)^3ln(1+\frac{\Lambda^2}{v^2}), \tag{86}$$

$$L_6=L_7=0,$$

$$L_8=\frac{N_c}{2304\pi^2}(\frac{2N_f}{N_c}-1)^2ln(1+\frac{\Lambda^2}{v^2}),$$

$$L_9=\frac{N_c}{432\pi^2}\{4(1-\frac{N_f}{2N_c})(1+\frac{N_f}{N_c})ln(1+\frac{\Lambda^2}{v^2})+(\frac{2N_f}{N_c}-1)^2\},$$

$$L_{10}=-\frac{N_c}{432\pi^2}\{4(1-\frac{N_f}{2N_c})(1+\frac{N_f}{N_c})ln(1+\frac{\Lambda^2}{v^2})+\frac{1}{2}(\frac{2N_f}{N_c}-1)^2\},$$

$$H_1=-\frac{N_c}{1728\pi^2}\{2(5-2\frac{N_f}{N_c}+2\frac{N_f^2}{N_c^2})ln(1+\frac{\Lambda^2}{v^2})-(\frac{2N_f}{N_c}-1)^2\},$$

$$H_2 = -\frac{N_c}{1152\pi^2}(\frac{2N_f}{N_c}-1)^2\frac{(8N_c-5N_f)}{N_f}ln(1+\frac{\Lambda^2}{v^2}).$$

Several remarks are in order:

i. Compared with (78), there is an extra factor $\frac{1}{3}(\frac{2N_f}{N_c}-1)$ in the formula for F_0^2 in (86). The positivity of F_0^2 requires that $2N_f > N_c$. This condition is only due to the SPA and will be modified by the quantum corrections. For $N_c = N_f = 3$, the requirement is satisfied and this extra factor further increases Λ/F_0 by a factor of $\sqrt{3}$ which makes Λ just greater than twice of the constituent s-quark mass if one requires F_0^2 to fit the experiment. Therefore this result is quite reasonable.

ii. Different from (78),the positive term in L_8 in eq.(86) does not vanish at $N_f = N_c$. Therefore the SPA result of L_8 has a correct sign as compared with the experiment[1].

iii. In both VEVA and SPA, the obtained L_7 is zero even with $O(g_s^{-2})$ corrections. This is not surprising because it can be shown that L_7 is related to the $\eta'-\eta$ mass difference,

$$m_{\eta'}^2 - m_\eta^2 = 16N_f F_0^{-2} m_\eta^4 L_7 \ . \tag{87}$$

The $\eta'-\eta$ mass difference is related to the $U_A(1)$ breaking mechanism, say the topological nontrivial solution [25], which is not considered in the present calculation. Therefore L_7 should be calculated by further taking into account the topological nontrivial solution.

V. Phenomenological Predictions

We shall first determine the two fundamental parameters Λ and g_s in the theory by taking two suitable experimental data as inputs, and then make phenomenological predictions from the obtained effective Lagrangian. Since the SPA is a better

approximation than the simple VEVA, we shall take the SPA formulae through out this section.

1.Determination of Λ and g_s

The present theory considers three flavors of quarks. Thus it is better to chose two well measured quantities concerning all the three flavors as the inputs. In this paper we take the well measured pion and kaon masses as the inputs.We neglect the small isospin breaking effect so that we take the isospin averaged m_π and m_K which are [26]

$$\begin{aligned} m_\pi &= \frac{1}{3}(2m_{\pi^\pm} + m_{\pi^0}) = 138.0367\pm0.0007MeV \ , \\ m_K &= \frac{1}{2}(m_{K^\pm} + m_{K^0}) = 495.659\pm0.020MeV \ . \end{aligned} \tag{88}$$

The theoretical formulae for m_π and m_K obtained from the chiral Lagrangian including the pseudoscalar meson loop corrections have been given in Ref.[1]. Our present theory is an effective field theory so that the obtained quantities are at the scale $\mu = \Lambda$ [5]. Therefore the formulae for m_π and m_K in Ref.[1] read

$$\begin{aligned} m_\pi^2 &= 2\overline{m}B_0(1+\mu_\pi - \frac{1}{3}\mu_\eta + 2\overline{m}K_3 + K_4) \ , \\ m_K^2 &= (\overline{m}+m_s)B_0(1+\frac{2}{3}\mu_\eta + (\overline{m}+m_s)K_3 + K_4) \ , \end{aligned} \tag{89}$$

where

$$\begin{aligned} \overline{m} &= \frac{1}{2}(m_u + m_d) \ , \\ \mu_\pi &= \frac{2mB_0}{32\pi^2F_0^2}ln\frac{2mB_0}{\Lambda^2} \ , \\ \mu_\eta &= \frac{\frac{2}{3}(m++2m_s)B_0}{32\pi^2F_0^2}ln\frac{\frac{2}{3}(m+2m_s)B_0}{\Lambda^2} \ , \\ K_3 &= \frac{8B_0}{F_0^2}(2L_8 - L_5) \ , \\ K_4 &= \frac{16B_0}{F_0^2}(2m+m_s)(2L_6 - L_5) \ . \end{aligned} \tag{90}$$

The current quark masses have been determined from the current algebra by Weinberg [27] which are $m_u = 4.2MeV$, $m_d = 7.5MeV$, $m_s = 150MeV$. Taking our SPA formula for F_0,B_0,L_5,L_6,and L_8 up to $O(1/g_s^2)$ on the right hand side of (89) together with eq.(68) with the same accuracy, we obtain from the input data

$$\begin{aligned} \Lambda &= 830MeV \ , \\ \alpha_s &= \frac{g_s^2}{4\pi} = 2.13\times10^4 \ , \\ \frac{v^2}{\Lambda^2} &= 0.398 \ . \end{aligned} \tag{91}$$

This value of Λ is quite reasonable since it is just higher than twice of the constituent s-quark mass and is still in the few hundred MeV region in which the QCD coupling is strong. Furthermore, as a rough estimate, the tree level $SU(3)$ averaged vector meson mass $\overline{m}_V$ in our theory can be read out from (60),(47) and (58), which is

$$\overline{m}_V^2 = \frac{M^2 - \frac{N_c\Lambda^2}{32\pi^2}}{\left(\frac{1}{6}a_{14} - \frac{5}{24}a_2 - \frac{1}{192}a_4 + \frac{1}{24}a_8\right)} \ . \tag{92}$$

With the values (91) we get $\overline{m}_V = 946MeV$ which is higher than Λ. This is consistent with the idea that the vector meson field should be integrated out in the effective field theory with the cut-off Λ (The value $\overline{m}_V = 946MeV$ is not far from the corresponding experimental value [26] $\overline{m}_V = (\frac{3m_\rho^2+4m_{K^*}^2+m_\omega^2+m_\phi^2}{9})^{1/2} = 837MeV$.).

The determined α_s in (91) is much larger than the critical value α_c (cf.(70)), thus chiral symmetry breaking really takes place, and the $1/g_s^2$ expansion converges rather quickly, so that our formalism is self-consistent. The largeness of α_s is due to the fact that to leading order ($g_s \to \infty$) the predicted values of m_π and m_K from (89) are already very close to the experimental values (88). We would like to emphasize again that this g_s should not be identified with the renormalized coupling constant in any of the conventional renormalization schemes in the full- range QCD theory [5]. It is also different from the effective coupling constant in the chiral quark model [4] in which the quark and gluon fields are not integrated out.

2.Predictions for the Gasser-Leutwyler Coefficients

With (91), the values of the Gasser-Leutwyler coefficients obtained from (85) up to $O(1/g_s^2)$ are

$$\begin{aligned} &F_0 = 92.1MeV \ , && B_0 = 1614MeV \ , && L_1 = 0.561\times10^{-3} \ , \\ &L_2 = 1.12\times10^{-3} \ , && L_3 = -3.15\times10^{-3} \ , && L_4 = 1.38\times10^{-6} \ , \\ &L_5 = 0.272\times10^{-3} \ , && L_6 = -5.77\times10^{-7} \ , && L_7 = 0 \ , \\ &L_8 = 0.233\times10^{-3} \ , && L_9 = 4.46\times10^{-3} \ , && L_{10} = -3.45\times10^{-3} \ , \\ &H_1 = -1.90\times10^{-3} \ , && H_2 = -0.955\times10^{-3} \ . \end{aligned} \tag{93}$$

In the sense of the effective field theory, these numbers should be understood as the values evaluated at the scale $\mu = \Lambda$, therefore they should not be compared directly with the experimental values given in Ref.[1] in which the scale is $\mu = m_\eta$. The evaluation of the L_i's considering the U-field loop contributions has been calculated in Ref.[1] which is

$$L_i^r(\mu_2) = L_i^r(\mu_1) + \frac{\Gamma_i}{16\pi^2} ln\frac{\mu_1}{\mu_2} \ , \quad i = 1,2,\cdots,10, \tag{94}$$

where L_i^r is the renormalized coefficient in the minimal subtraction (MS) scheme in dimensional regularization, and the values of Γ_i's are given in Ref.[1]. It is known that, up to 1-loop, if we replace Λ by μ in the momentum cut- off regularization, the obtained results are the same as those in the modified minimal subtraction ($\overline{MS}$) scheme [28]. In our present approximation, we keep only the divergent part (as $\Lambda \to \infty$) in the Λ-dependent terms in the coefficients a_n's. Therefore there is no distinction between the $\overline{MS}$ scheme and the MS scheme in the present approximation. Thus the values in (93) can be regarded as L_i^r's at the scale $\mu = \Lambda$, and hence we can used (94) to evolve them down to the scale $\mu = m_\eta$. These are listed in TABLE I together with the experimental values given in Ref.[1]. We see that L_1,L_2, L_3,L_4,L_6 are in good agreement with experiment; L_5,L_8,L_9,and L_{10} are smaller than the experimental values but are still reasonable. The problem of L_7 has been discussed in the last section. Although some of the predicted coefficients

lie outside the experimental error bars, this deviation does not cause much effect on the predictions for most of the physical observables since the L_i's are coefficients of the $O(p^4)$ terms which contribute only a small portion to the physical observables. In the following subsections, we shall calculate the main physical observables from our theory and compare them directly with the experiments. In the calculation of the low energy pion properties, the coefficients $\bar{l}_1,\bar{l}_2,\cdots,\bar{l}_7$ in the $N_f=2$ theory are usually used [1]. The relation between these coefficients and the L_i's has been given in Ref.[1]. In TABLE II, we list our predicted $\bar{l}_1$, $\bar{l}_2,\cdots,\bar{l}_7$ and the corresponding experimental values [1]. We see that the agreement is also nice.

3. Predictions for m_η,F_π,F_K,and F_η

In this paper, we simply regard the η-meson as a member of the flavor SU(3) octet 0^- mesons and denote its mass and decay constant by $m_{\eta 8}$ and $F_{\eta 8}$, respectively.

At tree level, the formulae for $m_{\eta 8}$,F_π,F_K, $F_{\eta 8}$ are simply [1]

$$\begin{aligned} m_{\eta 8}^2 &= \frac{2}{3}(\overline{m}+2m_s)B_0 \ , \\ F_\pi &= F_K = F_{\eta 8} = F_0 \ . \end{aligned} \tag{95}$$

With (93),we obtain

$$m_{\eta 8} = 574MeV \ , \quad F_\pi = F_K = F_{\eta 8} = 92.1MeV \ . \tag{96}$$

Taking into account of the U-field loop contribution, the formulae become [1]

$$\begin{aligned} m_{\eta 8}^2 &= \frac{2}{3}(\overline{m}+2m_s)B_0[1+2\mu_K-\frac{4}{3}\mu_\eta+\frac{2}{3}(\overline{m}+2m_s)K_3+K_4]+2\overline{m}B_0[-\mu_\pi+\frac{2}{3}\mu_K+\frac{1}{3}\mu_\eta] \ , \\ F_\pi &= F_0[1-2\mu_\pi-\mu_K+2\overline{m}K_6+K_7] \\ F_K &= F_0[1-\frac{3}{4}\mu_\pi-\frac{3}{2}\mu_K-\frac{3}{4}\mu_\eta+(\overline{m}+m_s)K_6+K_7] \ , \\ F_{\eta 8} &= F_0[1-3\mu_K+\frac{2}{3}(\overline{m}+2m_s)K_6+K_7] \ , \end{aligned} \tag{97}$$

where $\overline{m},\mu_\pi,\mu_\eta,K_3$ and K_4 are defined in (90), and

$$\mu_K = \frac{B_0}{32\pi^2F_0^2}(\overline{m}+m_s)ln\frac{(\overline{m}+m_s)B_0}{\Lambda^2} \ ,$$

$$K_5 = \frac{128}{9}\frac{B_0^2}{F_0^2}(m_s-\overline{m})^2(3L_7+L_8) \ , \quad K_6 = \frac{4B_0}{F_0^2}L_5 \ , \quad K_7 = \frac{8B_0}{F_0^2}(2\overline{m}+m_s)L_4 \ . \tag{98}$$

Up to $O(1/g_s^2)$,our SPA predicted values are

$$\begin{aligned} m_{\eta 8} = 580MeV \ , \quad F_\pi = 107MeV \ , \\ F_K = 116MeV \ , \quad F_{\eta 8} = 122MeV \ . \end{aligned} \tag{99}$$

Thus the U-field loop contribution increases all these quantities and makes $F_{\eta 8} > F_K > F_\pi$.The predicted F_π and F_K can be compared directly to the experimental values [26] $F_{\pi\pm} = 92.5\pm0.2MeV$ and $F_{K\pm} = 113.0\pm1.0MeV$ (We have multiplied the values in Ref.[27] by $1/\sqrt{2}$ according to the present definition).They are in reasonable agreement.Our predicted $m_{\eta 8}$ is higher than the experimental value of the physical η-meson mass $m_\eta = 547.45\pm0.19MeV$ [26].The deviation is 6%.There is a model-independent relation between $F_{\eta 8}$,F_π, and F_K,

$$\frac{F_{\eta 8}}{F_\pi}-1 = \frac{4}{3}(\frac{F_K}{F_\pi}-1) \ , \tag{100}$$

derived from the $SU(3)_L\times SU(3)_R$ chiral perturbation [29].The numberrs in (99) satisfy this relation with a deviation ~20%.Recently there have been new experimental data for the physical F_η: $F_\eta = 94\pm7MeV$ [30] and $F_\eta = 91\pm6MeV$ [31].These are considerably smaller than the predicted $F_{\eta 8}$.The conventional interpretation is that there is a significant $\eta-\eta'$ mixing with which the theoretical values of the physical m_η and F_η are all close to the experimental values [1][31].

4.Predictions for the Quark Condensates

The formula for $<\bar{q}q>$ is given in (34).In the chiral limit, at tree level,

$$<\bar{u}u>=<\bar{d}d>=<\bar{s}s>=\frac{1}{N_f}<\bar{q}q>=-2M^2v=-F_0^2B_0 \ , \tag{101}$$

where F_0 and B_0 are given in (85).With (93),our SPA predicts

$$< \bar{u}u >=< \bar{d}d >=< \bar{s}s >= -(0.24GeV)^3 \ . \tag{102}$$

This is very close to the original QCD sum rule determined value $-(0.25GeV)^3$ [32]. Taking into account the U-field loop correction [1] and neglecting the isospin breaking effect,we have

$$\begin{aligned} < \bar{u}u >=< \bar{d}d > &= -F_0^2 B_0[1 - 3\mu_\pi - 2\mu_K - \frac{1}{3}\mu_\eta + \overline{m}K_1 + K_2] \ , \\ < \bar{s}s > &= -F_0^2 B_0[1 - 4\mu_\pi - \frac{4}{3}\mu_\eta + m_s K_1 + K_2] \ , \end{aligned} \tag{103}$$

where

$$K_1 = \frac{8B_0}{F_0^2}(2L_8 + H_2) \quad , K_2 = \frac{32B_0}{F_0^2}(2\overline{m} + m_s)L_6 \ , \tag{104}$$

and we have ignored the $\pi^0 - \eta$ mixing in (103). With (93), the SPA predicts

$$< \bar{u}u >=< \bar{d}d >= -(0.26GeV)^3 \quad , \quad < \bar{s}s >= -(0.26GeV)^3 \ . \tag{105}$$

These are also close to the QCD sum rule determined value.

There are also other QCD sum rule determined values of $< \bar{u}u >$ corresponding to different values of $\overline{m}$ [33] which are rather diverse from each other. We know that $< \overline{m}\ \bar{u}u >$ is renormalization group invariant [32], therefore we can compare the theory with the experiments on $< \overline{m}\ \bar{u}u >$. Various QCD sum rule determined $< \overline{m}\ \bar{u}u >$ range from $-(0.093GeV)^4$ to $-(0.097GeV)^4$ [33]. Our SPA predicted $< \overline{m}\ \bar{u}u >$ (cf.(105)) is $-(0.101GeV)^4$. The agreement is quite reasonable.

In view of the fact that the QCD sum rule determination of higher dimensional condensates, e.g. the gluon condensate, the four-quark condensate, etc, is rather uncertain [35], we are not going to consider the higher dimensional condensates in this paper.

5.Predictions for the Pion-Pion Scattering Amplitude

At low energies,the partial wave $\pi - \pi$ scattering amplitudes $t_l^I(p^2)$ can be expressed in terms of the scattering lengths a_l^I and the slope parameters b_l^I, where I and l are, respectively, the isospin and orbital angular momentum quantum numbers [1]. From the $SU(2)_L \times SU(2)_R$ chiral Lagrangian, the formulae for a_l^I and b_l^I are [1]

$$\begin{aligned} a_0^0 &= \frac{7(2\overline{m}B_0)}{32\pi F_0^2}\{1 + \frac{5}{84\pi^2}\frac{2\overline{m}B_0}{F_0^2}(\bar{l}_1 + 2\bar{l}_2 - \frac{9}{10}\bar{l}_3 + \frac{21}{8})\} \ , \\ b_0^0 &= \frac{1}{4\pi F_0^2}\{1 + \frac{1}{12\pi^2}\frac{2\overline{m}B_0}{F_0^2}(2\bar{l}_1 + 3\bar{l}_2 - \frac{13}{16})\} \ , \\ a_0^2 &= -\frac{2\overline{m}B_0}{16\pi F_0^2}\{1 - \frac{1}{12\pi^2}\frac{2\overline{m}B_0}{F_0^2}(\bar{l}_1 + 2\bar{l}_2 + \frac{3}{8})\} \ , \\ b_0^2 &= -\frac{1}{8\pi F_0^2}\{1 - \frac{1}{12\pi^2}\frac{2\overline{m}B_0}{F_0^2}(\bar{l}_1 + 3\bar{l}_2 - \frac{5}{16})\} \ , \\ a_1^1 &= \frac{1}{24\pi F_0^2}\{1 - \frac{1}{12\pi^2}\frac{2\overline{m}B_0}{F_0^2}(\bar{l}_1 - \bar{l}_2 + \frac{65}{48}\} \ , \\ b_1^1 &= \frac{1}{288\pi^3 F_0^4}(-\bar{l}_1 + \bar{l}_2 + \frac{97}{120}) \ , \\ a_2^0 &= \frac{1}{1440\pi^3 F_0^4}(\bar{l}_1 + 4\bar{l}_2 - \frac{53}{8}) \ , \\ a_2^2 &= \frac{1}{1440\pi^3 F_0^4}(\bar{l}_1 + \bar{l}_2 - \frac{103}{40}) \ . \end{aligned} \tag{106}$$

The SPA predicted $\bar{l}_1, \bar{l}_2, \cdots, \bar{l}_7$ are listed in TABLE II with which we obtain our a_l^I and b_l^I. The comparison of the predicted a_l^I and b_l^I with the experimental values taken from Ref.[1] is listed in TABLE III. Some of them are in good agreement. All the deviations are within 30%.

6.Predictions for the Pseudoscalar-Meson Form Factors

Various form factors of the pseudoscalar mesons have been systematically studied from the chiral Lagrangian in Ref.[1]. We just quote the related formulae here and make our predictions.

Let j_μ be the electromagnetic current of quarks. The vector form factors are defined by

$$
\begin{aligned}
&< \pi^+|j_\mu \mid \pi^+ >= (p'_\mu + p_\mu)F_V^\pi(t) \ , \\
&< K^+|j_\mu|K^+ >= (p'_\mu + p_\mu)F_V^{K^+}(t) \ , \\
&< K^0|j_\mu|K^0 >= (p'_\mu + p_\mu)F_V^{K^0}(t) \ , \\
&< K^+|\bar{u}\gamma_\mu s \mid \pi^0 >= \sqrt{\tfrac{1}{2}}\{(p'_\mu + p_\mu)f_+^{K\pi}(t) + (p'_\mu - p_\mu)f_-^{K\pi}(t)\} \ ,
\end{aligned} \tag{107}
$$

where p_μ and p'_μ are the momenta of the initial- and final-state, respectively, and $t\equiv(p'-p)^2$. For low space- like t, these vector form factors can be well approximated by

$$
\begin{aligned}
&F_V^\pi(t) = 1 + \tfrac{1}{6} < r^2 >_V^\pi t + \cdots \ , \\
&F^{K^+}(t) = 1 + \tfrac{1}{6} < r^2 >_V^{K^+} t + \cdots \ , \\
&F^{K^0}(t) = \tfrac{1}{6} < r^2 >_V^{K^0} t + \cdots \ , \\
&f_+^{K\pi}(t) = f_+^{K\pi}(0)\{1 + \tfrac{1}{6} < r^2 >_V^{K\pi} t + \cdots\} \ .
\end{aligned} \tag{108}
$$

The square root of $< r^2 >_V$ measures the charge radius of the pseudoscalar meson.

The scalar form factor

$$
f_0^{K\pi}(t) = f_+^{K\pi}(t) + \frac{1}{m_K^2 - m_\pi^2} f_-^{K\pi}(t) \ , \tag{109}
$$

can also be approximated in terns of the scalar radius $< r^2 >_S^{K\pi}$ by

$$
f_0^{K\pi}(t) = f_+^{K\pi}(0)\{1 + \frac{1}{6} < r^2 >_S^{K\pi} t + \cdots\} \ . \tag{110}
$$

Furthermore there are also form factors of the scalar densities

$$
\begin{aligned}
< \pi^i \mid \bar{u}u + \bar{d}d - 2\bar{s}s \mid \pi^k > &= \delta^{ik}F^8(t) \ , \\
< \pi^i \mid \bar{u}u + \bar{d}d + \bar{s}s \mid \pi^k > &= \delta^{ik}F^0(t) \ , \\
< \pi^i \mid \bar{u}u + \bar{d}d \mid \pi^k > &= \delta^{ik}F_S^\pi(t) \ ,
\end{aligned} \tag{111}
$$

and they can also be approximated by

$$
F^8(t) = F^8(0)\{1 + \frac{1}{6} < r^2 >_S^8 t + \cdots\} \ ,
$$

$$
\begin{aligned}
F^0(t) &= F^0(0)\{1 + \frac{1}{6} < r^2 >_S^0 t + \cdots\} \ , \\
F_S^\pi(t) &= F_S^\pi(0)\{1 + \frac{1}{6} < r^2 >_S^\pi t + \cdots\} \ .
\end{aligned} \tag{112}
$$

In terms of the parameters in the chiral Lagrangian, the vector and scalar radii can be expressed by [1]

$$
\begin{aligned}
< r^2 >_V^\pi &= \frac{12L_9}{F_0^2} - \frac{1}{3\pi^2F_0^2}[2ln\frac{m_\pi^2}{\Lambda^2} + ln\frac{m_K^2}{\Lambda^2} + 3] \ , \\
< r^2 >_V^{K^0} &= -\frac{1}{16\pi^2F_0^2}ln\frac{m_K}{m_\eta} \ , \qquad < r^2 >_V^{K^+} = < r^2 >_V^\pi + < r^2 >_V^{K^0} \ , \\
< r^2 >_V^{K\pi} &= < r^2 >_V^\pi - \frac{1}{64\pi^2F_0^2}\{3h_1(\frac{m_\pi^2}{m_K^2}) + 3h_1(\frac{m_\eta^2}{m_K^2}) + \frac{5}{2}ln\frac{m_K^2}{m_\pi^2} + \frac{3}{2}ln\frac{m_\eta^2}{m_K^2} - 6\} \ , \\
< r^2 >_S^{K\pi} &= \frac{6}{m_K^2 - m_\pi^2}(\frac{F_K}{F_\pi} - 1) - \frac{1}{192\pi^2F_0^2}\{15h_2(\frac{m_\pi^2}{m_K^2}) + \frac{19m_K^2 + 3m_\eta^2}{m_K^2 + m_\eta^2}h_2(\frac{m_{eta}^2}{m_K^2}) - 18\} \ , \\
< r^2 >_S^8 &= \frac{6}{m_K^2 - m_\pi^2}(\frac{F_K}{F_\pi} - 1) + \frac{1}{64\pi^2F_0^2}\frac{1}{m_K^2 - m_\pi^2}\{6(2m_K^2 - m_\pi^2)ln\frac{m_K}{m_\pi} + \\
&\quad +9m_\eta^2 ln\frac{m_\eta^2}{m_\pi^2} - 2(m_K^2 - m_\pi^2)(10 + \frac{1}{3}\frac{m_\pi^2}{m_\eta^2})\} \ , \\
< r^2 >_S^0 &= < r^2 >_S^8 + \frac{6}{F_0^2}\{12L_4 - \frac{3}{64\pi^2}(1 + ln\frac{m_K^2}{\Lambda^2}) + \frac{m_\pi^2}{288\pi^2}\frac{1}{m_\eta^2}\} \ , \\
< r^2 >_S^\pi &= < r^2 >_S^8 + \frac{4}{F_0^2}\{12L_4 - \frac{3}{64\pi^2}(1 + ln\frac{m_K^2}{\Lambda^2}) + \frac{m_\pi^2}{288\pi^2}\frac{1}{m_\eta^2}\} \ ,
\end{aligned} \tag{113}
$$

where

$$
\begin{aligned}
h_1(x) &\equiv \frac{1}{2}\frac{(x^3 - 3x^2 - 3x + 1)}{(x-1)^3}lnx + \frac{1}{2}(\frac{x+1}{x-1})^2 - \frac{1}{3} \ , \\
h_2(x) &\equiv \frac{1}{2}(\frac{1+x}{1-x})^2 + \frac{3x(1+x)}{(1-x)^3}lnx \ .
\end{aligned} \tag{114}
$$

With (93), we can predict all these radii up to $O(1/g_*^2)$. They are listed in TABLE IV together with the experimental values taken from Ref.[1] . (The experimental values of $< r^2 >_S^{K\pi}$,$< r^2 >_S^0$,and $< r^2 >_S^8$ are obtained from theoretical relations with certain experimental inputs.). The agreement is reasonable.

VI. Conclusions

To summarize, we have derived the effective Lagrangian for the low lying pseudoscalar mesons up to $O(p^4)$ in the framework of the effective field theory with a physical cut-off Λ from the cut-off QCD Lagrangian $\mathcal{L}_{QCD}^{(\Lambda)}$ given in (5) which leads to the gauge invariant form of the gluon effective Lagrangian after integrating out the quark fields. We are working in the strong coupling region, so that we make $1/g_s^2$ expansion. To leading order, the integration over gluon fields leads to a NJL type four-fermion interaction between quarks. After introducing auxiliary fields for mesons and expanding the action S_{eff} in power series of the external momentum,we obtain a general effective Lagrangian (60) for the low lying pseudoscalar, scalar, vector, and axial-vector mesons up to $O(p^4)$ which contains 44 terms expressed in terms of 29 coefficients a_1,a_2,$\cdots$,a_{29} given in (59). The exact WZW term is also included in the imaginary part of S_{eff}. The coefficients a_1, a_2, $\cdots$, a_{29} can be calculated order by order in $1/g_s^2$ expansion as functions of the two fundamental parameters Λ and g_s. The gap equation (68) derived from the vacuum stabillity condition leads to chiral symmetry breaking provided $2N_c > N_f$ and $\alpha_s > \alpha_c$ (cf.(70)), and the quark condensates are calculable in this theory. We then take the SPA to integrate out the scalar, vector, and axial-vector-meson fields,and an effective Lagrangian for the low lying pseudoscalar mesons alone is obtained up tp $O(p^4)$. It is of the form of (77) which contains the complete Gasser-Leutwyler chiral Lagrangian with the coefficients given in (85), which are all calculable functions of Λ and g_s. The main theoretical uncertainty in this approach comes from: (a) neglecting the quantum fluctuations beyond SPA, (b) neglecting the non-divergent terms in the momentum integrations which diverge as $\Lambda \to \infty$. The two parameters Λ and g_s are then determined by taking the well measured values of m_π and m_K as inputs. The obtained $\Lambda = 830MeV$ is quite reasonable and the obtained g_s is really larger than

α_c so that chiral symmetry breaking really takes place,and our approach is self-consistent. With the determined Λ and g_s,the predicted physical observables in low energy pseudoscalar-meson physics shown in Sec.V are all in reasonable agreement with experiments.

Several improvements of the approach should be considered in the future investigations:

(a) To get a nonvanishing L_7,topologically nontrivial solutions should be further taken into account. This will give rise to the $\eta - \eta'$ mass splitting, the correction to the gap equation and will induce the tensor meson in the theory [12].

(b) To reduce the theoretical uncertainty, we should take into account the quantum corrections to the SPA and keep more terns in the momentum integrations which diverge as $\Lambda \to \infty$. The latter requires a carefull investigation of the corresponding terms in $\mathcal{L}_H^{eff}$ to preserve the local $U(N_f)_L \times U(N_f)_R$ symmetry.

The encouraging phenomenological results in this paper imply that the present approach may be generalized to the study of other interesting problems such as the strongly interacting electro-weak symmetry breaking mechanisms. For instance,it may be applied to study the dynamics of the technicolor, walking technicolor,and even non-QCD-like technicolor type theories. It is known that when applying the pure Gasser-Leutwyler Lagrangian to the study of the technicolor theory, unitarity will be violated when the energy exceeds $1TeV$ [7]. In the published papers [7],certain phenomenological unitarization models imitating the techni-ρ-meson resonance is put in by hand to avoid the violation of unitarity.The advantage of the present theory is that the low lying vector mesons are already included in the theory and we should just keep them unintegrated to avoid unitarity violation. This kind

of investigattions will be presented in separate papers.

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Appendix A

Consider a chiral transformation with the group elements $g_R(x) = g(x)$, $g_L(x) = g^\dagger(x)$, where $g(x)$ is a unitary matrix. The quark field transforms as

$$q \to q' = [g(x)P_R + g^\dagger(x)P_L]q \quad , \quad \overline{q} \to \overline{q}' = \overline{q}[g(x)P_R + g^\dagger(x)P_L] \ . \qquad (A1)$$

Then the functional measure becomes

$$\begin{aligned} \mathcal{D}q\mathcal{D}\overline{q} &= \mathcal{D}q'\mathcal{D}\overline{q}'\{Det[g^\dagger(x)g^\dagger(x)P_R + g(x)g(x)P_L]\}^{N_c} \\ &= \mathcal{D}q'\mathcal{D}\overline{q}'exp\{-Trln[g(x)g(x)P_R + g^\dagger(x)g^\dagger(x)P_L]\} \ . \end{aligned} \qquad (A2)$$

To evaluate $Trln[g(x)g(x)P_R + g^\dagger(x)g^\dagger(x)P_L]$,we consider further a variation $g(x) \to g(x) + \delta g(x)$. Then

$$\begin{aligned} &\delta Trln[g(x)g(x)P_R + g^\dagger(x)g^\dagger(x)P_L] = 2Tr[g^\dagger(x)\delta g(x)P_R + g(x)\delta g^\dagger(x)P_L] \\ &= 2Tr[g^\dagger(x)\delta g(x)P_R - \delta g(x)g^\dagger(x)P_L] = Tr\{[g^\dagger(x)\delta g(x) + \delta g(x)g^\dagger(x)]\gamma_5\} \ . \end{aligned} \qquad (A3)$$

This functional trace is actually not mathematically well-defined. We take Fujikawa's technique [18] to regularize it,i.e.

$$Tr\{[g^\dagger(x)\delta g(x) + \delta g(x)g^\dagger(x)]\gamma_5\} = \frac{1}{2}Tr\{[g^\dagger(x)\delta g(x) + \delta g(x)g^\dagger(x)]\gamma_5(e^{-\frac{D'^2}{\Lambda^2}} + e^{-\frac{(D'^\dagger)^2}{\Lambda^2}})\} \ , \qquad (A4)$$

where

$$iD' \equiv i\partial\!\!\!/ + a(x) + \not{C}(x) + \not{J}^{V'}(x) + \not{J}^{A'}(x)\gamma_5 \ , \qquad (A5)$$

in which $a(x)$ is an arbitrary scalar function, $C_\mu(x)$ is an $N_c \times N_c$ matrix, $J_\mu^{V'}(x)$ and $J_\mu^{A'}(x)$ are $N_f \times N_f$ matrices given in eq.(15) in the text. The evaluation of (A4) has been given in Ref.[10], and we just quote their result. The difference between our D' and the corresponding Dirac operator in Ref.[10] is that we have included an extra term $\not{C}(x)$ in (A5). Therefore the corresponding result of (A4) is

$$Tr\{[g^\dagger(x)\delta g(x) + \delta g(x)g^\dagger(x)]\gamma_5\} = Tr'\{[g^\dagger(x)\delta g(x) + \delta g(x)g^\dagger(x)]G(K, J^{V'}, J^{A'})\} \ , \qquad (A6)$$

where

$$\begin{aligned} G(K, J^{V'}, J^{A'}) = \frac{N_c}{\pi^2}\epsilon^{\mu\nu\rho\sigma}[&\frac{1}{3}J_\mu^A J_\nu^A J_\rho^A J_\sigma^A + \frac{1}{32}(F_{V\mu\nu}F_{V\rho\sigma} + \frac{1}{3}F_{A\mu\nu}F_{A\rho\sigma}) - \\ &-\frac{1}{12}(J_\mu^A J_\nu^A F_{V\rho\sigma} + F_{V\mu\nu}J_\rho^A J_\sigma^A + J_\mu^A F_{V\nu\rho}J_\sigma^A)] + \frac{1}{16\pi^2}tr_c(C_{\mu\nu}\tilde{C}^{\mu\nu}) \ , \end{aligned} \qquad (A7)$$

where

$$\begin{aligned} F_{V\mu\nu} &\equiv \partial_\mu J_{V\nu} - \partial_\nu J_{V\mu} + i[J_{V\mu}, J_{V\nu}] + i[J_{A\mu}, J_{A\nu}] \\ F_{A\mu\nu} &\equiv \partial_\mu J_{A\nu} - \partial_\nu J_{A\mu} + i[J_{V\mu}, J_{A\nu}] + i[J_{A\mu}, J_{V\nu}] \\ C_{\mu\nu} &\equiv \partial_\mu C_\nu - \partial_\nu C_\mu + i[C_\mu, C_\nu] \ . \end{aligned} \qquad (A8)$$

Integrating (A3) over $\delta g(x)$, we can get $Trln[g(x)g(x)P_R + g^\dagger(x)g^\dagger(x)P_L]$. The result has been given in Ref.[10], and the correspond result here is

$$\begin{aligned} Trln[g(x)g(x)P_R + g^\dagger(x)g^\dagger(x)P_L] = i\frac{N_c}{240\pi^2}\int_{D_5} tr'(u^{-1}du)^5 + \\ +i\int_{S^4}\mathcal{A}(u, J_L, J_R) - \int d^4x indet(u)\ \frac{1}{16\pi^2}tr_c(C_{\mu\nu}\tilde{C}^{\mu\nu}) \ , \end{aligned} \qquad (A9)$$

where $u(x) \equiv g(x)g(x)$, D_5 is a five-dimensional disc bounded by S^4,and $\mathcal{A}$ is given in eq.(11) in the text. Substituting (A9) into (A2), and regarding C_μ as B_μ or $M^{-2}I_\mu$, we get eq.(9) or eq.(41) in the text.

Appendix B

We sketch here the evaluation of $ReTrln(i\not{D})$. In the Euclidean space- time,

$$ReTrln(i\not{D}) = \frac{1}{2}Trln\{-\gamma_\mu\gamma_\nu(i\partial^\mu + V_I^\mu - A^\mu\gamma_5)(i\partial^\nu + V_I^\nu - A^\nu\gamma_5)$$

$$-[(i\partial^\mu + V_I^\mu - A^\mu\gamma_5), \gamma_\mu(S + iP\gamma_5)]_- + (S - iP\gamma_5)(S + iP\gamma_5)\} \ . \qquad (B1)$$

Using the technique in Ref.[24], (B1) can be written as

$$ReTrln(i\not{D}) = \frac{1}{2}\int d^4x \int \frac{d^4k}{(2\pi)^4} trln\{-\gamma_\mu\gamma_\nu(i\partial^\mu + k^\mu + V_I^\mu - A^\mu\gamma_5)(i\partial^\nu + k^\nu + V_I^\nu - A^\nu\gamma_5)$$

$$-[(i\partial^\mu + k^\mu + V_I^\mu - A^\mu\gamma_5), \gamma_\mu(S + iP\gamma_5)]_- + (S - iP\gamma_5)(S + iP\gamma_5)\} \ , \qquad (B2)$$

in which tr is the trace in the space of color, flavor and Lorentz spinor, and the derivative ∂^μ operates only on the fields V_I^μ,A^μ,S,P and the external source I^μ. We introduce the following covariant derivatives

$$D^\mu(F) \equiv \partial^\mu \mid_{on \ \ S,P,V^\mu,A^\mu} \ -i(V^\mu - A^\mu\gamma_5) \ ,$$
$$D_I^\mu(I) \equiv \partial^\mu \mid_{on \ \ I^\mu} \ -iM^{-2}I^\mu \ . \qquad (B3)$$

In terms of these covariant derivatives, (B2) can be written as

$$ReTrln(i\not{D}) = \frac{1}{2}\int d^4x \int \frac{d^4k}{(2\pi)^4} trln\{ \ v^2 - k_\mu k^\mu + \gamma_\mu\gamma_\nu D_I^\mu(I)D_I^\nu(I) - 2ik_\mu D_I^\mu(I) +$$

$$+ D_\mu(F)D^\mu(F) + 2D_I^\mu(I)D_\mu(F) - 2ik_\mu D^\mu(F) + \phi \ \} \ , \qquad (B4)$$

where

$$\phi \equiv \frac{1}{2}[\gamma^\mu D_\mu(F), \gamma^\nu D_\nu(F)]_- - i[D^\mu(F), \gamma_\mu(S + iP\gamma_5)]_- + (S - iP\gamma_5)(S + iP\gamma_5) - v^2 \ . \quad (B5)$$

The momentum integration in (B4) is understood to be taken below the cut-off Λ. With the cut-off Λ, the invariance of $ReTrln(i\not{D})$ under the local $U(N_f)_L \times U(N_f)_R$

transformation cannot be completely maintained. In this paper we take the approximations: (a) for momentum integrations which diverge as $\Lambda \to \infty$, we keep only the divergent terms (For $ln(1 + \frac{\Lambda^2}{v^2})$,we keep its full expression without expanding it in powers of $\frac{v^2}{\Lambda^2}$.); (b) for momentum integrations which converge as $\Lambda \to \infty$, we simply take $\Lambda \to \infty$ (This is equivalent to taking a finite Λ and taking account of the necessary terms in $\mathcal{L}_H^{eff}$ to preserve the local $U(N_f)_L \times U(N_f)_R$ symmetry [21].). In this case the only term contributing to $S_{eff}^{(M2)}$ and violating the local $U(N_f)_L \times U(N_f)_R$ symmetry induced by the finite Λ is in the term $-\frac{\Lambda^2}{32\pi^2}\int d^4x tr(V_\mu V^\mu + A_\mu A^\mu)$ which will be cancelled by a corresponding term in $\mathcal{L}_H^{eff}$ [21], so that the local $U(N_f)_L \times U(N_f)_R$ symmetry is preserved.

Doing the low energy expansion up to $O(p^4)$,we obtain

$$ReTrln(i\not{D}) = \frac{1}{2}\sum_{l=0}^{14} Q_l \quad , \qquad (B6)$$

where

$$Q_0 \equiv \int d^4x \int \frac{d^4k}{(2\pi)^4} tr \ ln \ \{v^2 + k^2 + \gamma_\mu\gamma_\nu D_I^\mu(I)D_I^\nu(I) - 2ik_\mu D_I^\mu(I)\} \ ,$$

$$Q_1 \equiv \int d^4x \int \frac{d^4k}{(2\pi)^4} tr(\Delta_k\phi) \ ,$$

$$Q_2 \equiv 2\int d^4x \int \frac{d^4k}{(2\pi)^4} tr\{\Delta_k(D_I^\mu(I) - ik^\mu)D_\mu(F)\} \ ,$$

$$Q_3 \equiv -\frac{1}{2}\int d^4x \int \frac{d^4k}{(2\pi)^4} tr(\Delta_k^2\phi^2) \ ,$$

$$Q_4 \equiv -\int d^4x \int \frac{d^4k}{(2\pi)^4} tr\{\Delta_k^2(D_I^\mu(I) - ik^\mu)\phi D_\mu(F) + \Delta_k(D_I^\mu(I) - ik^\mu)\Delta_k D_\mu(F)\phi\} \ ,$$

$$Q_5 \equiv \int d^4x \int \frac{d^4k}{(2\pi)^4} tr\{[\Delta_k g^{\mu\nu} - 2\Delta_k(D_I^\mu(I) - ik^\mu)\Delta_k(D_I^\nu(I) - ik^\nu)]D_\mu(F)D_\nu(F)\} \ ,$$

$$Q_6 \equiv \frac{1}{3}\int d^4x \int \frac{d^4k}{(2\pi)^4} tr(\Delta_k^3\phi^3) \ ,$$

$$Q_7 \equiv \frac{2}{3}\int d^4x \int \frac{d^4k}{(2\pi)^4} tr\{\Delta_k^3(D_I^\mu(I) - ik^\mu)\phi^2 D_\mu(F) + \Delta_k^2(D_I^\mu(I) - ik^\mu)\Delta_k\phi D_\mu(F)\phi +$$

$$+\Delta_k(D_I^\mu(I)-ik^\mu)\Delta_k^2 D_\mu(F)\phi^2\} \ , \qquad (B7)$$

$$Q_8 \equiv -\frac{1}{2}\int d^4x \int \frac{d^4k}{(2\pi)^4} tr\{\Delta_k^2\phi D_\mu(F)D^\mu(F)+\Delta_k^2 D_\mu(F)D^\mu(F)\phi-$$
$$-\frac{8}{3}[\Delta_k^2(D_I^\mu(I)-ik^\mu)\Delta_k(D_I^\nu(I)-ik^\nu)\phi D_\mu(F)D_\nu(F)+$$
$$+\Delta_k(D_I^\mu(I)-ik^\mu)\Delta_k^2(D_I^\nu(I)-ik^\nu)D_\mu(F)\phi D_\nu(F)+$$
$$+\Delta_k(D_I^\mu(I)-ik^\mu)\Delta_k(D_I^\nu(I)-ik^\nu)\Delta_k D_\mu(F)D_\nu(F)\phi]\} \ ,$$

$$Q_9 \equiv -\int d^4x \int \frac{d^4k}{(2\pi)^4} tr\{[\Delta_k^2(D_I^\lambda(I)-ik^\lambda)g^{\mu\nu}+\Delta_k(D_I^\mu(I)-ik^\mu)\Delta_k g^{\nu\lambda}-$$
$$-\frac{8}{3}\Delta_k(D_I^\mu(I)-ik^\mu)\Delta_k(D_I^\nu(I)-ik^\nu)\Delta_k(D_I^\lambda(I)-ik^\lambda)]D_\mu(F)D_\nu(F)D_\lambda(F)\} \ ,$$

$$Q_{10} \equiv -\frac{1}{4}\int d^4x \int \frac{d^4k}{(2\pi)^4} tr(\Delta_k^4\phi^4) \ ,$$

$$Q_{11} \equiv -\frac{1}{2}\int d^4x \int \frac{d^4k}{(2\pi)^4} tr\{\Delta_k^4(D_I^\mu(I)-ik^\mu)\phi^3 D_\mu(F)+\Delta_k^3(D_I^\mu(I)-ik^\mu)\Delta_k\phi^2 D_\mu(F)\phi+$$
$$+\Delta_k^2(D_I^\mu(I)-ik^\mu)\Delta_k^2\phi^2 D_\mu(F)\phi^2+\Delta_k(D_I^\mu(I)-ik^\mu)\Delta_k^3 D_\mu(F)\phi^3\} \ ,$$

$$Q_{12} \equiv \frac{1}{3}\int d^4x \int \frac{d^4k}{(2\pi)^4} tr\{\Delta_k^3\phi^2 D^\mu(F)D_\mu(F)+\Delta_k^3\phi D^\mu(F)D_\mu(F)\phi+\Delta_k^3 D^\mu(F)D_\mu(F)\phi^2-$$
$$-3\Delta_k^3(D_I^\mu(I)-ik^\mu)\Delta_k(D_I^\nu(I)-ik^\nu)\phi^2 D_\mu(F)D_\nu(F)-$$
$$-3\Delta_k^2(D_I^\mu(I)-ik^\mu)\Delta_k^2(D_I^\nu(I)-ik^\nu)\phi D_\mu(F)\phi D_\nu(F)-$$
$$-3\Delta_k^2(D_I^\mu(I)-ik^\mu)\Delta_k(D_I^\nu(I)-ik^\nu)\Delta_k\phi^2 D_\mu(F)D_\nu(F)-$$
$$-3\Delta_k(D_I^\mu(I)-ik^\mu)\Delta_k^3(D_I^\nu(I)-ik^\nu)D_\mu(F)\phi^2 D_\nu(F)-$$
$$-3\Delta_k(D_I^\mu(I)-ik^\mu)\Delta_k^2(D_I^\nu(I)-ik^\nu)\Delta_k D_\mu(F)\phi D_\nu(F)\phi-$$
$$-3\Delta_k(D_I^\mu(I)-ik^\mu)\Delta_k(D_I^\nu(I)-ik^\nu)\Delta_k^2 D_\mu(F)D_\nu(F)\phi^2\} \ ,$$

$$Q_{13} \equiv -\frac{1}{2}\int d^4x \int \frac{d^4k}{(2\pi)^4} tr\{\Delta_k^2 g^{\mu\nu}g^{\rho\sigma}-\frac{8}{3}[\Delta_k^2(D_I^\sigma(I)-ik^\sigma)\Delta_k(D_I^\rho(I)-ik^\rho)g^{\mu\nu}+$$
$$+\Delta_k(D_I^\mu(I)-ik^\mu)\Delta_k^2(D_I^\rho(I)-ik^\rho)g^{\nu\sigma}+\Delta_k(D_I^\mu(I)-ik^\mu)\Delta_k(D_I^\nu(I)-ik^\nu)\Delta_k g^{\rho\sigma}]+$$
$$+8\Delta_k(D_I^\mu(I)-ik^\mu)\Delta_k(D_I^\nu I-ik^\nu)\Delta_k(D_I^\sigma(I)-ik^\sigma)\Delta_k(D_I^\rho(I)-ik^\rho)\}D_\mu(F)D_\nu(F)D_\sigma(F)D_\rho(F) \ ,$$

$$Q_{14} \equiv \frac{2}{3}\int d^4x \int \frac{d^4k}{(2\pi)^4} tr\{\Delta_k^3(D_I^\mu(I)-ik^\mu)\phi D^\nu(F)D_\nu(F)D_\mu(F)+$$

$$+\Delta_k^2(D_I^\mu(I)-ik^\mu)\Delta_k\phi D_\mu(F)D^\nu(F)D_\nu(F)+$$
$$+\Delta_k^3(D_I^\mu(I)-ik^\mu)D^\nu(F)D_\nu(F)\phi D_\mu(F)+\Delta_k^2(D_I^\mu(I)-ik^\mu)\Delta_k D^\nu(F)D_\nu(F)D_\mu(F)\phi+$$
$$+\Delta_k(D_I^\mu(I)-ik^\mu)\Delta_k^2 D_\mu(F)\phi D^\nu(F)D_\nu(F)+\Delta_k(D_I^\mu(I)-ik^\mu)\Delta_k^2 D_\mu(F)D^\nu(F)D_\nu(F)\phi-$$
$$-3\Delta_k^2(D_I^\mu(I)-ik^\mu)\Delta_k(D_I^\nu(I)-ik^\nu)\Delta_k(D_I^\lambda(I)-ik^\lambda)\phi D_\mu(F)D_\nu(F)D_\lambda(F)-$$
$$-3\Delta_k(D_I^\mu(I)-ik^\mu)\Delta_k^2(D_I^\nu(I)-ik^\nu)\Delta_k(D_I^\lambda(I)-ik^\lambda)D_\mu(F)\phi D_\nu(F)D_\lambda(F)-$$
$$-3\Delta_k(D_I^\mu(I)-ik^\mu)\Delta_k(D_I^\nu(I)-ik^\nu)\Delta_k^2(D_I^\lambda(I)-ik^\lambda)D_\mu(F)D_\nu(F)\phi D_\lambda(F)-$$
$$-3\Delta_k(D_I^\mu(I)-ik^\mu)\Delta_k(D_I^\nu(I)-ik^\nu)\Delta_k(D_I^\lambda(I)-ik^\lambda)\Delta_k D_\mu(F)D_\nu(F)D_\lambda(F)\phi\} \ ,$$

where Δ_k is defined in (54). It can be shown that terms linear in Q_2,Q_4,Q_7,Q_9,Q_{11} and Q_{14} do not give nonvanishing contributions to $S_{eff}^{(M2)}$. Furthermore, it can be shown from local $U(N_f)_L \times U(N_f)_R$ invariance that, in $S_{eff}^{(M2)}$,the nonvanishing term coming from Q_8 can be written as

$$-\frac{1}{12}\int d^4x \int \frac{d^4k}{(2\pi)^4} tr\{\Delta_k(D_I^\sigma(I)-ik^\sigma)\Delta_k(D_I^\rho(I)-ik^\rho)\Delta_k[\gamma_\sigma,\gamma_\rho]_-\times$$
$$\times[D^\mu(F),D^\nu(F)]_-[D_\mu(F),D_\nu(F)]_-\phi\} \ , \qquad (B8)$$

the one from Q_{12} can be written as

$$\frac{1}{48}\int d^4x \int \frac{d^4k}{(2\pi)^4} tr\{\Delta_k\gamma^\sigma\Delta_k^2\gamma_\sigma([D_\mu(F),\phi]_-[D^\mu(F),\phi]_--[D_\mu(F),\gamma_5\phi]_-[D^\mu(F),\gamma_5\phi]_-)-$$
$$-2\Delta_k(D_I^\lambda(I)-ik^\lambda)\Delta(D_I^\kappa(I)-ik^\kappa)\Delta_k\gamma^\sigma\Delta_k(g_{\lambda\sigma}\gamma_\kappa-g_{\kappa\sigma}\gamma_\lambda)\phi(\phi\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu\phi)[D_\mu(F),D_\nu(F)]_-+$$
$$+4\Delta_k^3([D_\mu(F),\phi]_-[D^\mu(F),\phi]_-+[D_\mu(F),\gamma_5\phi]_-[D^\mu(F),\gamma_5\phi]_-)\} \ , \qquad (B9)$$

and the one coming from Q_{13} can be written as

$$-\frac{5}{12}\int d^4x \int \frac{d^4k}{(2\pi)^4} tr\{\Delta_k^2-\frac{4}{5}\Delta_k(D_I^\sigma(I)-ik^\sigma)\Delta_k(D_{I\sigma}(I)-ik_\sigma)\Delta_k(D_I^\rho(I)-ik^\rho)\Delta_k(D_{I\rho}(I)-ik_\rho)\times$$
$$\times[D^\mu(F),D^\nu(F)]_-[D_\mu(F),D_\nu(F)]_-\} \ . \qquad (B10)$$

Substituting all these into (B6),we get (52)-(53) in Sec.III.

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20. Here we simply treat the $\int \mathcal{D}B^a_\mu$ integration in (16) as a Gaussian integration , i.e. we ignore the cut-off function in the $\frac{1}{2}M^2 B^a_\mu B^{a\mu}$ term (the cut-off function in other terms are understood to be included in $J^{a\mu}_B$ and $\overline{q}\frac{\lambda^a}{2}\gamma^\mu q$.).This means that we have included the high momentum piece of the gluon field at this step of integration. Note that the cut-off function in $\mathcal{L}_G(B)$ is still kept.

21. This invariance is for the full-range theory.Once the momentum cut-off Λ is imposed in $ReTrlni\mathcal{D}$,a term $-\frac{\Lambda^2}{32\pi^2}\int tr(V_\mu V^\mu + A_\mu A^\mu)$ will emerge which violates the local $U(N_f)_L \times U(N_f)_R$ invariance.However,if the external sources are introduced in the original full-range theory,the integration over the high momentum piece of the quark field must contribute a term $\frac{\Lambda^2}{32\pi^2}\int tr(2J_{V\mu}V^\mu + 2J_{A\mu}A^\mu - J_{V\mu}J^\mu_V - J_{A\mu}J^\mu_A)$ (similar to the $\frac{1}{2}M^2 B^a_\mu B^{a\mu}$ in $\mathcal{L}^{eff}_H$) in $\mathcal{L}^{eff}_H$ which cancels the above flavor gauge- noninvariant part from $ReTrlni\mathcal{D}\,|_\Lambda$ to maintain the invariance of the entire theory.In the following calculations,the automatic cancellation of the above gauge-noninvariant terms will be understood without specially mentioning it.

22. At first sight,it looks that terms like Σ^n/Λ^n are not necessarily small at low energies.However, the Σ-field is to be integrated out in the effective field theory at hand.After the integration,the dimension-n field operator corresponding to Σ^n will be expressed in terms of the nth power of covariant derivatives of U (Note that U is dimensionless.),so that such terms are really suppressed at low energies.

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TABLE I. The predicted Gasser-Leutwyler coefficients L_1, $L_2, \cdots, L_{10}$ (in 10^{-3}) in SPA up to $O(g_s^{-2})$ at the scale $\mu = m_\eta$.The experimental values are taken from Ref.[1]

	L_1	L_2	L_3	L_4	L_5
Theory	0.8	1.6	-3.2	0.3	1.3
Expt.	0.9±0.3	1.7±0.7	-4.4±2.5	0±0.5	2.2±0.5
	L_6	L_7	L_8	L_9	L_{10}
Theory	0.2	0	0.5	5.1	-4.1
Expt.	0±0.3	-0.4±0.15	1.1±0.3	7.4±0.7	-6.0±0.7

TABLE II. The predicted coefficients $\bar{l}_1, \cdots, \bar{l}_7$ in $N_f = 2$ theory in SPA up to $O(g_s^{-2})$ at the scale $\mu = m_\eta$.The experimental values are taken from Ref.[1]

	$\bar{l}_1$	$\bar{l}_2$	$\bar{l}_3$	$\bar{l}_4$	$\bar{l}_5$	$\bar{l}_6$	$\bar{l}_7$
Theory	-0.25	5.7	3.0	3.8	10.1	12.1	2×10^{-3}
Expt.	-2.3±3.7	6.0±1.3	2.9±2.4	4.3±0.9	13.9±1.3	16.5±1.1	$O(5\times10^{-3})$

TABLE III. Comparison of the SPA predicted values of a_l^I and b_l^I (up to $O(g_s^{-2})$) with the experimental values taken from Ref.[1]. The dimensionful parameters $b_0^0, b_0^2, a_1^1, b_1^1, a_2^0$,and a_2^2 are given in the unit of $m_{\pi^\pm}$.

	a_0^0	b_0^0	a_0^2	$2a_0^0 - 5a_0^2$	b_0^2
Theory	0.18	0.24	-0.035	0.53	-0.063
Expt.	0.26±0.05	0.25±0.03		0.614±0.028	-0.082±0.008
	a_1^1	b_1^1	a_2^0	a_2^2	
Theory	0.033	0.0040	19×10^{-4}	3.5×10^{-4}	
Expt.	0.038±0.002		$(17\pm3)\times10^{-4}$	$(1.3\pm3)\times10^{-4}$	

TABLE IV. Comparison of the SPA predicted vector and scalar radii (up to $O(g_s^2)$) with the experimental values taken from Ref.[1](in fm^2).

	$<r^2>_V^\pi$	$<r^2>_V^{K^0}$	$<r^2>_V^{K^+}$	$<r^2>_V^{K\pi}$
Theory	0.32	-0.037	0.28	0.26
Expt.	0.439±0.03	-0.054±0.026	0.28±0.07	0.36±0.02
	$<r^2>_S^{K\pi}$	$<r^2>_S^0$	$<r^2>_S^8$	$<r^2>_S^\pi$
Theory	0.076	0.47	0.46	0.47
Expt.	0.20 ± 0.05^a	0.7 ± 0.3^b	0.55 ± 0.10^a	0.7±0.2

a. obtained from the low energy theorem with the experimental value of F_K/F_π as input.

b. obtained from the experimental value of $<r^2>_S^\pi$ and the theoretical estimate of $<\delta r^2>_S$.