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UMN-TH-1409-95
September 1995

NON-PERTURBATIVE ELECTROWEAK PHYSICS

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Lectures at the 1995 Trieste Summer School on High Energy Physics and Cosmology, Trieste, Italy, July 1995

Abstract

These lectures are an introduction to phenomena in electroweak physics, which require a non-perturbative analysis: the baryon and lepton number violation and processes involving many interacting bosons.

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1. Introduction

The $SU(2)\times U(1)$ electroweak interactions in the Standard Model provide a typical example of a theory with weak coupling, where the perturbation theory analysis is perfectly applicable to all thus far observed processes. If also the yet unknown Higgs sector of the Standard Model is weakly coupled, which implies that there are no heavy Higgs bosons with masses of order 1000 GeV, the whole electroweak sector falls into the realm of perturbation theory. The perturbation theory, as applied to the electroweak interactions, however is known to be incomplete in at least two aspects: at very short distances and in very high order.

The incompleteness at short distances arising from the Landau – Pomeranchuk pole due to lack of asymptotic freedom in all of the couplings is a problem of energies, which are exponentially large in the non-asymptotically-free coupling g^2: $E_{LP} \sim \exp(\text{const}/g^2)$ and can be possibly remedied by merging our Standard Model into a larger 'new physics' theory, like it is the case for the $U(1)$ gauge coupling in grand unification models. In this sense this difficulty can be termed as fundamental or principal, since its resolution requires principally new knowledge of the Nature.

The other difficulty of the perturbation series expansion, first pointed out by Dyson[1], is that the expansion in fact never converges and the series is asymptotic. In other words, in large order n of the expansion the asymptotic behavior of the terms in the series is in general given by $n!\,(cg^2)^n$ with c being a constant. Thus starting from $n \sim 1/(cg^2)$ the successive terms in the expansion grow in magnitude and the perturbation theory looses its predictive power. This behavior, well known in QED, is mostly of a purely theoretical concern for the quantities, which receive contribution in low orders of perturbation theory, like the anomalous magnetic moment. The inability in principle to calculate such quantities with arbitrary precision is certainly somewhat disappointing, however it does not have phenomenological implications, since few first orders deliver precision sufficient for all practical purposes. The situation changes considerably for the processes, which involve $n > O(g^{-2})$ interacting particles. An application of the lowest-order perturbative estimates suggests that the cross section for such processes may be large, since the growth of $n!$ overcomes the high power of the coupling constant g^{2n}. However for these processes the lowest-order amplitudes already are beyond the applicability of the perturbation theory, which thus does not offer any credible estimate of the magnitude of the amplitudes. For the electroweak W, Z bosons this domain of essentially non-perturbative behavior starts at energies at which $O(100)$ bosons can be produced, which corresponds to conceivably accessible range of few tens or hundreds TeV. Thus the problem of calculating multi-particle amplitudes is an internal technical problem of the Standard Model, and it is undoubtedly a challenge to our theoretical understanding of the field dynamics to find working methods for calculating these amplitudes.

The problem of multiparticle processes is closely related to the intriguing subject of the baryon B and lepton L number violation by the electroweak interaction. This property of the Standard Model arising[2] from the chiral anomaly has practically no

significance in few-body processes, however it may become significant in processes involving $O(\text{const}/g^2) \sim O(100)$ weakly interacting particles[3-7]. Such processes were the ones contributing to the evolution of the baryon and lepton numbers in the early universe at high temperature[8]. Also they give rise to instability[9] of dense cold plasma of quarks and leptons. In the laboratory conditions the only possibility of observing the electroweak B and L non-conservation arises if the cross section for the $B+L$ violating processes associated with production of multiparticle states is sufficiently large. Therefore only by understanding the multiparticle electroweak dynamics at the non-perturbative level one might be able to assess the feasibility of observing the B and L violation in future accelerator experiments.

2. Baryon and lepton number violation

2.1. Anomaly in the $B+L$ current

The baryon and lepton currents in the Standard Model

$$B_\mu = \sum_q \frac{1}{3}(\bar{q}\gamma_\mu q)\,, \qquad L_\mu = \sum_l (\bar{l}\gamma_\mu l) \tag{1}$$

are conserved at the classical level, i.e. $\partial_\mu B_\mu = \partial_\mu L_\mu = 0$ as a result of the classical field equations. However already at the one-loop level, due to the fact that in regularized theory it is impossible to maintain both the gauge invariance and the conservation of these currents, the divergence of the baryon and the lepton current receives a non-zero contribution[1]:

$$\partial_\mu B_\mu = \partial_\mu L_\mu = N_f \left(\frac{g^2}{32\pi^2} W^a_{\mu\nu}\tilde{W}^a_{\mu\nu} - \frac{g'^2}{32\pi^2} V_{\mu\nu}\tilde{V}_{\mu\nu} \right)\,, \tag{2}$$

where g (g') is the SU(2) (U(1)) coupling constant, $W^a_{\mu\nu}$ ($V_{\mu\nu}$) is the corresponding gauge field strength tensor, and $\tilde{V}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}V_{\lambda\sigma}$. The factor N_f in Eq.2 is the number of quark-lepton families, for which we have all reasons to believe that in reality $N_f = 3$. In what follows, we shall consider for simplicity the limit, where the $U(1)$ interaction can be neglected, i.e. the limit of $\sin^2\theta_W = 0$.

Equation 2 can be rewritten in such a way that its right hand side takes the form of divergence of a (gauge non-invariant) topological current:

$$\partial_\mu B_\mu = \partial_\mu L_\mu = N_f\,\partial_\mu K_\mu\,, \tag{3}$$

where in the limit $g' \to 0$ the current K_μ is given by

$$K_\mu = \frac{g^2}{16\pi^2}\epsilon_{\mu\nu\lambda\sigma}\left(W^a_\nu \partial_\lambda W^a_\sigma + \frac{g}{3}\epsilon^{abc}W^a_\nu W^b_\lambda W^c_\sigma \right)\,. \tag{4}$$

[1]Here most of the points related to the chiral anomaly and to topological properties of gauge theories are only briefly mentioned. There are many good detailed reviews of these topics. One of rather detailed discussions can be found in the book 10.

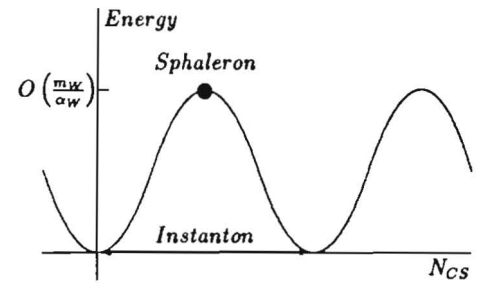

Figure 1: The dependence of minimal energy of the electroweak gauge field on its winding number in a theory without fermions.

Integration of Eq.3 over space-time leads to the final conclusion that the change in the total baryon and lepton charge is determined by the change in the Chern-Simons number $N_{CS} = \int K_0\, d^3x$:

$$\Delta B = \Delta L = N_f\,\Delta N_{CS}\,. \tag{5}$$

The latter equation reveals that in the Standard Model the baryon and the lepton numbers are not conserved in the same amount. This can be formulated that their difference is conserved, $\Delta(B-L) = 0$, while their sum is not: $\Delta(B+L) = 2N_f\,\Delta N_{CS}$. However from the definition of the topological current K one may see that in order to have $|\Delta N_{CS}| \geq 1$ a process should involve configurations of the gauge field with amplitude A proportional to $1/g$, which clearly makes such process a non-perturbative one. The typical energy of the gauge field of the required amplitude is given by $m_W A^2 \sim m_W/\alpha_W$.

As mentioned, the current K_μ is gauge non-invariant. By a non-singular gauge transformation one can shift the definition of the Chern-Simons number N_{CS} by an integer number. Therefore in purely bosonic theory the energy of a field configuration should be periodic in N_{CS} with the period equal to one. If one fixes the value of N_{CS} and minimizes the field energy with respect to other degrees of freedom of the bosonic fields, one gets a periodic behavior of the energy vs. N_{CS} as shown in Figure 1. The overlap between the wave functions of the field located in two neighboring minima, the "tunneling amplitude", is described by the instanton solution[11] to the Euclidean space field equations and the field configuration[12], corresponding to the top of the barrier between two minima, is called sphaleron[13, 14].

In a theory with fermions, like the Standard Model, one should make a clear distinction between the two ways in which N_{CS} can be changed:

- gauge transformation, which changes only N_{CS} without changing $B+L$;

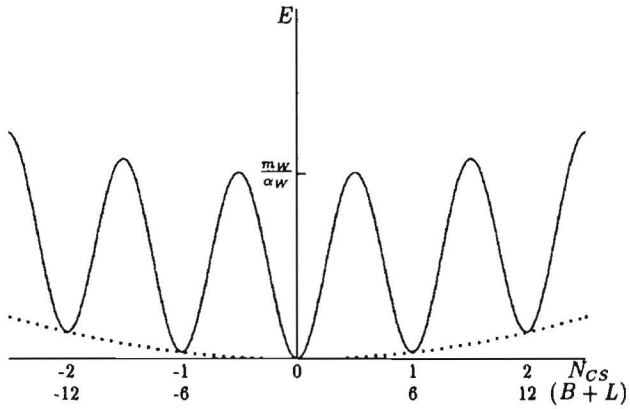

Figure 2: The dependence of minimal energy of the electroweak gauge field on its winding number in a theory with fermions (Standard Model). The envelope of the minima corresponds to the energy of produced (anti)fermions.

- dynamical variation of N_{CS}, which changes $B+L$ in such a way that $\Delta(B+L) = 2\,N_f\,\Delta N_{CS}$.

The latter variation is called dynamical, since only this can occur in the evolution of the system, since the "charge" $(B+L) - 2\,N_f\,N_{CS}$ is conserved (commutes with the Hamiltonian). The energy is periodic with respect to only the first type of variation and is not periodic under the latter. Considering the one-particle fermion states in the gauge field one may wonder what the difference is, since one can follow a path in the field space connecting two gauge copies of the gauge field with the winding number differing by an integer, then the energy levels of the fermions should be the same for both copies. The answer is that the occupation numbers for the one-particle levels are different, corresponding to different values of $B+L$. Production or absorption of the fermions changes the energy, hence the energy is not periodic in N_{CS}. The behavior under the variation of the second type is illustrated in Figure 2 for the Standard Model with three fermion families. The envelope of the energy oscillations corresponds to the energy of fermions.

2.2 $B+L$ violation at low energy

At energies much smaller than the height of the barrier ($\sim m_W/\alpha_W$), separating the local minima in Figure 2, the only possible mechanism for $B+L$ violating processes is by the quantum tunneling through the barrier. Neglecting the difference

in the energy of two neighboring local minima one can approximate the tunneling path by that in a theory without fermions, i.e. by the instanton configuration of the bosonic field in the Euclidean space-time. Thus the tunneling amplitude contains a huge exponential suppression $\exp(-S_I)$, where $S_I = 8\pi^2/g^2 = 2\pi/\alpha_W$ is the instanton Euclidean action. Therefore the probability of the process is suppressed[2] by the factor $\exp(-4\pi/\alpha_W) \sim \exp(-170)$, which makes completely impractical any discussion of phenomenological implications of the anomalous $(B+L)$ non-conservation at low energy in the Standard Model. The situation however significantly changes under extremal conditions: high density, high temperature and, arguably, at high energy in particle collisions.

2.3. Instability of dense quark-lepton matter

Due to the Pauli exclusion principle, if there is a high density of fermions, it costs a large energy: chemical potential μ, to add a fermion to the system, or, conversely, a large energy μ can be gained by removing a fermion from the system. Therefore the envelope in the Figure 2, if continued to very high values of $|B+L|$ will eventually become steeper than the slopes of the bosonic energy barrier. Clearly the condition for this behavior is that μ should be larger than $\sim m_W/\alpha_W$, which corresponds to fermion density larger than $\sim (m_W/\alpha_W)^3$. For such and larger densities there are no local maxima on the curve of energy vs. $(B+L)$. Thus the system classically rolls down to smaller values of the density of fermions. In other words, the quark-lepton matter with chemical potential μ larger than $\sim m_W/\alpha_W$ would be absolutely unstable[9].

2.4 $B+L$ violation at high temperature

In thermal equilibrium at a temperature T, the transitions between two minima of the energy at neighboring values of N_{CS} can proceed due to the thermal fluctuations rather than due to the quantum ones[8]. If the temperature is lower than the height of the barrier, the probability of a thermally activated transitions is given by the standard formula

$$\Gamma_{B+L} \equiv -\frac{1}{B+L}\frac{d(B+L)}{dt} \propto e^{-E_{Sp}/T} \quad , \tag{6}$$

where E_{Sp} is the energy at the top of the barrier. The new element in the standard model is that the sphaleron energy itself is a function of temperature: $E_{Sp} = E_{Sp}(T) = \text{const}\cdot v(T)/g$, with $v(T)$ being the temperature-dependent vacuum mean value of the Higgs field. The Higgs system however is known[15] to undergo a phase transition at a critical temperature T_c, so that the v.e.v. vanishes at $T > T_c$. Thus the exponential barrier suppression in Eq.6 is not present at such temperatures, and a different consideration[16] of the rate Γ_{B+L} should be applied.

At a temperature above T_c the gauge bosons are massless, and the energy of the barrier between local minima at different values of N_{CS} is determined by the size of the field configurations. In order to have $\Delta N_{CS} \sim 1$ at a distance scale L the

gauge field should have amplitude $W \sim (g\,L)^{-1}$. The energy of such configuration can be estimated as $E \sim (\omega^2, \mathbf{k}^2)\, W^2\, L^3 \sim W^2\, L \sim (g^2\, L)^{-1}$, where the characteristic frequency ω and momentum $\mathbf{k}$ are of order L^{-1}. To have the rate of fluctuation of the field to such configuration to be of order one, one needs the energy E to be of the order of the temperature: $E \sim T$. Thus one finds that the characteristic size of the $(B+L)$ violating fluctuations is $L \sim (\alpha_W\, T)^{-1}$. Therefore the number of $(B+L)$ violating transitions per unit time per unit volume can be estimated as[16]

$$\frac{d(B+L)}{d^3x\, dt} \sim L^{-4} \sim (\alpha_W\, T)^4 \ . \tag{7}$$

2.5. Baryon asymmetry of the universe

The rate given by Eq.7, though smaller than a typical unsuppressed thermal rate of T^4, is high enough to completely change the baryon and lepton number history of the early universe[8]. During the time, when the temperature of the universe is above the EW phase transition, two scenaria are possible:

- If an asymmetry in $B-L$ has been generated by the Grand Unification interactions, the $B-L$ is conserved by the EW interactions. The only effect of the latter interactions is that they make $B=-L$, which minimizes the free energy of fermions.
- If there is no $B-L$ asymmetry generated at the GUT scale, or if the asymmetry is small, e.g. due to an inflation period, any possible initial baryon asymmetry has been washed out by the EW interactions, and it has to be regenerated to the presently observed value at the EW phase transition.

The mechanism, which is currently deemed viable, for the baryogenesis at the EW phase transition is operative if the transition is of the first order. The first order phase transition proceeds through nucleation and subsequent expansion of bubbles filled with the phase with non-zero v.e.v. of the Higgs field. Outside the bubbles in the unbroken symmetry phase all particles are massless and the $(B+L)$ violating processes go at high rate given by Eq.7, so that any asymmetry is quickly washed out outside a bubble. Inside the bubble the $(B+L)$ violation is exponentially suppressed, as described by Eq.6. Thus if when traversing the bubble wall there are more quarks getting inside the bubble than antiquarks, the baryon asymmetry inside the bubble would be "frozen" and eventually it will be carried over to the whole volume of the universe by the expansion of the bubbles. The main dynamic ingredient in this picture is the mechanism by which the bubble wall transmits more quarks than antiquarks from outside. Any such mechanism requires the presence of a sufficiently effective CP violation in the theory. In particular, the minimal Standard Model with only one Higgs doublet and the only source of CP violation being the Kobayashi-Maskawa phase in the quark mixing matrix does not provide sufficient CP violation for creating the necessary bias in the transmission coefficient between quarks and antiquarks.

However a minimal extension, e.g. with at least two Higgs multiplets, allows enough freedom for having the necessary CP violation in the EW sector[17,18]. Whatever the eventual understanding of the mechanism for the baryon asymmetry may be, the Standard Model definitely has the necessary ingredients for baryogenesis, formulated by Sakharov[19]:

- non-conservation of the baryon number
- non-conservation of CP and C
- non-equilibrium stage in the thermal history of the universe: first order phase transition.

Therefore one may expect that a reliable quantitative understanding of the baryon asymmetry can be found with an appropriate extension of the minimal Standard Model.

3. $B+L$ violating scattering ?

3.1 Ringwald-Espinosa calculation

The previously discussed extremal conditions of high density or high temperature can not be directly studied in laboratory experiments. However the absence of the WKB suppression in the probability of $B+L$ violation under those conditions tells us that there are at least some high-energy microscopic processes in which the $B+L$ violation is not suppressed. This observation legitimizes the question whether such processes can be induced with an observable cross section in high energy collisions of two particles.

As follows from Eq.5 the minimal process, corresponding to $\Delta N_{CS}=1$, involves the change by three units of the baryon and lepton number. Thus such minimal process should involve production (or absorption) of simultaneously 3 leptons and 9 quarks (one from each electroweak doublet, counting also the quark color). Also the anomalous process can be accompanied by emission/absorption of n_W electroweak gauge bosons and n_H scalar Higgs bosons[2]. Any two of these n_W+n_H+12 particles can be counted as incoming and the rest as outgoing.

A calculation of the scattering amplitude for this generic process can be started with considering the multi-point Green function in the coordinate space

$$\begin{aligned} &G(x_1,\ldots,x_{12},y_1,\ldots,y_{n_H},w_1,\ldots,w_{n_W}) \equiv \\ &\langle 0|T\left(\psi(x_1)\cdots\psi(x_{12})\,\sigma(y_1)\cdots\sigma(y_{n_H})\,W^{a_1}_{\mu_1}(w_1)\cdots W^{a_{n_W}}_{\mu_{n_W}}(w_{n_W})\right)|0\rangle \ , \end{aligned} \tag{8}$$

where ψ, σ and W the fermion, the Higgs boson and the W field operators respectively. The Green function is an analytical function of its arguments, therefore the calculation

[2]Here we again restrict the discussion to pure SU(2) theory, i.e. to the limit $g'=0$, thus the gauge bosons are generically referred to as W bosons.

can be done in the Euclidean space using the path integral representation:

$$G(\{x\},\{y\},\{w\}) = \int \prod_i \psi(x_i) \prod_j \sigma(y_j) \prod_k W(w_k)\, e^{-S[W,\sigma,\psi]}\, \mathcal{D}W\, \mathcal{D}\sigma\, \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\ . \quad (9)$$

The classical bosonic field configuration with $\Delta N_{CS} = 1$ in the Euclidean space is given by instanton. Therefore, as a starting point, the path integral in Eq.9 is calculated in terms of the expansion around the saddle-point configuration given by instanton. In the leading saddle-point approximation the field factors in the path integrals are replaced[5,6] by their values in the instanton solution. The fields of W and σ are replaced by the classical values and those of fermions by the amplitudes of the fermionic zero modes[3]. The integral then also involves the integration over the instanton size ρ, over the gauge group measure $d\mu_g$ and over the position of the instanton z:

$$G_0 = \int e^{-S(\rho)}\, D(\rho)\, d\rho \int d\mu_g \int \prod_i \psi_i(x_i - z) \prod_j \sigma(y_j - z) \prod_k W^{a_k}_{\mu_k}(w_k - z) \quad (10)$$

with the ρ-dependent action

$$S(\rho) = \frac{8\pi^2}{g^2} + \pi^2 v^2 \rho^2\ . \quad (11)$$

The on-shell scattering amplitude is obtained from the Green function by performing the Fourier transform to the momentum space in all the variables and then applying the standard reduction formula, i.e. finding the residue of the on-shell pole in simultaneously all the external momenta squared. In doing the Fourier transform the integral over the position z of the instanton results in the factor $(2\pi)^4\, \delta^{(4)}(\sum_a p_a)$, which, as usual ensures the energy-momentum conservation, while the rest of the expression splits into product of individual Fourier transforms of each of the field functions. Since at the Euclidean infinity these functions fall off as free fields, their Fourier transforms display the on-shell poles with non-vanishing residues. thus the final structure of the scattering amplitude is[5,6]

$$A \propto \int e^{-S(\rho)}\, D(\rho)\, d\rho\, \left(\rho\, F_f(m_f^2)\right)^{12} \left(-v\, \rho^2\, F_\sigma(m_\sigma^2)\right)^{n_H} \left(\frac{\rho^2}{g}\, F_W(m_W^2)\right)^{n_W} \cdot \mathcal{K}\ , \quad (12)$$

where the factor $\mathcal{K}$ contains a group and kinematical structure and the factors $F(m^2)$ are the on-shell residues of the respective individual Fourier transforms. The most important feature of this structure is that the form-factors F depend only on the individual momenta squared and there is no non-trivial dependence on the correlation, like $(p_i \cdot p_j)$, of the external momenta. In other words the multi-particle instanton-induced amplitude in this approximation is equivalent on the mass shell to a point-like

[3]For a tutorial on calculations in the instanton background see e.g. the review 20.

amplitude. Moreover, the ρ integration in Eq.12 introduces a factorial dependence on the boson multiplicity n_W, n_H. As a result, the amplitude in this approximation, taken literally, gives rise to a rapidly growing with energy cross-section[7], associated with production of high-multiplicity states. This growth not only compensates the WKB suppression factor $e^{-4\pi/\alpha_W}$, but also violates unitarity bounds at energies of the order of $E_{Sp} \sim m_W/\alpha_W$, where it is dominated by the final states with multiplicity $n \sim 1/\alpha_W$.

3.2 Beyond the Ringwald-Espinosa approximation

The described leading approximation at the instanton saddle-point configuration in fact is very poorly applicable. The subsequent terms arise, when one expands the field factors in Eq.9 into the classical and quantum parts, e.g. $\sigma(y) = \sigma_c(y) + \sigma_q(y)$, and keeps the quantum parts. In the discussed approximation only the classical part is retained. The next step is to replace for any pair of the field factors their classical fields by the correlator, say $\langle \sigma_q(y_i)\sigma_q(y_j)\rangle$, in the instanton field. The higher terms in this expansion correspond to taking into account higher correlators of the quantum parts in the instanton background. It turns out that at $E \sim E_{Sp}$ all terms of this expansion are equally important, so that any truncation of the series is not justified.

The problem gets a more organized formulation if one uses the optical theorem and considers the total cross section due to the instanton-induced processes in terms of the imaginary part of the instanton-antiinstanton contribution to the forward scattering amplitude of two particles[21,22]. The contribution of the instanton-antiinstanton pair to the forward scattering amplitude at energy E is also found in the saddle-point approximation by extremizing the expression for this contribution:

$$A_{forw} \propto \exp\left(-S_{I\bar{I}}(T) + ET\right)\ . \quad (13)$$

Here T is the separation between the instanton and the antiinstanton and $S_{I\bar{I}}(T)$ is the Euclidean action for the configuration. In general the $I\bar{I}$ configuration is not well defined, however if T is much larger than the size of either I or $\bar{I}$ the instantons only weakly deform each other and the problem is tractable. Moreover those are the effects of the mutual deformation, which correspond to the higher-order correlators in the direct calculation of the instanton-induced amplitudes. Thus the method is to fix the separation T and to minimize the action with respect to all other variables, leaving in Eq.13 only the T dependent minimized value of the action (the so-called "valley method"[4]).

One of the application of this method is that it provides a straightforward derivation of a scaling law for the discussed total cross section. Indeed, introduce dimensionless coordinates $\tilde{x}$ and field variables $\tilde{\psi}$, $\tilde{\sigma}$ and $\tilde{W}$ as

$$\tilde{x} = g\, v\, x\ ,\quad \psi(x) \to g^{\frac{1}{2}} v^{\frac{3}{2}} \tilde{\psi}(\tilde{x})\ ,\quad \sigma(x) \to v\, \tilde{\sigma}(\tilde{x})\ ,\quad W(x) \to v\, \tilde{W}(\tilde{x})\ . \quad (14)$$

[4]For a review of the valley method see e.g. 23.

Then the action can in general be written as

$$S = \frac{1}{g^2} \int \tilde{L} \left[\tilde{W}(\tilde{x}), \tilde{\sigma}(\tilde{x}), \tilde{\psi}(\tilde{x}); \frac{\lambda}{g^2} \right] d^4\tilde{x} \ , \tag{15}$$

where λ is the Higgs boson self-coupling and $\tilde{L}$ is the dimensionless Lagrangian function. In the dimensionless variables it is obvious that the instanton-antiinstanton action has the following scaling form

$$S_{I\bar{I}}(T) = \frac{1}{g^2} \Phi \left(g\, v\, T; \frac{\lambda}{g^2} \right) \tag{16}$$

with a dimensionless function Φ. The extremum T^* in T of the expression in the exponent in Eq.13 is then found from the equation $\frac{v}{g} \Phi'(g\, v\, T^*; \frac{\lambda}{g^2}) = E$, where Φ' is the derivative of Φ with respect to its first argument. This implies the following scaling law for T^*:

$$T^* = \frac{1}{g\, v} f \left(\frac{E\, g}{v}; \frac{\lambda}{g^2} \right) \tag{17}$$

with f being also a dimensionless function. Thus the energy dependence of the extremal value in Eq.13 is given by the following scaling behavior[24,25]

$$\sigma_{tot} \sim \exp \left[\frac{4\pi}{\alpha_W} F \left(\frac{E}{E_{Sp}}; \frac{\lambda}{g^2} \right) \right] , \tag{18}$$

where, to remind, $E_{Sp} \sim m_W/\alpha_W \sim v/g$. The function F is referred to in the literature as the 'holy grail' function. Indeed, the whole issue of the feasibility of observing $B + L$ violation in a high energy scattering amounts to the problem of whether the function F, starting at the value -1 at $E = 0$, ever approaches zero at $E/E_{Sp} = O(1)$.

Unfortunately the valley method in principle can not provide a clue to the behavior of the holy grail function at $E \sim E_{Sp}$: the separation T^* between the instanton and the antiinstanton becomes comparable to their size, so that the notion of an $I\bar{I}$ configuration becomes ambiguous. Thus far this method has resulted in calculation of few first terms in the expansion in E/E_{Sp} (more precisely in powers of $(E/E_{Sp})^{2/3}$)[26-30]:

$$F = -1 + \frac{9}{8}\epsilon^{4/3} - \frac{9}{16}\epsilon^2 - \frac{3}{32}\left(4 - 3\frac{m_H^2}{m_W^2}\right)\epsilon^{8/3} \ln\epsilon + \dots \ , \tag{19}$$

where $\epsilon = E/E_0$ with $E_0 = \sqrt{6}\,\pi\, m_W/\alpha_W \approx 18\, TeV$. At the level of these terms there is no dependence on the initial state in the scattering. This dependence however arises in the next $\epsilon^{10/3}$ term[31,32].

It has been advocated[33-35] that the holy grail function is in fact bounded from above by the value -1/2 due to the so-called "premature unitarization". The argument

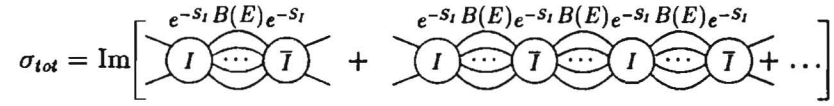

Figure 3: Unitarization in the s-channel according to the model of premature unitarization.

is based on considering the interplay in the s-channel unitarity of the processes *few* $\to$ *many* and *many* $\to$ *many*. The former processes are argued to be still weak (exponentially suppressed) when the processes *many* $\to$ *many* are at the unitarity limit, which effectively shuts off the further growth of the *few* $\to$ *many* cross section.

A somewhat simplified picture of this behavior is shown in Figure 3. According to the model of "premature unitarization"[35] the total amplitude is given by summation over instanton - antiinstanton chains iterated in the s-channel, i.e. where all the total energy flows through the additional (anti)instantons. Each additional $I - \bar{I}$ pair brings in the factor $e^{-2S_I} B(E)$, where the "bond function" $B(E)$ is the multi-boson enhancement of the one-instanton-induced cross section observed in 5-7. The summation over the $I - \bar{I}$ chains (Figure 3) gives

$$\sigma_{tot} \sim e^{-S_I} \, \mathrm{Im} \left[\frac{B(E) e^{-S_I}}{1 + \eta \left(B(E) e^{-S_I} \right)^2} \right] , \tag{20}$$

where $\eta = O(1)$ is a rescattering factor. If given by Eq.20, the cross section reaches its maximum when $B(E)e^{-S_I} = O(1)$ and its value at the maximum is of order e^{-S_I}, which corresponds to the upper bound of $-1/2$ for the function F. The presented reasoning is however oversimplified: it assumes that all the (anti)instantons in the chains have same fixed size. Relaxing this assumption leads[35] to an upper bound for F, which is generally different from $-1/2$.

At still higher energies the formula in Eq.20 gives a falling cross section. However this regime is unphysical: initial particle can shake off energy by emitting one or few hard bosons, so that the energy in the collision gets back to the one corresponding to the maximum. (Emission of hard bosons suppresses the cross section by a few powers of the coupling constant, while the gain in the non-perturbative amplitude is exponential.) If indeed the "holy grail" function has a minimum at some energy, this would imply that above that energy the process can not be described by semiclassical methods, since emission of hard quanta becomes essential.

It turns out however, that the "premature unitarization" and thus the simple picture of the s-channel iteration of instanton-antiinstanton correlations is not mandatory and apparently depends on specifics of the theory. The known examples of simplified models, where the "holy grail" function is indeed bounded from above by -1/2 are the Quantum Mechanical problem with a double well potential[36,37] and the soft contribution to the scattering through a bounce[38,39] in a (1+1) dimensional

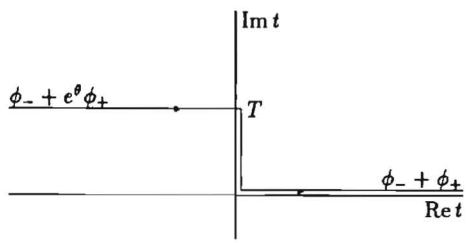

Figure 4: Contour in the complex time plane and boundary conditions for a classical solution, describing the scattering of a semiclassical initial state into multiparticle final states.

model of one real field with metastable vacuum[40]. (It has been pointed out[41,42] that in the latter model there is also a hard contribution to the bounce-induced scattering, for which the "holy grail" function goes to zero at the analog of the sphaleron energy.) Another example, where the "holy grail" function is bounded by a value, larger than -1/2, namely -0.160, is the problem of catalysis of false vacuum decay in (3+1) dimensions by collision of two (or few) particles[43]. In this problem the semiclassical probability reaches maximum at the top of the energy barrier.

3.3. Rubakov - Tinyakov approach

The main difficulty in developing a semiclassical approach to the *few* $\to$ *many* scattering is the presence of hard quanta in the initial state, which state is thus not a semiclassical one. It has been suggested[44,45] to circumvent this difficulty by considering a scattering, where a finite small number of particles in the initial state is replaced by $n_{i(nitial)} = \nu/g^2$, where g is the coupling constant in the theory and ν is a parameter. For a finite ν the initial state of this kind can be treated semiclassically, and in the end the limit of the probability at $\nu \to 0$, or $\nu \to \text{const} \cdot g^2$ is to be considered in order to relate to the process *few* $\to$ *many*. Within such setting the "holy grail" function depends on ν: $F(\epsilon, \nu)$ and it is conjectured that its limit at $\nu \to 0$ is smooth, which conjecture is supported by high-order perturbative calculations around the instanton[46]. The central point of this approach is that the function $F(\epsilon, \nu)$ is determined from a solution to a well-defined boundary value problem[47] for classical field equations, although in essentially complex time.

The classical solution that describes the path of largest probability in a model with one real field ϕ is evolving along the contour in the complex time plane shown in Figure 4. At $\mathrm{Re}\, t \to +\infty$ the solution is required to be real, thus its momentum components should be of the form $\phi(\mathbf{k}) = b_{\mathbf{k}}\, e^{-i\omega_k t} + b^*_{-\mathbf{k}}\, e^{i\omega_k t}$, while at $\mathrm{Re}\, t \to -\infty$ the positive

frequency part is rescaled by the parameter e^θ: $\phi(\mathbf{k}) = f_{\mathbf{k}}\, e^{-i\omega_k t} + e^\theta f^*_{-\mathbf{k}}\, e^{i\omega_k t}$. The parameter θ in the boundary condition and the parameter T of the contour (cf. Figure 4) are Legendre-conjugate of respectively the multiplicity n_i and the total energy E of initial particles. Namely, if iS is the classical action on the whole contour (thus S is defined in the way that it is real in the Euclidean space), one finds[47]

$$n_i = 2\,\frac{\partial S}{\partial \theta}\,, \qquad E = 2\,\frac{\partial S}{\partial T}\,. \tag{21}$$

Furthermore the "holy grail" function, entering the WKB estimate of the total cross section as $\sigma^{tot} \sim \exp\left(-g^{-2}\, F(\epsilon, \nu)\right)$ is given by the Legendre transform of the action:

$$\frac{1}{g^2}\, F(\epsilon, \nu) = 2\,S - ET - n_i\theta\ . \tag{22}$$

Quite naturally, the formulated classical boundary value problem is not easily solvable, and a sufficiently good approximation to the solution is known only in a few models[48,49]. In particular the model, considered in 49, describes one scalar field in (1+1) dimensions with the potential

$$V(\phi) = \frac{m^2}{2}\phi^2 - \frac{m^2 v^2}{2} \exp\left[2\lambda\left(\frac{\phi}{v} - 1\right)\right]\,, \tag{23}$$

where v and λ are dimensionless constants, which both are assumed to be large. The parameter $1/v$ is the small coupling constant of the perturbation theory in the vacuum $\phi = 0$. The negative sign of the interaction term implies that the energy is unbounded from below at large ϕ, thus the vacuum $\phi = 0$ is metastable, and is separated from the decreasing part of the potential by a barrier located at $\phi \approx v$, provided that $\lambda \gg 1$. Beyond the maximum the potential rapidly goes down, so that the potential essentially is a quadratic well with a "cliff"[49]. The metastability of the perturbative vacuum at $\phi = 0$ does not show up in calculations of the scattering amplitudes to any finite order of the perturbation theory, and it only arises through a non-perturbative effect: unitary "shadow" from the false vacuum decay[40,50], which makes this contribution analogous to instanton-induced scattering amplitudes in a Yang-Mills theory. The analog of the sphaleron energy is the height of the barrier separating two phases: $E_{Sp} = \text{const} \cdot mv^2$.

At large λ the potential in Eq.23 contains a sharp matching of the quadratic part (free field) and a steep exponential "cliff", which enables[49] one to solve the boundary value problem in the leading order in $1/\lambda$ and also to clarify the contribution of multi-instanton (multi-bounce) configurations. It has been found[49] that the multi-instanton configurations in this model are still not important when the one-instanton contribution becomes large. As a result the "holy grail" function, as shown in Figure 5, reaches zero at finite energy, which energy increases when the semiclassical parameter of the initial state multiplicity $\nu = n_i\, v^2$ decreases. In Figure 5 is also

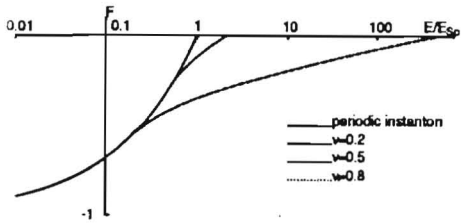

Figure 5: The "holy grail" function in the (1+1) dimensional model with exponential interaction.

shown the behavior corresponding to the periodic instanton, which maximizes over n_i the rate of tunneling through the barrier in the processes $n_i \to n_f$ at given energy E[25]

4. Multi-boson processes in topologically trivial sector

The growth with energy of the rate of the instanton-induced processes is associated with production of multiboson states until at high multiplicity $n \sim 1/g^2$ the final state becomes not tractable perturbatively. The same dynamic complexity is inherent[51,52] in the processes without the $B+L$ violation, arising in the topologically trivial sector of the theory. Therefore it makes perfect sense to study the riddle of multiparticle states for such processes, which are of interest on their own and for which some simplification may be expected due to absence of the non-trivial topological structure. For the most part we will use a further simplification and consider a model of one scalar field ϕ with a $\lambda\,\phi^4$ interaction, i.e. with the potential

$$V(\phi) = \frac{m^2}{2}\,\phi^2 + \frac{\lambda}{4}\,\phi^4\,. \tag{24}$$

If m^2 is positive the field has one vacuum state at $\langle 0|\phi|0\rangle = 0$ and the symmetry under sign reflection: $\phi \to -\phi$ is unbroken, while at negative m^2 there are two degenerate vacua $\langle 0|\phi|0\rangle = \pm v$ with $v = |m|/\sqrt{\lambda}$, which situation describes the spontaneous symmetry breaking (SSB).

4.1. Solvable case: Multiboson amplitude at zero energy and momentum

The simplest problem concerning multiboson amplitudes is, perhaps, that of calculating connected n-boson off-shell scattering amplitudes A_n, in which all the external particles have zero energy and momentum[53,54]. The amplitude A_n can be written in terms of the connected part of the Euclidean-space correlator:

$$\frac{\int \left(\int \sigma(x)\,d^dx\right)^n \exp(-S[\phi])\,\mathcal{D}\phi}{\int \exp(-S[\phi])\,\mathcal{D}\phi}\,, \tag{25}$$

where $\sigma(x)$ is the deviation of the field ϕ from its vacuum mean value $\sigma(x) = \phi(x) - \langle 0|\phi|0\rangle$, and the integral $\int \sigma(x) d^dx$ is understood as the $p \to 0$ limit of the Fourier transform $\int \sigma(x) e^{ipx} d^dx$. Furthermore the connected part of the correlator in Eq.25 is conveniently given by the n-th logarithmic derivative of the generating functional $Z(j) = \int \exp(-S[\phi] + \int j\sigma(x)\,d^dx)\,\mathcal{D}\phi$ with a constant source j:

$$A_n = \left(\frac{d}{dj}\right)^n \ln Z(j)\Big|_{j=0}\,. \tag{26}$$

Introduction of a constant source is equivalent to replacing the original potential $V(\phi)$ by $V(\phi) - j\,\sigma$. Furthermore, for a constant source $\ln Z(j)$ is related to the energy density of the vacuum $E(j)$ in the presence of j: $\ln Z(j) = -VT\,E(j)$, where VT is the normalization space-time volume. Thus according to Eq.26 the asymptotic at large n behavior of the amplitudes A_n is related to the position j_c of the nearest to $j = 0$ singularity of $E(j)$ in the complex j plane: $A_n \sim n! j_c^{-n}$. At the classical level the position of the singularity is determined by the value of j, at which two solutions of the equilibrium equation $dV/d\phi = j$ coincide. For the potential in Eq.24 this happens at $j_c = \pm i\sqrt{4/27\lambda}\,m^3$, which determines the asymptotic behavior[53,54] of the tree-level amplitudes A_n:

$$|A_n^{\rm tree}| \sim n!\left(\frac{27}{4}\,\frac{\lambda}{|m|^6}\right)^{n/2}\,. \tag{27}$$

In a theory with unbroken symmetry the quantum loops modify $E(j)$ according to the Coleman-Weinberg potential[55] thus shifting and modifying the singularity in the j plane. However these corrections neither eliminate the singularity nor bring it to $j = 0$. The shift of the position can be absorbed in normalization of λ and m, while the modification of the type of the singularity only affects sub-leading in n factors, so that the leading behavior in Eq.27 is not modified by quantum corrections in a theory with unbroken symmetry.

The situation with quantum effects in a theory with SSB is drastically different: non-perturbatively the point $j = 0$ is in fact a branch point of the vacuum energy $E(j)$ for either of the vacua. Indeed, if, for definiteness, one choses to consider the amplitudes A_n in the 'left' vacuum: $\langle 0|\phi|0\rangle = -v$ with $v = |m|/\sqrt{\lambda}$, and follows the dependence of its energy on j, one finds that this state is stable for real $j < 0$ and is metastable at arbitrarily small positive j. Thus at $j > 0$ the energy $E(j)$ acquires an imaginary part given by the decay rate of the metastable vacuum. In this situation the Taylor expansion of $E(j)$ is asymptotic and the coefficients are determined by the decay rate in the presence of an infinitesimal positive source term. In this situation the calculation[56] of the false vacuum decay rate in the thin wall approximation is applicable exactly. Thus one can readily find the exact non-perturbative asymptotic behavior of the amplitudes A_n at large n in a theory in d space-time dimensions[54]:

$$A_n \sim \left(\frac{n\,d}{d-1}\right)!\left(\frac{c_d\,\lambda}{|m|^6}\right)^{\frac{n}{2}}\left(\frac{\lambda}{|m|^{4-d}}\right)^{\frac{n}{d-1}} \tag{28}$$

with

$$c_d = \left[\frac{3^d}{2^{2d-1}} \frac{\Gamma(d/2)}{d^{d-1}\,\pi^{d/2}} \right]^{\frac{2}{d-1}} .$$

The factorial behavior in Eq.28, if interpreted in terms of loop graphs in perturbation theory, corresponds to contribution of graphs with $n/(d-1)$ loops.

The considered off-shell amplitudes A_n are not physical. However one can draw from the described exercise at least two, possibly important, theoretical conclusions about multiboson amplitudes:

- the $n!$ behavior suggested by the tree-level analysis is not necessarily eliminated and may even be enhanced in the exact result, and
- the large n behavior of multiboson amplitudes does not have to be universal and may in fact be very sensitive to details of the theory.

4.2. Tree-level amplitudes for production of on-shell bosons at threshold

More physical, than the previously discussed off-shell amplitudes, are the amplitudes of processes, where n on-shell bosons are produced by a highly virtual field ϕ ($1 \to n$ process): $a_n = \langle n|\phi(0)|0\rangle$. (These e.g. can be related to the reaction $e^+e^- \to n\,H$, in which the Higgs-electron coupling is considered to be small, thus it is not iterated. Then the process is considered as going through one virtual Higgs particle: $e^+e^- \to H^* \to n\,H$.) As will be explained, it turns out that one can explicitly find the sum of all tree graphs and all one-loop graphs for these amplitudes at any n, provided that the final bosons are exactly at rest in the c.m. system. Also the summation of two- and higher- loop graphs is in principle possible for this kinematical arrangement, however a calculation with a finite number of loops is inevitably plagued by the breakdown of the perturbation theory at large n. Thus far three methods have been used in calculation of the threshold amplitudes of the $1 \to n$ processes: the Landau WKB method, the recursion equations, and the functional technique.

Landau WKB method[57,58] is used in Quantum Mechanics for calculating transition matrix elements between strongly different levels. (For a field theory derivation of this technique see 59.) In the tree graphs for the threshold $1 \to n$ amplitudes all the external and internal lines carry no spatial momentum. Thus the problem is reduced to dynamics of only one mode of the field with spatial momentum $\mathbf{p} = 0$, i.e. to a Quantum Mechanical problem. This approach yields the result[60] for the sum of the tree graphs at the threshold with accuracy up to terms $O(1/n^2)$ at large n. (Application of the Landau WKB technique in the problem of multiboson amplitudes is also discussed in 36, 37, 61, 62.)

Recursion equations[54] for the amplitudes $a(n)$ arise from inspecting the construction of Feynman graphs. For the simplest case of tree graphs in $\lambda\phi^4$ theory without spontaneous symmetry breaking there is only a quartic vertex, and the graphic form

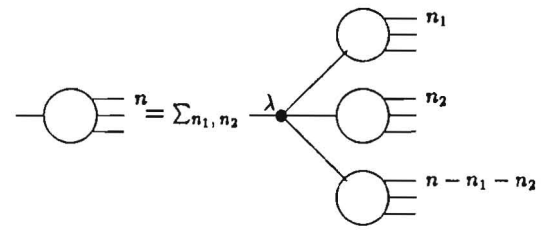

Figure 6: Graphic representation of recursion equations for tree amplitudes in $\lambda\,\phi^4$ theory without spontaneous symmetry breaking. The circles correspond to the sum of all tree graphs with the specified number of external particles.

of the recursion equations is shown in Figure 6 and their algebraic form is

$$(n^2-1)\,\frac{a(n)}{n!} = \lambda \sum_{odd\ n_1,n_2} \frac{a(n_1)}{n_1!}\,\frac{a(n_2)}{n_2!}\,\frac{a(n-n_1-n_2)}{(n-n_1-n_2)!} , \tag{29}$$

where the sum runs over odd n_1 and n_2 as well as n is odd, since due to the unbroken sign reflection symmetry the parity of the number of particles is conserved. Also the mass m in Eq.29 is set to one, since it can be restored in the final result from dimensional counting. The solution to Eq.29 reads as[54]

$$a(n) = n!\,(\lambda/8m^2)^{\frac{n-1}{2}} . \tag{30}$$

The solution in Eq.30 to the recursion equations 29 can be found by applying the regular method of generating functions[63]. The generating function $f(z)$ of an auxiliary variable z is related to the amplitudes $a(n)$ as

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}\,a(n) , \tag{31}$$

so that $a(n) = (d/dz)^n\,f(z)|_{z=0}$. Eq.29 is then equivalent to the n-th term of the Taylor expansion in z of the differential equation

$$z^2\,\frac{d^2}{dz^2}\,f(z) + z\,\frac{d}{dz}\,f(z) - f(z) = \lambda\,f(z)^3 . \tag{32}$$

The Cauchy data for this equation are determined by the relations $a(0) \equiv \langle 0|\phi(0)|0\rangle = 0$ and $a(1)\langle 1|\phi(0)|0\rangle = 1$ (one-particle state normalization), which in terms of the generating function imply $f(0) = 0$ and $f'(0) = 1$. The solution to Eq.32 satisfying these initial conditions reads as

$$f(z) = \frac{z}{1-(\lambda/8)\,z^2} , \tag{33}$$

which generates the expression for the amplitudes in Eq.30. (Notice that Taylor expansion of $f(z)$ contains only odd powers of z, as it should.)

For the theory with SSB the recursion equations are modified by the presence of cubic vertices. The result for the generating function in this case is

$$f(z) = -v\,\frac{1+\frac{z}{2v}}{1-\frac{z}{2v}}\ , \tag{34}$$

which generates the tree-level amplitudes in the theory with SSB as[63]

$$a(n) = -n!\,(2\,v)^{1-n} \tag{35}$$

The recursion method can be extended to other theories[64] as well as to loop graphs[65,66] and to an analysis of higher loops[67]. However a more convenient method for further analysis is the one suggested by Brown[68] and is based on a functional technique. Before proceeding to discussing this method and its further applications we will briefly discuss the estimates of the tree amplitudes above the threshold and thus of the total probability of the processes $1 \to n$ at a high energy E.

4.3. Lower bound on tree graphs above the threshold

The tree graphs for the processes $1 \to n$ in a $\lambda\phi^4$ theory all have the same sign[51,52]. The decrease of the amplitude above the threshold is thus determined by the increasing virtuality of the propagators in those graphs, which depends on the kinematics of the final state. One can thus find a lower bound on the tree amplitudes above the threshold in a restricted part of the final-particle phase space[69,70], which gives a lower bound on the total probability of the process. In particular, if the kinematical restriction is chosen[69] as the condition that the c.m. energy of each individual particle in the final state does not exceed ω, then in this region of the phase space the virtuality of each propagator can be bounded in terms of ω. Indeed, for a virtual line, which gives rise to a subtree terminating in k final particles all being at rest the denominator of the propagator is completely fixed in terms of the physical mass M of the bosons: $s_k - M^2 = M^2(k^2-1)$. Above the threshold, when the final particles are allowed to have non-vanishing momenta, the invariant mass squared s_k of the same k particles, is bounded from above as $s_k < k^2\omega^2$, if each particle has energy less than ω. Taking into account that the minimal number k_{min} of particles originating from a virtual line is equal to 3 in the theory without SSB and is equal to 2 in the theory with SSB one readily finds the lower bounds for the propagator corresponding to the considered virtual line:

$$\frac{1}{s_k^2 - M^2} > \frac{1}{k^2\omega^2 - M^2} > \frac{1}{c\,\omega^2\,(k^2-1)}\ , \tag{36}$$

where $c = k_{min}^2/(k_{min}^2-1)$: $c = \frac{9}{8}$ for the case of no SSB (in which case $M = m$) and $c = \frac{4}{3}$ for the theory with SSB (where $M = \sqrt{2}|m|$). One sees, that the latter

expression in Eq.36 is exactly the one for the corresponding graph at the threshold with M^2 being replaced by $c\omega^2$. Thus one finds the lower bound for the amplitude above the threshold in the restricted part of the phase space:

$$A(1 \to n) > a(n)|_{M^2 \to c\,\omega^2}\ . \tag{37}$$

The cut off energy ω should be optimized for each value of the total energy E and multiplicity n in order to find the largest lower bound on the total probability

$$\sigma_n = \int |A(1 \to n)|^2\, d\tau_n \tag{38}$$

given by the integral over the ω-restricted part of the phase space τ_n. As a result the lower bound on σ_n is found[69] in the scaling form

$$\sigma_n > \exp\left[\frac{4\pi^2 c}{\lambda} f(\epsilon,\nu)\right]\ , \tag{39}$$

where $\nu = n\,M/E$ is the ratio of the multiplicity n to its maximal possible value E/M, $\epsilon = E/E_0$ with E_0 being an analog of the 'sphaleron' energy: $E_0 = 4\pi^2\,r\,M/\lambda$, and the constant c in these formulas is $r = 9$ ($r = 8/3$) for a theory without (with) SSB. The calculated[69] behavior of the function $f(\epsilon,\nu)$ is shown in Figure 7, which thus illustrates and quantifies the interplay between the $n!$ and the power of small coupling constant discussed in the Introduction. The function $f(\epsilon,\nu)$ displays a normal perturbative maximum at zero multiplicity. However, as energy grows, and production of high multiplicity states becomes unsuppressed kinematically, this function develops a second maximum, which at larger energies eventually crosses zero with an apparent violation of unitarity.

It is interesting to notice that the kinematical suppression of multiparticle final states is quite essential and shifts the energy, at which the tree graphs violate unitarity significantly higher than one would guess from a simple estimate $E_{crit} \approx 4\pi\,M/\lambda$. If applied to a multi-Higgs production in the Standard Model the lower bound in Eq.14 breaks unitarity at $E_{crit} \approx 15.5\,(32\pi^2 M_H/\lambda) \approx 1000\,\mathrm{GeV}(200\mathrm{GeV}/M_H)$.

It should be also mentioned that the scaling behavior (Eq.39) at a given large multiplicity n also holds for the actual cross section[71-73]. The function $f(\epsilon,\nu)$ can thus be called differential in n "holy grail" function. Most recently a better lower bound for the tree-level total cross section was found[74] using the newly developed method of singular classical solutions[73].

4.4 Generating field technique.

A more convenient and more conceptually transparent technique for dealing with tree level threshold multiboson amplitudes was suggested by Brown[68] and was later extended to calculation of one-loop[75,76] and higher quantum effects[77,78] in these amplitudes. The technique is based on the standard reduction formula representation

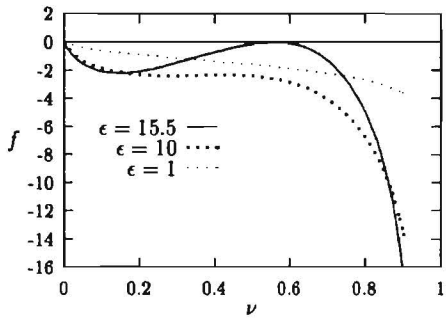

Figure 7: The function $f(\epsilon,\nu)$ vs. ν at several characteristic values of energy: low energy, no secondary maximum ($\epsilon = 1$), the secondary maximum just developed ($\epsilon = 10$), the secondary maximum becomes global and is just above the unitarity limit ($\epsilon = 15.5$).

of the amplitude through the response of the system to an external source $\rho(x)$, which enters the term $\rho\phi$ added to the Lagrangian.

$$\langle n|\phi(x)|0\rangle = \left[\prod_{a=1}^{n} \lim_{p_a^2\to m^2} \int d^4x_a\, e^{ip_a x_a}(m^2-p_a^2)\,\frac{\delta}{\delta\rho(x_a)}\right] \langle 0_{out}|\phi(x)|0_{in}\rangle^{\rho}|_{\rho=0} \quad , \tag{40}$$

the tree-level amplitude being generated by the response in the classical approximation, i.e. by the classical solution $\phi_0(x)$ of the field equations in the presence of the source.

For all the spatial momenta of the final particles equal to zero it is sufficient to consider the response to a spatially uniform time-dependent source $\rho(t) = \rho_0(\omega)\, e^{i\omega t}$ and take the on-mass-shell limit in Eq.40 by tending ω to m. The spatial integrals in Eq.40 then give the usual factors with the normalization spatial volume, which as usual is set to one, while the time dependence on one common frequency ω implies that the propagator factors and the functional derivatives enter in the combination

$$(m^2-p_a^2)\frac{\delta}{\delta\rho(x_a)} \to (m^2-\omega^2)\frac{\delta}{\delta\rho(t)} = \frac{\delta}{\delta z(t)} \quad , \tag{41}$$

where

$$z(t) = \frac{\rho_0(\omega)\, e^{i\omega t}}{m^2 - i\epsilon - \omega^2} \tag{42}$$

coincides with the response of the field to the external source in the limit of absence of the interaction, i.e. of $\lambda = 0$. For a finite amplitude ρ_0 of the source the response

$z(t)$ is singular in the limit $\omega \to m$. The crucial observation by Brown[68] is that, since according to Eq.41 we need the dependence of the response of the interacting field ϕ only in terms of $z(t)$, one can take the limit $\rho_0(\omega) \to 0$ simultaneously with $\omega \to m$ in such a way that $z(t)$ is finite: $z(t) \to z_0 e^{imt}$.

Furthermore, to find the classical solution $\phi_0(x)$ in this limit one does not have to go through this limiting procedure, but rather consider directly the on-shell limit with vanishing source. The field equation with zero source in $\lambda\phi^4$ theory without SSB is

$$\partial^2\phi + m^2\phi + \lambda\phi^3 = 0 \quad . \tag{43}$$

For the purpose of calculating the matrix element in Eq.40 at the threshold one looks for a solution of this equation which depends only on time and contains only the positive frequency part with all harmonics being multiples of e^{imt}, which condition is equivalent to requiring that $\phi(t) \to 0$ as $\mathrm{Im}\, t \to +\infty$. This condition implies that the energy integral is vanishing $\frac{1}{2}\dot{\phi}^2 + V(\phi) = 0$, since energy is conserved over all complex plane of time. This reduces the problem to a trivial integration and the solution satisfying these conditions reads as[68]

$$\phi_0(t) = \frac{z(t)}{1-(\lambda/8m^2)z(t)^2} \tag{44}$$

According to Eq.41 and Eq.40 the n-th derivative of this solution with respect to z gives the matrix element $\langle n|\phi(0)|0\rangle$ at the threshold in the tree approximation:

$$\langle 2k+1|\phi(0)|0\rangle_0 = \left(\frac{\partial}{\partial z}\right)^{2k+1} \phi_0\Big|_{z=0} \quad , \tag{45}$$

which reproduces the result in Eq.30.

For the case of theory with SSB the solution reads as

$$\phi_0(t) = -v\frac{1+z/2v}{1-z/2v} \quad , \tag{46}$$

which reproduces the tree amplitudes in Eq.35. In this case $z(t) = e^{iMt}$, where $M = \sqrt{2}\,|m|$ is the mass of physical scalar boson. Clearly, in both cases the solutions for the generating field in Eq.44 and Eq.46 coincide with the expressions for the generating functions in Eq.33 and Eq.34 for the solutions to the recursion equations.

5. Quantum effects in multi-boson production

5.1. One-loop corrections

In order to advance the calculation within the generating field technique to the quantum loop level one has to replace the classical field ϕ_0 in calculating the amplitudes from Eq.40 and Eq.41 by the mean value of the full field

$$\phi(x) = \phi_0(x) + \phi_q(x) \quad , \tag{47}$$

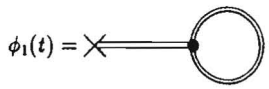

Figure 8: The tadpole graph in the background field ϕ_0 for calculating one-loop correction to ϕ_0. The double lines and the heavy dot represent respectively the Green function and the vertex in the background field.

where $\phi_q(x)$ is the quantum part of the field. The mean value is subject to the condition that it expands in positive powers of $z(t)$. If one writes this mean value as $\langle 0|\phi(x)|0\rangle = \phi_0(x) + \phi_1(x)$ with $\phi_1(x)$ being the first quantum correction, then the expansion of Eq.43 to the one-loop order results in the following non-homogeneous equation for ϕ_1:

$$\left(\partial^2 + m^2 + 3\lambda\,\phi_0(x)^2\right)\,\phi_1(x) + 3\lambda\,\phi_0(x)\,\langle\phi_q(x)\,\phi_q(x)\rangle = 0\ , \tag{48}$$

where $\langle\phi_q(x)\phi_q(x)\rangle$ is the limit of the Green function in the classical background field ϕ_0

$$G(x_1, x_2) = \langle T(\phi_q(x_1)\phi_q(x_2))\rangle \tag{49}$$

when its arguments are at the same point x. Graphically the calculation of the one-loop correction ϕ_1 can be represented by the tadpole graph of Figure 8.

The Green function $G(x;\,x')$ from Eq.49 satisfies the equation

$$\left(\partial^2 + m^2 + 3\lambda\,\phi_0(t)^2\right)G(x;\,x') = -i\,\delta(x - x')\ , \tag{50}$$

in which the differential operator in the Minkowski time contains explicitly complex field ϕ_0 (cf. Eq.44 or Eq.46). A straightforward rotation to the Euclidean time, $it \to \tau$, is problematic, since the background field then develops a pole at a real τ. The acceptable solution is achieved by simultaneously rotating and shifting the time axis in Eq.50 in such a way that $-\lambda\,z(t)^2/8m^2 \to \exp(2m\tau)$ for the theory without SSB, and $-z(t)/2v \to \exp(M\tau)$ for the theory with SSB. In terms of thus defined τ the background field has the form $\phi_0(\tau) = i\sqrt{2/\lambda}\,m/\cosh(m\tau)$ (no SSB) and $\phi_0(\tau) = v\,\tanh(M\tau/2)$ (with SSB). In both cases the term $\phi_0(t)^2$ in Ref.50 is real and non-singular. After applying the standard decomposition of the Green's function over the conserved in the background $\phi_0(t)$ spatial momentum $\mathbf{k}$:

$$G(\tau,\,\mathbf{x};\,\tau',\,\mathbf{x}') = \int G_\omega(\tau,\,\tau')\,e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\,\frac{d^3k}{(2\pi)^3}\ , \tag{51}$$

one arrives for the case of no SSB at the well-known in Quantum Mechanics equation

$$\left(-\frac{d^2}{d\tau^2} + \omega^2 - \frac{6}{(\cosh\tau)^2}\right)G(\tau,\,\tau') = \delta(\tau - \tau') \tag{52}$$

with $\omega = \sqrt{\mathbf{k}^2 + 1}$ and the mass m set to one. (For the theory with SSB one gets the same equation with a rescaled ω [76].)

The solution to Eq.52 is found in the standard way in terms of solutions for a homogeneous equation. These solutions are:
the solution regular at $\tau \to +\infty$

$$f_1(u(\tau)) = \frac{2 - 3\,\omega + \omega^2 - 8\,u^2 + 2\,\omega^2\,u^2 + 2\,u^4 + 3\,\omega\,u^4 + \omega^2\,u^4}{u^\omega\,(1 + u^2)^2}\ ; \tag{53}$$

and the solution regular at $\tau \to -\infty$

$$f_2(u(\tau)) = f_1(1/u(\tau)) = \frac{u^\omega\,(2 + 3\,\omega + \omega^2 - 8\,u^2 + 2\,\omega^2\,u^2 + 2\,u^4 - 3\,\omega\,u^4 + \omega^2\,u^4)}{(1 + u^2)^2} \tag{54}$$

with $u(\tau) = e^\tau$. The Wronskian of these solutions is given by

$$W = f_1(\tau)f_2'(\tau) - f_1'(\tau)f_2(\tau) = 2\omega(\omega^2 - 1)(\omega^2 - 4)\ . \tag{55}$$

Then the Green function is explicitly given in terms of f_1, f_2 and W as:

$$G_\omega(\tau_1, \tau_2) = \frac{1}{W}\,\left(f_1(\tau_1)\,f_2(\tau_2)\theta(\tau_1 - \tau_2) + f_1(\tau_2)\,f_2(\tau_1)\,\theta(\tau_2 - \tau_1)\right)\ , \tag{56}$$

where $\theta(\tau)$ is the step function.

The partial (in the wave with momentum $\mathbf{k}$) Green function at coinciding points is then given by

$$g_\omega(\tau) = \frac{1}{2\omega} + \frac{6\,u^2}{\omega\,(\omega^2 - 1)\,(1 + u^2)^2} + \frac{72\,u^4}{\omega\,(\omega^2 - 1)\,(\omega^2 - 4)\,(1 + u^2)^4}\ . \tag{57}$$

In order to calculate the Green function at coinciding points one needs to integrate this expression over $d^3k/(2\pi)^3$:

$$G(\tau,\,\mathbf{x};\,\tau,\,\mathbf{x}) = \int g_\omega(\tau)\,\frac{d^3k}{(2\pi)^3} = \frac{1}{2\,\pi^2}\int_1^\infty g_\omega(\tau)\,\omega\,\sqrt{\omega^2 - 1}\,d\omega\ . \tag{58}$$

The first two terms in Eq.57 give rise to a quadratically and logarithmically divergent integrals, which can be regularized in any standard way, the most straightforward being the Pauli-Villars regularization. Upon substitution into Eq.48 for the mean field with the quantum correction the quadratically divergent part gives rise to a term linear in the classical field ϕ_0 while the logarithmically divergent part results in a correction to the term with $\lambda\phi_0^3$. Therefore these terms can be dumped into the definition of the renormalized mass $\bar{m}$ and the coupling constant $\bar{\lambda}$ according to

$$\begin{aligned}\bar{m}^2 &= m^2 + \frac{3\lambda}{2}I_1\\ \bar{\lambda} &= \lambda - \frac{9\lambda^2}{4}I_3\ ,\end{aligned} \tag{59}$$

where

$$I_1 = \frac{1}{2\pi^2}\int \sqrt{\omega^2-1}\,d\omega \quad \text{and} \quad I_3 = \frac{1}{2\pi^2}\int \frac{d\omega}{\sqrt{\omega^2-1}}\,. \tag{60}$$

The third term in Eq.57 gives rise to an ultraviolet-finite integral in Eq.58. However it contains on-shell singularities, which need to be briefly discussed. The on-shell singularities correspond to the zeros of the Wronskian (Eq.55) at $\omega^2 = 1$ and $\omega^2 = 4$. The first of these corresponds to the translational zero mode of the classical solution ϕ_0 and in fact produces no problem in the integral in Eq.58, since the singularity at $\omega^2 = 1$ is integrable. The pole at $\omega^2 = 4m^2$ (the dependence on mass is restored) is dealt with using the Feynman's $i\epsilon$ rule i.e. by shifting the pole to the negative half-plane $m^2 \to m^2 - i\epsilon$. The integral then develops imaginary part, which in the end corresponds to the dynamical imaginary part of the one-loop graphs, dictated by the unitarity. Proceeding in this way one readily finds the regularized inhomogeneous term in Eq.48 and that the equation takes the explicit form

$$\left(\frac{d^2}{d\tau^2} - 1 + \frac{24u^2}{(1+u^2)^2}\right)\phi_1 = -i18\lambda\sqrt{\frac{8}{\lambda}}F\frac{u^5}{(1+u^2)^5} \tag{61}$$

with

$$F = \frac{\sqrt{3}}{2\pi^2}\left(\ln\frac{2+\sqrt{3}}{2-\sqrt{3}} - i\,\pi\right)\,. \tag{62}$$

The condition on the appropriate solution to Eq.61 is that its expansion in u starts with the fifth power, since only starting from final states with five particles the threshold amplitudes develop an imaginary part, which in this calculation originates in the imaginary part of F. The solution satisfying this condition is

$$\phi_1(\tau) = -i\frac{3\lambda}{4}\sqrt{\frac{8}{\lambda}}F\frac{u^5}{(1+u^2)^3}\,. \tag{63}$$

Therefore the mean value of the field in terms of $z(t)$ with the first quantum correction included is given by

$$\phi_{0+1}(t) = \frac{z(t)}{1-(\bar{\lambda}/8\bar{m}^2)z(t)^2}\left(1 - \frac{3\lambda}{4}F\frac{(\lambda/8m^2)^2 z(t)^4}{(1-(\lambda/8m^2)z(t)^2)^2}\right)\,. \tag{64}$$

Thus for the threshold amplitudes the result with the one-loop correction included reads as[75]

$$\langle 2k+1|\phi(0)|0\rangle = (2k+1)!\left(\frac{\bar{\lambda}}{8\bar{m}^2}\right)^k\left[1 - k(k-1)\frac{3^{3/2}\lambda}{16\pi^2}\left(\ln\frac{2+\sqrt{3}}{2-\sqrt{3}} - i\,\pi\right)\right]\,. \tag{65}$$

The corresponding one-loop result[76] in the theory with SSB is

$$a(n) = -n!\,(2\bar{v})^{1-n}\left[1 + n\,(n-1)\frac{\sqrt{3}\,\lambda}{8\,\pi}\right]\,. \tag{66}$$

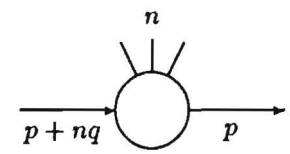

Figure 9: The propagator $D_n(p)$ with emission of n on-mass-shell particles all being at rest. The circle represents the sum of all tree graphs.

We see that the relative magnitude of the one-loop corrections grows as $n^2\lambda$, which quantitatively confirms that the standard perturbation theory is of no help at large values of n. Namely, the presented results suggest that the loop expansion requires the condition $n^2\lambda \ll 1$. However the situation can be somewhat improved by considering the leading in $n^2\lambda$ terms in the loop expansion. As a result of such inspection it has been shown[72] that in fact the $n^2\lambda$ terms exponentiate. More generally, it has been argued[71-73], that the general structure of the amplitude is determined in the limit $\lambda \to 0$, $n\lambda$ fixed and E/n fixed, by the scaling law in Eq.39. Therefore the results in Eq.65 and Eq.66 can be viewed as corrections of order $(n\lambda)^2$ to the 'holy grail' function in the limit $n\lambda \ll 1$.

5.2. Nullification

Before proceeding with discussion of quantum effects in multi-boson production, it is worthwhile to make a detour and discuss an unusual phenomenon, which is revealed by the results in Eq.65 and Eq.66: in spite of the presence of an intermediate state with two bosons in one-loop graphs, their contribution to the amplitudes in the case of SSB is real, while the factor F in Eq.65 is an easily recognizable threshold factor for the $2 \to 4$ process with no indication of presence of other thresholds. Using the unitarity relation for the imaginary part of the loop graphs, one immediately concludes that this can only be if the tree amplitudes of the on-shell processes $2 \to n$ are all zero at the threshold for $n > 4$ in the theory without SSB[75,79] and for $n > 2$ in the theory with SSB[76].

This behavior can be traced to the special properties of the reflectionless potential $-6/(\cosh\tau)^2$ in Eq.52 and generalized[80,66] to other theories, where the problem of the $2 \to n$ scattering is reduced to finding the Green's function in the reflectionless potential $-N(N+1)/(\cosh\tau)^2$ with integer N. In order to illustrate the nullification we consider here a somewhat generic case of interaction between two fields ϕ and χ described by the interaction term in the potential $\xi\,\chi^2\phi^2/2$. The processes of interest are where a collision of two χ particles produce n bosons of the field ϕ. To calculate the tree-level amplitudes of these processes at the threshold we consider the propagator $D_n(p)$ of the particle χ with emission of n on-mass-shell bosons of the of the field ϕ all being at rest and p being the final momentum in the propagator after

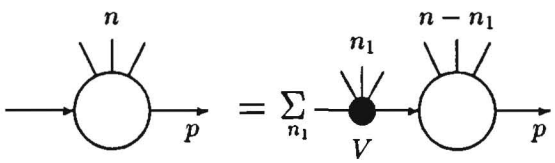

Figure 10: The recursion equation for the propagators $D_n(p)$. The filled circle corresponds to the sum of all tree graphs originating from one-fold interaction, described by the vertex function $V(\phi)$, of the propagating particle with the field ϕ.

the emission (see Figure 9). This propagator can be found by the recursion technique. The recursion equations for the propagators $D_n(p)$ arise from graph shown in Figure 10 and can be written as

$$D_n(p) = -i\, d(p+nq) \sum_{n_1} \frac{n!}{n_1!\,(n-n_1)!} V_{n_1} D_{n-n_1}(p) \ , \tag{67}$$

where q is the momentum of each of the produced ϕ-bosons, in their rest frame $q = (M, \mathbf{0})$ with M being the mass of the ϕ boson, $d(p)$ is the propagator in the absence of interaction with the field ϕ: $d(p) = i/(p^2 - m_{0\chi}^2)$, with $m_{0\chi}$ being the mass of the χ bosons in the absence of interaction with ϕ, and V_n are the matrix elements for production of n bosons at rest by the operator $\xi\,\phi(x)^2$:

$$V_n = \langle n|\xi\,\phi(0)^2|0\rangle \ . \tag{68}$$

Notice, that in a theory where the field ϕ develops a vacuum expectation value $v = \langle 0|\phi|0\rangle$ the "bare" propagator $d(p)$ does not coincide with the propagator

$$D_0(p) = \frac{i}{p^2 - m_{0\chi}^2 - V_0} \tag{69}$$

since the physical mass of the χ boson receives also a contribution from $V_0 = \xi\, v^2$: $m_\chi^2 = m_{0\chi}^2 + V_0$.

The recursion equations (Eq.67) are converted into a differential equation by introducing generating functions $\mathcal{V}(z)$ and $\mathcal{D}(p;\ z)$ related to the V_n and $D_n(p)$ as

$$\mathcal{V}(z) = \sum_{n=0} \frac{z^n}{n!} V_n \qquad \text{and} \qquad \mathcal{D}(p;\ z) = -i \sum_{n=0} \frac{z^n}{n!} D_n(p) \ . \tag{70}$$

In terms of the generating functions Eq.29 is equivalent to the n-th term of the Taylor series expansion of the differential equation

$$\left(M^2 z^2 \frac{d^2}{dz^2} + (2(p\cdot q) + M^2) z \frac{d}{dz} + p^2 - m_{0\chi}^2 - \mathcal{V}(z) \right) \mathcal{D}(p;\ z) = 1 \tag{71}$$

with the initial condition at $z = 0$ that $\mathcal{D}(p;\ 0) = D_0(p)$ (the consistency of this condition is guaranteed by the inhomogeneous term in the right hand side) and that the solution has Taylor expansion in positive powers of z. Alternatively one can specify the latter condition by calculating explicitly the propagator $D_1(p)$, i.e. with emission of one particle, which determines the first derivative $d\mathcal{D}(p;\ z)/dz$ at $z = 0$.

At the tree level the generating function $\mathcal{V}(z)$ is determined by the generating function $f(z)$ for the matrix elements of the field ϕ itself (Eq.33 and Eq.34), $\mathcal{V}(z) = \xi\, f(z)^2$.

Let us first consider the unbroken symmetry case. Upon substitution into Eq.71 of the generating function $\mathcal{V}(z)$ as determined by Eq.33 the equation takes the form

$$\left(M^2 z^2 \frac{d^2}{dz^2} + (2(p\cdot q) + M^2) z \frac{d}{dz} + p^2 - m_{0\chi}^2 - \frac{\xi z^2}{[1 - (\lambda/8M^2)\, z^2]^2} \right) \mathcal{D}(p;\ z) = 1 \ . \tag{72}$$

It is convenient to use the notation $\epsilon = (p\cdot q)/M^2$ and $\omega = \sqrt{\epsilon^2 - (p^2 - m_{0\chi}^2)/M^2}$, where the square root of a positive number is taken to be positive. In the rest frame of the produced bosons ϵ is the energy p_0 of the final particle in the propagator in units of M and ω^2 is related to its spatial momentum $\mathbf{p}$, namely it is $\mathbf{p}^2 + m_{0\chi}^2$ in units of M^2. Thus the difference $\epsilon^2 - \omega^2$ is the measure of by how much the momentum p is off shell.

The solution of Eq.72 can be sought in the form

$$\mathcal{D}(p;\ z) = y^{-\epsilon/2} f(p;\ y) \ , \tag{73}$$

where $y = -\lambda z^2/(8M^2)$, in terms of which Eq.72 is rewritten as

$$\left(4y^2 \frac{d^2}{dy^2} + 4y \frac{d}{dy} - \omega^2 + 8\frac{\xi}{\lambda} \frac{y}{(1+y)^2} \right) f(p;\ y) = \frac{y^{\epsilon/2}}{M^2} \ . \tag{74}$$

For the broken symmetry case, using in the same manner the generating function in Eq.34 and seeking the solution in the form of Eq.73 in terms of the variable $y = -z/(2v)$, one arrives at the equation essentially identical to Eq.74 up to rescaling of some terms:

$$\left(4y^2 \frac{d^2}{dy^2} + 4y \frac{d}{dy} - 4\bar{\omega}^2 + 8\frac{\xi}{\lambda} \frac{y}{(1+y)^2} \right) f(p;\ y) = 4\frac{y^{\epsilon/2}}{M^2} \ . \tag{75}$$

where $\bar{\omega}^2 = \epsilon^2 - (p^2 - m_{0\chi}^2 - \xi v^2)/M^2$, and it can be also reminded that the expression for M in terms of $|m|$ differs from that in the case of unbroken symmetry by factor $\sqrt{2}$. Therefore the solution of the equation 75 is obtained from that of Eq.74 by replacing $\omega \to 2\bar{\omega}$ and by overall rescaling of the solution by the factor 4.

The operator in the homogeneous left hand side of Eq.74 is related to the well known exactly solvable Schrödinger operator with the Pöschl-Teller potential. Namely, the substitution $y = e^{2\tau}$ converts the operator into

$$\frac{d^2}{d\tau^2} - \omega^2 + \frac{s(s+1)}{(\cosh\tau)^2} \tag{76}$$

with $s(s+1) = 2\xi/\lambda$.

The properties of the operator in Eq.76 are well known from Quantum Mechanics. In particular, these properties are quite special when the parameter s is integer, $s = N$, and there is no reflection of waves in the one-dimensional Quantum Mechanical problem. It is in this special case of integer s that the infinite series of zeros of the threshold amplitudes arises and which will be considered here. The solution of Eq.72 for $\mathcal{D}(p;\ z)$ can be written explicitly[58] in terms of the hypergeometric function $F(-s, \omega - s, \omega + 1, -y)$, which in the case $s = N$ is reduced to a Jacobi polynomial of the N-th power. We therefore define the function

$$F_N(\omega, y) = \frac{\Gamma(1+\omega)}{\Gamma(N+1+\omega)} y^{-\omega/2}(1+y)^{N+1} \frac{d^N}{dy^N}\left(\frac{y^{\omega+N}}{(1+y)^{N+1}}\right) \tag{77}$$

and write the explicit formula for $\mathcal{D}(p;\ z)$ in the unbroken symmetry case in terms of the variable y in the form

$$\mathcal{D}(p, z(y)) =$$
$$\frac{y^{-\epsilon/2}}{4\omega M^2}\left[F_N(-\omega, y)\int_0^y u^{\frac{\epsilon}{2}-1} F_N(\omega, u)\, du + F_N(\omega, y)\int_y^\infty u^{\frac{\epsilon}{2}-1} F_N(-\omega, u)\, du\right] , \tag{78}$$

where y is assumed to be positive. Notice, that $F_N(\omega, y)$ is regular at small y: $F_N(\omega, y) \approx y^{\omega/2}$, while $F_N(-\omega, y)$ is singular at $y \to 0$. At $y \to \infty$ the behavior of these functions is switched. Also in its dependence on ω the function $F_N(-\omega, y)$ develops simple poles at $\omega = 1,\ 2, \ldots,\ N$ due to the factor $\Gamma(1-\omega)/\Gamma(N+1-\omega)$ in its definition.

We can now proceed to considering the amplitude of production n ϕ-bosons by two incoming particles. In terms of the propagator $D_n(p)$ this amplitude corresponds to negative p_0 (the final line in the propagator is that of an incoming particle), so that $\epsilon = -n/2$. The on-mass-shell amplitude is given by the double pole of this propagator when $\omega = -\epsilon = n/2$. The only possibility for the expression in the right hand side of Eq.78 to develop a double pole for positive ω is when one pole term comes from the function $F_N(-\omega, y)$ and the other one comes from the divergence of the integral. At negative ϵ the first of the integrals in Eq.78 indeed develops single poles, so the double poles are possible only for the values $\omega = 1,\ 2, \ldots,\ N$, i.e. only as long as $n \leq 2N$ and thus the on-mass-shell threshold amplitudes are all zero for $n > 2N$

For the case of broken symmetry as we have seen the generating function $\mathcal{D}$ for the propagators is determined by the same operator as for the unbroken symmetry case with ω rescaled by factor 2 (Eq.75). Therefore the same consideration leads one to the conclusion that in this case the on-mass-shell threshold amplitudes are zero for all n larger than N.

This nullification of the amplitudes at the thresholds is the one observed for production of the ϕ bosons by particles of the same field. Indeed, expanding the field ϕ into the sum of modes with zero spatial momentum (denoted as ϕ in this discussion) and with a non-zero one (corresponding to χ), one easily identifies that the $\lambda\phi^4/4$

term corresponds to $\xi\,\phi^2\chi^2/2$ interaction with $\xi = 3\lambda$, so that $2\xi/\lambda = 6$ and thus $N = 2$ both in the case of no spontaneous symmetry breaking and in the case of broken symmetry. Other known applications corresponding to different integer values $s = N$ in the situation just considered are[80]

- Linear sigma model: the interaction of the Goldstone bosons – pions with the massive σ bosons corresponds to $s = 1$, thus the tree-level amplitudes of the scattering $\pi\pi \to n\sigma$ are all zero at the corresponding thresholds for $n > 1$.
- Transversal vector bosons with Higgs generated mass: for these components the Lagarngian in the Proca gauge is equivalent to that of a scalar χ interacting with the Higgs field with the constant ξ such that $2\xi/\lambda = 4\,m_V^2/m_H^2$. Therefore if $4\,m_V^2/m_H^2 = N(N+1)$ with integer N, the tree-level amplitudes for the process $V_T V_T \to n\,H$ vanish at the threshold for $n > N$.

Additional known cases, which can be derived by the same technique as described here are:

- Longitudinal vector bosons with Higgs generated mass, satisfying the condition $4\,m_V^2/m_H^2 = N(N+1)$ with integer N. The tree-level threshold amplitudes of $V_L V_L \to n\,H$ are all zero for $n > N+1$ [66] and also for $n = N$ [81].
- Fermions with Higgs-generated mass[80]: if $2m_f/m_H = N$ (integer), then all tree-level amplitudes of $f\bar{f} \to n\,H$ are zero at threshold for all $n \geq N$.

The observed behavior of the threshold amplitudes somewhat resembles the nullification of inelastic amplitudes in the Sine-Gordon theory, where it is a consequence of a symmetry and is a deep property of the theory. In the theories considered here this is a much weaker property, which holds only at threshold and, generally, only at the tree level[82]. However the nullification in this case can be a consequence of a hidden symmetry, which holds at the classical level and/or has a more restricted scope. Thus far such symmetry has been revealed[83] only for the case of $N = 1$, where it can be traced to the symmetry of a system of two anharmonic oscillators, described by the potential

$$V(x, y) = \frac{\omega_1^2}{2}x^2 + \frac{\omega_2^2}{2}y^2 + \frac{\lambda}{4}(x^2+y^2)^2 . \tag{79}$$

If the frequencies ω_1 and ω_2 were equal, the model would have an O(2) symmetry, corresponding to conservation of the angular momentum $Q = \dot{x}y - x\dot{y}$. However even for $\omega_1 \neq \omega_2$ the symmetry persists[83] corresponding to conservation of the invariant

$$\frac{\lambda}{4}Q^2 + (\omega_1^2 - \omega_2^2)(\frac{1}{2}\dot{y}^2 + \frac{\omega_2^2}{2}y^2 + \frac{\lambda}{4}y^4 + \frac{\lambda}{4}x^2y^2) .$$

It should be also noted that if the ratio $2\xi/\lambda$ does not satisfy the above mentioned condition, the threshold amplitudes of the processes $2\chi \to n\,\phi$ display[84] a 'normal' factorial growth with n: $a_n \propto \Gamma(\frac{n}{2} - s)/\Gamma(-s)$ for the case of no SSB and $a_n \propto \Gamma(n-s)/\Gamma(-s)$ for a theory with SSB.

5.3. General properties in a theory with SSB

The failure of the standard perturbation theory at large n makes it imperative to consider alternative methods of calculating multiparticle amplitudes. Here we will discuss few considerations in a theory with SSB, possibly allowing to get some insight into behavior of these amplitudes beyond the standard loop expansion.

Both the classical approximation and the loop corrections in the theory of one scalar field with SSB make the problem of calculating the threshold amplitudes equivalent to the following Euclidean space problem:

- Calculate the quantum average $\Phi(t)$ of the field ϕ in the Euclidean time t,

$$\Phi(\tau) = \frac{\int \phi(x)\, e^{-S[\phi]}\, \mathcal{D}\phi}{\int e^{-S[\phi]}\, \mathcal{D}\phi} \tag{80}$$

  with the kink-type boundary conditions, i.e. that $\phi \to \pm v$ as $t \to \pm\infty$.

- Expand $\Phi(t)$ at $t \to -\infty$ in powers of e^{Mt}:

$$\Phi(t) = \sum_{n=0}^{\infty} c_n\, e^{nMt} \tag{81}$$

  where, obviously, $c_0 = -v$.

- Find the threshold amplitudes as

$$\langle n|\phi(0)|0\rangle = n!\, c_n / c_1^n \tag{82}$$

  (The coefficient $c_1 = \langle 1|\phi(0)|0\rangle$ is the normalization for one-particle state, thus it appears in Eq.82.)

At the tree level the amplitudes are given by the solution to the classical field equations with the kink boundary conditions $\Phi(t) = v\,\tanh(Mt/2)$, and Eq.82 reproduces the result for the amplitudes in Eq.35. The same correspondence is true for the first loop correction[76]. However, this new formulation of the problem also allows for general considerations beyond the perturbation theory. One of such considerations is that the classical kink solution provides the absolute minimum of the action in the class of real fields with the kink boundary conditions. Therefore the path integral in Eq.80 converges on real fields and thus the average field $\Phi(t)$ is real in any finite number of loops. Therefore the coefficients c_n in the expansion in Eq.81 are all real, hence the amplitudes $a(n)$ are also real to any order of perturbation theory. The latter property is far not trivial, by unitarity the imaginary part of the amplitudes is associated with physical intermediate states:

$$2\,\mathrm{Im}A(1 \to n) = \sum_k A(1 \to k) * A(k \to n)\ , \tag{83}$$

where the asterisk denotes multiplication and integration over the phase space of the intermediate state of k on-shell particles. At the one-loop level the only cut of the graphs possible is across a two-particle state, $k = 2$. Since $A(1 \to 2)$ is not zero, one concludes that all amplitudes $A(2 \to n)$ should be zero at the n particle thresholds. In higher loops the fact that $\mathrm{Im}A(1 \to n) = 0$ in Eq.83 in general leads to relations between amplitudes with different k. There is however one special case on $n = 3$. In this case the only intermediate state with a non-vanishing phase space is that corresponding to $k = 2$. Therefore we conclude that $A(2 \to 3)$ is zero at the three-particle threshold in any order of perturbation theory.

The factorial behavior of the amplitudes, as we have seen, can be viewed as being due to the finite radius of convergence of the series in Eq.81 in the variable $z = e^{Mt}$, which behavior at the classical level is due to the poles of the classical solution at finite complex t. One may ask whether the singularity at finite t is present in the exact average field in Eq.80 and thus whether the factorial growth of the amplitudes is preserved by quantum effects. To analyze this problem, we will assume that the series in Eq.81 has infinite radius of convergence and will show that this assumption is inconsistent, thus concluding that the factorial behavior is not eliminated by the higher quantum effects.

If the Taylor series in Eq.81 for the mean field $\Phi(t)$ has infinite radius of convergence, the same is true for the quantity

$$\Delta(t) = v^2 - \Phi^2(t) \tag{84}$$

(notice that it is the square of the mean field entering here and not the mean value of the square of the field). The asymptotic behavior of $\Delta(t)$ at $t \to -\infty$ is $\Delta(t) \to 2v^2 e^{Mt}$. At $t \to +\infty$ the asymptotic behavior is determined by the boundary condition on $\Phi(t)$ and by the fact that M is the lowest energy in the spectrum: $\Delta(t) \to b e^{-Mt}$ where b is a constant.

The series in Eq.81 defines $\Phi(t)$ and thus also $\Delta(t)$ as a periodic function of complex t with the period $2\pi/M$. Let us consider the contour C in the complex plane of t shown in Figure 11. Two side links of the contour run along the lines separated by one period: $\mathrm{Im}\, t = 0$ and $\mathrm{Im}\, t = 2\pi/M$, and the horizontal links between the two lines are chosen sufficiently far at negative and positive $\mathrm{Re}\, t$, so that one can use the asymptotic expression for $\Delta(t)$ on those links. One can now consider calculation of the index of the function $\Delta(t)$ on the contour C:

$$I = \frac{1}{2\pi i} \oint_C \frac{d\Delta(t)/dt}{\Delta(t)}\, dt\ , \tag{85}$$

which gives the difference between the number of zeros and poles of the function $\Delta(t)$ inside the contour: $I = Z - P$, where Z is the number of zeros and P is the number of poles inside the contour. Since the function $\Delta(t)$ is assumed to have no singularities at finite complex t, the only obstacle to calculating the index integral can be presence of isolated zeros of $\Delta(t)$ at the contour C. This obstacle however can

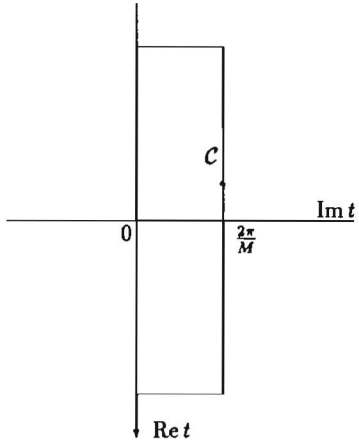

Figure 11: The contour C on which the index of the function $\Delta(t)$ is calculated.

be easily removed by slightly shifting the contour away from each of the isolated zeros. (Clearly, a condensation of zeros of $\Delta(t)$ at a finite t would either contradict the assumption that $\Delta(t)$ is entire function, or imply that the function is identically zero, which obviously is not the case.) One can readily see, that the boundary condition that $\Delta(t)$ falls exponentially at both infinities in $\mathrm{Re}\, t$ completely determines the index in Eq.85 : $I = -2$. Therefore $\Delta(t)$ necessarily has a non-zero number of poles inside the contour C, which is inconsistent with the assumption that the series in Eq.81 has infinite radius of convergence. This completes the proof that the amplitudes have a factorial growth in n in the full quantum theory.

5.4. In a search for correct saddle point

The dependence of the quantum corrections on the parameter $n^2\lambda$ hints that the standard perturbation theory corresponds to an expansion of the path integral in Eq.80 around a wrong saddle point, namely, around the saddle point for the action only, not taking into account the pre-exponent, which becomes essential at large n. Therefore the large 'quantum' effects may merely be simply the difference between the values of two quaziclassical exponents: the one corresponding to the true saddle point in the problem and the one for the e^{-S} only. Thus the problem of multiparticle amplitudes may have a quaziclassical solution, albeit in terms of different saddle point configuration. This behavior has been observed in a two-dimensional model problem[40] and here we will discuss the search for the true saddle point in a theory with SSB[78].

In order to find the true saddle point configuration one can compare the contri-

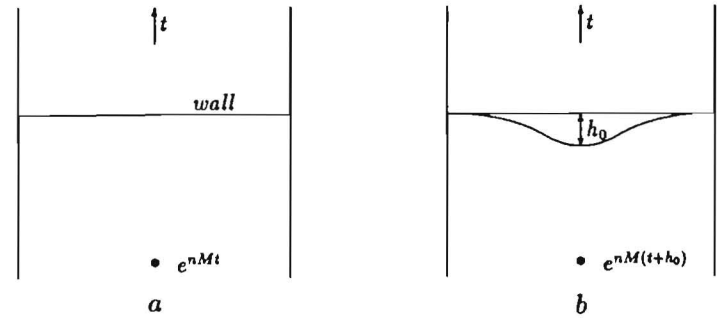

Figure 12: A configuration of the field, corresponding to the flat inter-phase boundary (a) and to the same boundary bent to the maximal deflection h_0 (b). The evolution of the field between the point at a negative time t (heavy dot) and the boundary proceeds over the time $|t| - h_0 + \delta$, where δ is of the order of the thickness of the domain wall. The vertical lines at the edges are the world lines of the boundaries of the spatial bounding box.

bution to the coefficient of the n harmonics of the field at a large negative Euclidean time t of the two configurations shown in Figure 12. The first one corresponds to the classical solution for the action and is a flat wall separating the phase with ϕ positive and negative. The other configuration has the same inter-phase wall bent to a maximum deflection denoted by h_0. The shape of the wall (the surface corresponding to $\phi(\mathbf{x}, t) = 0$) can be described by its $\mathbf{x}$-dependent deviation from $t = 0$, i.e. by the solution $t = -h(\mathbf{x})$ of the equation $\phi(\mathbf{x}, t) = 0$. Assume now that one fixes the shape of the boundary corresponding to a particular function $h(\mathbf{x})$ and minimizes the action with respect to all other variables of the field by solving the Euler-Lagrange equations with the surface of zeros being fixed. Then for large negative t at the point $\mathbf{x}_0$ corresponding to the maximum of $h(\mathbf{x})$ the n-th harmonics of the field is given in the leading exponential approximation by

$$\exp\left(nM(t + h(\mathbf{x}_0) + O(nM\delta)\right) \sim \exp(nMh(\mathbf{x}_0))\, e^{nMt} \,. \tag{86}$$

This behavior can be understood by considering that at the point $\mathbf{x}_0$ the evolution of the field in time from $\phi = 0$ towards $\phi = -v$ proceeds over the time $|t| - h(\mathbf{x}_0) + \delta$, where δ reflects the uncertainty related to the curvature of the surface of the inter-phase boundary. This uncertainty is of a subleading importance in situations where $h(\mathbf{x}_0)$ is large, which as will be seen is the case for $n \gg 1/\lambda$.

Equation 86 tells that the coefficient c_n gets enhanced by the factor $\exp(nMh(\mathbf{x}_0))$. Thus in the leading WKB approximation this coefficient can be evaluated as

$$c_n \sim \max\left[\exp\left(nMh(\mathbf{x}_0) - S[h] + S_0\right)\right] \,, \tag{87}$$

where $S[h]$ is the action for the field configuration described by the shape $h(\mathbf{x})$ of the inter-phase boundary and $S_0 = S[h=0]$ is the action of the unperturbed classical solution, and all the pre-exponential factors are omitted.

The appearance of the action S_0 with the plus sign in Eq.87 is due to the fact that in Eq.80 for the mean field the path integrals in both numerator and the denominator are calculated with the kink-type boundary conditions. Thus the factor $\exp(S_0)$ appears from the saddle point value of the denominator.

If $h(\mathbf{x})$ varies at scale larger than the thickness of the wall, M^{-1}, the action $S[h]$ can be calculated in the thin-wall approximation[56] as the surface tension of the wall μ times its area A:

$$S[h] = \mu A[h] \ , \tag{88}$$

where $\mu = M^3/(3\lambda)$.

Finding the maximum of the expression in Eq.87 with the action from Eq.88 is equivalent to solving a surface tension problem for a d-dimensional film in $(d+1)$-dimensions. The edges of the film are fixed at the boundary of the bounding box: $h(boundary) = 0$, and at the point $\mathbf{x}_0$ the force equal to nM is applied downwards. The maximal deviation $h_0 = h(\mathbf{x}_0)$ of the film will be largest if the force is applied to the center of the film, therefore we set $\mathbf{x}_0 = 0$. (In fact for $d > 2$ the equilibrium shape of the film does not depend on $\mathbf{x}_0$ if this point is sufficiently far from the edges. For $d \leq 2$ there is an infrared behavior in this problem, so that the equilibrium deviation explicitly depends on the size of the bounding box. Also it is explicitly assumed throughout this paper that $d > 1$. A one-dimensional 'film' lacks intrinsic curvature, which makes most of the formulas in this paper singular in the formal limit $d \to 1$, i.e. that of a (1+1) - dimensional field theory, for which the present analysis is thus not directly applicable.) Assuming that the bounding box is spherically symmetrical in the spatial d dimensions with a large radius R, one concludes that the shape, which the film takes under the force applied at the center, is also spherically symmetrical and can be characterized by the radius $r(t)$ of its slice at $t = const$ if the slice is positioned at an instant t such that $-h(0) < t < 0$. The boundary conditions for $r(t)$ being $r(-h_0) = 0$ and $r(0) = R$.

In terms of $r(t)$ the quantity $S[h] - n\,M\,h_0$ entering the expression in Eq.87 can be written as

$$s[r] = \int_{-h_0}^{0} l_d\,\mu\,r^{d-1}\sqrt{1+\dot{r}^2}\,dt - Eh_0 \ , \tag{89}$$

where the factor nM is identified as the total energy E, $\dot{r} = dr/dt$, and l_d is the $(d-1)$ dimensional volume of unit sphere S_{d-1}: $l_d = 2\,\pi^{d/2}/\Gamma(d/2)$. The integral in Eq.89 is nothing else than the Euclidean action for a spherical bubble in the thin wall approximation[56] in the theory with degenerate vacua. Since E is the conserved value of the Hamiltonian for the classical trajectory $r(t)$, the functional $s[r]$ is identified as the truncated action

$$s[r] = \int p_E\,dr \ , \tag{90}$$

where p_E is the Euclidean momentum conjugate of r:

$$p_E = \frac{l_d\,\mu\,r^{d-1}\dot{r}}{\sqrt{1+\dot{r}^2}} \ . \tag{91}$$

On an Euclidean-space classical trajectory the value E of the Hamiltonian is related to p_E and r as

$$E^2 + p_E^2 = (l_d\,\mu\,r^{d-1})^2 \ . \tag{92}$$

The solution of the latter equation for p_E in terms of E and r reads as

$$p_E = \sqrt{(l_d\,\mu\,r^{d-1})^2 - E^2} = l_d\,\mu\sqrt{r^{2d-2} - r_0^{2d-2}} \ , \tag{93}$$

where

$$r_0 = (E/(l_d\,\mu))^{\frac{1}{d-1}} \ . \tag{94}$$

This solution immediately reveals an important point: there is no real solution for p_E and thus for $r(t)$ at $r < r_0$. In the equivalent surface tension problem the origin of this behavior is obvious: there is a minimal radius equal to r_0 of a slice of the surface with surface tension μ that can support the force E. In terms of quantum mechanics with the action in Eq.90 the point r_0 corresponds to the classical turning point, and for $r < r_0$ the evolution of the system proceeds in the Minkowski time. We are thus compelled to consider the evolution of the radius of the bubble along the complex time trajectory, on which the part of the trajectory with $r < r_0$ evolves along imaginary Euclidean, i.e. real Minkowski, time. In the Minkowski space this part of the trajectory describes the bubble oscillating from size $r = 0$ to the classical turning point $r = r_0$. Naturally, the oscillations of a bubble are subject to energy dissipation due to emission of particles. However, it was found in the numerical studies of mid - 70's[85-87] that the bubbles display a remarkable stability: they dissipate energy over several, sometimes many, oscillations. Recently there is renewed interest to this problem[88]. This makes it meaningful to ignore the instability of the bubbles in the present calculation and to calculate the coefficients c_n from Eq.87 considering bubbles as capable of undergoing arbitrary number of oscillations between $r = 0$ and $r = r_0$. This picture produces an infinite number of saddle points for the effective action in Eq.87, corresponding to $k + \frac{1}{2}$ periods of oscillation with arbitrary integer k. Since the Minkowski part of trajectory results in the phase factor e^{iS_M} with S_M being the Minkowski space action, the contribution of each of these saddle points differs only by a phase factor, and all these contributions should be summed up. Proceeding in this way, we find

$$c_n \sim e^{s_E}\sum_{k=0}^{\infty}\exp[i\,(k+1/2)\,I(E)] = e^{s_E}\,\frac{\exp(iI(E)/2)}{1-\exp(iI(E))} \ , \tag{95}$$

where $I(E)$ is the Minkowski truncated action over a full period of oscillation and s_E is the Euclidean truncated action on the trajectory going from the turning point

$r = r_0$ to $r = \infty$. The explicit expressions for the latter terms are[78]:

$$I(E) = l_d \frac{\sqrt{\pi}\,\Gamma[1/(2d-2)]}{2(d-1)\,\Gamma[3/2+1/(2d-2)]}\,\mu\, r_0^d \tag{96}$$

and

$$s_E = \frac{\sqrt{\pi}\,\Gamma[1/2-1/(2d-2)]}{2\,d\,\Gamma[1-1/(2d-2)]}\,E\left(\frac{E}{l_d\mu}\right)^{\frac{1}{d-1}} \tag{97}$$

for a theory in $(d+1)$ dimensions.

The expression in Eq.95 implies that the amplitudes develop poles at the values of energy, satisfying the condition $I(E) = 2\pi N$ with integer N. This condition is nothing else than the Bohr–Sommerfeld quantization condition (within the accuracy of present calculation) for the quantized levels of the bubble oscillations. Therefore the amplitudes of the n particle production can be viewed as arising from a two-stage process: the virtual field ϕ produces a bubble and then the bubble couples to the state of n particles, all being at rest:

$$(n|\phi(0)|0\rangle = (B_N|\phi(0)|0> \cdot A(B_N \to n)\ . \tag{98}$$

The amplitude of production of a bubble with energy E by a virtual field can be found[78] by the Landau-WKB method and is exponentially small:

$$|(n|\phi(0)|0\rangle| \sim e^{-s_E} \tag{99}$$

with s_E given by Eq.97. On the other hand, the product of the amplitudes of formation and decay of the bubble is exponentially large in $E^{d/(d-1)}$ (Eq.95). Thus the amplitude of the coupling of the bubble to n bosons, which are all being at rest contains the doubled positive exponent:

$$|A(B_N \to n)| \sim \left(\frac{E_N}{m}\right)!\ \exp(2\,s_E)\ . \tag{100}$$

To reconcile this extremely strong coupling of the bubble to the state of n bosons, in which they all have exactly zero spatial momenta, with a total decay rate that is not exponentially large, one inevitably has to assume that the coupling of the bubble to bosons develops a form factor which sharply decreases above the threshold and thus is capable of suppressing the double-exponential and the factorial energy growth of the amplitude at the threshold. In view of this observation it is extremely likely that in the processes $1 \to n$, whose amplitude at the threshold has only single exponential growth factor, the same form factor makes the total probability exponentially suppressed at high energy.

The calculations discussed in this section suggest the following hierarchy of the amplitudes for transitions between one highly virtual particle, the state of n bosons, all having zero spatial momenta, and a state of a spherical bubble with energy $E = nm$:

$$A(1 \to B) \sim e^{-s_E}\ ,$$

$$A(1 \to n) \sim A(1 \to B)\cdot A(B \to n) \sim e^{s_E}\ ,$$
$$A(B \to n) \sim e^{2\,s_E}\ . \tag{101}$$

When the particles have non-zero momenta a form factor arises, due to the size r_0 of the bubble which cuts off the phase space integration. The suppression due to the form factor can be evaluated from the fact that the decay rate of a large bubble is not exponentially large in its energy, i.e. in this exponential scale the rate is $\Gamma(B \to many) \sim O(1)$. If the processes $1 \to many$ at ultra-high energy are going through the bubbles, as it is strongly indicated by Eq.95, their rate should be cut off by the same form factor, which implies

$$\Gamma(1 \to many) \sim |A(1 \to B)|^2\,\Gamma(B \to many) \sim e^{-2s_E} \tag{102}$$

and thus these processes are strongly suppressed.

This hierarchy can be extended to processes $many \to many$, which potentially can go at high temperature. If such scattering processes are also mediated by the bubbles, one can estimate

$$A(n \to n) \sim A(n \to B)\cdot A(B \to n) \sim e^{4\,s_E}\ . \tag{103}$$

A kinetical calculation of the rate in a thermal equilibrium in this case involves the form factor for both the final and the initial states. Thus the rate of the multi-particle processes at high temperature should be $\Gamma(many \to many) \sim O(1)$, which is in agreement with standard thermodynamical calculations.

6. Conclusions

At present there appears to be general understanding that the processes of the type $many \to many$ go unsuppressed, thus in particular giving rise to the $B+L$ violation in the Standard Model at high temperature or at a high density of fermions. The problem of calculating the cross section of the processes $few \to many$ at high energy remains open and awaits new non-perturbative methods to be developed both for the processes with and without the $B+L$ violation. These methods can also bring unexpected results in study of field dynamics, like the discussed property of nullification of the tree-level threshold amplitudes. As is argued in the previous section, the cross section most probably remains small in the theory of one scalar field with SSB. However the reasoning there uses special properties of the model, which are not present, say in a theory without SSB. Most recently within a new approach[73] based on singular classical configurations in a complexified space-time a solution was suggested[73] in which the threshold amplitudes in a theory without SSB are exponentially small. Nevertheless, in general it is not known yet, whether there is any universality across theories for the non-perturbative behavior of the multi-boson processes.

7. Acknowledgments

I am thankful to the International Centre for Theoretical Physics for hospitality. This work is supported in part by the DOE grant DE-AC02-83ER40105.

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