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False Vacuum Decay

M.B. Voloshin
Theoretical Physics Institute, University of Minnesota
Minneapolis, MN 55455, USA
and
Institute of Theoretical and Experimental Physics
Moscow 117259, Russia

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Abstract

This lecture is a review of the theory of spontaneous decay of metastable vacuum states of fields due to quantum fluctuations, as well as of the current understanding of the catalysis of the false vacuum decay by particles and by their collisions.

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1. Introduction

The notion of metastable, or false, vacuum arises in the situation where in addition to the global vacuum state of the quantum field system, which provides the absolute minimum of the energy, there exists at least one 'local vacuum' state, which has energy higher than the true vacuum and which provides a local minimum of energy with respect to small variations of the fields. The simplest model with a false vacuum state is that of one real scalar field ϕ with a tilted double-well potential

$$V(\phi) = \frac{\lambda}{4}\left(\phi^2 - v^2\right)^2 + \frac{\epsilon}{2v}\phi \quad . \tag{1}$$

For $\epsilon = 0$ the two vacua $\phi_+ = +v$ and $\phi_- = -v$ are degenerate, while at a small positive ϵ the state ϕ_- becomes the true vacuum and that at ϕ_+ becomes a metastable vacuum state (Figure 1), whose energy density is by the amount ϵ larger than that of the true vacuum.

The false vacuum state is stable under small variations of the field. In particular the fact that there exists a vacuum state with a lower energy does not show up in any finite order of the perturbation theory for scattering in the vacuum ϕ_+. However large non-perturbative fluctuations of the field eventually destroy the false vacuum causing its transition to the lower energy state. The false vacuum state can thus have quite long lifetime in as much as large fluctuations have small probability. Calculation of the probability of such process i.e. of the rate of the decay of false vacuum is the subject of the present lecture.

The large fluctuations resulting in decay of the false vacuum can either be induced by particles present in the false vacuum, or be a part of the quantum fluctuations of the field in the vacuum ϕ_+. The case of decay induced by particles present in the vacuum includes also the decay at a finite temperature, which can be considered by the standard methods of thermodynamics[1,2], and at a high temperature becomes a purely classical process driven by thermal rather than quantum fluctuations. In the course of this lecture we will mainly concentrate on an opposite extreme when the decay is driven by purely quantum effects.

Since a thermal decay of a metastable vacuum state in this setting is nothing else than a classical first order phase transition, it is of no surprise that the subject of the false vacuum decay originates in the condensed matter physics[1,2], where it was also extended to the purely quantum decay mechanism of 'phase transitions at zero temperature'[3] and of other similar processes, like motion of dislocations in a crystal[4].

In the relativistic field theory the subject of false vacuum decay was initiated by Lee and Wick[5], who have pointed out the possibility of existence of metastable vacuum states of relativistic quantum fields and also have formulated the problem of calculating the decay rate for such states. Since then this subject appeared many times both in the context of a temperature-driven transition and of a purely quantum one in the cosmology of the early universe in considering the stability and metastability of the vacuum in the parameter space of various models of particle interactions

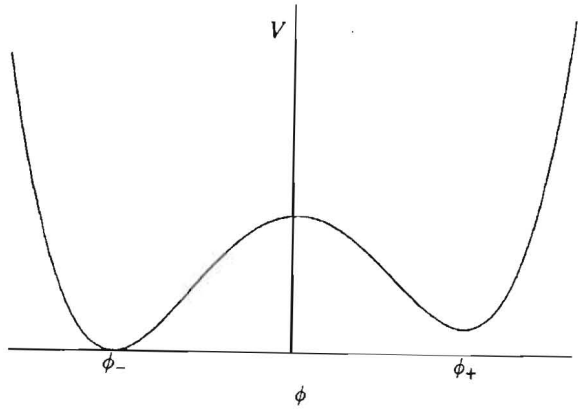

Figure 1: True (ϕ_-) and false (ϕ_+) vacua in a theory of one real scalar field.

and as a tool in considering non-perturbative phenomena in the dynamics of quantum fields. In addition to its possible applications in cosmology and in particle and condensed matter physics the problem of false vacuum decay is by itself an extremely interesting exercise in quantum field theory and can be quite useful in developing the theoretical methods for a deeper understanding of quantum dynamics of fields.

2. Decay of false vacuum through critical bubbles.

2.1. Critical bubbles

The mechanism[6] of the spontaneous transition between the false and the true vacuum parallels that of ordinary phase transitions (like in boiling water): First, bubbles of the more stable phase are formed in the bulk of the metastable phase, which bubbles then expand and eventually destroy the metastable phase. Classical existence and expansion of the bubbles is however allowed only starting from a certain minimal size. Indeed, though the volume energy of the interior of the bubble is negative with respect to the outer false vacuum: $E_{\rm Volume} = -\epsilon \cdot (Volume)$, there is a positive energy associated with the transition region on the surface of the bubble: $E_{\rm Surface} = \mu \cdot (Surface\ Area) + E_{\rm kin}$, where $E_{\rm kin}$ is the kinetic energy due to motion of the surface in the expansion and μ is the surface energy per unit area: surface tension. Since the total energy of the spontaneously nucleated bubble with respect to the initial phase should be equal to zero, one concludes that the minimal size is achieved for a spherical bubble (maximizing the volume to surface area ratio), which starts expansion having its wall at rest ($E_{\rm kin} = 0$) with the critical radius

$$R_c = 3\mu/\epsilon \quad . \tag{2}$$

It was conjectured[6] (quite naturally) that the spherical bubbles of the critical radius R_c are the minimal fluctuations, which drive the decay of false vacuum and that therefore the rate of the decay is that of spontaneous nucleation of the critical bubbles. A formal proof of this conjecture was given later[7] in the course of further development of the formalism, which will be discussed in this lecture.

It should be also emphasized that in the described reasoning concerning the critical bubbles it is assumed that the thickness of the transition region, the bubble wall, is much less than the size of the bubble all the way down to the critical radius, so that there is a distinct separation between the volume and the surface energy. This is the so called thin wall approximation (TWA), which is often used in the problem of false vacuum decay, since in many cases it allows to find general results, which are sensitive to the details of the underlying field theory only through the parameters μ and ϵ. The condition for this approximation to be valid is $R_c/\delta \gg 1$, where δ is the characteristic thickness of the bubble wall. In general $\delta \sim m_+^{-1} + m_-^{-1}$, where m_+ (m_-) is the smallest of the masses of relevant particles in the false (true) vacuum. Due to Eq.2 in a theory with finite smallest relevant mass TWA is always applicable in the limit of small ϵ.

It is a useful reference to illustrate explicitly the parameters involved for the scalar field model with the potential in Eq. 1. In the limit $\epsilon \to 0$ the model describes standard scalar theory with spontaneous breaking of the discrete symmetry under the change of the sign of ϕ. The latter model describes scalar bosons with the mass $m = \sqrt{2\lambda}\, v$ and the classical solution interpolating between the two degenerate vacua is a flat wall with the profile of the field across it given by

$$\phi(x) = v \tanh(m\, x/2) \quad . \tag{3}$$

The energy of the wall per unit area is

$$\mu = \frac{\sqrt{8\lambda}}{3} v^3 = \frac{m^3}{3\lambda} \quad . \tag{4}$$

For small ϵ, such that $R_c\, m \gg 1$, the curvature of the bubble wall can be neglected, thus can be neglected the effect of non-zero ϵ on the surface tension of the wall and also on its profile (thickness). Thus, up to terms of order ϵ the surface tension is given by Eq.4, and the critical radius is $R_c = 3\mu/\epsilon = \sqrt{8\lambda}\, v^3/\epsilon = m^3/(\lambda\, \epsilon)$.

2.2. Dynamics of spherical bubbles

In order to find the probability of quantum nucleation of a spherically symmetrical critical bubble it is useful first to consider the classical dynamics of spherical bubbles. In TWA a spherical bubble is characterized by only one dynamical variable: its radius r. One can readily write an effective Lagrangian for this variable, using purely classical reasoning[6]. Indeed the relativistic Lagrangian for such object consists of the 'kinetic' part associated with the motion of the mass $4\pi\, r^2\, \mu$ and the potential part,

corresponding to the negative volume energy $-4\pi r^3 \epsilon/3$:

$$L = -4\pi r^2 \mu \sqrt{1-\dot{r}^2} + 4\pi r^3 \epsilon/3 \quad , \tag{5}$$

where $\dot{r} = dr/dt$ is the speed of the wall. Using this Lagrangian it is straightforward to derive the canonical momentum p

$$p = 4\pi r^2 \dot{r} \mu/\sqrt{1-\dot{r}^2} \tag{6}$$

and, finally, the relation for the Hamiltonian H in terms of r and p:

$$\left(H + \frac{4}{3}\pi r^3 \epsilon\right)^2 - p^2 = \left(4\pi r^2 \mu\right)^2 \quad . \tag{7}$$

As is already mentioned the energy of a spontaneously nucleated bubble should be equal to zero. Thus the expansion law can be derived by setting $H = 0$ and finding from Eq.7

$$p = 4\pi r^2 \mu \sqrt{\frac{r^2}{R_c^2} - 1} \quad . \tag{8}$$

One can then easily see from Eq.6 that the dependence of the radius r on time is given by

$$r = \sqrt{R_c^2 + t^2} \quad . \tag{9}$$

It can be noticed that this expansion law can be rewritten in manifestly Lorenz-invariant form $r^2 - t^2 = R_c^2$. Therefore the trajectory of the bubble wall spans an invariant hyperboloid in the space-time. This in fact implies that the expanding critical bubble is a Lorenz-invariant object[8]. In particular its center is at rest for any inertial observer.

One may have noticed that reduction of evolution of the field to dynamics of one degree of freedom, bubble radius, besides the thin wall approximation also heavily relies on the assumption that in the process of expansion other degrees of freedom of the field are not excited. In other words, it is assumed that the energy does not dissipate into production of particles in the course of evolution of the bubble. In this case all the energy released by conversion in the volume swept by the bubble of the false vacuum into the true one is spent on accelerating the bubble walls. This assumption, perfectly self-consistent within the discussed reduction procedure, has been analyzed and proven to be correct to a great degree of accuracy: dissipation of energy of the bubble is strongly suppressed and negligible both due to the classical generation of waves [9,10] as well as due to quantum creation of particles[11]. A detailed discussion of this point is however beyond the scope of the present lecture.

2.3. Quantum tunneling rate

One can see from Eq.8 that there is no real solution for the momentum if the radius is less than R_c, so that the region $r < R_c$ is classically forbidden. Thus the

path from $r = 0$ (initial state of the false vacuum with no true vacuum present) to $r = R_c$ corresponds to the quantum tunneling through the classically forbidden region, and we can apply the standard quantum mechanical formula for the barrier penetration rate. The rate is given by $\Gamma \sim \exp(-B)$, where $B = 2\int_0^{R_c} |p(r)|\, dr$ with $p(r)$ being the result of the analytical continuation of the classical expression for the momentum (in our case given by Eq.8) to the classically forbidden domain. Using Eq.8, we find for the barrier factor

$$B = 2\int_0^{R_c=3\mu/\epsilon} 4\pi r^2 \mu \sqrt{1 - \frac{r^2}{R_c^2}}\, dr = \frac{\pi^2}{2}\mu R_c^3 = \frac{27}{2}\pi^2 \frac{\mu^4}{\epsilon^3} \quad , \tag{10}$$

which represents the final result for the exponential suppression factor in the decay rate of the false vacuum in TWA. This formula is derived for the realistic case of a (3+1) dimensional theory. One can however readily derive similar formulas for any number of space dimensions accounting for the appropriate volume and surface area expressions in terms of the bubble radius r.

One can notice that the final expression in Eq.10 in fact contains a very large numerical factor. Indeed, in terms of the simple scalar model with the potential in Eq.1 the factor B can be written as

$$B = \frac{2^{11}\pi^2}{3}\lambda^{-1}\left(\frac{V_{max}}{\epsilon}\right)^3 \approx 6737.6\,\lambda^{-1}\left(\frac{V_{max}}{\epsilon}\right)^3 \quad , \tag{11}$$

where $V_{max} \approx \lambda v^4/4$ is the maximal energy density of the field ϕ in between the two vacua (i.e. near $\phi = 0$). Since small ϵ implies in particular $\epsilon \ll V_{max}$ and also it is reasonable to imply that λ is at least not bigger than of order one, it can be concluded that the barrier factor is very large and that the decay rate of the false vacuum is extremely small.

3. Euclidean space formalism

3.1. Euclidean path integral formulation. Bounce solution.

The derivation of the exponential factor in the decay rate described so far heavily relies on TWA in order to reduce the problem to that of tunneling in quantum mechanics of one degree of freedom. A more general approach[8,12] is based on a re-formulation of the problem in the Euclidean space time and is based on a path integral calculation of the imaginary part of the energy E_{vac} of the false vacuum. The imaginary part of the energy is related to the decay rate in the standard way: $\Gamma = -2\,\mathrm{Im}\,E_{vac}$. The vacuum energy is expressed through the partition function Z as $E_{vac}T = -\ln(Z)$, with T being the (large) normalization time interval, and the partition function itself is calculated by means of the path integral:

$$Z = \int e^{-S[\phi]}\mathcal{D}\phi \quad , \tag{12}$$

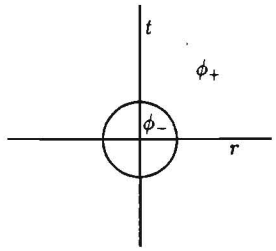

Figure 2: An $O(4)$ symmetrical solution to the Euclidean field equations in TWA. A spherical thin wall separates the outer region of the false vacuum from the inner one filled with the true vacuum.

where the integration is performed over all configurations of the fields, generically denoted as ϕ, that approach their false-vacuum values at the Euclidean space-time infinity. (The latter condition ensures that it is the energy of the false vacuum that is calculated.)

The path integral in this setting has a non-trivial saddle point configuration, solution to the classical field equations, associated with the instability of the false vacuum. It is easiest to see the existence of such configuration in TWA and then to discuss a more general situation. The expansion law for the critical bubble in Eq. 9 when continued to Euclidean time ($t \to i\,t$) corresponds to an $O(4)$ symmetrical configuration $r^2 + t^2 = R_c^2$, which describes the wall separating the false and the true vacua wrapped into a sphere S_3 in the four-dimensional Euclidean space (Figure 2).

It is straightforward to find in TWA the action of an $O(4)$ symmetrical configuration in which the wall is wrapped into a sphere S_3 with radius R. The Euclidean action for such configuration can be written as

$$S = \mu \cdot (Surface\ Area) - \epsilon \cdot (Volume) = 2\,\pi^2\,R^3\,\mu - \frac{\pi^2}{2}\,R^4\,\epsilon \quad . \tag{13}$$

The action has an extremum ($dS/dR = 0$) at $R = R_c = 3\mu/\epsilon$ where the value of the action is $S_0 = 27\,\pi^2\,\mu^4/(2\,\epsilon^3)$, which coincides with the barrier factor B found in Eq. 10.

The latter 'coincidence' is of no surprise, since in fact the Euclidean configuration describes the motion of the bubble wall 'under the barrier': the bubble starts with zero radius at the Euclidean time $-R_c$, reaches the classical turning point $r = R_c$ at $t = 0$, where it "bounces" back and contracts to zero. For this reason the saddle point configuration in this problem is dubbed[8] "bounce". Clearly at the turning point the evolution of the bubble radius can be matched, instead of with contraction trajectory in the Euclidean space, with an expansion trajectory in the Minkowski

space, described by the evolution law in Eq.9. In other words the bounce solution in the Euclidean space and the classical expansion in the Minkowski space are parts of one analytical solution in the complex time.

One can also notice that the extremum of the action provided by the bounce of radius R_c is in fact not a minimum of the action but rather a maximum in the variable R. This implies that the second variation of the action at this configuration has at least one negative eigenmode corresponding to variation of R. It turns out that this is the only negative mode, and, as will be discussed, it is due to this negative mode the energy of the false vacuum acquires an imaginary part, corresponding to the decay rate.

The Euclidean-space formalism allows in general to go beyond TWA and consider a general case when the asymmetry ϵ between the vacua is not small. It was conjectured[8] and later formally proven[7] that the relevant saddle point configuration for the action is always $O(4)$ symmetric[1]. The symmetry then reduces the field equations to ordinary differential equations in one variable $z = \sqrt{r^2 + t^2}$, which are more tractable numerically. We will discuss some considerations regarding such calculations later in the lecture.

As stated before the path integral in Eq.12 receives contribution from all field configurations approaching the false vacuum at infinity. These configurations include also the multi-bounce ones. However if among these only the configurations with far separated bounces are essential their contribution can be summed up[12] in a standard way in the exponent of a one-bounce contribution Z_1: $Z = Z_0 \exp(Z_1)$, where Z_0 is the contribution of the zero-bounce configurations. The assumption that the bounce effects in the partition function are dominated by widely separated independent bounces, which do not interfere with each other, is justified when the probability of the false vacuum decay is small and the critical bubbles nucleate independently from one another in the volume of the false vacuum. The effects of interfering bounces would become essential in the situation where the bubbles nucleate on top of each other, i.e. when the probability of nucleation is high and the metastable vacuum is in fact on the verge of total instability. This regime has not been studied thus far, and most probably it is beyond the applicability of the quaziclassical method. Moreover it is not clear whether in such situation the whole problem makes physical sense, since only in the case os sufficiently long lifetime it is justified to approximately consider a metastable state as one quantum state. In the rest of this lecture we will discuss under the reasonable assumption that the rate of decay of the false vacuum is small enough, so that nucleation of a critical bubble is a rare occurrence and simultane-

[1] In TWA the O(4) symmetry of the saddle point for the action wralitten in terms of the surface area and the volume is obvious. Also the $O(4)$ symmetry of the bounce can be argued on physical grounds: The expanding configuration in Minkowski space is an analytical continuation of the Euclidean bounce solution. The $O(4)$ symmetry of the bounce then corresponds to the Lorenz-invariance of the Minkowski space configuration. If the spontaneously nucleating configurations were not Lorenz-invariant, one would have to integrate over the non-compact Lorenz group, which would lead to a meaningless divergent result.

ous nucleation of two or several bubbles overlapping in space and time can be safely neglected. Thus our working formula for the decay rate in terms of the one-bounce contribution will read as

$$\Gamma = 2\,\mathrm{Im}\, Z_1/T \quad . \tag{14}$$

3.2. Pre-exponential factor in the decay rate

The action S_B at the saddle point, the bounce, determines the exponential factor $\exp(-S_B)$ in the decay rate. The standard procedure for the saddle-point calculation of the path integral gives in the next order of expansion around the stationary configuration the expression also for the pre-exponential factor:

$$Z_1 = N\,\left[\det(S^{(2)})\right]^{-1/2} \exp(-S_B) \quad , \tag{15}$$

where $S^{(2)}$ is the operator of the second variation of the action around the saddle point configuration $\phi(x) = \bar{\phi}(x)$: $S^{(2)} = \delta^2 S[\phi]/\delta\phi^2|_{\phi=\bar{\phi}}$. The factor N in Eq.15 is, as usual, the normalization factor in the path integral. For the problem discussed this factor is set by the perturbation theory in the false vacuum, so that in the same one-loop order as Eq.15 this factor is[12] $N = \left[\det(S_0^{(2)})\right]^{1/2}$ with $S_0^{(2)}$ being the second variation of the action around 'flat' false vacuum, i.e. around the trivial configuration where the fields have their false vacuum values.

The formula in Eq.15 in the problem under discussion however requires certain modification[12]. The most essential one arises from the fact that the second variation of the action around the bounce has, as discussed, a negative mode. Formally this would mean that the path integral for the partition function in the false vacuum does not converge on real fields. As explained in detail by Callan and Coleman[12] this is due to the fact that strictly speaking the false vacuum state does not exist as a quantum state of the system (similar to any resonance state) and its energy can be calculated only in the sense of analytical continuation of the parameters of the theory from the domain, where this state is stable. A consideration of such analytical continuation results in the prescription, which allows to calculate the imaginary part of the energy of such resonance state: The imaginary part arises from the steepest descent integration over the amplitude of the negative mode, for which the steepest descent path runs parallel to the imaginary axis (hence the imaginary part). The integration runs from the real axis, where the saddle point value resides, thus it spans only one half of the Gaussian bell curve. Therefore the contribution of the negative mode comes with additional factor 1/2 as compared to positive modes. As a result the correct formula for the decay rate at the one loop level reads as

$$\Gamma = \frac{1}{T}\left|\frac{\det(S^{(2)})}{\det(S_0^{(2)})}\right|^{-1/2} \exp(-S_B) \quad . \tag{16}$$

Notice that the factor 2 from Eq.14 is cancelled by the discussed additional factor 1/2 in the contribution of the negative mode. Also the sign is chosen to be positive: this

is a usual ambiguity in the choice of the imaginary part of the energy of the resonance state which is resolved by the physical requirement that we want the state to decay in time rather than to be spontaneously created.

In addition to the negative mode the bounce configuration has d zero modes, with d being the number of the space-time dimensions, which modes are due to the translational invariance of the field equations. Thus the profile of these modes is manifestly given by $\partial_\mu\bar{\phi}(x)$, where $\bar{\phi}(x)$ is the profile of the fields in the bounce solution. Integration over these modes is not Gaussian and applying the Gaussian formula in Eq.16 would result in an infinite expression due to these modes, which make the determinant $\det(S^{(2)})$ vanish. However the integration over the space-time coordinates of the bounce can be done explicitly[12], rather than in the Gaussian approximation, with the well known result, which also has numerous applications in the instanton calculus and other problems related to classical solutions of the field equations[2]. The integration over the space-time coordinates results in the factor proportional to the normalization space-time volume VT and the coefficient (the integration measure) is expressed trough the action of the classical solution and is $(S_B/(2\pi))^{1/2}$ per each dimension. Thus in the (3+1) dimensional case the final formula for the rate of nucleation of the critical bubbles in the false vacuum reads as[12]

$$\frac{\Gamma}{V} = \left(\frac{S_B}{2\pi}\right)^2 \left|\frac{\det'(S^{(2)})}{\det(S_0^{(2)})}\right|^{-1/2} \exp(-S_B) \quad , \tag{17}$$

where the notation det′ means that the translational zero modes are removed from the spectrum of the operator in calculation of the determinant. One can note that Eq.17 reflects the natural fact that the probability of nucleation of a critical bubble is proportional to the space-time volume, thus a finite physical quantity is the probability per unit time per unit volume, i.e. the rate per unit volume.

3.3 Pre-exponent in (1+1) dimensional theory.

The formula in Eq.17 solves the problem of calculating the pre-exponent in a closed but a somewhat symbolic form, since it is not quite straightforward to calculate the functional determinant around a solution, which is known only approximately. I am not aware of any analytical calculation of this determinant in any theory in (3+1) dimensions. However it turns out that in (1+1) dimensions the pre-exponent can be found in a closed and universal form.

First direct calculation of the pre-exponential factor using the Callan-Coleman formula in Eq.17 was done[14] for the theory described by the potential in Eq.1 in (1+1) dimensions in TWA. The result of this calculation takes a remarkably simple form when expressed in terms of the one-loop renormalized parameters μ and ϵ:

$$\frac{\Gamma}{V} = \frac{\epsilon}{2\pi}\, e^{-\pi\mu^2/\epsilon} \quad . \tag{18}$$

[2]For a pedagogical-style discussion of the integration over symmetry modes of a classical solution see e.g. the review 13.

The simplicity of the result invites a suggestion that it can be derived in a quite universal way. Indeed, it turns out[15] that this expression for the decay rate is true for any (1+1) dimensional theory, and also beyond one loop to any finite order of perturbation theory in the small parameter ϵ/μ^2, though the applicability of this formula is still restricted by the TWA. It can be reminded in this connection that in a (1+1) dimensional theory the requirement of renormalizability does not restrict the potential to the at most quartic form as in Eq.1, and one can have arbitrarily complicated expression for the potential in terms of ϕ which can have many local minima. The formula in Eq.18 is applicable for transition between any two neighboring vacua.

To derive Eq.18 in a model-independent way, consider the bounce calculation in a 2-dimensional Euclidean space. In TWA the bounce is determined by the boundary between the phases ϕ_+ and ϕ_-, which boundary has the form of a circle of the radius $R_c = \mu/\epsilon$. The spectrum of modes around the bounce consists of two distinct parts:

hard modes, whose eigenvalues are set by the boson mass m^2 and which correspond to waves propagating in the bulk of the phases and to excitations of the profile of the field across the bounce wall and

soft modes, whose eigenvalues are of order R_c^{-2} (thus these eigenvalues are much smaller than in the hard series in view of the TWA condition $R_c\, m \gg 1$) and which correspond to deformations of the shape of the bounce wall generally into some closed curve γ; the latter modes also include the negative mode and the two translational zero modes.

Using this separation of the modes one can consider the path integral in the one-bounce sector in two steps: fix the closed curve γ describing the position of the wall and integrate first over all the rest variations of the field, corresponding to hard modes. Then one is left with the integral over the closed curves γ. The latter integral after the hard modes are integrated out in general has the form

$$Z_1 = \int e^{-S_{eff}[\gamma]}\, J[\gamma]\, \mathcal{D}\gamma \ , \tag{19}$$

where $S_{eff}[\gamma]$ is the effective action depending on the shape of the curve γ arising after integration over the hard modes, and $J[\gamma]$ is the appropriate integration measure, which is yet to be determined. In general the effective action for the closed curve on a plane can be expanded as

$$S_{eff} = \mu \cdot Perimeter[\gamma] - \epsilon \cdot Area[\gamma] + (\text{curvature dependent terms}) \ . \tag{20}$$

The curvature-dependent terms come with coefficients containing powers of $(R_c\, m)^{-1}$ and disappear in the TWA. Thus only the first two terms with the perimeter of the curve and the area enclosed by it remain in this approximation, which gives the effective action for the soft modes. According to the Appelquist-Carazzone decoupling theorem[16] the path integral is given by that in the low-energy effective theory with the parameters of the effective theory being renormalized by the high-energy sector, provided that the low energy theory is renormalizable by itself. In this case the

condition of renormalizability is definitely fulfilled, since the theory of curves on a plane is equivalent to quantum mechanics of one degree of freedom and there in fact is no renormalization at all of the parameters μ and ϵ coming from the low-energy sector. By comparing the effective action in Eq. 20 with that used before in the general consideration of the false vacuum decay in Eq.13 one identifies μ as the surface tension of the bounce wall and ϵ as the difference in energy density between the two vacua. Naturally, in two dimensions the perimeter of the curve replaces the surface area of the bounce and the area enclosed by the curve replaces the volume inside the bounce wall. Also due to the (1+1) dimensional specifics, μ can be called surface tension only in the sense of action per unit length in the space-time. In the Minkowski space picture instead of an extended wall of a critical bubble one has a localized one dimensional soliton, kink, interpolating between two vacua, and μ is the mass of kink, while the other wall of the bubble is an anti-kink. The expansion of the bubble in the one dimensional space is in fact a kink and anti-kink accelerating away from each other.

To calculate the path integral over the closed curves γ with the effective action from Eq.20 let us use the parametrization of the curves in the polar coordinates: $r(\theta)$, where θ is the polar angle. This parametrization does not allow to take into account the curves which are strongly deformed from a circle, but this parametrization is sufficient for all curves in a finite neighborhood of a circle, which will be sufficient for our present purpose. The low energy action then takes the form

$$S_{eff} = \int_0^{2\pi} (\mu\sqrt{r^2 + \dot{r}^2} - \frac{1}{2}\epsilon\, r^2)\, d\theta \tag{21}$$

with $\dot{r} = dr/d\theta$.

Consider now the problem as a standard mechanical problem in which θ is (periodic) time. Then the path integral for Z_1 can be written together with the proper measure if it is rewritten in the canonical form of path integral in the phase space:

$$Z_1 = \int \exp\left(-\int p\, dq + \int H\, d\theta\right) \frac{\mathcal{D}p\,\mathcal{D}r}{2\pi} \ , \tag{22}$$

where the canonical momentum and the Hamiltonian are found from the Lagrangian in Eq.21:

$$\begin{aligned} p &= \mu\, \dot{r}/\sqrt{r^2 + \dot{r}^2} \\ H &= \frac{1}{2}\epsilon\, r^2 - r\sqrt{\mu^2 - p^2} \ . \end{aligned} \tag{23}$$

Clearly the expression for p implies the restriction $|p| \le \mu$, which in fact is the limitation due to the used polar parametrization of the curves. This limitation however leaves a finite domain of variables around the saddle point located at $p = 0,\ r = \mu/\epsilon$.

The Hamiltonian in Eq.23 can in fact be transformed to a purely quadratic form by a simple canonical transformation $(r,\ p) \to (q,\ p)$ with $q = r - \sqrt{\mu^2 - p^2}/\epsilon$:

$$H(q,p) = \frac{p^2}{2\epsilon} + \frac{\epsilon}{2} q^2 - \frac{\mu^2}{2\epsilon} \ , \tag{24}$$

which up to the constant $-\mu^2/(2\epsilon)$ is a Euclidean-space Hamiltonian for a harmonic oscillator with the frequency $\omega^2 = -1$.

For a general frequency ω and the evolution time $T = 2\pi$ the partition function Z_1 could be readily read off that for a harmonic oscillator:

$$Z_1(\omega, T = 2\pi) = \frac{1}{2}\,\frac{1}{2\sinh(\omega\, T/2)}\, e^{-\pi\,\mu^2/\epsilon} = \frac{1}{4\sinh(\omega\,\pi)}\, e^{-\pi\,\mu^2/\epsilon} \quad , \tag{25}$$

where the extra factor 1/2 is due to the discussed halving of the contribution of the negative mode and the factor $\exp(-\pi\,\mu^2/\epsilon)$ is obviously due to the constant term in the Hamiltonian.

However one cannot set ω equal to i in the latter expression in Eq.25 because then the denominator becomes zero. This divergence is due to the two zero modes $r = R_c + a_1 \cos\theta$ and $r = R_c + a_2 \sin\theta$, which correspond to translation of the center of the bounce by the vector $a = (a_1, a_2)$. One way, not directly appealing to the standard technique of integration over zero modes, of dealing with this infrared divergence is the following. Let us regularize this divergence by artificially introducing the dependence of the probability of the critical bubble nucleation proportional to $\exp(-\lambda\, x^2)$, where λ is a small regularizing parameter. Then integrating the rate per unit volume Γ/V with this weight factor over the Euclidean space gives

$$\int \frac{\Gamma}{V}\, e^{-\lambda\, x^2}\, d^2x = \frac{\pi}{\lambda}\,\frac{\Gamma}{V} \quad . \tag{26}$$

On the other hand introducing the position dependence in the path integral for Z_1, i.e. the dependence on the shift vector a amounts to shifting the frequency of the equivalent oscillator from $\omega^2 = -1$ to $\omega^2 = -1 + 2\lambda/(\pi\,\epsilon)$. Thus using Eq. 25 one finds the regularized Z_1:

$$Z_1 = \frac{1}{4\sinh\left[i\,\pi\,\left(1 - \frac{\lambda}{\pi\epsilon}\right)\right]}\, e^{-\pi\,\mu^2/\epsilon} \quad . \tag{27}$$

Comparing now at small λ this expression with Eq.26 one arrives at the final result given by Eq.18.

Notice that due to the purely quadratic form of the Hamiltonian in Eq. 24 the path integration is purely Gaussian in a finite neighborhood of the saddle point $p = 0$, $q = 0$. Therefore the result of Eq.18 is not subject to power series corrections in the dimensionless parameter ϵ/μ^2 of the low-energy effective theory. The only corrections, which can arise in the low-energy theory, in particular due to the restriction of the region of integration, are exponentially suppressed in this parameter, i.e. they contain higher powers of the exponent $\exp(-\pi\,\mu^2/\epsilon)$. It is quite natural that such corrections should appear due to the effects of overlapping critical bubbles. Their existence is also confirmed by one exact result[17] for the (1+1) dimensional theory of the "Sine-Gordon staircase", i.e. in the theory with the potential

$$V(\phi) = \frac{\alpha}{\beta^2}\,(1 - \cos(\beta\,\phi)) - c\,\phi \quad . \tag{28}$$

In this model each 'vacuum' state is metastable and its decay rate can be found by the "fermionization" technique i.e. from the formal equivalence of the Sine-Gordon theory to the massive Thirring model of fermions. Within this equivalence the tilt term $c\,\phi$ is equivalent to an external electric field. The decay rate is then given by the rate of creation of the fermion - antifermion pairs by the external electric field. For a particular value of the coupling $\beta^2 = 4\pi$ the interaction in the equivalent Thirring model vanishes, thus the problem is reduced to calculating the rate of creating free fermion pairs by an external classical electric field, which can be found exactly. The result[17] in terms of μ and ϵ is

$$\frac{\Gamma}{V} = -\frac{\epsilon}{2\pi}\,\ln\left(1 - e^{-\pi\,\mu^2/\epsilon}\right) \quad , \tag{29}$$

which manifestly displays the one-bubble expression of Eq.18 as well as higher exponential terms.

3.4. Influence of soft modes of other fields

As discussed above the decay of the false vacuum in a (1+1) dimensional theory amounts to creation of a soliton - antisoliton pair due to the excess in the false vacuum energy. It is not infrequent in multifield models that some bosonic or/and fermionic fields develop zero of soft modes localized on the soliton (but still it is assumed that the particles are sufficiently massive in the bulk of either of the vacuum phases in order to justify TWA). For example, a fermion field necessarily develops a zero energy mode on the kink, if the fermion mass term changes sign between the two vacua. In such situation the effective low-energy theory considered in the previous calculation has to be modified to include these soft modes.

One can view the situation with a soft level on the soliton as that the theory possesses a spectrum of solitons, corresponding to different values of the occupation number n of the soft level, with masses

$$\mu_n = \mu + E_n \ , \tag{30}$$

where E_n is the energy of the state with occupation number n. The final result[18] for the influence of the soft modes on the rate of the false vacuum decay is that this rate is given by the sum of contributions of the whole spectrum of the solitons:

$$\frac{\Gamma}{V} = \sum_n \frac{\Gamma_n}{V} = \frac{\epsilon}{2\pi}\sum_n \exp(-\pi\mu_n^2/\epsilon) \quad . \tag{31}$$

In particular for the situation with fermion zero mode, where the occupation number is $n = 0$ or $n = 1$ with $E_n = 0$, the probability of the decay of false vacuum doubles[19].

In order to derive the formula in Eq.31 consider the effective low-energy expression for the partition function Z_1 after integration over the hard modes. This expression contains integration over the closed curves γ as well as over the fields χ of soft modes:

$$Z_1 = \int \exp(-S_{eff}[\gamma] - S[\chi, \gamma])\, J[\gamma]\, \mathcal{D}\chi\, \mathcal{D}\gamma \quad . \tag{32}$$

Consider now the integration over the fields χ with fixed γ. The fields χ depend on the length parameter l along γ and are periodic (antiperiodic) for bosons (fermions) in l with the period equal to the perimeter P of the curve γ. Therefore the path integral over the soft modes is formally equal to the partition function of the soft modes at the temperature $T = P^{-1}$. Thus one can write the result of the integration over the χ degrees of freedom as

$$Z_1 = \sum_n \int \exp(-S_{eff}[\gamma] - E_n P)\, J[\gamma]\, \mathcal{D}\gamma \quad . \tag{33}$$

Each term in the sum is an integral identical to that in Eq.19 with μ being replaced by $\mu_n = \mu + E_n$. Therefore one immediately gets the formula in Eq.31.

3.5. Beyond the thin wall and multi-field case

As discussed above, to find the bounce beyond the TWA in a d dimensional theory one should in general consider an $O(d)$ symmetrical solution to the classical field equations, which approaches the false vacuum at the Euclidean infinity. The $O(d)$ symmetry means that the solution depends only on one variable $z = \sqrt{r^2 + t^2}$ and in the case of one scalar field ϕ with the potential $V(\phi)$ this amounts to solving the equation

$$\frac{d^2\phi}{dz^2} + \frac{(d-1)}{z}\frac{d\phi}{dz} = \frac{dV}{d\phi} \quad . \tag{34}$$

The boundary conditions being $d\phi/dz = 0$ at $z = 0$ and $\phi(\infty) = \phi_+$. The existence of a non-trivial solution to this equation can be deduced from a mechanical analogy in which z plays the role of time and ϕ is the coordinate of a particle, moving in the *inverted* potential V from z to $z = \infty$ with a z dependent friction term $\frac{(d-1)}{z}\frac{d\phi}{dz}$ (Figure 3). In the inverted potential the two minima of V turn into two hills, the one corresponding to the false vacuum ϕ_+ being lower than that corresponding to ϕ_-. At $z = \infty$ (Euclidean infinity) the 'particle' should stop at the lower hill. If at $z = 0$ the motion starts with zero velocity very close to the top or the higher hill the 'particle' spends long 'time' z having small velocity and will dissipate too little energy, so that it will overshoot the lower hill. If on the contrary the motion starts at $z = 0$ too low down the upper hill it quickly acquires large speed and dissipates too much energy by friction, so that it will never reach the top of the lower hill. Clearly, there exists a point near the top of the upper hill, starting from which the 'particle' dissipates due to friction just the right amount of energy to reach the top of the lower hill at $z = \infty$. It is quite straightforward to implement this trial method as a 'shooting' method in a numerical integration of Eq.34.

The problem of numerically finding the profile of the bounce becomes much more complicated once the vacuum states are determined by a combination of several fields. Already in the case of two real scalar fields ϕ_1 and ϕ_2 the potential $V(\phi_1, \phi_2)$ is a two-dimensional relief in the field space and the shooting method can not be applied for finding the appropriate starting point at $z = 0$ in the vicinity of the true vacuum

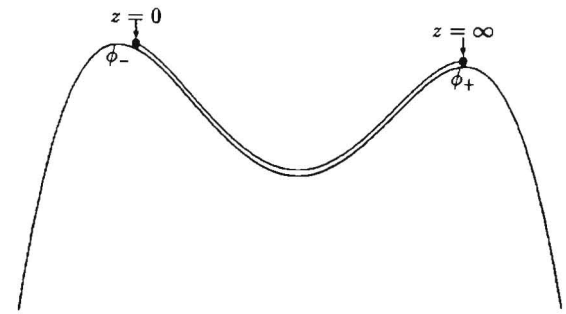

Figure 3: Solution to Eq.34 in a mechanical analogy: The 'particle' starts having zero velocity somewhat to the right of ϕ_- at the 'time' $z = 0$ and slides down the left hill loosing due to friction just the right amount of energy in order to end up exactly on top of the right hill at $z = \infty$.

in such a way that at $z = \infty$ the motion ends up exactly in the false vacuum. The problem of numerically finding the trajectory is further complicated by the fact that the sought trajectory in the field space is fundamentally unstable: it does not correspond to the minimum of the action. A recently suggested technique[20] to resolve the instability problem is based on replacing the initial action by a modified functional, for which the bounce solution for the initial action provides a local minimum. Once this goal is achieved the problem of finding numerically the bounce solution can be done by a relaxation method.

This technique is based on special virial properties of the bounce solution. Consider the bounce problem in a theory of n real scalar fields ϕ_i, $i = 1, \ldots, n$ with the potential $V(\phi_i)$. The bounce in four space-time dimensions is a solution $\bar{\phi}_i(z)$ to the field equations

$$\frac{d^2\phi_i}{dz^2} + \frac{3}{z}\frac{d\phi_i}{dz} = \frac{\partial V}{\partial \phi_i} \tag{35}$$

with the boundary conditions $d\bar{\phi}_i/dz = 0$ at $z = 0$ and $\bar{\phi}_i(\infty)$ is at one of the (possibly many) local minima of V. By shifting the definition of the fields one can always set to zero the value of the fields in the local minimum considered (the particular false vacuum): $\bar{\phi}_i(\infty) = 0$. Then by also removing an additive constant in the potential, one can write the most generic renormalizable potential in a 4-dimensional field theory as a sum of quadratic, cubic and quartic parts: $V(\phi_i) = V_2(\phi_i) + V_3(\phi_i) + V_4(\phi_i)$ (the linear part vanishes after the shift of the fields setting a local minimum of V at $\phi_i = 0$).

The action S_B for the bounce solution can be written as a sum $S = T + V_2 + V_3 + V_4$

of the terms coming respectively from the kinetic part: $T = \int \frac{1}{2} (\partial_\mu \phi_i)^2 \, d^4x$ and from the three parts of the potential of different power in the fields: $V_n = \int V_n(\phi_i) \, d^4x$. For the bounce solution there are certain relations between the terms in the action, which all generally follow from the fact that since the solution $\bar{\phi}$ extremizes the action, the first variation of the action around $\bar{\phi}$ is vanishing. This implies that if one chooses a continuous path in the field space $\phi(x;\lambda)$ parametrized by λ in such way that at $\lambda = 1$ one has $\phi(x;1) = \bar{\phi}$, the first derivative of action along the path with respect to λ vanishes: $\frac{d}{d\lambda} S[\phi(x;\lambda)]\,|_{\lambda=1} = 0$. In particular, choosing the family of the fields as $\phi_i(x;\lambda) = \lambda^p \, \bar{\phi}_i(\lambda^q \, x)$, so that in the 4-dimensional space-time

$$S[\phi_i(x;\lambda)] = \lambda^{2(p-q)} + \sum_{n=2}^{4} \lambda^{np-4q} \, V_n \quad , \tag{36}$$

and applying the condition that the derivative of action over λ vanishes at $\lambda = 1$, one finds the relations[20]

$$\begin{aligned} \Lambda_1 &\equiv T + 2(V_2 + V_3 + V_4) = 0 \\ \Lambda_2 &\equiv 2V_2 + V_3 = 0 \end{aligned} \tag{37}$$

The first equation is a quite known general one, while the second is specific to the particular polynomial form of the potential. Additional similar relations[20] arise from considering rotations of pairs of the fields: $\bar{\phi}_k(x) + i\,\bar{\phi}_l(x) \to e^{i\lambda}(\bar{\phi}_k(x) + i\,\bar{\phi}_l(x))$. Vanishing of the first derivative of the action under these transformations requires the relations

$$\Lambda_3^{(k,l)} \equiv \int \left(\phi_k \frac{\partial}{\partial \phi_l} - \phi_l \frac{\partial}{\partial \phi_k} \right) V(\{\phi\}) \Big|_{\phi=\bar{\phi}(x)} d^4x = 0 \quad . \tag{38}$$

The functional $\tilde{S}$, which coincides with the original action S on the bounce solution and for which the bounce provides a local minimum[20], is constructed as

$$\tilde{S}[\phi] = S[\phi] + \sum_n \left(\alpha_{1,n} \, |\Lambda_1|^{p_n} + \alpha_{2,n} \, |\Lambda_2|^{p_n} + \alpha_{3,kl,n} \, |\Lambda_3^{(k,l)}|^{p_n} \right) \quad , \tag{39}$$

where all the coefficients α and powers p_n are non-negative. In fact for stabilizing the problem it is sufficient to choose $\tilde{S} = S + \alpha_1 \, |\Lambda_1|$ with a positive α_1 while the additional terms in Eq.39 may improve convergence in a specific numerical implementation.

4. Induced decay of false vacuum

If there are particles present in the false vacuum, their energy can be transferred to the degrees of freedom, corresponding to nucleation of bubbles, and thus the false vacuum decay may be catalyzed. At least two interesting cases of such catalysis can be discussed: false vacuum decay at finite temperature that is at the microscopic level

associated with multi-particle collisions and the decay catalysis by few-particle states. The decay of false vacuum in which particles are present in the thermal equilibrium at a temperature T is in fact a first-order phase transition and can be considered within the standard methods of thermodynamics[1,2]. By the well-known Matsubara technique this amounts to solving the bounce problem in Euclidean space-time with the condition that the bounce solution is periodic in the Euclidean time with the period $\beta = T^{-1}$ instead of requiring that the fields approach their false vacuum values at infinity in time as well as at infinity in space. The exponential factor in the transition rate is then given by $\exp(-S_\beta)$ with S_β being the action over the period β. Naturally at a finite temperature such solution does not any longer have an $O(4)$ symmetry but rather only $O(3)$, which corresponds to the symmetry of a thermal equilibrium state. Clearly for $T = 0$ the period β becomes infinite and one returns to the discussed problem at zero temperature. In the opposite extreme, when the temperature is much larger than R_c^{-1} the period β becomes much shorter than the radius of the bounce, so that effectively the time dependence disappears and the bounce is a time-independent solution. The latter limit corresponds to the classical limit, where the transition rate is governed by the height E_{max} of the barrier as $\exp(-E_{max}/T)$. ($E_{max}\beta$ is obviously the action of a static field configuration with energy E_{max} over the time β.) We will not pursue further the subject of the false vacuum decay at finite temperature. Instead in the remaining part of this lecture we will discuss the possibility of catalyzing the decay by few particles and their collisions.

4.1. Catalysis of the decay by one particle

The simplest case of a particle-induced catalysis is where just one particle may serve as a "center of condensation" for a critical bubble. A non-trivial effect arises when the field of the particle has a soft mode bound on the bubble wall. In particular this is the case with the bosons of the scalar field, which undergoes the transition as well as for fermions, whose mass term changes sign in the transition. Consider the evolution of the wave function of a massive particle in the false vacuum. With no bounce effects the main exponential behavior over a large time T is given by $\exp(-m\,T)$, where m is the mass in the false vacuum. If however there is a bounce in the particle path the particle propagates along the bounce walls and the exponential factor in the wave function becomes $\exp[-m\,(T-D)]$, where D is the maximal size (diameter) of the bounce in the time direction (Figure 4). Therefore the modification of the wave function due to the one-bounce contribution can be written as[21]

$$e^{-m\,T} \to e^{-m\,T} + \text{const} \cdot T \, e^{-S[\phi]} \, e^{-m\,(T-D)} + \ldots \quad , \tag{40}$$

where $S[\phi]$ is the action for the bounce configuration, and the factor T arises from integration over the position of the bubble along the line of propagation of the particle. The expression in Eq.40 can be compared with the expansion of the wave function with m being replaced by a corrected mass $m+\delta m$: $e^{-(m+\delta m)\,T} = e^{-m\,T}(1-\delta m\,T+\ldots)$. From this comparison we interpret the term associated with the bounce in Eq.40 as

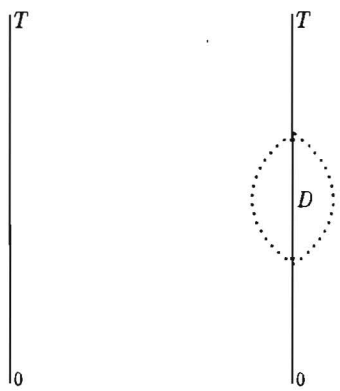

Figure 4: Propagation of a massive particle in the false vacuum without a bounce and through the bounce wall (the dotted line shows the bounce wall)

giving the mass correction δm, whose imaginary part relates to the rate of decay of the (one-particle + false vacuum) state:

$$\Gamma = -2\mathrm{Im}\, m \propto e^{-S[\phi]+m\,D} \quad . \tag{41}$$

Naturally, the factor of volume is not present in this decay rate, since the location of nucleation of the bubble is fixed by the position of the particle.

The expression in Eq.41 shows that the TWA action for the bounce problem is modified[21] with respect to the case of spontaneous decay by the term $m\,D$:

$$S_{eff} = S[\phi] - m\,D = \mu \cdot (Surface\ Area) - \epsilon \cdot (Volume) - m\,D \quad . \tag{42}$$

In general one should solve anew the problem of finding the extremum of this modified action. However for a sufficiently small m one can consider the term $m\,D$ as a perturbation, in which case in the first order its effect is given by its value on the unperturbed configuration of the spherical bounce: $m\,D = 2\,m\,R_c$. Thus in the first order in m one arrives at the estimate[22] of the catalysis factor by one particle as $\exp(2\,m\,R_c)$. The condition for applicability of this estimate is that $m\,R_c$ is small in comparison with the bounce action: $m\,R_c \ll S_B$, which still does not contradict the TWA condition $m\,R_c \gg 1$.

4.2. Induced decay in (1+1) dimensions

Beyond the first order in m the problem of finding the stationary configuration for the action in Eq.42 in d dimensions is equivalent to finding the equilibrium shape

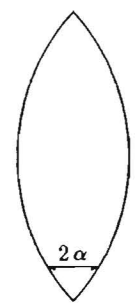

Figure 5: Equilibrium shape of the bounce for the effective action in Eq.42 in a two-dimensional theory.

of a bubble made of a $d-1$ dimensional film with the surface tension μ, having the inside pressure ϵ and also an expanding force m applied at two diametrically opposite points. In more than two dimensions this problem does not have a straightforward solution: if the film has more than one dimension it can not support any finite force applied at a point. (Physical two-dimensional films get punched in this situation.) The appropriate consideration in (3+1) dimensions will be considered later in this lecture. Here we rather discuss the equilibrium configuration of the one-dimensional bounce boundary in a two-dimensional theory, where the solution exists for a finite range of m.

In order to balance the pressure ϵ the boundary of the bounce in a two-dimensional theory should be an arc of a circle with the radius $R_c = \mu/\epsilon$, except for the points where the force m is applied (Figure 5). Then the Laplace pressure μ/R equalizes the internal one equal to ϵ. At the points, where the force is applied, the boundary makes an angle 2α such that the projection of the force due to the surface tension equalizes the external force:

$$2\,\mu\,\cos\alpha = m \quad . \tag{43}$$

Since these two conditions completely define the size and shape of the bounce, it is straightforward to find the modified effective action from Eq.42 for this extremal configuration[21]:

$$S_{eff} = \frac{\mu^2}{\epsilon}\left[2\,\arccos\left(\frac{m}{2\,\mu}\right) - \frac{m}{\mu}\sqrt{1-\frac{m^2}{4\,\mu^2}}\right] \quad . \tag{44}$$

One can readily see that for $m \ll \mu$ this expression reproduces the known limit $S_{eff} \approx \pi\,\mu^2/\epsilon - 2\,m\,\mu/\epsilon$. However with increasing m the effective action becomes

zero at $m = 2\mu$. This is interpreted as that the exponential suppression in the decay rate disappears starting from this value of the mass[3].

An interpretation of the result in Eq.44 also arises if one considers the two-dimensional version of Eq.7:

$$(H + 2\epsilon r) - p^2 = 4\mu^2 \quad . \tag{45}$$

The expression in Eq.44 is equal to the barrier rate exponential factor $2\int |p|\, dr$ calculated at the energy $H = m$. This implies that the energy stored in the particle mass m in the false vacuum is transferred without an additional exponential suppression to the soliton-antisoliton degree of freedom. Also there is no energy in the final state associated with the particle mass: the particle becomes a zero-energy mode localized on the soliton. When the particle mass m is equal to or above the soliton-antisoliton threshold 2μ it decays into the soliton pair without the WKB suppression.

It has been also shown[24, 25] that as far as the exponential in μ^2/ϵ factor is concerned the total energy E in a collision of particles acts in the same way as m in the considered case of one-particle induced decay. Therefore the exponential suppression in the parameter μ^2/ϵ in a two-dimensional theory disappears at the soliton-antisoliton threshold 2μ. However there are good reasons to believe that in the process of production of the soliton pairs by high-energy collisions of few particles there still is an exponential suppression in the coupling constant of the underlying field theory.

4.3. Induced decay in four dimensions

Extending the analysis of the induced decay of false vacuum to space-time dimension higher than two, even in the TWA, encounters the difficulty that the tunneling path from $r = 0$ to a value of r, from which a classical expansion can start does not entirely lie within the Euclidean space-time. One can see this from the plot in the Figure 6 of the 'potential energy' of a bubble in four dimensions, i.e. of the energy at $p = 0$ as given by Eq.7: $U(r) = \tilde{\mu}\, r^2 - \tilde{\epsilon}\, r^3$, where $\tilde{\mu} = 4\pi\mu$ and $\tilde{\epsilon} = 4\pi\epsilon/3$. At any energy above zero and below the top of the barrier: $E_c = \frac{4}{27}\frac{\tilde{\mu}^3}{\tilde{\epsilon}^2}$, in addition to the critical region to the right of the barrier, where the bubbles are infinitely expanding, there is a classically allowed region of subcritical bubbles, where they oscillate between expansion and contraction[4]. This is different from the behavior in two dimensions, where the potential energy derived from Eq.45 does not allow the subcritical region. Thus the trajectory for evolution of bubble from zero radius to the critical region at a positive energy goes in both Minkowski and Euclidean time, or rather in a complexified time.

[3]The pre-exponential factor in the particle-induced decay rate in two dimensions has also been calculated[23].

[4]It has been discovered in numerical simulations of mid-70s[26] that classically such bubbles oscillate many times before dissipating their energy into outgoing waves. Recently there appeared a renewed interest in the stability problem of oscillating bubbles[27].

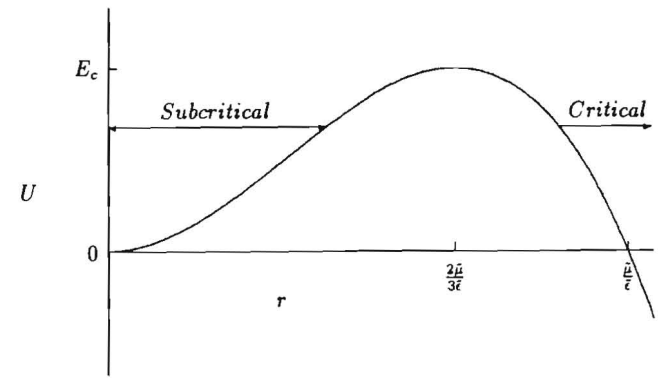

Figure 6: Potential energy of a bubble in four dimensions vs. the radius.

In order to deal with this behavior we return to the methods of traditional quantum mechanics for a system described by Eq.7 and apply the Landau WKB technique[28] to calculate the exponential power $-F(E)$ in the amplitude of creating a bubble with energy E by a highly virtual field ϕ: $|\langle B(E)|\phi|0\rangle|^2 \sim \exp(-F(E))$, with $|B(E)\rangle$ being a state of a bubble with energy E. The specific form of the field operator exciting the bubbles is chosen for definiteness: within the Landau method the exponential factor does not depend on this operator, provided that the operator is sufficiently 'smooth' in the sense, which will be discussed later. Therefore the result of this calculation[29] is applicable to situations, where the bubble is created by one particle or in a collision of few particles with energy E.

In contrast with the two-dimensional case, the result is such that the exponential suppression of the probability in the parameter $\tilde{\mu}^4/\epsilon^3$ does not disappear at any energy. The growth of the probability with energy saturates at the top of the barrier, where the magnitude of the negative exponential factor is only numerically smaller than at zero energy: $F(E_c)/F(0) \approx 0.160$.

According to Landau[28] for a system with the coordinates q the matrix element of an operator $f(q)$ between two strongly different states $|X(E_1)\rangle$ and $|Y(E_2)\rangle$ with energies E_1 and E_2: $\langle Y(E_2)|f|X(E_1)\rangle$ in the leading WKB approximation is given by

$$|\langle Y(E_2)|f|X(E_1)\rangle| \sim \exp\left[\mathrm{Re}\left(i\int_{qX}^{q^*} p(q;E_1)\, dq + i\int_{q^*}^{qY} p(q;E_2)\, dq\right)\right] \quad , \tag{46}$$

where q^* is the (generally complex) 'transition point', i.e. the point of stationary

phase of the expression

$$\exp\left(i\int_{q_X}^{q^*} p(q;E_1)\,dq + i\int_{q^*}^{q_Y} p(q;E_2)\,dq\right) \quad , \tag{47}$$

$p(q,E_1)$ ($p(q,E_2)$) are the momenta on the classical (generally complex) trajectory with energy E_1 (E_2), which runs between the points q_X and q^* (q^* and q_Y), and, finally, q_X and q_Y are points, chosen somewhere in the classically allowed regions for the states X and Y correspondingly. The particular choice of each of the latter points in a simply connected domain of the classically allowed region does not affect the real part of the integrals in Eq.46. The interpretation of the Landau formula in Eq.46 is straightforward within the approach consistently pursued in the Landau-Lifshits textbook in connection with the WKB calculation of various transition amplitudes: the amplitude is given by the exponent of the truncated classical action on the trajectory, which runs from the initial state to the final through a (complex) 'transition point'.

Few remarks are in order in connection with the application of Eq.46 in the problem discussed here. First is that this equation is written for the case, relevant to the present calculation, when the classical value of the operator f is not exponential at the 'transition point' q^*, so that the exponential factor, given by Eq.46 is not sensitive to the specific form of the operator. Second is that eq.(46) does not require the WKB approximation to be applicable for the wave functions of either of the states X and Y in the classically allowed region, i.e. where these wave functions are large. Thus it can be applied even if the lowest of the two energies, say E_1, is small, including the case $E_1 = 0$. The only condition for applicability of Eq.46 is that the states X and Y are 'strongly different' in the sense that the matrix element, given by this equation, contains large exponential power, i.e. that it is strongly exponentially suppressed. Third is that the branch of the function $p(q,E)$ in the complex plane is to be chosen so that the exponential power in Eq.46 is negative. Finally, if there are several 'transition points' q^*, only the contribution of the one which gives the maximal transition probability is to be retained.

In the matrix element $\langle B(E)|\phi|0\rangle$ the field operator with zero spatial momentum (c.m. system) translates in the effective theory of the thin-wall bubbles into the operator

$$\int(\phi(x)-\phi_+)\,d^3x \sim r^3 \quad . \tag{48}$$

Thus the whole problem can be reformulated in terms of the effective theory as a calculation of the matrix element $\langle B(E)|r^3|0\rangle$ for a system with the Hamiltonian determined by Eq.(7). Using the Landau formula one can write the exponential estimate for this matrix element as

$$|\langle B(E)|\phi|0\rangle| \sim$$
$$\exp\left[-\mathrm{Re}\left(\int_0^{r^*}\sqrt{(\tilde{\mu}\,r^2)^2-(\tilde{\epsilon}\,r^3)^2}\,dr + \int_{r^*}^{r(E)}\sqrt{(\tilde{\mu}\,r^2)^2-(\tilde{\epsilon}\,r^3+E)^2}\,dr\right)\right] =$$

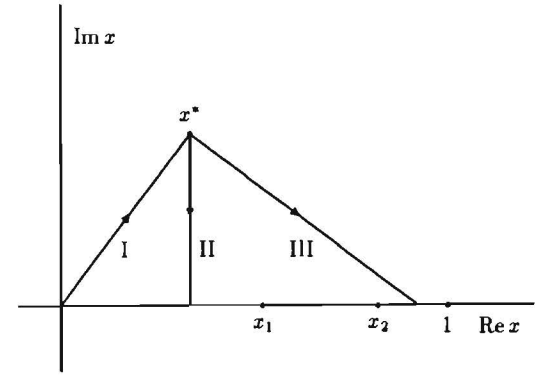

Figure 7: Classical turning points and the transition path in the Landau formula for the bubbles. $x = 1$ is the turning point on the right of the barrier at zero energy. x_1 and x_2 are the turning points on the left and on the right of the barrier at energy E. The transition trajectory starts at $x = 0$ and goes with energy $E = 0$ to the 'transition point' x^* (the link I), then it goes with energy E either to the subcritical region (the link II) or to the critical one (the link III).

$$\exp\left[-\frac{1}{\xi}\mathrm{Re}\left(\int_0^{x^*}\sqrt{x^4-x^6}\,dx + \int_{x^*}^{x(E)}\sqrt{x^4-(x^3+w)^2}\,dx\right)\right] \quad , \tag{49}$$

where instead of r and E the dimensionless variables x and w are introduced as $r = x\,\tilde{\mu}/\tilde{\epsilon}$ and $E = w\,\tilde{\mu}^3/\tilde{\epsilon}^2$ and $\xi = \tilde{\epsilon}^3/\tilde{\mu}^4$ is the small dimensionless constant in the effective theory of bubbles. In Figure 7 are shown the classical turning points for bubbles at zero energy and also for an energy $E < E_c$. At $E = 0$ the classically allowed domain consists of the region $x > 1$ and of the point $x = 0$. At a non-zero energy $E < E_c$ the classically allowed domain consists of two finite regions: to the left of barrier, $x < x_1(E)$, corresponding to subcritical bubbles, and to the right of the barrier, $x > x_2(E)$, which corresponds to infinitely expanding critical bubbles. Accordingly the final point $x(E)$ of the transition trajectory in Eq.49 can be chosen either in the subcritical domain (path I + II in Figure 7) or in the critical one (path I + III in Figure 7). The former choice produces the amplitude of the excitation of a subcritical bubble: $A_- = \langle B_{\mathrm{sub-c}}(E)|\phi|0\rangle$, while the latter choice gives the amplitude of production of an infinitely expanding critical bubble $A_+ = \langle B_c(E)|\phi|0\rangle$. In either case the transition path starts at the point $x = 0$ and with $E = 0$, which corresponds to absence of a bubble in the initial state. Strictly speaking, the thin-wall approximation is not applicable at $r = 0$. However, the inaccuracy of the approximation at the values of the radius of the order of the thickness of the wall does not affect the factors $\sim \exp(-\mathrm{const}/\xi)$, which are being considered in this calculation. In other words, the

expression in Eq.49 receives dominant contribution from the region of large r, and therefore is calculable within the thin-wall approximation. From the paths shown in Figure 7 it is clear that the amplitudes A_+ and A_- are related as

$$|A_+| = |A_-| \exp(-b(E)/\xi) \ , \tag{50}$$

where

$$b(E)/\xi = \int_{r_1(E)}^{r_2(E)} |p(r;\, E)|\, dr = \frac{1}{\xi}\int_{x_1(E)}^{x_2(E)} \sqrt{x^4 - (x^3+w)^2}\, dx \tag{51}$$

is the exponential power in the barrier penetration rate at energy E. The relation (50) can thus be interpreted as stating that the production of the critical bubble at $E < E_c$ proceeds through excitation of a subcritical one with subsequent tunneling through the barrier.

According to the expression (47) the 'transition point' x^* is determined by solution of the equation

$$\sqrt{x^4 - x^6} - \sqrt{x^4 - (x^3+w)^2} = 0 \ . \tag{52}$$

The solutions to this equation are given by the three values of the cubic root $(-\frac{w}{2})^{1/3}$. A simple inspection shows that as the appropriate 'transition point' one can choose either of the complex values of the root in the right half plane (choosing one instead of another gives the same result after proper redefinition of the branches of the expressions in Eq.49). The integrals in Eq.49 were evaluated[29] numerically to determine the functions $c(E)$ and $b(E)$, appearing in the amplitudes A_- and A_+:

$$|A_-| \sim \exp\left(-\frac{c(E)}{\xi}\right) , \qquad |A_+| \sim \exp\left(-\frac{c(E)+b(E)}{\xi}\right) . \tag{53}$$

The results of the numerical calculation are shown in Figure 8. At the critical energy E_c, corresponding to the top of the barrier, the barrier penetration term $b(E)$ vanishes. However the excitation term $c(E)$ at this energy has a finite value $c(E_c) \approx 0.0314 \approx 0.160\, b(0)$, where $b(0) = \pi/16$ is the value of the barrier penetration term for the spontaneous false vacuum decay. (In fact $c(E_c)$ can be found exactly in terms of elliptic integrals, but the final expression for the result is unusually cumbersome.)

The function $c(E)$ can be found analytically in the limit of large w as well as of small w. For large w one can neglect x^4 in comparison with x^6 and with $(x^3+w)^2$ in Eq.49 and thus find

$$c(E) = \frac{3\sqrt{3}}{4}\left|\frac{w}{2}\right|^{\frac{4}{3}} \qquad (w \gg 1) \ . \tag{54}$$

For $w \ll 1$ the expression in Eq.49 is determined by the region of x near the classical turning point $x_1(E)$. In this region one can neglect in Eq.49 x^6 in comparison with x^4 and also neglect x^3 in comparison with w. Then $c(E)$ can be found as

$$c(E) = \int_0^L x^2\, dx - \int_{\sqrt{w}}^L \sqrt{x^4 - w^2}\, dx = \frac{\sqrt{\pi}}{6}\frac{\Gamma(1/4)}{\Gamma(3/4)} w^{3/2} \approx 0.874 w^{3/2} \qquad (w \ll 1) \ , \tag{55}$$

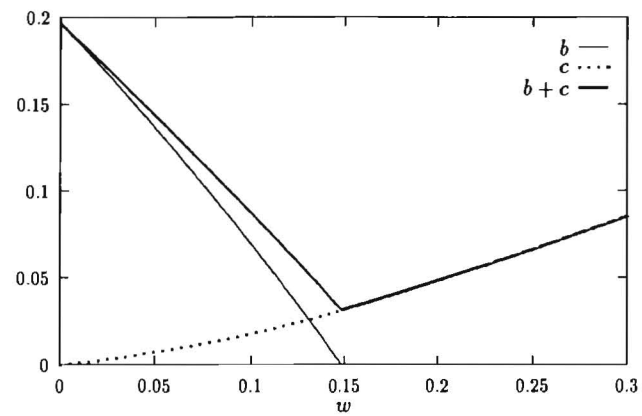

Figure 8: The barrier penetration function $b(E)$, the excitation function $c(E)$, and their sum vs. $w = E\,\tilde{\epsilon}^2/\tilde{\mu}^3$. At the point $w_c = 4/27$ and beyond the barrier disappears, hence $b(E) = 0$ and the sum coincides with $c(E)$.

where both integrals run along the real axis and L is a cutoff parameter, $L \gg \sqrt{w}$. The difference of the integrals is determined by the region $x \sim \sqrt{w}$, which substantiates the approximation, leading from Eq.49 to Eq.55. The full exponential power in the excitation amplitude A_- for small w is thus given by $c(E)/\xi = \text{const}\, E\sqrt{E/\tilde{\mu}}$ which coincides with the result for the amplitude[30] of excitation of a bubble with energy E in the case of degenerate vacua. (Clearly, in that case only subcritical bubbles exist). One should however keep in mind that the region of small w is limited from below by the condition of applicability of the thin-wall approximation, which implies that the characteristic size of the bubbles in the relevant region $r \sim \sqrt{w}\,\tilde{\mu}/\tilde{\epsilon}$ is larger than the thickness of the wall. In terms of E this translates into the condition[30] $E \gg \tilde{\mu}^{1/3}$.

It can be also noticed that at small energy the barrier penetration term

$$b(E) = \frac{\pi}{16} - w + o(w) \tag{56}$$

decreases faster than the $w^{3/2}$ growth of the $c(E)$. Therefore the probability of the induced decay of the false vacuum grows with energy in this region. As is seen from Figure 8, this behavior continues up to the top of the barrier, where $b(E)$ vanishes.

It should be also noted that though the behavior, shown in Figure 8, beyond the top of the barrier corresponds to a decreasing probability, this is an artefact of the reduction of the field theory problem to dynamics of just one degree of freedom. Indeed at an energy above E_c the system can decrease its energy down to E_c by emission of one or few hard quanta, which does not introduce an exponential behavior. Therefore

as far as the main exponential factor in the probability is concerned it remains at its value at E_c at all energies beyond the top of the barrier.

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NAME
ID NO.
MS.
BUDMAN
10795
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