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Background Gravitational Wave produced in Power-law Inflationary Universe

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Abstract

We quantize the linearized gravitational wave which arises from quantum fluctuations in the Power-law inflationary universe described by the Brans-Dicke gravity in the Einstein frame and obtain the power spectrum of the gravitational wave. As a result, we find that the obtained power spectrum of background gravitational wave differs from the previous result in the literature, which is calculated on the base of the analogy to the power spectrum of scalar field.

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1 Introduction

Since the anisotropies of the cosmic microwave background (CMB) were discovered by the COBE measurement[1], many authors have been interested in the background gravitational wave produced in the inflationary universe [2]. They suspect that the origin of the anisotropies is not only the density (scalar) perturbations but also the Sacks-Wolfe effect by the primordial gravitational waves. Unfortunately, we know that for the simplest standard (exponentially) inflation model, the contribution to CMB anisotropy from the density (scalar) perturbations is much greater than the one from the background gravitational wave [3]. We must investigate more complicated inflation models, e.g., the power-law inflation model [4]. However, many works [5] are short of detailed calculation based on the explicit description of the cosmological evolution of universe in the case of the power-law inflation models. Therefore, we consider a concrete cosmological evolution model which is described by the Brans-Dicke (BD) gravity [6] in the Einstein frame, and explore this subject in great detail.

Adopting the BD gravity, we naturally obtain the cosmological evolution of the expanding universe, which has the power-law inflation ($a \propto t^{\gamma} \sim \eta^{\frac{\gamma}{1-\gamma}}$, $\gamma > 1$), radiation dominating ($a \propto t^{1/2} \sim \eta$) and matter dominating ($a \propto t^{2/3} \sim \eta^2$) epoch. Here a is the scale factor and η is the conformal time. In this case, as well as the other power-law inflation cases [5], the power spectrum of gravitational wave is enhanced in the long wave length region. However, we find that the enhancement factor relative to the power spectrum in the standard (exponential) inflationary universe is not $\left(\frac{2H}{k}\right)^{\frac{1}{\gamma}}$ which is usually used [5], but $\left(\frac{2}{-k\eta_1}\right)^{\frac{1}{\gamma}}$, where H is Hubble constant at the inflation epoch and η_1 is the conformal time at the end of the inflation. In other words, the previous result is $(-H\eta_1)^{\frac{1}{\gamma}} \simeq 10^{\frac{20}{\gamma}}$ times greater than our result. Note that this difference has influence on the upper limit of the GUT mass, which can be inferred from refs. [2].

Contents of this paper are as follows. In section 2 we review the BD gravity and provide our power-law inflation model. In section 3 we explain the background gravitational wave and its quantization. In section 4 we investigate the power spectrum of the background gravitational wave.

The final section 5 is devoted to concluding remarks.

2 Power-law inflation

In this section we briefly review the power-law inflation. However, there are several versions of power-law inflation model [4]. We especially investigate the model using the BD gravity in the Einstein frame. The action for the BD gravity in the Einstein frame takes the form

$$S = \int d^4x\sqrt{-g}\left\{\frac{R}{16\pi G} - \frac{1}{2}g^{\mu\nu}\partial_\mu\Psi\partial_\nu\Psi + \exp(-\frac{2\Psi}{\Psi_0})\mathcal{L}_m(\phi)\right\}, \qquad (1)$$

where Ψ is the BD scalar field, Ψ_0 is the BD parameter which has mass dimension, and $\mathcal{L}_m(\phi)$ is the Lagrangian density of matter fields which lead to the first order phase transition.

We consider the homogeneous BD scalar field Ψ and the homogeneous isotropic universe described by the spatially flat FRW metric:

$$ds^2 = a(\eta)^2(-d\eta^2 + dx^2 + dy^2 + dz^2), \qquad (2)$$

where η is the conformal time. The equation of motion for Ψ and the Einstein equation are

$$\ddot{\Psi} + 2\frac{\dot{a}}{a}\dot{\Psi} = \frac{-2a^2}{\Psi_0}\exp(-\frac{2\Psi}{\Psi_0})\mathcal{L}_m(\phi). \qquad (3)$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}a^2\left\{\frac{1}{2}\dot{\Psi}^2 + \exp(-\frac{2\Psi}{\Psi_0})\mathcal{H}_m(\phi)\right\}, \qquad (4)$$

where the overdot denotes $\frac{d}{d\eta}$ and $\mathcal{H}_m(\phi)$ is the Hamiltonian density of the matter fields. We assume that the Lagrangian and Hamiltonian density are given by

$$\mathcal{L}_m(\phi) \simeq -M_{GUT}^4, \quad \mathcal{H}_m(\phi) \simeq M_{GUT}^4, \quad \eta < \eta_1, \qquad (5)$$

$$\mathcal{L}_m(\phi) \simeq 0, \quad \mathcal{H}_m(\phi) \propto a^{-4}, \quad \eta_1 < \eta < \eta_2, \qquad (6)$$

$$\mathcal{L}_m(\phi) \simeq 0, \quad \mathcal{H}_m(\phi) \propto a^{-3}, \quad \eta_2 < \eta, \qquad (7)$$

at the inflation ($\eta < \eta_1$), radiation dominating ($\eta_1 < \eta < \eta_2$) and matter dominating ($\eta_2 < \eta$) epochs, respectively. Here M_{GUT} is the GUT mass which is taken to be 10^{16} GeV [7].

To obtain the scale factor $a(\eta)$ and the BD scalar field Ψ, we impose the conditions that the scale factor, its first η-derivative and the amplitude of the BD scalar field are continuous at the transition time, $\eta = \eta_1$ or η_2. Then the scale factor and the BD scalar field are

$$a(\eta) = \left\{ \sqrt{\frac{3(\gamma-1)^2}{\gamma(3\gamma-1)}} h(-\eta) \right\}^{\frac{\gamma}{1-\gamma}}, \qquad \eta < \eta_1, \tag{8}$$

$$= \frac{a(\eta_1)}{(-\eta_1)} \left(\frac{\gamma}{\gamma-1}\right)(\eta - (2-\frac{1}{\gamma})\eta_1), \qquad \eta_1 < \eta < \eta_2, \tag{9}$$

$$= \frac{a(\eta_1)}{4(-\eta_1)} \left(\frac{\gamma}{\gamma-1}\right) \frac{(\eta - (4-\frac{2}{\gamma})\eta_1 + \eta_2)^2}{(\eta_2 - (2-\frac{1}{\gamma})\eta_1)}, \qquad \eta_2 < \eta, \tag{10}$$

$$\Psi(\eta) = \Psi_0 \ln \left(\frac{a(\eta)}{a_I}\right)^{\frac{1}{\gamma}}, \qquad \eta < \eta_1, \tag{11}$$

$$= \Psi(\eta_1), \qquad \eta_1 < \eta, \tag{12}$$

where $\gamma \equiv 4\pi G {\Psi_0}^2 \ (> 1)$, $h \equiv \sqrt{\frac{8\pi G}{3}} M_{GUT}^2 (a_I)^{\frac{1}{\gamma}}$ and a_I is an integration constant. To determine the integration constant a_I, we impose the following condition that the effective time-dependent gravitational constant $G_{eff}(\eta)$ coincides with the presently observed value G;

$$G_{eff}(\eta_0) \equiv e^{-2\Psi(\eta_0)/\Psi_0} G = \left(\frac{a(\eta_1)}{a_I}\right)^{\frac{-2}{\gamma}} G, \tag{13}$$

$$= G = (1.22 \times 10^{19} \text{ GeV})^{-2}, \tag{14}$$

where η_0 denotes the present time. From this condition (14), we see $a_I = a(\eta_1)$.

Now we can express the times η_2, η_0 in terms of the time η_1 from the following physical considerations. We demand that the reheating temperature at the end of the inflation is $T_1 = 10^{16}$ GeV $\simeq 1.16 \times 10^{29}$ K and that the temperature at which the matter density is equal to the radiation density is $T_2 \simeq 6.72 \times 10^4$K. According to the standard Big Bang cosmology the relation $a \times T = constant$ persists during the cosmological evolution [8]. From this relation we determine η_2 and η_0 as

$$\eta_2 \simeq -1.7 \times 10^{24} \eta_1, \tag{15}$$

$$\eta_0 \simeq -5.4 \times 10^{26} \eta_1. \tag{16}$$

Note that η_1 has a negative value.

3 Background gravitational wave

In this section we consider a deviation from the spatially flat FRW metric, which is denoted by $a(\eta)^2 h_{ij}$. We focus our attention to the case that the deviation h_{ij} exhibits the background gravitational wave in the weak field approximation. The approximate action for the deviation takes the form ($16\pi G \equiv 1$)

$$S_{BGW} = \int d^4x \frac{a(\eta)^2}{2} \left\{ \left(\frac{\partial h^{TT}}{\partial \eta}\right)^2 - (\nabla h^{TT})^2 \right\}, \tag{17}$$

where h^{TT} is a nonvanishing component of a transverse traceless part of the deviation h_{ij} [8].

From eq.(17), we easily perform the quantization of this system [9]. As a result, the field operator $\hat{h}^{TT}$ is given by

$$\hat{h}^{TT} = \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^{3/2}} \left\{ \hat{A}_k u_k(\eta) e^{i\vec{k}\cdot\vec{x}} + \hat{A}_k^\dagger u_k^*(\eta) e^{-i\vec{k}\cdot\vec{x}} \right\}, \tag{18}$$

where the wave number $\vec{k}$ is the comoving wave number, and $\hat{A}_k$ and $\hat{A}_k^\dagger$ are annihilation and creation operators, respectively, which satisfy the following commutation relations:

$$[\hat{A}_k, \hat{A}_{k'}^\dagger] = \delta^{(3)}(\vec{k} - \vec{k}'), \tag{19}$$

$$[\hat{A}_k, \hat{A}_{k'}] = [\hat{A}_k^\dagger, \hat{A}_{k'}^\dagger] = 0. \tag{20}$$

Note that the vacuum state $|0>$ is defined by $\hat{A}_k|0>= 0$.

The mode functions u_k are the solutions of the equation of motion:

$$\ddot{u}_k + 2\frac{\dot{a}}{a}\dot{u}_k + k^2 u_k = 0, \tag{21}$$

where the overdot denotes $\frac{d}{d\eta}$. We can explicitly solve this equation of motion (21) at the three stages of the expanding universe in our model, i.e., the inflation(I), radiation dominating(R) and matter dominating(M) epochs. The positive frequency parts of the mode functions are

$$u_{k(z)}^I = -\frac{\sqrt{\pi}}{2k^{1/2}} \frac{\sqrt{-z}}{a(z)} H^{(1)}_{\frac{3\gamma-1}{2(\gamma-1)}}(-z), \tag{22}$$

$$u_k^R(z) = \frac{\sqrt{\pi}}{2k^{1/2}} \frac{-z_1(1-\frac{1}{\gamma})}{a(z_1)} \frac{H^{(2)}_{\frac{1}{2}}(z-(2-\frac{1}{\gamma})z_1)}{\sqrt{z-(2-\frac{1}{\gamma})z_1}}, \tag{23}$$

$$u_k^M(z) = \frac{2\sqrt{\pi}}{k^{1/2}} \frac{-z_1(1-\frac{1}{\gamma})(z_2-(2-\frac{1}{\gamma})z_1)}{a(z_1)\left(z-(4-\frac{2}{\gamma})z_1+z_2\right)^{3/2}} H^{(2)}_{\frac{3}{2}}(z-(4-\frac{2}{\gamma})z_1+z_2), \tag{24}$$

where we have taken a new time variable z defined by $z \equiv k\eta$, and $H_\nu^{(1)}$ and $H_\nu^{(2)}$ are the Hankel functions. We also define that $z_1 \equiv k\eta_1$ and $z_2 \equiv k\eta_2$. The above equations (22)-(24) satisfy the normalization condition in the canonical quantization:

$$\int a(\eta)^2 [u_k e^{i\vec{k}\cdot\vec{x}}] \overleftrightarrow{\partial}_\eta [u_{k'}^\dagger e^{-i\vec{k}'\cdot\vec{x}}] d^3x = i(2\pi)^3 \delta^{(3)}(\vec{k}-\vec{k}'). \tag{25}$$

If the Bunch-Davies vacuum [10], which gives the same short-distance behavior as the one defined in the Minkowski space-time, is chosen for the initial condition, the mode functions are

$$\begin{aligned} u_{k(z)} &= u_k^I, & z < z_1, & \quad (26)\\ &= \alpha^{IR} u_k^R + \beta^{IR}(u_k^R)^*, & z_1 < z < z_2, & \quad (27)\\ &= \alpha^{IM} u_k^M + \beta^{IM}(u_k^M)^*, & z > z_2. & \quad (28) \end{aligned}$$

where α, β are the Bogoliubov coefficients which are given by

$$\alpha^{IR} \cong \frac{i}{4\sqrt{\pi}}\left(\frac{\gamma}{\gamma-1}\right)\Gamma\left(\frac{3\gamma-1}{2(\gamma-1)}\right)\left(\frac{2}{-z_1}\right)^{\frac{2\gamma-1}{\gamma-1}} \cong \beta^{IR}, \tag{29}$$

$$\begin{aligned} \alpha^{IM} &\cong \frac{\Gamma(\frac{3\gamma-1}{2(\gamma-1)})}{4\sqrt{\pi}}\left(\frac{\gamma}{\gamma-1}\right)\left(\frac{2}{-z_1}\right)^{\frac{2\gamma-1}{\gamma-1}}\left\{(1+\frac{i}{2z_2})e^{iz_2} + \frac{i\sin z_2}{4{z_2}^2}e^{2iz_2}\right\},\\ &\cong -(\beta^{IM})^*, \end{aligned} \tag{30}$$

where $\Gamma(\cdot)$ is a gamma function, and we have used the approximation, $z_1 \ll z_2$.

4 The power spectrum

The power spectrum of the relic gravitational wave at the present time $z_0 \equiv k\eta_0$, which is directly related to observations, is defined by

$$\Delta h^2(k) \equiv 16\pi G \frac{k^3}{(2\pi)^3} \int d^3r e^{i\vec{k}\cdot\vec{r}} < 0|h^{TT}(r,z_0)h^{TT}(0,z_0)|0>, \tag{31}$$

$$\begin{aligned} &= 16\pi G \frac{k^3}{(2\pi)^3}[(|\alpha^{IM}|^2 + |\beta^{IM}|^2)|u_k^M(z_0)|^2\\ &\quad + 2Re\{\alpha^{IM}(\beta^{IM})^*(u_k^M(z_0))^2\}]. \end{aligned} \tag{32}$$

From eq.(24) and eq.(30), we get

$$\Delta h(k)_{P.I.} \cong \frac{2\Gamma(\frac{3\gamma-1}{2(\gamma-1)})}{\sqrt{\pi}}\left(\frac{3(\gamma-1)^2}{\gamma(3\gamma-1)}\right)^{\frac{\gamma}{2(\gamma-1)}}\left(\frac{h}{H}\right)\left(\frac{2h}{k}\right)^{\frac{1}{\gamma-1}} \Delta h(k)_{S.I.}, \tag{33}$$

$$= \frac{2\Gamma(\frac{3\gamma-1}{2(\gamma-1)})}{\sqrt{\pi}}\sqrt{\frac{3(\gamma-1)^2}{\gamma(3\gamma-1)}}\left(\frac{2}{-z_1}\right)^{\frac{1}{\gamma-1}} \Delta h(k)_{S.I.}, \tag{34}$$

where $\Delta h(k)_{S.I.}$ is the power spectrum of background gravitational wave produced in the standard (exponential) inflationary universe [11], which is given by $\Delta h(k)_{P.I.}|_{\gamma\to\infty}$, and H is defined by $\sqrt{\frac{8\pi G}{3}}M_{GUT}^2$. This power spectrum is shown in Fig.1 and Fig.2. Here we identify the present Hubble horizon with 6000Mpc ($\simeq 1.6 \times 10^{-18}$ Hz). When $\gamma \gg 1$, the power spectrum $\Delta h(k)_{P.I.}$ at the horizon scale ($z_0 \simeq 2$) is about $10^{\frac{27}{\gamma}}$ times greater than $\Delta h(k)_{S.I.}$.

5 Conclusion and Discussion

We have investigated the background gravitational wave produced in the power-law inflationary universe described by BD gravity in the Einstein frame. As a result, we have obtained the power spectrum $\Delta h(k)_{P.I.}$, which is about $10^{\frac{27}{\gamma}}$ times greater than $\Delta h(k)_{S.I.}$ at horizon scale.

Our present result differs from the previous calculations by many authors [5]. They obtained the power spectrum for $\gamma \gg 1$

$$\Delta h(k)_{P.I.} \sim \left(\frac{2H}{k}\right)^{\frac{1}{\gamma}} \Delta h(k)_{S.I.}, \tag{35}$$

on the analogy of the power spectrum of scalar field. They concluded that the power spectrum $\Delta h(k)_{P.I.}$ at the horizon scale is about $10^{\frac{47}{\gamma}}$ times greater than $\Delta h(k)_{S.I.}$.

The difference arises from the treatment of integration constant a_I. Assuming that $h \simeq H$, i.e., $a_I \simeq 1$, from eq.(33) we would get the same

result. However, this assumption $a_I \simeq 1$ is wrong in our model because from eq.(13) the effective gravitational constant is

$$G_{eff(z_0)} = a_{(z_1)}{}^{\frac{-2}{\gamma}} G \simeq 10^{\frac{42}{\gamma}} \times G \neq G. \tag{36}$$

That is, to obtain the correct power spectrum, we need to provide a concrete cosmological evolution model of universe and to solve the equations of motions carefully. We must not use the analogy of the scalar field to simplify this problem.

Acknowledgments

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Figure captions

Fig.1

The power spectrum of the background gravitational wave $\Delta h_{(k)}$ is plotted as a function of the frequency, $k/2\pi$, for $\gamma = 50$.

Fig.2

The amplitude of the background gravitational wave whose wave length is equal to the horizon size is plotted as a function of the power index γ.

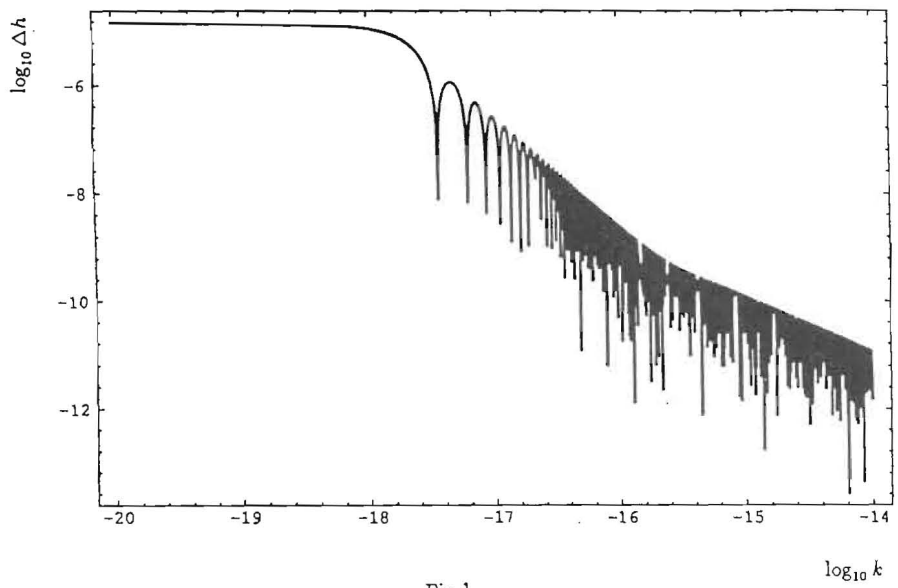
$\log_{10} \Delta h$
-6
-8
-10
-12
-20
-19
-18
-17
-16
-15
-14
$\log_{10} k$

Fig.1

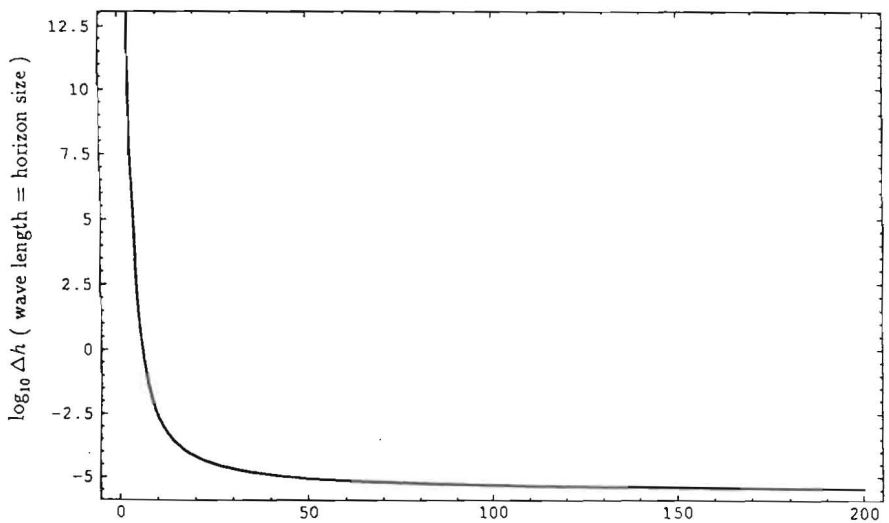
$\log_{10} \Delta h$ (wave length = horizon size)
12.5
10
7.5
5
2.5
0
-2.5
-5
0
50
100
150
200

Fig.2