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UNITARITY OF THE BLACK HOLE SCATTERING MATRIX*

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ABSTRACT

Approaches towards the problem of constructing an S-matrix for a black hole are outlined. An earlier proposal by this author showed that this S-matrix will be related to string theory amplitudes (though they are not identical). A new approach formulated here involves the entire black hole history, for which a topologically trivial Penrose diagram is constructed. We can then construct segments ("blocks") of the S-matrix, for which there is no problem of information drainage. Because of this it is suspected (though not proven) that the S-matrix constructed along such lines will be unitary.

1. Introduction

The problem of reconciling the theory of general relativity with the principles of quantum mechanics is one of the deepest and most fundamental ones of theoretical physics and it continues to mystify many of us. Now the procedure of replacing a "classical" theory by a corresponding "quantum mechanical" one is straightforward in many cases, in particular when we are dealing with relatively tiny interaction strengths or a small number of degrees of freedom. Indeed, if we consider circumstances where the gravitational force is weak and therefore accessible to a perturbative treatment we know fairly precisely how to perform this

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so-called "quantization procedure". The resulting theory, perturbative quantum gravity, turns out to be similar to any other gauge theory, except that when increasing accuracies are required new, undetermined physical parameters emerge: subtraction constants associated with unrenormalizable interactions. This complication, though of course a fundamental one, is relatively mild compared to the obstacles one encounters whenever a "non-perturbative" formalism is asked for. One then notices that any attempt even at giving a sensible frame for a description of what might happen will falter at distance scales smaller than the Planck length. A fundamentally new approach is needed.

One reason why any attempt based on the classical description of gravity must break down is a basic instability of the gravitational force: the possibility of gravitational collapse. As soon as too much energy is concentrated within one tiny volume element, a black hole - sometimes of considerable size - emerges. In a theory of quantized elementary particles something at least as complicated is expected to happen. But this would then be a drastic deviation from one of the basic starting points in quantum field theory, namely that particles can be treated, in a first approximation, as if they move independently of each other, as if particles states can be simply superimposed on top of each other. In the high energy regime this must be utterly false.

If we add to this the observation that the high energy regime of a particle theory is connected to the low energy regime by Lorentz transformations we see that this brings into doubt either the basic postulates of Lorentz invariance, or the superposition principle for quantized particle fields, the applicability of partial differential equations to these fields, and the like.

Whenever extremely strong gravitational fields come into play we encounter fundamental problems of this sort in our understanding of the basic laws of physics. The strongest gravitational fields possible occur in the vicinity of black holes. This brings us quite naturally to the consideration that indeed black holes are the prototype testing facilities for any quantum gravity theory. A proper incorporation of black holes in any theory of quantized gravity must be absolutely essential, since they form the natural asymptotic limit of the energy spectrum of "most pointlike" particles.

And most standard theories of gravity do not incorporate black holes properly. In a proper theory black holes, or at least objects that would behave like black holes in the limit of large mass and size, should occupy a natural position in Hilbert space, be included in the unitarity conditions of the S martix, and so on.

Instead, what is usually done is that black holes are treated in the so-called background formalism. One specifies the metric as if it were a classical one, and then performs quantum field theory with respect to this background. At first sight one would expect that this were a correct procedure, comparable to, for instance, the treatment of magnetic monopoles in a gauge theory for elementary particles. But the outcome is drastically, and catastrophically, different[1]. It is found that, when viewed this way, quantum black holes extract and destroy "quantum information"[2]. In terms of pure quantum states this means that when we start with two states that are orthogonal to each other in Hilbert space, for instance because they differ by the presence of one extra particle moving into the black hole, these states become indistinguishable after a while, and hence cannot continue to be orthogonal to each other; if they did, the number of possible states inside a black hole would rapidly surpass the total number of possible states in the universe. In a slightly different interpretation of the same mental exercise one would say that a quantum mechanically pure state evolves naturally into a quantum mechanically mixed state. *

One could try to maintain, as indeed is often done, that black holes must therefore be radically different from elementary particles, including solitons such as magnetic monopoles. But this is too rash a conclusion. It would imply that black holes are not even "quantum predictable", but only obey probabilistic laws. To this author such a lack of precisely defined physical equations such as the Schrödinger equation is not likely. Surely the background metric approach to black holes cannot be right, just because it assumes that the particle fields can be superimposed onto the background fields, and we had already concluded that this superposition principle cannot be correct.

We can pinpoint in another way the complication that was ignored: there are interactions, in particular gravitational ones, between the in- and outgoing particles. Now under normal situations this would not have been a great disaster. In quantum field theories one can easily correct for such interactions by adding a series of successively tiny perturbative corrections. But the gravitational interactions are not normal in this respect. If we want to know how the out-states react upon any

* A similar phenomenon seems to occur in theories of multiply connected universes. Here an uncertainty in the fundamental interactions arises on top and above the familiar quantum uncertainties. Pure states evolve into mixed states due to this uncertainty, but here this is clearly seen as a shortcoming in our information concerning the effective interactions. The uncertainty in question could be resolved for instance by performing accurate measurements.

variation among the ingoing particles at an earlier epoch, we find a disturbing divergence: the strength of the mutual gravitational interaction diverges *exponentially* with the time difference. Hence any perturbative approach is out of the question whenever we wish to follow the evolution of some configuration over any appreciable time interval.

In these notes I will skip the general introduction to black holes, which have been described abundantly in the literature[1-4]. One important aspect one has to remember is that the total number of states, or energy levels, of a black hole can be estimated using simple arguments from thermodynamics, assuming that a black hole carries a temperature as given by Hawking[1]:

$$kT = 1/8\pi M\,, \tag{1.1}$$

in units where $c = \hbar = G = 1$. The result is that the level density $\rho(M)$ as a function of the mass M is given by

$$\rho(M) = e^{4\pi M^2 + C}\,, \tag{1.2}$$

where C is an unknown constant. The point is that this number is small! If one counts the number of levels provided by the thermal particles in the vicinity of the black hole one finds that the particles further than about one Planck unit away from the horizon are sufficient to produce all the entropy corresponding to these levels. The ones closer to the horizon would provide an infinite contribution if we were allowed to use a linearized theory. Of course these particles do *not* obey a linearized theory, but the mechanism by which their contribution to the entropy is turned off is obscure.

For this reason we expect that incoming particles indeed do affect the details of the quantum state a black hole can be in, in the sense that they determine details of those emerging particles that were closer to the horizon than one Planck length when the incoming particle entered. Our best guess is then that the black hole is just one set of the possible intermediate states in an S-matrix. It should be in no fundamental way different from ordinary particles. Light particles have a Schwarzschild radius much smaller than their Compton wavelength; for black holes this radius is much bigger. This distinction must be a gradual one. And so we arrive at the "S-matrix Ansatz"[5] for the black hole. Once we assume that the black hole has an S-matrix, we can actually derive many of its properties, because many

of the relevant laws of physics are already known to us.

2. The Pseudostring

We first observe that the nature of the gravitational interactions between incoming and outcoming particles can very easily be characterized. Incoming particles produce a *horizon shift*. This horizon shift may be very tiny, but its effects upon the outgoing particles grow exponentially with time. They are also readily computable[6]. The wave functions of all outgoing particles are simply shifted, by an amount that depends on the angular location on the horizon.*

The quantum state is shifted, and hence the outgoing wave functions are all multiplied with factors $\exp(ip_{\rm out}\delta y)$, where $p_{\rm out}$ is the momentum in Kruskal coordinates and δy the horizon shift, a function depending explicitly upon the angular coordinates ϑ and φ. The effect of this operation would be a harmless multiplication if the outcoming particles were in a Kruskal momentum eigenstate, but of course, in more relevant circumstances they are not in such eigenstates. This way we conclude that any alteration of the form

$$|\psi\rangle_{\rm in} \to |\psi + \delta\psi\rangle_{\rm in}\,, \tag{2.1}$$

where $\delta\psi$ carries a given momentum $p_{\rm in}(\vartheta,\varphi)$, affects the outcoming state by the above given operation.

We can now repeat the argument as many times as we wish so that, in principle, we should obtain *all* other S-matrix elements. The procedure, and its results, are described in Ref[5]. They can be summarized as follows.

The momenta of in- and outgoing particles $p_{\rm in}(\vartheta,\varphi)$ and $p_{\rm out}(\vartheta,\varphi)$, are to be defined with respect to Kruskal coordinates, not Schwarzschild coordinates – this is a point of concern, to be discussed later. When specified at all angular positions (ϑ,φ) these momenta, and in addition some other quantities such as electric charge density $\rho(\vartheta,\varphi)$, these variables should *entirely* specify the quantum states of the in- and out- quantum states respectively. So we refer to these states as

$$|p_{\rm in}(\Omega)\,,\rho_{\rm in}(\Omega)\rangle \quad \text{and} \quad |p_{\rm out}(\Omega)\,,\rho_{\rm out}(\Omega)\rangle\,, \tag{2.2}$$

* This angular dependence is crucial for our arguments, since without such an angular dependence one could transform (practically) all its effects away. This is why one must be very careful in interpreting some popular two-dimensional toy models of black holes[7].

where Ω stands for (ϑ,φ). The resulting S-matrix can then be written as a functional integral

$$\langle p_{\rm out}(\Omega)\,,\rho_{\rm out}(\Omega)\,|\,p_{\rm in}(\Omega)\,,\rho_{\rm in}(\Omega)\rangle =$$
$$\mathcal{N}\int \mathcal{D}u_\mu(\Omega)\mathcal{D}\phi(\Omega)\exp i\int d^2\Omega\big(-\tfrac{1}{2}(\partial_\Omega u_\mu)^2 + p_\mu u_\mu - \tfrac{1}{2\kappa}(\partial_\Omega\phi)^2 + \phi(\rho_{\rm out}-\rho_{\rm in})\big)\,. \tag{2.3}$$

Here $\mathcal{N}$ is a normalization factor; the Lorentz index μ is defined such that $p_\mu = (\sqrt{2})^{-1}(p_{\rm in}+p_{\rm out}\,,0,0,p_{\rm in}-p_{\rm out})$, similarly u_μ, and κ is a constant defining the unit of electric charge. u_μ and ϕ are functional integration variables, depending on the two angular coordinates on the black hole horizon. u_μ are like the two transverse dynamic variables of a "string" whose world sheet is the intersection of the future and the past horizon of the black hole, which is a two-dimensional surface. ϕ is a periodic variable (it is defined as an angle *modulo* 2π). This is a consequence of electric charge quantization. Observe that in every respect electromagnetism appears to be represented here as a Kaluza-Klein theory. This was *not* put in but came out of our theory as a consequence of the S-matrix Ansatz.

The similarity between Eq. (2.3) and a string theory amplitude is striking. This resemblance becomes even closer if we represent the in- and out-states as particles in wave-packets. One then has to integrate over the coordinates (ϑ,φ) convoluted with a wave function, and these integrals then correspond to the Koba-Nielsen integrations. An important *difference* between (2.3) and string theory is the factor i in the exponent, which corresponds to a purely imaginary string constant*. Our interpretation of this observation is that the black hole horizon can in some repects be regarded as the world sheet of a virtual closed string. The external particles are inserted there as vertex insertions in the usual sense.

We discovered that one can start with several kinds of fundamental interactions in one's favorite standard model and observe that these are reproduced in the functional integral (2.3) on the horizon. Electromagnetism, here represented by the variable ϕ, being just an example. Non-Abelian interactions give rise to more complex variables in two dimensions. Quite generally however the following picture emerges: The *gauge transformation generators* of the 4-dimensional theory correspond to the *dynamical variables* in the 2-dimensional one. Therefore the spin of a physical degrees of freedom in 2 dimensions is one less than the corresponding one in 4 dimensions.

* The fact that the string constant comes out imaginary should not be seen as a departure from unitarity, as was asserted by one author, but rather as a *consequence* of unitarity as required in our formalism.

Scalar and Dirac-spinor fields seem not to generate anything in 2 dimensions. An exception to this is the occurrence of spontaneous symmetry breaking: if in four dimensions a symmetry is broken spontaneously, the corresponding symmetry in 2 dimensions is *explicitly* broken: the scalar field in 4 dimensions maps into a "spurion" field in 2 dimensions (spurions were used in the '60's to describe explicit symmetry breaking interactions). Indeed one may view the value of the scalar fields at the horizon intersection point as being the spurion parameter.

A dual transformation in 4 dimensions corresponds to a similar dual transformation in 2 dimensions. Thus, magnetic monopoles entering the black holes generate a topological kink in the two-dimensional system; furthermore, quark confinement in 4 dimensions can be seen to correspond to an explicit symmetry breaking in terms of the scalar disorder parameter in two dimensions.

Proceeding along these lines it is natural to suspect that a *gravitino* in four dimensions corresponds to a Dirac spinor in the 2-dimensional theory. What we have not understood at present however is how to incorporate effects of Dirac spinors in four dimensions in the 2-dimensional theory; they seem to leave no trace.

For more details of the string picture of black holes we refer to Ref[5].

3. Problems with Unitarity

Is our scattering matrix (2.3) unitary? A strange new problem arises. One may observe that the scattering matrix will indeed be unitary, but only so in a very unconventional Hilbert space. Two states that have exactly the same momentum (and charge) distribution for the ingoing - or outgoing - particles, cannot be distinguished any other way and therefore must be identical. In particular the *number* of particles entering or leaving at a given spot on the horizon cannot be specified. This implies that the Fock space of elementary particles will eventually look very different from what it used to be in elementary particle physics. For instance, the in- and out- states will carry no label specifying their baryon number. Consequently the black hole scattering matrix cannot possibly obey baryon number conservation. Clearly continuous global symmetries in our fundamental particle interactions cannot be reproduced in the black hole scattering matrix.

Another apparent problem with unitarity arises if the shift $\delta y(\vartheta, \varphi)$ at some values of ϑ and φ becomes too large. It could then be that a particle, originally destined to emerge in the out-state when the in- wave function was $|\psi\rangle$, is shifted beyond the horizon when the in-state is $|\psi + \delta\psi\rangle$. This is a consequence of the

fact that we had been forced to define momenta in Kruskal coordinates instead of Schwarzschild coordinates. A shift in Kruskal space can bring a particle behind the horizon.

We should stress that this latter problem is only an apparent one. There is no real contradiction with unitarity here because we imagine the total set of allowed out- states to be much smaller than the Hilbert space spanned by *all* possible waves of outgoing particles. The shift δy does not affect one single particle but an infinite series of particles emerging at all times. So if one or several of these disappear behind the horizon there are always enough others left to enable us to distinguish this shifted out- state from other out- states. Thus, our problem is more of a practical nature than fundamental. It tells us that the standard way to build up a Hilbert space in terms of plane wave of particles cannot be used here.

These problems must be related to another practical problem: even the set of all functions $p(\vartheta, \varphi)$ and $\rho(\vartheta, \varphi)$ is too large. Our entropy aruments suggest that there should be no more than about one Boolean variable per unit of surface area on the horizon in Planck units. This is as if these functions p and ρ have a cut-off. Components of their Fourier transforms in the transverse directions with momentum larger than a Planck unit should be removed or considered redundant. On the other hand lots of details on a distance scale just a bit larger than the Planck length are described by as yet unknown parts of the standard particles interactions. These details will be essential in the definitions of inner products in our Hilbert spaces, yet they are not yet accessible to us because the particle interactions at those scales are not yet known.

All this may seem to be extremely unconventional and inaccessible physics. But it is not quite that bad. We emphasize that the mathematical situation here is exactly as in string theories. In string theory also it is not the entire Hilbert space but rather the scattering matrix that is constructed. If particles are identified as vertex insertions on a string world sheet then exactly the same features do show up in string theory. Consider namely the Koba-Nielsen integrand with a given array of vertex insertions, for a given N particle amplitude. If in this integrand two vertex insertions occur at the same spot on the world sheet then this is indistinguishable from the integrand for the $N-1$ particle amplitude. Replace the string world sheet by the horizon. The indistinguishability of two particles on the same spot on the horizon, or rather the fact that this state cannot be distinguished from a single particle state at that spot, has the same mathematical origin.

4. Unitarity in Complete Black Hole Histories

Our scattering matrix Ansatz tells us to assume as a starting point the existence of a scattering matrix for a black hole. And then we can deduce information about this matrix by applying all physical laws we know. The only reason why this does not work completely is that we only know the interactions between elementary particles at low energies, or, equivalently, at large distance scales. So we do not know how to characterize the very small distance features of our scattering matrix, and since inner products of states depend crucially also on the small distance features, we run into problems as described in the previous section. The general strategy we are trying to implement is to use the known laws in as many forms as possible to reduce these uncertainties as much as possible. Also we can try certain assumptions concerning the small distance interactions to check which of these produce a consistent theory (we saw for instance that baryon number conservation must not be a symmetry of our basic interactions).

With this strategy in mind we now proceed to consider a sector of the scattering matrix different from the one considered before, namely the transition amplitude from a black hole just formed into a black hole exploding into expanding dust shells. Thus we consider a completely specified in-state, $|\text{in}\rangle$, a completely specified out-state, $|\text{out}\rangle$, and assume that one single amplitude $\langle\text{out}|\text{in}\rangle$ is given. As before, the question is to deduce other amplitudes

$$\langle \text{out} + \delta_{\text{out}} | \text{in} + \delta_{\text{in}} \rangle \,, \tag{4.1}$$

where δ_{out} and δ_{in} are tiny alterations. We now proceed in a way very different from Section 2, namely by first postulating a singularity free, topologically trivial space-time metric corresponding to the original amplitude. That this is possible at all is surprising and requires some discussion. The trick is to assume the outcoming matter to be due to some unspecified interaction process very near the horizon which gave rise to extremely strong curvature there. This curvature would not be directly detectable for ingoing or outgoing observers and therefore its presence does not contradict anything we know. If it *were* observable it *would* contradict the ordinary laws of physics. This is a necessary aspect of the S-matrix Ansatz. The mere assumption that an amplitude $\langle\text{out}|\text{in}\rangle$ exists does contradict "normal" laws of physics. So we are forced to assume something out of the ordinary there, and the least harmful way to do this is to postulate a *conical singularity* (actually it is not a singularity but just a region of very strong curvature, because the singularity will be slightly smoothened). The presence of such a singularity will only be visible when

devices sent into the (approximate) black hole and reappearing somewhat damaged after the black hole decayed, are compared to apparatus that stayed just outside. But such an experiment will be impossible classically. Instead of these "devices" we will just consider infinitesimal additions $\delta\psi$ to our wave functions and study the effects on these.

The Penrose diagram is now the one pictured in Fig. 1a. It is topologically trivial. Apart from a mild (very slightly smeared) singularity at the point S there are no further singularities. The dotted lines are very much like horizons, but of course they are not horizons, they replace them. At the point S the standard laws of physics seem to be not obeyed. The curvature there is the one produced by a very violent "interaction" that caused the incoming shell of matter to turn around and go outwards. It is as if a "chemical" explosion takes place there which was just strong enough to avert the gravitational implosion. Let us stress again that an observer who stays outside the black hole (or "pseudo-black-hole") can never detect this curvature, so that from his point of view all laws of physics are obeyed.

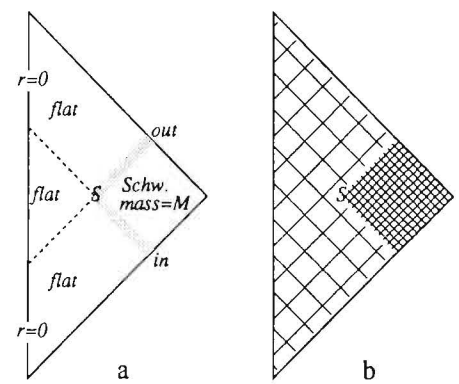

Fig. 1. a) Non-singular Penrose diagram for an entire black hole history.
b) Coordinate frame in the outside region of the black hole is more dense.
At the point S there is a conical singularity.

What we claim now is that this process may well be reconciled completely with the known laws of physics, even at S, by studying quantum field theoretical effects caused by the curvature at S. In the next Section we shall prove that the singularity is such that if one starts off with a local vacuum, a nearly infinite spectrum of particles will be created there. We will then argue that if on the dotted

lines in Fig. 1. we require the *absence* of particles, there *must* be particles in the gray area. Originally we had "postulated" that there are particles there; we can now derive that the postulate may well be correct. So the whole picture may become self-consistent.

In our simplified model we replace all incoming and all outcoming matter by single "dust shells". Upon careful inspection one finds that this is hardly an approximation, see Fig. 1, where all matter coming in is squeezed towards the "far past" and everything coming out towards the "far future". Near S the most regular coordinate frame is a "temporary" Kruskal frame, and hence all matter in our space-time diagram is very strongly Lorentz boosted.

Our model then is of the same type as some of the systems studied by T. Dray and the author in Ref[8]. Here we studied the effect of in- and outgoing dust shells on the Schwarzschild metric. But in this work we explicitly postulated the absence of conical singularities at points such as S, so that the occurrence of typical black hole singularities at $r = 0$ both in the past and in the future is inevitable. Now we take the same models *with* conical singularity at S, chosen in such a way that the singularities at $r = 0$ go away. The metric one then gets fits naturally with the S-matrix Ansatz. The strategy is now simple. In Fig. 1 we postulate space-time to be flat in all of the interior region, except in the quadrant where an outside observer sees the black hole. There we have the Schwarzschild metric corresponding to a mass M. Consider the Kruskal coordinates x and y . Let the physical quadrant be given by $x > 0, y > 0$. Very near the Schwarzschild horizons, at the line $x = x_0$ and the line $y = y_0$, where x_0 and y_0 are very small but positive, we have the matter shells. At those shells we glue the Schwarzschild metric against the flat space-time metric such that the Schwarzschild r parameter matches with the flat space r parameter. The metric is then C^0.

But a singularity develops at S . This we see as follows. Suppose we use Penrose coordinates, that is, coordinates such that the local light cones have a width of exactly 45°. One then finds that the glueing procedure just described forces us to scale down the Schwarzschild solution (as written in Kruskal coordinates) to a very small size, and to blow up the internal region of the black hole to large sizes. This is sketched in Fig. 1b by drawing dense coordinate lines in the outside region and wide coordinate lines inside.

In Fig. 2 we illustrate what happens to geodesics near such a point. At the point B in Fig. 2c we make the transition to Lorentz transformed coordinates, but because the orthogonal coordinate is scaled in the wrong direction a geodesic

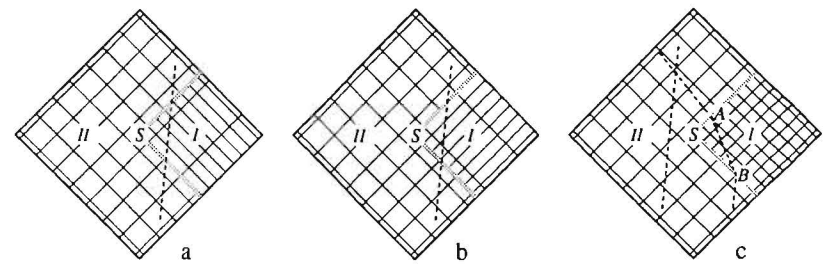

Fig.2. In a) and b) the region I is Lorentz boosted compared to region II. Since this is just a coordinate reparametrization a geodesic (dotted line) goes straight. In c) we performed the transformation of a) at the point A and the one of b) at the point B. At both points, the orthogonal coordinate was squeezed. Consequently, a geodesic going through region I is now bent over.

crossing at B is bent over. The same happens in A. Thus, two particles with equal velocities may end up having different velocities if they pass the point S at opposite sides. Thus the singularity at S has the effect of a Lorentz transformation if one follows a loop around it.

For a black hole with lifetime long compared to its size the Lorentz boost across S is extremely large. For the remainder of our considerations we prefer to concentrate on the case that this Lorentz boost is not so extremely large. This happens either if one considers very tiny black holes, or black holes with an extremely "unlikely" history. The only reason then why this history is unlikely for large black holes is that the amplitude is too small after multiplication with the appropriate phase space factor, which is also too small, so that other processes (giving the hole a lifetime of order M^3) are more probable. We just point out that this is not at all an objection against considering the amplitudes for such "unlikely" histories.

Thus, we concentrate on Fig. 1 where the region very close to the origin, S, is described by Fig. 2c. let the total Lorentz boost along a closed curve be given by the parameter ϕ in the boost matrix

$$L = \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{4.2}$$

The local effect of the shells of matter is small compared to the effect of the conical singularity.

5. Particle Creation by a Conical Singularity

We now consider the effect a conical singularity of the sort described in the previous Section has on a quantized state in field theory. Since the metric has no timelike Killing vector there is no conserved energy. If we begin with the vacuum state at $t = -\infty$ the state at $t = +\infty$ will in general contain particles. The computation is not hard. Observe that, in contrast with the familiar calculation of the Hawking-Unruh effect there will be no information loss. Later we will be interested in different initial states, but let us begin with the vacuum.

For simplicity we take the field to be scalar. The local operators $\varphi(\mathbf{x}, t)$ and $\dot{\varphi}(\mathbf{x}, t)$ are given by

$$\varphi(\mathbf{x}, t) = \sum_k \frac{1}{\sqrt{(2Vk^0)}} \left(a_k e^{ikx} + a_k^\dagger e^{-ikx}\right), \tag{5.1}$$

and

$$\dot{\varphi}(\mathbf{x}, t) = \sum_k \sqrt{\frac{k^0}{2V}} \left(-ia_k e^{ikx} + ia_k^\dagger e^{-ikx}\right), \tag{5.2}$$

where a_k and $a_k^\dagger$ are annihilation and creation operators at given three-momentum $\mathbf{k}$. As usual we define $k^0 = \sqrt{(\mathbf{k}^2 + m^2)}$ and $kx = \mathbf{kx} - k^0 t$.

We take Eqs (5.1) and (5.2) to hold at time $t < 0$, before the singularity S occurred. At time $t > 0$ we take the fields to be

$$\varphi(y) = \sum_k \frac{1}{\sqrt{(2Vk^0)}} \left(b_k e^{iky} + b_k^\dagger e^{-iky}\right), \tag{5.3}$$

where y are Cartesian space-time coordinates at $t > 0$. They are related to the $\mathbf{x}, t$-coordinates by

$$y = x \quad \text{if} \quad x_1 < 0 \quad , \quad y = L^{-1}x \quad \text{if} \quad x_1 > 0 \,, \tag{5.4}$$

where L is the Lorentz transformation (4.2).

One finds that

$$b_p = \sum_k A_{pk}^+ a_k + \sum_k A_{pk}^- a_k^\dagger \,, \tag{5.5}$$

where A_{pk}^+ and A_{pk}^- are coefficients. From now on the variables p and k are only the x-components of the momenta, the ones that transform non-trivially under the

Lorentz transformation (4.2). p^0 and k^0 are the usual time components of the momenta. Also we write $x = x_1$. Let us furthermore use the shorthand notation

$$\cosh\phi = c \quad , \quad \sinh\phi = s \,, \tag{5.6}$$

where ϕ is the Lorentz boost parameter. We will use a finite-volume formulation so that the momenta are discrete. The coefficients are then computed to be

$$A_{pk}^\pm = \frac{1}{2V}\sqrt{\frac{p^0}{k^0}} \int_0^\infty dx \left(\left(1 \pm \frac{k^0}{p^0}\right) e^{-i(k-p)x} + \left(1 \pm \frac{ck^0 - sk}{p^0}\right) e^{i(ck - sk^0 - p)x} \right), \tag{5.7}$$

where V is the volume (soon to be sent to infinity).

The integral over x can of course be calculated:

$$A_{pk}^\pm = \frac{1}{2V\sqrt{p^0 k^0}} \left(\frac{-i(p^0 \pm k^0)}{k - p - i\epsilon} + \frac{i(p^0 \pm (ck^0 - sk))}{ck - sk^0 - p + i\epsilon} \right). \tag{5.8}$$

It is illustrative to compute the occupation number $\langle b_p^\dagger b_p \rangle_0$, where $\langle\rangle_0$ corresponds to the vacuum of the annihilation operators a_k. It is found to be

$$\begin{aligned} \langle b_p^\dagger b_p \rangle_0 &= \sum_k |A_{pk}^-|^2 = \\ &= \frac{1}{8\pi V p^0} \int \frac{dk}{k^0} \left(\frac{(p^0 - k^0)(ck - sk^0 - p) + (p - k)(p^0 - ck^0 + sk)}{(k - p)(ck - sk^0 - p)} \right)^2 , \end{aligned} \tag{5.9}$$

where the summation was replaced by the integral for $V \to \infty$, and in the integral we must insert

$$k^0 = \sqrt{k^2 + \tilde{k}^2 + m^2} \,, \tag{5.10}$$

and similarly for p^0. Here $\tilde{k}$ is the transverse part of the momentum k.

The rest is straightforward arithmetic. All integrals can be performed and the result is

$$\langle b_p^\dagger b_p \rangle_0 = \frac{1}{2\pi V p^0} \left[\frac{\phi}{\tanh \frac{1}{2}\phi} - 2 \right]. \tag{5.11}$$

For small ϕ the quantity between square brackets is

$$[\ldots] = \frac{\phi^2}{6} - \frac{\phi^4}{360} + \ldots \,, \tag{5.12}$$

and if ϕ is large then it approaches

$$[\ldots] = |\phi| - 2 + 2|\phi| e^{-|\phi|} + \ldots \,. \tag{5.13}$$

Note that the p dependence is $d^3p/2p^0 = d^4p\delta(p^2+m^2)$, which is Lorentz invariant. Invariance under Lorentz transformations in the x direction is not surprising. But the invariance in the transverse direction is an accident. The coefficients $A^{\pm}$ themselves do not have this latter invariance. Also, the fact that Eq. (5.11) is independent of the sign of ϕ is an accident.

The coefficients of Eq. (5.8) were computed for given 3-momenta. The calculations simplify however if we go to lightcone coordinates instead. The outcome, such as Eq. (5.11), of course stays the same.

6. Conclusion

We propose to use the metric of Fig. 1 to compute amplitudes (4.1) if one single amplitude $\langle \text{out}|\text{in}\rangle$ is given. The conical singularity S is not strong enough to cause any loss of information. If S were infinitely sharp a vacuum in-state would cause an unlimited particle production into the out-state. We can put a bound on this particle production by smearing the singularity a bit. We showed that calculating the evolution of the state that started out as a local vacuum is straightforward. But what actually will be needed is the evolution of a state that has no particles coming from ($r = 0$) (the lower dotted line in Fig. 1a) into a state that has no particles moving towards ($r = 0$) (the upper dotted line in Fig. 1a). In general these states may have particles on the other side (the gray areas in Fig. 1a). Computation of these transitions is much harder because the distinction between left-goers and right-goers is to some extent arbitrary and hence difficult to implement.

We note that the in- and outcoming particles caused by the singularity essentially imply that our Ansatz for the metric is self-consistent. Somewhat more precisely, we propose the following. In contradistinction to the procedure we proposed previously, and which was recapitulated in Section 2, we now assume that the variations $\delta\psi$ of both incoming and outgoing states are too small to have any direct gravitational effect, so that here we can superimpose quantum states in the usual way. We will refer to the particles in $\delta\psi$ as "soft" particles. All particles whose gravitational effects we wish not to ignore (the "hard" particles) we put in the original states $|\text{out}\rangle$ and $|\text{in}\rangle$. So in each of these "gravitational windows" we can compute a block

$$\langle \text{out} + \delta_{\text{out}}|\text{in} + \delta_{\text{in}}\rangle . \tag{6.1}$$

Indeed all these amplitudes are uniquely defined up to one overall multiplicative constant. There is no drain of information. On the other hand however, there is

a divergence: if S is infinitely sharp the majority of transitions will contain huge numbers of particles modifying the already heavily populated in- and out-states. Just because we wish to consider only soft particles in $\delta\psi$ we must accept a cut-off for the singularity S. The exact location of the cut-off, the transition region between soft and hard particles, must to some extent be irrelevant.

Note that the transitions $\delta_{\text{in}} \to \delta_{\text{out}}$ themselves will not violate any of the symmetries of our standard interactions. However in the entire block (6.1) the hard particles will violate all global symmetries, but for the entire block this violation will be the same.

We believe that our new proposal will open up different elements of the black hole scattering matrix and allow us to study this matrix further. Ultimately all procedures should be combined into one single theory, but we are not yet that far. By construction it seems that there cannot be any violation of unitarity for this matrix, but we should admit that this has not yet been demonstrated. The problem is now that the S-matrix describing the soft particles alone, after the cut-off, will be unitary. But without cut-off the blocks (6.1) that we have are each *different* parts of *different* S-matrices. Each of theses matrices separately are unitary, but whether this combination will again be unitary remains to be seen. A delicate study of the various limiting procedures involved will be needed to answer such questions.

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