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Multiplicative Functions, Dirichlet Convolution, and Quantum Systems

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ABSTRACT

The number theoretic notions of multiplicative functions and of Dirichlet convolution are shown to have natural physical interpretations in the context of certain quantum systems, the prototype of which is Einstein's oscillator model of solids. Various number theoretic results are obtained using physical insights, and Dirichlet convolution is shown to correspond to the loss of information when distinguishable excitations are treated as indistinguishable. An especially appealing feature is that the concepts involved are both mathematically fundamental and physically natural.

One of the basic sets of functions in number theory is the set of arithmetic functions, namely functions whose domain is the positive integers. Within this set, the multiplicative functions play a fundamental rôle. The multiplicative functions are arithmetic functions that satisfy the additional property that

$$f(a)\,f(b) = f(ab) \quad \text{for } (a,b)=1 \ , \tag{1}$$

where (a,b) denotes the greatest common divisor of the integers a and b. In this paper, I will describe how multiplicative functions aris naturally when one considers quantum systems which are, essentially, generalizations of Einstein's oscillator model of solids. It will then follow that a variety of very natural physical consideration will correspond to some standard mathematical constructions, and physical reasoning will lead to various basic mathematical results.

These results shed useful light on the relationship between physic and mathematics. Overall, the idea that quantum systems provide a natural arena for number theoretic considerations is a very promisin one, rich in its potential applications. Already there has been some work in this general area [1,2,3]. Typical results previously obtained include the explanation of the Möbius inversion formula by means of supersymmetry, and the suggestion of how the validity of the Riemann hypothesis [4] might be ascertained by studying certain quantum mechanical systems. The isomorphism introduced in this paper between multiplicative functions and certain quantum operator therefore seems quite worthwhile, as it extends the basis from which one can continue to understand from new vantage points, and to obtain new results in, both number theory and physics.

To begin, consider first Einstein's model for the specific heat of solids [5]. Let H_{osc} represent the Hamiltonian for a single one-dimensional quantum harmonic oscillator; then Einstein treated the

Hamiltonian of a solid as a set of non-interacting oscillators, three non-interacting oscillators acting at each lattice site. Thus the full Hamiltonian is $H_{solid} = \sum_{k=1}^{3N} H_j$, where $H_j = H_{osc}$ for all $j = 1, 2, \ldots, 3N$.[1]

In this paper, I will consider generalizations of this system. In particular, I will consider the Hamiltonian $H = \sum_{k=1}^{\infty} H_k$, where each H_k is any quantum mechanical Hamiltonian with a purely discrete spectrum. In contrast to Einstein's model, all the H_k's can, and in general will, be different from one another. Such a system is a very simple example of a quantum field theory. For simplicity, we will always take the H_k to be Hamiltonians for one-dimensional systems. However, considering d-dimensional Hamiltonians is a straightforward generalization, should one wish (or have) to deal with degeneracies. I will refer to the individual H_k's as the "site Hamiltonians" and to H as the "full Hamiltonian". In contrast to Einstein's model, all the H_k's can, and in general will, be different from one another.

To make contact with number theory, we label states as follows. Denote the prime numbers in order of increasing magnitude as p_1, p_2, p_3, *etc.* Thus, $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, *etc.* The Hilbert space $\mathcal{H}$ of H is the tensor product of the Hilbert spaces $\mathcal{H}_k$ of the individual H_k. Label the eigenstates of H_k in order of increasing energy as $|(p_k)^n>$, $n = 0, 1, 2, \ldots$; thus the ground state of H_k is $|1>$, the first excited state of H_k is $|p_k>$, and $|(p_k)^{n_1}>$ has higher energy than $|(p_k)^{n_2}>$ whenever $n_1 > n_2$. Then one can employ the notation that the eigenstates of the full Hamiltonian H are $|N>$, where if the prime decomposition of $N = \prod_{k=1}^{\infty} (p_k)^{n_k}$ for the non-negative integers n_k, the

[1]Strictly speaking, one should write $1 \times 1 \times \cdots \times 1 \times H_j \times 1 \times \cdots 1$ in the summation rather than H_j, but it will be clearer simply to leave such matters implicit.

state $|N>$ is defined as

$$|N> = \bigotimes_{k=1}^{\infty} |(p_k)^{n_k}> . \tag{2}$$

(On the left side of (2) is a state in $\mathcal{H}$; on the right side of (2), the same state is written as the tensor product of states, with one state in each of the Hilbert spaces $\mathcal{H}_k$.) With these definitions and conventions, the states $|N>$, $N = 1, 2, 3, \ldots$, span the full Hilbert space of the Hamiltonian H.[2] For convenience, we always take our states to be normalized.

Of course, since the site Hamiltonians are non-interacting, the partition function of the full system is just the product of the constituent partition functions,

$$Z = tr\ e^{-\beta H} = \prod_{k=1}^{\infty} tr\ e^{-\beta H_k} = \prod_{k=1}^{\infty} Z_k . \tag{3}$$

Furthermore, there is also a very natural set of operators, namely operators $\mathbf{A}$ such that $\mathbf{A} = \bigotimes_{k=1}^{\infty} \mathbf{A_k}$, where each $\mathbf{A_k}$ is an operator acting on the Hilbert space of the Hamiltonian H_k. Thus, for example,

$$<N|\mathbf{A}|N> = \prod_{k=1}^{\infty} <(p_k)^{n_k}|\mathbf{A_k}|(p_k)^{n_k}> . \tag{4}$$

I will call an operator $\mathbf{A}$ that decomposes à la (4) a *multiplicative operator*. For convenience, I will always consider multiplicative operators which are diagonalized in the energy eigenstate basis I am using; this simplification will be sufficient for the purposes of this paper. The simplest example of a multiplicative operator is $exp(-\beta H)$ this fact is related to (3).

(Often in quantum mechanics one considers operators $\mathbf{a}$ such tha one has an equation analogous to (4), but with the infinite product replaced by an infinite sum. These operators correspond to additive

[2]Any state that would correspond to a divergent infinite value of N in this scheme is a non-physical state of infinite energy.

functions. Given such an operator **a**, one can construct an operator $\mathbf{A} = exp(\mathbf{a})$ which does obey (4).[3])

The above set-up has not been required because the physics is complicated. Quite the opposite is true: the physics is remarkably simple. It is merely that the labelling of states used above is unconventional that it has taken a few paragraphs to lay the groundwork. If the reader is comfortable with the labelling of states used, the remaining results will follow quite straightforwardly. This labelling of states is important, as it is the tool we use to identify the positive integers in a natural way with quantum states, and hence to relate arithmetic number theory to quantum mechanical systems.

Given a quantum system as described above, with states labelled as indicated, it is possible to define an arithmetic function

$$A(N) = \frac{<N|\mathbf{A}|N>}{<1|\mathbf{A}|1>} . \tag{5}$$

Remember that **A** is a multiplicative operator. The key observation is that the requirement that the individual site Hamiltonians be non-interacting implies that the function $A(N)$ is a multiplicative function! (The normalization in (5), *i.e.*, dividing by the vacuum expectation value $<1|\mathbf{A}|1>$, is necessary here.) Note that two integers N_1 and N_2 are relatively prime if and only if the corresponding states $|N_1>$ and $|N_2>$ are such that every site that is excited in $|N_1>$ is in its ground state in $|N_2>$, and vice versa. By (4), then, $A(N)$ is a multiplicative function, according to the definition (1). Conversely, for any multiplicative function $A(N)$, one can define a multiplicative operator **A** that, when used in (5), reproduces $A(N)$; in particular,

[3]In considering invariances, operators such as **a** correspond to generators of the algebra which generate infinitesimal transformations, while operators such as **A** correspond to group elements which implement finite transformations.

one can define **A** by

$$<N|\mathbf{A}|M> = A(N)\delta_{N,M} . \tag{6}$$

Thus there is a correspondence between multiplicative functions and certain quantum operators in systems of non-interacting site Hamiltonians. (For real-valued multiplicative functions, these operators are observables.) While interesting in and of itself, this result is also important for the power it gives one to devise physical arguments to obtain number theoretic results.

Suppose there are two multiplicative operators **A** and **B**; when are these two operators equivalent? Clearly, it is necessary that for all β,

$$tr\, \mathbf{A}\, e^{-\beta H} = tr\, \mathbf{B}\, e^{-\beta H} . \tag{7}$$

However, since the constituent Hamiltonians in H are independent, **A** and **B** will be equivalent if and only if at each site **A** and **B** have identical thermal expectation values, *i.e.*, provided that, for all positive β and all positive integers k,

$$\sum_{n=1}^{\infty} <(p_k)^n|\mathbf{A}\, e^{-\beta H}|(p_k)^n> = \sum_{n=1}^{\infty} <(p_k)^n|\mathbf{B}\, e^{-\beta H}|(p_k)^n> . \tag{8}$$

Mathematically, (8) implies that two multiplicative functions are identical whenever

$$\sum_{n=1}^{\infty} A(p^n)\, e^{-\beta E_{p^n}} = \sum_{n=1}^{\infty} B(p^n)\, e^{-\beta E_{p^n}} , \tag{9}$$

for the set of real numbers E_N corresponding to the spectrum of any system of site Hamiltonians ($E_N \equiv <N|H|N>$).

This result (9) generalizes a well-known mathematical result. Define the Bell series modulo p (p is a prime number) associated with the function A to be [6]

$$A_p(x) = \sum_{n=1}^{\infty} A(p^n)\, x^n . \tag{10}$$

In the special case in which the site Hamiltonian H_k corresponds to a

harmonic oscillator of frequency ωk (with the ground state energies normalized to zero), the energy eigenvalue of the state $|N>$ is simply ωN. For this system, the two sides of (9) become the Bell series modulo p associated with A and B, respectively. Consequently, using this harmonic oscillator system, we see that the two operators **A** and **B** are equivalent, and thus that the two multiplicative functions A and B are equivalent, provided that the functional equality

$$A_p(x) = B_p(x) \tag{11}$$

holds for every prime p. Thus we obtain the standard number theoretic result that two multiplicative functions A and B are identical if and only if for each prime p their Bell series are equal [6]. We note also that (8) actually provides a generalization of Bell series and results about them, as the harmonic oscillator system is only one of an infinite number of Hamiltonians for which the considerations of this paper apply.

We now consider a more general scenario. Consider a situation in which one has two Hamiltonians $\hat{H}$ and $\tilde{H}$, such that $\hat{H}$ and $\tilde{H}$ are formally the same, each built out of an identical set of site Hamiltonians. At site k, the Hamiltonian is the sum $\hat{H}_k + \tilde{H}_k$, with the two terms duplicating each other. We imagine that there is a quantum number (such as flavor) that distinguishes between the excitations of $\hat{H}$ and $\tilde{H}$, but that this quantum number has no (or negligible) other effect on the physics.

Now consider two physicists, one who has an experiment that can distinguish between the excitations of $\hat{H}$ and $\tilde{H}$, and one who has an experiment that cannot. To make this idea more specific, imagine that physicist #1 is aware that there are two types of creation operators, say $a^\dagger$ and $b^\dagger$, while physicist #2 believes there is

only one type of creation operator, say $c^\dagger$. So physicist #2 will interpret any $a^\dagger$ <u>or</u> $b^\dagger$ excitation as another $c^\dagger$ excitation.

The first physicist might independently measure two observables (or other operators): **f**, which acts on the Hilbert space $\hat{\mathcal{H}}$ of $\hat{H}$; and **g**, which acts on the Hilbert space $\tilde{\mathcal{H}}$ of $\tilde{H}$. Both **f** and **g** are taken to be multiplicative operators in their respective Hilbert spaces, and to be diagonalized by the energy eigenstates. Of course the full Hilbert space is the tensor product of these two constituent Hilbert spaces, and in this space one can consider the operator $\mathbf{f}\otimes\mathbf{g}$. The thermal expectation value of this operator is given by

$$<\mathbf{f}\otimes\mathbf{g}> = \frac{tr[\mathbf{f}\, exp(-\beta\hat{H})]}{tr[exp(-\beta\hat{H})]}\,\frac{tr[\mathbf{g}\, exp(-\beta\tilde{H})]}{tr[exp(-\beta\tilde{H})]} \tag{12}$$

Now what can the second physicist measure? This physicist is unable to distinguish between the excitations of $\hat{H}$ and $\tilde{H}$, and so can only measure operators without reference to $\hat{H}$ and $\tilde{H}$. The second physicist would label the states of the system by integers $|N>$; but each of these states would, in general, correspond to several states which are distinguishable only to the first physicist.

To be more precise, the first physicist would label states as $|\hat{N}, \tilde{N}>$, where the integer $\hat{N}$ ($\tilde{N}$) refers to the state in the Hilbert space $\hat{\mathcal{H}}$ ($\tilde{\mathcal{H}}$), according to the numbering scheme introduced in the previous portion of this paper. The second physicist would call this same state $|N>$, where $N = \hat{N}\cdot\tilde{N}$. The fact that this mapping from states $|\hat{N}, \tilde{N}>$ to states $|N>$ is many-to-one reflects the loss of information in going from the first to the second physicist's experiments.

We are now led to the natural question: What operator for the second physicist is equivalent to the operator $\mathbf{f}\otimes\mathbf{g}$? And what are its

properties? We first discuss these questions physically,, and then provide number theoretic interpretations of the results.

Let us denote by $\mathbf{\Phi}$ the operator to which $\mathbf{f}\otimes\mathbf{g}$ reduces in the second physicist's experiment. Then it is clear that

$$<N|\mathbf{\Phi}|N> = \sum_{\hat{N}\times\tilde{N}=N} <\hat{N},\tilde{N}|\mathbf{f}\otimes\mathbf{g}|\hat{N},\tilde{N}> = \sum_{\hat{N}\times\tilde{N}=N} <\hat{N}|\mathbf{f}|\hat{N}><\tilde{N}|\mathbf{g}|\tilde{N}> , \tag{13}$$

where the sum is restricted to those values of $\hat{N}$ and $\tilde{N}$ whose product is N. Furthermore, the thermal expectation values of these operators have to be related by

$$<\mathbf{\Phi}>_\beta = <\mathbf{f}>_\beta^{\hat{}} \cdot <\mathbf{g}>_\beta^{\tilde{}} , \tag{14}$$

where $<\cdot>_\beta$ is the thermal expectation value at inverse temperature β in the full system, and $<\cdot>_\beta^{\hat{}}$ ($<\cdot>_\beta^{\tilde{}}$) is the thermal expectation value restricted to the Hilbert space $\hat{\mathcal{H}}$ ($\tilde{\mathcal{H}}$).

Finally, we note that the operator $\mathbf{\Phi}$ is itself also a multiplicative operator whenever $\mathbf{f}$ and $\mathbf{g}$ are. In the Hilbert space constructed by physicist #2, $\mathbf{\Phi}$ still does not couple independent sites, and it is clear that, under the labelling of states used by physicist #2, $\mathbf{\Phi}$ meets all the conditions to be a multiplicative operator. The detailed discussion of physicist #1 is, in fact, essentially superfluous in recognizing that $\mathbf{\Phi}$ is a multiplicative operator.

We now translate these results into mathematical terms. Upon defining the three arithmetic functions $\phi(N) = <N|\mathbf{\Phi}|N>/<1|\mathbf{\Phi}|1>$, $f(\hat{N}) = <\hat{N}|\mathbf{f}|\hat{N}>/<1|\mathbf{f}|1>$, and $g(\tilde{N}) = <\tilde{N}|\mathbf{g}|\tilde{N}>/<1|\mathbf{g}|1>$, we see from (13) that (switching to the conventional mathematical notation)

$$\phi(N) = \sum_{d|N}^{\infty} f(d)\,g(N/d) , \tag{15}$$

where "$\sum_{d|N}$" means the sum over all integers d that are divisors of N. The composition law given in (15) is called Dirichlet convolution

by mathematicians; Dirichlet convolution is conventionally denoted by $*$, and so (15) may be rewritten as $\phi = f*g$. Thus we see that Dirichlet convolution has a natural physical interpretation; it arises in the situation in which there is a quantum number that distinguishes between two systems that are otherwise indistinguishable, and one wishes to compare the results of a physicist who is sensitive to this distinction with the results of a physicist who is unaware of it. (Of course, historically, one expects that objects thought identical will later turn out to be distinguishable, but not the other way around!)

It was noted above that $\mathbf{\Phi}$ is itself a multiplicative operator as long as $\mathbf{f}$ and $\mathbf{g}$ are. Mathematically, then, this becomes the statement that the function ϕ is multiplicative whenever f and g are, *i.e.*, that the Dirichlet convolution of two multiplicative functions is itself a multiplicative function. It is quite illuminating to have a physical understanding of this result. It is worth noting that this mathematical result is one of the reasons multiplicative functions play a special rôle in number theory; it enables the multiplicative functions to form a group under Dirichlet convolution.

In addition, result (14) is a generalization of a famous result in the theory of Dirichlet convolution. One can re-write (14) as

$$tr[\mathbf{\Phi}\,exp(-\beta(\hat{H}+\tilde{H}))] = tr[\mathbf{f}\,exp(-\beta\hat{H})]\;\; tr[\mathbf{g}\,exp(-\beta\tilde{H})] , \tag{16}$$

using the fact that

$$tr[exp(-\beta(\hat{H}+\tilde{H})] = tr[exp(-\beta\hat{H})]\cdot tr[exp(-\beta\tilde{H})] . \tag{17}$$

Let us specialize to Hamiltonians $\hat{H}$ and $\tilde{H}$ such that the state $|\hat{N},\tilde{N}>$ has energy $\omega[log(\hat{N})+log(\tilde{N})]$. In this case, in all three Hilbert spaces $\mathcal{H}$, $\hat{\mathcal{H}}$, and $\tilde{\mathcal{H}}$, the state $|M>$ has energy $\omega\, log M$. For this particular system, (16) reduces to

$$\sum_{n=1}^{\infty}\frac{f*g(n)}{n^s} = \sum_{j=1}^{\infty}\frac{f(j)}{j^s}\cdot\sum_{k=1}^{\infty}\frac{g(k)}{k^s} , \tag{18}$$

where $s=\beta\omega$, and where we have invoked the result (15) that $\phi = f*g$. The infinite sums in (18) are called Dirichlet series; *i.e.*, "$\sum_{k=1}^{\infty}\{g(k)/k^s\}$" is the Dirichlet series associated with the function g. Thus the result (18) is the statement that the Dirichlet series of the convolution of two multiplicative functions is equal to the product of the Dirichlet series of the two multiplicative functions. One should note also that (18) is just a special case of the more general result (14), and so, once again, this time in the guise of equation (14), quantum mechanics has led us to a generalization of a standard number theoretic result.

It is worth observing that (18) is true for all arithmetic functions f and g, not just multiplicative ones. The attentive reader will notice that nowhere in the arguments leading to (18) was it essential for f and g to be multiplicative; that was only essential for the claim that ϕ is multiplicative. Hence there is no undue restriction in the results given above. The interested reader can work out the details.

Thus we have seen that some of the central ideas of arithmetic number theory have simple physical interpretations and motivations. In this paper, I have shown that the notion of a multiplicative function arises naturally when considering systems that consist of several non-interacting sites. The prototype of all such systems is Einstein's oscillator model which he used to compute the specific heats of solids. Furthermore, I have shown that composing two functions by means of Dirichlet convolution represents the loss of information when one neglects a quantum number that distinguishes between what is otherwise a twofold degeneracy. Furthermore, on physical grounds we obtained the results that the Dirichlet convolution of two multiplicative functions is itself a multiplicative function, and that the product of the Dirichlet series of two arithmetic functions is the

Dirichlet series of the Dirichlet convolution of the two functions. These results, along with those of [1,2,3], clearly justify the claim that arithmetic number theory has a very natural relationship to the physical properties of quantum systems. Importantly, not only is there a mapping between number theoretic and physical concepts, but also ideas which are natural and/or important in one language are likewise natural and/or important in the other language

This is a very promising situation. In results such as (8) and (14), physical reasoning has led not just to the standard number theoretic results (which arise when one considers certain particular Hamiltonians), but in fact to generalizations of those results. Thus there is every reason to expect that continued application of physical ideas to number theory will lead to a fresh understanding of number theory. Similarly, the long history of number theory has produced many deep results, and these, in turn, should have an impact on physics. It is already known, for example, that the mathematical theory of partitions can be used to obtain examples of bose–fermi equivalence, as well as to study the spectrum of states in string theory. With the continued development of identifications between arithmetic number theory and quantum mechanics such as is described in this paper, the prospects are promising for continued fruitful cross–fertilization.

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