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THE ROUGHENING TRANSITION IN SURFACE MODELS WITH NON-LOCAL POTENTIALS*

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A thesis presented

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to

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Abstract

The high temperature behavior of a class of surface models with exponentially decaying non-local potentials is studied. The existence of a phase in which fluctuations in the interface described by the model grow logarithmically with the separation between the points of observation is proved. Since these models presumably possess a low temperature phase with uniformly bounded fluctuations, this establishes the occurrence of a roughening transition for these models. The possible extension of these results to models of surfaces with overhangs is briefly discussed.

Both upper and lower bounds on the fluctuations of the surface are derived. The upper bound combines a Mayer expansion with a duality transformation to rewrite the model as a spin system whose correlations may be bounded by complex translations. The lower bound combines a Mayer expansion with renormalization group ideas.

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CHAPTER I

INTRODUCTION

I.1. Background

This thesis is concerned with the study of a class of models obtained by introducing non-local potentials into the solid on solid (SOS) model. A certain class of such potentials is shown to be made up of "irrelevant operators" in the sense that there exists a high temperature "Kosterlitz Thouless" phase in which fluctuations in the height of the interface described by the model grow logarithmically with the separation between the points of observation. More specifically, we prove the existence of a range of inverse temperature, β, wherein there exist positive constants $c_u(\beta,J)$ and $c_L(\beta,J)$ such that

$$c_L(\beta,J)\ln(1+|x|) \le \langle (h_o-h_x)^2\rangle_\Lambda \le c_u(\beta,J)\ln(1+|x|) \quad . \quad (I.1)$$

In addition to extending the class of models in which a roughening transition is known to occur, we show in the next section that these models represent a first step toward understanding the existence of roughening transitions in models of surfaces with overhangs, like the three dimensional anisotropic Ising model. A heuristic but enlightening argument that such models should possess a roughening transition was given by Burton, Cabrera and Frank [10]. Consider an Ising model on a cubic lattice. Choose coordinate axes so that the $z = 0$ plane lies

between the two center planes of spins. Choose boundary conditions so that the boundary spins are forced to be +1 if the site under consideration lies above the $z = 0$ plane, and -1 if it lies below the $z = 0$ plane. At temperature $T = 0$ we expect all randomness to be "frozen" out of the system and we get a separation into two regions with all up or down spins. This gives a flat interface between the two phases, and is represented in Figure 1(a).

Now consider what happens when we raise the temperature slightly above zero. In the layer of spins just above the boundary between the two phases, we expect the effects of positive spins from above the interface to approximately cancel the effects of negative spins from below. Thus this plane should behave like a two dimensional Ising model, which for sufficiently small temperature still possesses a net magnetization, and gives rise to a nearly flat interface, as depicted in Figure 1(b). In Figure 1(c) we examine what happens in this model for temperatures above the critical temperature of the two dimensional Ising model, but below the bulk critical temperature for the three dimensional Ising model. Although far from the $z = 0$ plane, we still expect a net magnetization, whose sign depends on whether we are in the upper or lower half of the system, near the interface, where the behavior is similar to the two dimensional Ising systems, the net magnetization will disappear, and the interface between the two phases will become quite convoluted. This signals the occurrence of the roughening transition.

For the two dimensional Ising model, this argument predicts that the interface should remain rough even at very low temperatures, a fact

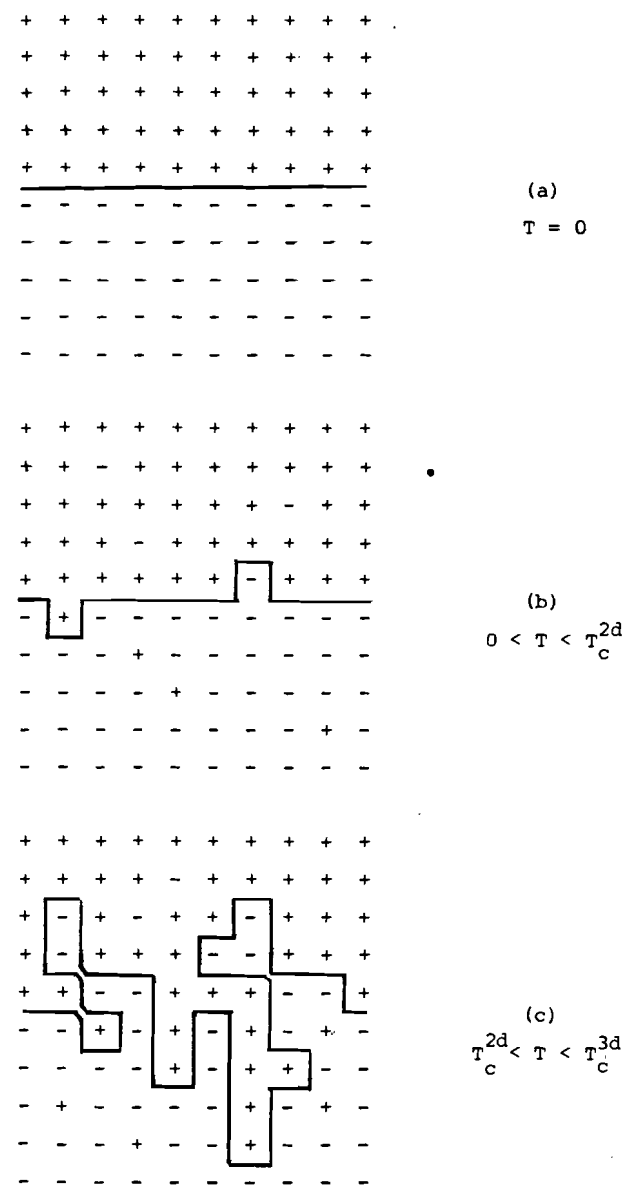


Figure I.1. The Burton, Cabrera and Frank argument.

which was verified by Gallavotti [16].

Our proof of the lower bound of (I.1) combines the renormalization group method of Fröhlich and Spencer [14,15] with Mayer expansion techniques. The mathematical understanding of the renormalization group is still at a preliminary stage, but its uses to date fall into two broad categories. In the first of these, the effect of some transformation on the dynamical function of the theory (e.g., the Hamiltonian of a statistical mechanical system) is considered. Fixed points of this (renormalization group) transformation are sought in some appropriate function space, and then critical exponents, scaling relations, and other information about the system near its critical point may be derived. Examples of systems in which this procedure has been carried out are provided by [6, 11, 12, 18, 19, 20, 21].

In the second group of studies, more modest goals are pursued. In these works the inductive ideas which underly the renormalization group are used to control some aspect of the theory in a finite volume, but one does not attempt to answer the more difficult questions, such as whether or not this inductive procedure leads to a renormalization group fixed point. Examples of this sort of use of renormalization group ideas are provided by [2, 3, 5, 14, 15, 24].

The lower bound of the present work falls into the second class of renormalization group studies. Our models are first rewritten as a "gas" of interacting charge densities. Unfortunately, certain of these charge densities possess very large entropy factors which make correlations in these models difficult to estimate. We control these entropy factors by using the complex translation technique, developed by

Fröhlich and Spencer [15], to extract some fraction of the self-energy of these charges on larger and larger distance scales. This inductive procedure is closely tied to the renormalization group [28, 29]. The small factors which result from this procedure are shown to dominate the entropy of the charge densities for some range of temperature. Because of the non-local potentials present in our model, long range interactions exist between the renormalized charge densities. The principle contribution of this section of the thesis is to show how the Mayer expansion techniques [9, 27], useful in the study of systems far from a critical point, may be used here to control the interactions in this critical model.

The upper bounds we prove on fluctuations in these models combine the complex translation techniques of McBryan and Spencer [30] with the duality transformation [15, 31] relating the SOS and discrete Gaussian models to two dimensional spin systems with nearest neighbor interactions. The duality transformation depends on the fact that the interactions between columns in the SOS model involve only nearest neighbor sites and hence it is not immediately obvious how these ideas would be applicable to models which contain non-local potentials. It is demonstrated that although one cannot transform these models into spin systems with strictly nearest neighbor interactions, the dual transform of the partition function of the perturbed SOS model can be written, using a Mayer expansion combined with the algebraic formalism of [4, 7, 17,25,26] as a two dimensional spin system with a set of very complicated non-local interactions. Despite these complications, the Mayer expansion yields sufficient control over the potentials that one generates to

prove the logarithmic upper bound.

In both the upper and lower bound, the use of the Mayer expansion is somewhat novel in that one has less control over the state about which one is expanding than is typically the case. Usually, one expands about an exactly solvable (e.g., ideal gas) state, or about some state over which one has strong control by means of a cluster expansion. In the present case neither of these applies. It seems that the applicability of the expansion is not much affected by this restriction, however, but rather one just limits the sort of questions one is allowed to ask about the resulting system.

I.2. The Definition of the Models

We now define the class of models we consider. Let Λ be a square region in $\mathbb{R}^2$, centered at the origin, of side length $(2m+1)$, $m \in \mathbb{Z}$. We shall require for technical reasons that $|\Lambda| > \Lambda_o$, where $|\Lambda|$ is the number of sites in $\mathbb{Z}^2 \cap \Lambda$, and Λ_o is some $O(1)$ constant defined in Appendix I. We consider models whose partition functions are given by

$$Z_\Lambda = \sum_{\{h\}} e^{-\beta \sum_{\langle i,j\rangle} |h_i - h_j| + \sum_{X \subset \Lambda} V_X^J(\{h\}|_X)} . \qquad (2.1)$$

The sum over $\{h\}$, runs over all configurations of integer valued fields on $\mathbb{Z}^2$, which obey the boundary conditions $h_i \equiv 0$ for all sites $i \in \mathbb{Z}^2 \cap \Lambda^c$. Here, and throughout this work, $\langle i,j\rangle$ will denote a pair of nearest neighbor sites in $\mathbb{Z}^2$. Because of our boundary conditions, the sum over nearest neighbor pairs may be thought of as running over all pairs in $\mathbb{Z}^2$, or merely over those pairs which intersect Λ. The sum over $X \subset \Lambda$ runs over all connected sets of plaquettes in the dual lattice, $\mathbb{Z}^{2*} = (\mathbb{Z} + \frac{1}{2})^2$, which are contained in Λ. The non-local potentials $V_X^J(\cdot)$ are assumed to depend only on the values of the field $\{h\}$ at sites in $\mathbb{Z}^2 \cap X$. We denote this restriction of the field configuration by $\{h\}|_X$.

Our results require that the functions $V_X^J(\cdot)$ obey certain restrictions. Specifically we require

(a) Inversion Symmetry: We assume that $V_X^J(\cdot)$ is invariant under the change in the field configuration, $\{h\} \to -\{h\}$.

(b) Nearest Neighbor Dependence: We assume that $V_X^J(\cdot)$ depends not on the absolute values of the field $\{h\}|_X$ but only on the differences in height at adjacent sites in $\mathbb{Z}^2 \cap X$. Referring to Figure 2 this means that $V_X^J(\cdot)$ depends only on h_1-h_2, h_2-h_3, and h_3-h_4. We denote this fact by writing (in a slight abuse of notation)

$$|V_X^J(\{h\}|_X)| = V_X^J(\{h_i-h_j\}|_X) = V_X^J(\{\partial h\}|_X) \quad . \tag{2.2}$$

(c) $V_X^J(\{h\}|_X) \le \exp(-J|X|)$, independent of $\{h\}$.

We shall also assume in Chapter IV that $V_X^J(\cdot) = 0$ unless $|X| \ge 3$. This, however, is not a real restriction since if $V_X^J(\{h\}|_X) \ne 0$ for $|X| = 1,2$ we can always absorb these potentials into $V_Y^J(\cdot)$, for Y the set with $|Y| = 3$ and $X \subset Y$, and then redefine J if necessary.

We now discuss the origin of these models. Previous to this work the only models in which the existence of a roughening transition had been rigorously demonstrated were the Discrete Gaussian and SOS models [15]. These models both attempt to describe the behavior of surfaces--presumably at the interface between two phases of some system. In each case the shape of the interface is specified by associating to each site in $\mathbb{Z}^2$ an integer and assuming that the integer h_i specifies the height of the interface at the point i. These models possess two crucial restrictions. First, they prohibit direct interaction between lattice sites more than one lattice spacing distant from one another.

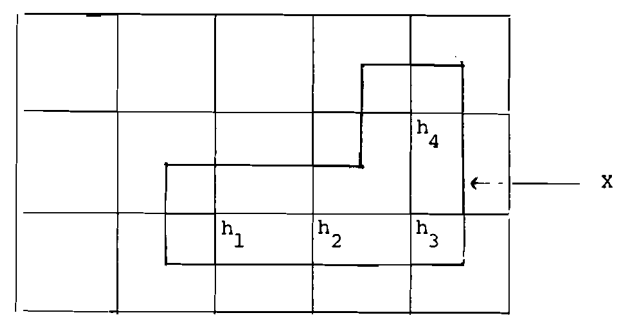
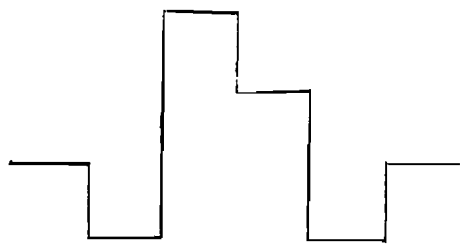
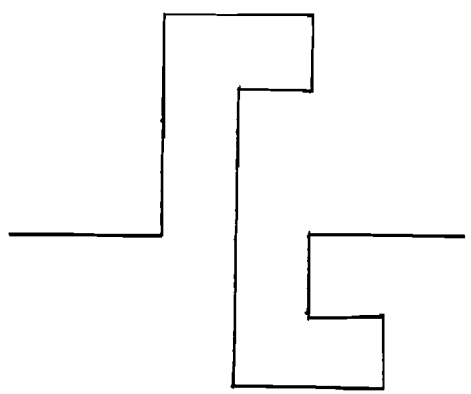


Figure I.2. The non-local potential $V_X^J(\cdot)$ depends on the variables; h_1-h_2, h_2-h_3, and h_3-h_4.



(a)



(b)

Figure I.3. Sections of interfaces: (a) An interface for the SOS model. (b) Because of the overhangs, this interface could not appear in the SOS model.

Second, they do not allow for any overhanging parts of the interface. Our models certainly overcome the first of these restrictions and since, as we shall see below, these two restrictions are related, our results represent a first step towards removing the second restriction as well.

One more realistic model for interfaces is provided by the three dimensional Ising model introduced in Section 1. In this case the interface is the Peierls contour separating the upper and lower halves of the system. As Figure 1 shows these interfaces do have overhangs. One might expect on heuristic grounds that the effect of overhanging configurations would be similar to that of the non-local potentials Certainly the sort of overhangs allowed depend on the shape of the remainder of the interface, just because of the requirement that the interface be non-intersecting.

Consider the statistical mechanical model obtained in the following fashion. An interface in the SOS model may be represented as a connected set of plaquettes in $(\mathbb{Z}^3)^*$. Because of our boundary conditions, there is a horizontal plane whose intersection with the interface contains an infinite number of plaquettes. Call this plane the zero plane. Let Λ be the square of side length $2m+1$ centered at the origin of the zero plane. Given an interface specified by the set of plaquettes Γ, define

$\Gamma_W = \{$vertical plaquettes in $\Gamma\} \equiv$ walls of Γ ;

$\Gamma_F = \{$horizontal plaquettes in Γ whose projection into the zero plane intersects $\Lambda\} \equiv$ floors of Γ.

These geometrical constructions are similar to those of [8, 13]. In this representation, the Boltzman factor of the SOS model becomes

$$e^{-\beta H(\Gamma)} = e^{-\beta|\Gamma_W|} ,$$

where $|\Gamma_W|$ denotes the number of plaquettes in the walls of Γ. One then defines an overhang to be some closed connected set of plaquettes, I, whose projection, P(I), onto the zero plane is contained in Λ. One defines the allowed interfaces in our new model by inserting overhangs into SOS interfaces. Given some SOS interface Γ, and a collection of non-intersecting overhangs $\{I_1,\ldots,I_k\}$, we define an interface $\Gamma' = \Gamma \cup \{\cup_1^k I_j\} \setminus \cup_1^k \{\Gamma \cap I_j\}$, provided such an interface is an allowed interface for the phase separated Ising model, and provided $\Gamma'' = (\Gamma \cup I_j) \setminus (\Gamma \cap I_j)$ is not an allowed interface for the SOS model for any $j = 1,\ldots,k$ and $\Gamma_W \cap I_j \neq \phi$, for any $j = 1,\ldots,k$. We assign Γ' a statistical weight

$$e^{-\beta H(\Gamma')} = e^{-\beta|\Gamma_W|} \cdot e^{-\beta J_z|\Gamma'_F|} .$$

This corresponds to the Boltzman factor of the interface in an anisotropic Ising model whose coupling in the x and y directions is 1, and whose coupling in the z direction is J_z. Summing over all sets of insertions consistent with a given SOS interface and then summing over all possible SOS interfaces, one arrives at a partition function of the form (2.1). One can then ask to what extent the conditions (2.2) are verified for the non-local potentials we generate in this procedure. Conditions (2.2)(a) and (2.2)(b) are satisfied. Condition (a) reflects the fact that the number, shape, and size of the overhangs which can be inserted into a given interface are unchanged if

one inverts the whole interface. Similarly, condition (b) is just a reflection of the fact that one can translate the whole SOS interface (or some section of it) vertically, without changing the kind of overhangs which one can attach to it. Condition (c) is the one where our estimates break down at present. The assumed exponential decay is valid. It arises from the fact that any overhang must contain at least twice as many horizontal plaquettes as the number of plaquettes in P(I). Since $V_X^J(\cdot)$ is associated with sets of overhangs for which $\cup_1^k P(I_j) = X$, and since each horizontal plaquette carries with it a factor of $e^{-\beta J_z}$, we obtain decay of the sort claimed in (2.2)(c), with J proportional to βJ_z. The problem which arises is that this decay is not uniform in $\{h\}$. The reason for this can be seen in Figure 3(b). As the vertical "wall" into which we insert the overhang becomes higher and higher, the number of places available to insert the overhang grows, and thus $V_X^J(\cdot)$ should also grow. This is found to be the case. This is the principle restriction which prevents one from applying the results of this thesis to the anisotropic Ising model. It should be noted that opposing this growth in the non-local potentials due to the increasing height of the walls is the fact that the SOS part of the partition function becomes exponentially small as the height of the walls becomes large. Work is in progress which attempts to exploit this fact.

I.3. Statement of Results and Notational Conventions

We now state our principle results on the correlations in models whose finite volume partition function is given by (2.1).

Theorem I.1. There exist constants $0 < \beta_1 < \beta_2$ and $J_o(\beta_1) > 0$ such that for $\beta \in (\beta_1,\beta_2)$ and all $J > J_o(\beta_1)$, there exist positive constants $c_u(\beta,J)$ and $c_L(\beta,J)$ such that if $x = (n,o) \in \mathbb{Z}^2$ with $0 < n < |\Lambda|^{1/8}$, one has

$$c_L(\beta,J)\ln(1+|x|) \le \langle (h_o-h_x)^2\rangle_\Lambda \le c_u(\beta,J)\ln(1+|x|) \quad . \tag{3.1}$$

This theorem establishes the existence of some range of inverse temperatures wherein (for J sufficiently large) the fluctuations in the interface described by the model grow logarithmically with the separation between the points of observation. Heuristically one would not expect that once the surface became "rough" further increase in temperature (decrease in β) would cause the surface to become "smooth" again. Thus one would expect to be able to take $\beta_1 = 0$ in Theorem I.1. The fact that this is not the case appears to be a technical limitation of the method used to renormalize the nonlocal potentials.

The restriction in this theorem that requires the site x to lie on the horizontal axis is not an essential one. It could be removed at the expense of complicating the proof of theorem.

Theorem I.1 follows from the next two results.

Theorem I.2. There exist constants $0 < \beta_1 < \beta_2$ and $J_1(\beta_1) > 0$ such

that for all $\beta \in (\beta_1,\beta_2)$ and $J > J_1(\beta_1)$, there exists a constant $c_L(\beta,J) > 0$ such that for all sites $x \in \mathbb{Z}^2$ with $|x| \le |\Lambda|^{1/8}$ one has

$$\langle e^{\varepsilon(h_o-h_x)}\rangle_\Lambda \ge e^{\varepsilon^2 c_L(\beta,J)\ln(1+|x|)} \quad . \tag{3.2}$$

(By $|\Lambda|$ we mean the number of sites in $\mathbb{Z}^2 \cap \Lambda$.)

Theorem I.3. For every $\beta > 0$ there exists a constant $J_2(\beta) > 0$ such that for all $J > J_2(\beta)$ there exists a positive constant, $c_u(\beta,J)$, such that for all $x = (n,o) \in \mathbb{Z}^2$ with $0 < n < |\Lambda|^{1/8}$ one has

$$\langle e^{\varepsilon(h_o-h_x)}\rangle_\Lambda \le e^{\varepsilon^2 c_u(\beta,J)\ln(1+|x|)} \quad . \tag{3.3}$$

Furthermore, $C_u(\beta,J)$ may be uniformly bounded by some constant $K(\beta)$, for all $J > J_2(\beta)$. A few remarks are in order.

(1) Theorem I.1 follows from Theorem I.2 and Theorem I.3 by subtracting one from both sides of inequality (3.2) and (3.3), expanding to third order in ε, dividing by ε^2, and taking the limit $\varepsilon \to 0$. Note that in both cases, the term proportional to ε on the l.h.s. of (3.2) and (3.3) vanishes due to the invariance of these models under $\{h\} \to -\{h\}$.

(2) The constant $J_o(\beta_1)$ in Theorem I.1 may be taken to be $\max(J_1(\beta_1), \sup_{\beta\in(\beta_1,\beta_2)} J_2(\beta))$.

(3) Although the details of the proof have not been worked out,

slight extensions of the results of [7, 8, 13] should lead to a proof of the existence of a "smooth" low temperature (large β) phase for these models, which would be characterized by some x-independent constant K' with

$$\langle (h_o - h_x)^2 \rangle_\Lambda \le K' \quad .$$

(4) The bounds of (3.1) are uniform in the volume Λ of the system. One may therefore construct (if necessary by considering some subsequence of regions $\Lambda_n \uparrow \mathbb{R}^2$) the infinite volume limit for these models, and the stated bounds hold for that limit as well. Note further that in this limit the restriction that $|x| < |\Lambda|^{1/8}$ disappears.

(5) Our methods work equally well for perturbations of the discrete Gaussian model. All the above results hold if we replace the models defined by (2.1) by models with finite volume partition functions

$$Z_\Lambda = \sum_{\{h\}} e^{-\beta \sum_{\langle i,j \rangle} (h_i - h_j)^2 + \sum_{X \subset \Lambda} V_X^J(\{h\}|_X)} \quad . \tag{3.4}$$

(6) Note that the SOS model is just the $J \to \infty$ limit of the models defined by (2.1). Thus one has

<u>Corollary I.4</u>. For every $\beta > 0$, there exists a constant $K(\beta)$ such that for all $x = (n,o) \in \mathbb{Z}^2$, with $0 < n < |\Lambda|^{1/8}$, the fluctuations in the SOS model are bounded by

$$\langle (h_o - h_x)^2 \rangle_\Lambda \le K(\beta) \ln(1+|x|) \quad . \tag{3.5}$$

This estimate complements the lower bound on fluctuations in this

model, proven by Fröhlich and Spencer [15]. We shall in fact, for ease of exposition, prove Corollary I.4 first and then extend the proof to include the models with non-local potentials.

We close this chapter with some comments on notation.

(1) At numerous points in the proofs of these theorems, it will be necessary to introduce constants independent of β and J. These will be generically denoted as $c, c', \ldots$. They need not denote the same constant in their different appearances.

(2) At a number of points in the proofs in later chapters, the phrase "for J sufficiently large" appears. This should always be interpreted as "there exists $J_i(\beta) > 0$ (with $i = 1,2$ depending on the context) such that for all $J > J_i(\beta)$ the above statement is true." Similarly the phrase, "for β sufficiently small" means "there exist constants $0 < \beta_1 < \beta_2$ such that for all $\beta \in (\beta_1, \beta_2)$ the above statement is true."

(3) The notation $|\Gamma|$ will mean "cardinality of the set Γ." Any deviations from this usage will be noted.

(4) For convenience we suppress the dependence of the potentials on J, i.e., we write $V_X^J(\cdot) = V_X(\cdot)$.

(5) There are a large (but finite) number of situations in this thesis in which we will be required to choose $J_i(\beta)$ large (i is again either 1 or 2) and $\beta_2 > \beta_1 > 0$ small. In order to see that these choices may all be made consistently, we take $\beta_1 = \beta_2/2$. Then choose the β_2 which appears in Theorem I.2 to be the minimum of all the choices of β_2, it was necessary to make in the course of the proof

of the theorem. We then choose $J_1(\beta_1)$ to be the maximum value of $J_1(\beta_1)$ among the choices we had to make in the course of the proof of the theorem, given the value of β_1 chosen above. Similarly, in Theorem I.3 we choose $J_2(\beta)$ to be the maximum among the finite number of choices of $J_2(\beta)$ we made in the course of proving the theorem.

(6) In referring to equations within a given chapter we shall give only the section number, and the equation number within that section (e.g., (2.2) refers to equation 2 in section 2 of this chapter). In referring to equations in other chapters, we shall precede the section and equation numbers by a chapter number (e.g., (II.2.2) refers to equation 2 in section 2 of chapter II). Similar considerations apply to figure numbers.

CHAPTER II

THE MAYER EXPANSION FOR THE LOWER BOUND

In this chapter we prove Theorem I.2. The proof breaks naturally into four sections. In the first section, we exchange our integral variables $\{h\}$ for the continuous variables $\{\phi\}$. The estimate of Theorem I.2 is shown to follow from an estimate on a ratio of partition functions in the $\{\phi\}$ variables. The second section describes the renormalization of these partition functions using the method of complex translations developed by Fröhlich and Spencer [15]. We diverge from their treatment because one cannot write the sum which results from the renormalization procedure as a product over "renormalized" but independent charge densities, due to the non-local potentials $V_x(\cdot)$. This problem is solved in Section 3, where the renormalized charge densities are decoupled from one another via a Mayer expansion. The form of this expansion is similar to that of [27]. The output of the expansion is then exponentiated using algebraic techniques like those of [4, 7, 17, 25].

The exponentiated expression leads to a positive measure to which we apply Jensen's inequality in Section 4. Using bounds which come from the convergence of the Mayer expansion, we demonstrate the requisite lower bound on the ratio of partition functions.

II.1 The Sine-Gordon Transformation

In this section we rewrite the partition function of (I.2.1) as an integral of analytic functions. The first step is to replace the functions $e^{-\beta|h_i-h_j|}$ and $e^{V_X(\{h\}|_X)}$ in the partition function with analytic functions whose values are equal to those of the above functions when their arguments are real and integral. The functions we use come from the following propositions.

Proposition II.1 (Fröhlich and Spencer [15]). There exists a function $I_\beta(\phi)$, analytic and non-zero in the strip $|\mathrm{Im}\phi| < \mathrm{const.}\beta^{-1/3}$, which satisfies

(a) $I_\beta(\phi=n) = e^{-\beta|n|}$, for all $n\in\mathbb{Z}$,

(b) $I_\beta(\cdot)$ is even, positive and real on the real axis,

(c) $|I_\beta(\phi+ia)/I_\beta(\phi)| \le e^{\beta^{1/3} g(a)}$, for all a with $|a| \le \mathrm{const.}\ \beta^{-1/3}$,

(d) $\left|\frac{d^m}{d\phi^m} \log I_\beta(\phi+ia)\right| \le c\beta^{1/3}(g(a)+1)$, with c a constant independent of ϕ and β, for all a with $|a| \le \mathrm{const.}\beta^{-1/3}$, $m = 1,2$. (1.1)

Here,

$$0 < g(a) \le \begin{cases} \mathrm{const.}\ a^2 & |a| \le 1 \\ \mathrm{const.}\ e^{2\pi|a|} & 1 \le |a| \le \mathrm{const.}\ \beta^{-1/3} \end{cases} .$$

Define functions

$$\tilde{V}_X(\{\phi\}|_X) = \sum_{\substack{\{h\}|_X: \\ h_i = \text{constant} \\ \text{for some } i\in X}} V_X(\{h\}|_X) \prod_{\langle i,j\rangle\subset X} \frac{\sin^2\pi(\phi_i-\phi_j+h_i-h_j)}{\pi^2(\phi_i-\phi_j+h_i-h_j)^2} \tag{1.2}$$

Because of (I.2.2)(b), $\tilde{V}_X(\{\phi\}|_X)$ is independent of the site $i\in X$ at which we choose to fix the value of the field h_i, and also of the value of h_i which we choose.

Proposition II.2. The functions $\tilde{V}_X(\cdot)$ satisfy

(a) $\tilde{V}_X(\cdot)$ is analytic in each of its arguments,

(b) $\tilde{V}_X(\{\phi\}|_X) = \tilde{V}_X(-\{\phi\}|_X)$,

(c) $\tilde{V}_X(\cdot)$ depends in fact only on the differences in the value of the field ϕ at nearest neighbor sites contained in X, not on the absolute value of ϕ at any site,

(d) $\tilde{V}_X(\{\phi=n\}|_X) = V_X(\{n\}|_X)$ if each $n_i\in\{n\}|_X$ is an integer. (1.3)

Proof. Properties (a)-(c) follow by inspection of the definition (1.2). Property (d) follows because $\frac{\sin^2\pi x}{\pi^2 x^2}$ is equal to 1 if $x=0$, and zero for any other integral value of x. Conditions (1.1)(a) and (1.3)(d) insure that (I.2.1) may be rewritten as

$$Z_\Lambda = \sum_{\{h\}} \prod_{\langle i,j\rangle} I_\beta(h_i-h_j)\ e^{\sum_{X\subset\Lambda} \tilde{V}_X(\{h\}|_X)} . \tag{1.4}$$

Recall that our boundary conditions require that $h_i \equiv 0$ for $i\in\Lambda^c\cap\mathbb{Z}^2$.

Rewrite (1.4) via the Poisson summation formula as:

$$Z_\Lambda = (\prod_i \int d\phi_i)(\prod_i \sum_{m_i=-\infty}^{\infty} \delta(\phi_i - m_i))$$

$$\times [\prod_{<i,j>} I_\beta(\phi_i-\phi_j)] e^{\sum_{X\subset\Lambda} \tilde{v}_X(\{\phi\}|_X)}$$

$$= (\prod_i \int d\phi_i)(\sum_{\{m\}} e^{i2\pi m_i\phi_i})(\prod_{<i,j>} I_\beta(\phi_i-\phi_j)) e^{\sum_{X\subset\Lambda} \tilde{v}_X(\{\phi\}|_X)}$$

$$= (\prod_i \int d\phi_i)\prod_i (1+2\sum_{m_i=1}^{\infty} \cos 2\pi m_i\phi_i) \prod_{<i,j>} I_\beta(\phi_i-\phi_j)$$

$$\times e^{\sum_{X\subset\Lambda} \tilde{v}_X(\{\phi\}|_X)} \quad . \tag{1.5}$$

Here, $\prod_i$ runs over all sites in $\mathbb{Z}^2\cap\Lambda$. Now use the multipole expansion of [15] to rewrite the last expression. Define a charge density, ρ, to be a function on $\mathbb{Z}^2\cap\Lambda$, which takes values in $2\pi\mathbb{Z}$. We define an ensemble N of charge densities to be a collection of charge densities with disjoint support.

<u>Theorem II.3</u> (Fröhlich and Spencer [15]). There exists a set of ensembles of charge densities, $\{N\}$, and a set of positive constants $\{c_N\}$ such that

$$\prod_j (1+2\sum_{m_j=1}^{\infty} \cos 2\pi m_j\phi_j) = \sum_N c_N \prod_{\rho\in N} (1+K(\rho)\cos\phi(\rho)) \quad . \tag{1.6}$$

Here, $\phi(\rho) = \sum_{i\in\Lambda} \phi_i\rho(i)$. The charge densities ρ in each N satisfy:

(a) If $\rho \neq \rho'$ then $\mathrm{dist}(\rho,\rho') \geq M[\min(d_\Lambda(\rho), d_\Lambda(\rho')]^\alpha$,

(b) if $\rho_1 \subset \rho \in N$ satisfies $\mathrm{dist}(\rho_1, \rho-\rho_1) \geq 2Md(\rho_1)^\alpha$, then $Q(\rho_1) \neq 0$ and $2M\mathrm{dist}(\rho_1, \Lambda^c)^\alpha \geq \mathrm{dist}(\rho_1, \rho-\rho_1)$,

(c) the constants $K(\rho)$ satisfy $0 < K(\rho) \leq \prod_{i\in \mathrm{supp}\,\rho} (4\rho(i))^4 \cdot e^{S(\rho)}$ where $0 < S(\rho) < c\cdot \sum_{k=0}^{n(\rho)} A_k(\rho)$ for some constant c. Here, $n(\rho)$ = smallest integer $\geq \ln_2(Md_\Lambda(\rho)^\alpha)$, and $A_k(\rho)$ is equal to the minimal number of $2^k\times 2^k$ squares necessary to cover supp ρ. $(A_0(\rho) \equiv |\mathrm{supp}\,\rho|)$ (1.7)

In this theorem, $Q(\rho) \equiv \sum_{i\in\mathbb{Z}^2} \rho(i)$, $\mathrm{dist}(\rho,\rho')$ = distance between the support of ρ and ρ', M and α are constants with $3/2 < \alpha < 2$ and $M > 1$, and

$$d_\Lambda(\rho) = \begin{cases} d(\rho) = \text{diameter of supp } \rho & \text{if } Q(\rho) = 0 \\ \max[\mathrm{dist}(\rho,\Lambda^c), d(\rho)] & \text{if } Q(\rho) \neq 0 \end{cases} \quad . \tag{1.8}$$

<u>Proof</u>. An outline of the proof of this theorem is provided in Appendix II.

Now perform the Sine-Gordon transformation on the unnormalized expectation $Z_\Lambda \langle e^{\varepsilon(h_o-h_x)}\rangle_\Lambda$, as well as on the partition function arriving at

$$\langle e^{\varepsilon(h_o-h_x)}\rangle_\Lambda = \frac{\sum_N c_N \int \prod_{\rho\in N} (1+K(\rho)\cos\phi(\rho)) e^{\varepsilon(\phi_o-\phi_x)} e^{\sum_{X\subset\Lambda} \tilde{v}_X(\{\phi\}|_X)} d\mu_\Lambda(\phi)}{\sum_N c_N \int \prod_{\rho\in N} (1+K(\rho)\cos\phi(\rho)) e^{\sum_{X\subset\Lambda} \tilde{v}_X(\{\phi\}|_X)} d\mu_\Lambda(\phi)} \quad . \tag{1.9}$$

Here, $$d\mu_\Lambda(\phi) = \prod_{<i,j>} I_\beta(\phi_i-\phi_j) \prod_{i\in\Lambda\cap\mathbb{Z}^2} d\phi_i \quad .$$

Define the function

$$\sigma_i = \varepsilon|\ln\beta|[C_D(i,0)-C_D(i,x)] = \varepsilon|\ln\beta|(C_D f)(i) \quad . \qquad (1.10)$$

C_D is the inverse of minus the two dimensional Laplacian with Dirichlet boundary conditions on $\partial\Lambda$ and $f(i) = \delta_{i,0} - \delta_{i,x}$.

Lemma II.4.

$$<e^{\varepsilon(h_0-h_x)}>_\Lambda = e^{\varepsilon(\sigma_o-\sigma_x)} \frac{\sum_N c_N Z_\Lambda(\sigma,\varepsilon)}{\sum_N c_N Z_\Lambda(0,0)} \quad , \qquad (1.11)$$

with

$$Z_N(\sigma,\varepsilon) = \sum_\Gamma \sum_{\{\zeta\}} (\tfrac{1}{2})^{|\Gamma|} \Big(\prod_{\rho\in\Gamma} K(\rho) e^{i\zeta_\rho(\phi+\sigma)(\rho)} \Big) e^{\varepsilon(\phi_o-\phi_x)}$$
$$\times\, e^{\sum_{X\subset\Lambda} \tilde{V}_X(\{\phi+\sigma\}|_X)} \Big(\prod_{<i,j>} \frac{I_\beta(\phi_i-\phi_j+\sigma_i-\sigma_j)}{I_\beta(\phi_i-\phi_j)} \Big) d\mu_\Lambda(\phi) \quad . (1.12)$$

The sum over Γ runs over all subsets of N, and $\sum_{\{\zeta\}}$ runs over all $\zeta_\rho = \pm 1$, $\rho \in \Gamma$. $Z_N(0,0)$ is exactly the same expression with $\sigma_i \equiv 0$, and the observable $e^{\varepsilon(\phi_o-\phi_x)}$ deleted.

Proof. The lemma follows from a change of variables

$$\phi_i \to \phi_i + \sigma_i \quad , \qquad (1.13)$$

in the numerator of (1.9). Such a change of variables respects our boundary conditions. One obtains

$$Z_\Lambda <e^{\varepsilon(h_o-h_x)}>_\Lambda = \sum_N c_N \int \prod_{\rho\in N} (1+\cos(\phi+\sigma)(\rho)) \cdot e^{\varepsilon(\sigma_o-\sigma_x)+\varepsilon(\phi_o-\phi_x)}$$
$$\times\, e^{\sum_{X\subset\Lambda} \tilde{V}_X(\{\phi+\sigma\}|_X)} \Big(\prod_{<i,j>} \frac{I_\beta(\phi_i-\phi_j+\sigma_i-\sigma_j)}{I_\beta(\phi_i-\phi_j)} \Big) d\mu_\Lambda(\phi) \quad . \qquad (1.14)$$

To obtain (1.12), first expand the product over $\rho \in N$. This yields the sum over Γ. Then rewrite the cosine functions as exponentials, giving rise to the factors of $(\frac{1}{2})$ and the sum over $\{\zeta\}$. The denominator in (1.11) is obtained by expanding the product over ρ in the partition function and rewriting the cosine functions as exponentials, but not translating by σ.

The remainder of the chapter will be devoted to proving

Theorem II.5. For any ensemble of charge densities N satisfying conditions (1.7)(a)-(c), there exists $0 < \beta_1 < \beta_2$ and $J_1(\beta_1) > 0$, independent of N, such that for all $\beta \in (\beta_1,\beta_2)$ and $J > J_1(\beta_1)$, there exists a constant $k(\beta,J)$ independent of N such that

$$\frac{Z_N(\sigma,\varepsilon)}{Z_N(0,0)} \geq e^{-\varepsilon^2 k(\beta,J)\ln(1+|x|)} \cdot e^{-2\varepsilon^2} \quad . \qquad (1.15)$$

Furthermore, $k(\beta,J)$ can be made arbitrarily small by making β_2 sufficiently small and $J_1(\beta_1)$ sufficiently large. This theorem, coupled with (1.11) and the fact that $Z_N(0,0) > 0$, which we demonstrate later in this chapter, yields

$$<e^{\varepsilon(h_o-h_x)}>_\Lambda \geq e^{\varepsilon(\sigma_o-\sigma_x)} \cdot e^{-\varepsilon^2 k(\beta,J)\ln(1+|x|)} e^{-2\varepsilon^2} \quad . \qquad (1.16)$$

Since

$$\sigma_o-\sigma_x = \varepsilon|\ln\beta|[C_D(0,0)+C_D(x,x)-C_D(0,x)-C_D(x,0)]$$

$$\geq \varepsilon|\ln\beta|\cdot C\cdot\ln(1+|x|) \quad , \tag{1.17}$$

for some $C > 0$, a fact which we demonstrate in Appendix I, Theorem I.2 follows immediately upon application of Lemma II.4 and Theorem II.5.

II.2 Renormalization via Complex Translation

In this section we renormalize the expressions for $Z_N(\sigma,\varepsilon)$ and $Z_N(0,0)$ by the method of complex translations. We follow closely [15]. The renormalization proceeds in an inductive fashion on increasing scales of size 2^k. Let N be some ensemble of charge densities. We first choose $a_\rho^0(\cdot)$, the translating functions for each $\rho\in N$ on the 2^0 scale. Divide the support of each $\rho\in N$ into two subsets T_1^ρ and T_2^ρ such that no site in T_1^ρ is a nearest neighbor of any site in T_2^ρ. Let T_0^ρ be either T_1^ρ or T_2^ρ depending on which one has $\sum_{j\in T_i^\rho}\rho^2(i)$ largest for $i=1,2$. Since $|\rho(j)|\geq 1$ for $j\in\operatorname{supp}\rho$, $\sum_{j\in T_0^\rho}\rho^2(i)\geq 1/2|\operatorname{supp}\rho|$. Now choose

$$a_\rho^0(i) = \begin{cases} 0 & \text{if} \quad i\notin T_0^\rho \\ \gamma\ \operatorname{sgn}\rho(i) & \text{if} \quad i\in T_0^\rho \end{cases} \tag{2.1}$$

We continue to choose our translating functions on larger and larger distance scales. Borrowing the notation of Fröhlich and Spencer, let $S_k(\rho)$ be some minimal covering of $\operatorname{supp}\rho$ by squares of size $2^k\times 2^k$, $k>1$. We then choose $S_k'(\rho)\subseteq S_k(\rho)$ to be the squares $s'\in S_k(\rho)$ satisfying

$$\operatorname{dist}(s's)\geq\sqrt{2}2M\ 2^{\alpha k} \quad \forall s\in S_k(\rho), \quad s\neq s'$$

and (2.2)

$$\operatorname{dist}(s',\Lambda^c)\geq 2^{k+2} \quad ,$$

(for $k=0$ we set $|S_k'(\rho)|=A_0(\rho)=|\operatorname{supp}\rho|$.)

The first restriction insures that we only translate charged components

of ρ (see (1.7)(b)) while the second makes sure that we don't translate so close to the edge of Λ that we violate our boundary conditions.

Let k_0 be the first $k > 0$ for which $S'_{k_0}(\rho)$ is non-empty for some $\rho \in N$. Let s' be an element of $S'_{k_0}(\rho)$, and assume for simplicity that s' is centered at the origin, and let $\bar{s}'$ be the square of side length $2k_0$, also centered at the origin. Define

$$\tilde{a}_\rho^{ks'}(j) = \begin{cases} 0 & \text{if } |i| \geq \sqrt{2}\cdot 2^k \\ \gamma(\operatorname{sgn} q_\rho^{ks'})\ln_2[\sqrt{2}|i|^{-1}\cdot 2^k] & \text{if } 2^k \leq |i| \leq \sqrt{2}\cdot 2^k \\ \gamma\ln_2\sqrt{2}(\operatorname{sgn} q_\rho^{ks'}) & \text{if } |i| \leq 2^k \end{cases} , \qquad (2.3)$$

where
$$q_\rho^{ks'} = \sum_{i\in(\operatorname{supp}\rho)\cap s'} \rho(i) .$$

Repeat this procedure for each $s' \in S_{k_0}(\rho)$. Note that the supports of $\tilde{a}_\rho^{k_0 s'_1}$ and $\tilde{a}_\rho^{k_0 s'_2}$ are disjoint if $s'_1 \neq s'_2$ by (2.2). Finally define

$$a_\rho^k = \sum_{s'\in S'_k(\rho)} \tilde{a}_\rho^{ks'} . \qquad (2.4)$$

The choice of translating functions on successively larger length scales is similar to this, the only difference being that one must exercise some caution not to destroy the renormalization which has already taken place on shorter scales. For this purpose some further notation is introduced (see [15]).

Let $D(\rho)$ be all those sites in $\mathbb{Z}^2$ contained in a disk in $\mathbb{R}^2$ of radius $2d_\Lambda(\rho)$ centered at a point $x(\rho)$ such that $\operatorname{supp}\rho \subset D(\rho)$ and $\operatorname{dist}(\operatorname{supp}\rho, \partial D(\rho)) = d_\Lambda(\rho)$. Here $\partial D(\rho)$ refers to the outer boundary of $D(\rho)$.

Now consider some $k > k_0$ and some ρ_1 with $S'_k(\rho_1) \neq \phi$. Let

$$N^1 = \{\rho : \rho \in N, d_\Lambda(\rho) \leq 2d_\Lambda(\rho_1), \rho \neq \rho_1\} . \qquad (2.5)$$

Let $\{E_{1,\ell}\}$ be the connected components of $\bigcup_{\rho\in N^1} D(\rho)$. Heuristically, we just choose $\{E_{1,\ell}\}$ so that we cover the support of all the translating functions for "little" charge densities close to s'. Define

$$a_{\rho_1}^{ks'}(i) = \begin{cases} \tilde{a}_{\rho_1}^{ks'}(i) & \text{if } i \notin \{E_{1,\ell}\} \\ \tilde{a}_{\rho_1}^{ks'}(i_\ell) & \text{if } i\in E_{1,\ell} \text{ for some } \ell \text{ where } i_\ell = x(\rho) \\ & \text{in the definition of } D(\rho), \text{ and } \rho \\ & \text{is the largest (i.e., maximal} \\ & d_\Lambda(\rho)) \text{ charge density in } N^1 \\ & \text{which intersects } E_{1,\ell} . \end{cases} \qquad (2.6)$$

For all neutral ρ's (i.e., $Q(\rho) = 0$) one continues this process until $|S_k(\rho)| = 1$. If $Q(\rho) \neq 0$, we continue up to a scale $k = \ln_2[\operatorname{dist}(\rho, \Lambda^c)] - 3$. Now proceed to the renormalization itself.

<u>Lemma II.6</u>. Fix some integer $k \geq 0$. After the renormalization has been performed on all scales less than or equal to 2^k, one has

$$Z(\sigma,\varepsilon) = \sum_{\Gamma}\sum_{\{\zeta\}}\int (\tfrac{1}{2})^{|\Gamma|}\Big(\prod_{\rho\in\Gamma} K(\rho)e^{i\zeta_\rho(\phi+\sigma)(\rho)}\Big)e^{\varepsilon(\phi_0-\phi_x)}\Big(\prod_{\rho\in\Gamma} e^{-Q_{k,\rho}}\Big)$$

$$\times\, e^{i\varepsilon\sum_{\rho\in\Gamma}\sum_{m=0}^{k}(a_\rho^m(0)-a_\rho^m(x))\zeta_\rho}\times e^{\sum_{X\subset\Lambda}\tilde{V}_X(\{\phi+\sigma+i\sum_{\rho\in\Gamma}\sum_{m=0}^{k}\zeta_\rho a_\rho^m\}|_X)}$$

$$\times\left(\prod_{\langle i,j\rangle}\frac{I_\beta(\phi_i-\phi_j+\sigma_i-\sigma_j+i\sum_{\rho\in\Gamma}\sum_{m=0}^{k}\zeta_\rho(a_\rho^m(i)-a_\rho^m(j)))}{I_\beta(\phi_i-\phi_j)}\right)d\mu_\Lambda(\phi) \tag{2.7}$$

with

$$q_{k,\rho} = \gamma\sum_{\rho\in\Gamma}\sum_{i\in\pi_0^\rho}|\rho(i)| + \gamma\ln_2\sqrt{2}\sum_{\rho\in\Gamma}\sum_{m=1}^{k}\sum_{s'\in S_m'(\rho)}|q_\rho^{m,s'}| \quad . \tag{2.8}$$

<u>Proof</u>. The proof is by induction.

<u>Case 1</u>: $k = 0$. The analyticity of the functions $I_\beta(\phi)$ and $\tilde{V}_X(\{\phi\}|_X)$ allows one to make the change of variables

$$\phi_j \to \phi_j + i\sum_\rho \zeta_\rho a_\rho^0(j)$$

in the expression for $Z_N(\sigma,\varepsilon)$ if the constant γ in (2.1) is not chosen too large. (We shall choose $\gamma = c|\ln\beta|$, which by Proposition II.1 is a permissible choice.) Thus

$$Z_N(\sigma,\varepsilon) = \sum_{\Gamma}\sum_{\{\zeta\}}\int (\tfrac{1}{2})^{|\Gamma|}\Big(\prod_{\rho\in\Gamma} K(\rho)e^{i\zeta_\rho(\phi+\sigma)(\rho)}\Big)e^{\varepsilon(\phi_0-\phi_x)}$$

Eq. (2.9) to be continued

$$\times\, e^{i\sum_{\rho\in\Gamma}\zeta_\rho(a_\rho^0(0)-a_\rho^0(x))}\times e^{-\sum_{\rho\in\Gamma}\zeta_\rho^2\sum_{i\in\mathrm{supp}\, a_\rho^0}a_\rho^0(i)}$$

$$\times\, e^{\sum_{X\subset\Lambda}\tilde{V}_X(\{\phi+\sigma+i\sum_{\rho\in\Gamma}\zeta_\rho a_\rho^0\}|_X)}$$

$$\times\prod_{\langle i,j\rangle} I_\beta(\phi_i-\phi_j+\sigma_i-\sigma_j+i\sum_{\rho\in\Gamma}\zeta_\rho(a_\rho^0(i)-a_\rho^0(j)))\times\Big(\prod_{i\in\Lambda}d\phi_i\Big) \quad . \tag{2.9}$$

Using $\zeta_\rho^2 = 1$, and substituting the definition (2.1) for $a_\rho^0(i)$ into (2.9), and also using the definition of $d\mu_\Lambda(\phi)$, we obtain (2.7) for the case $k = 0$.

Assume (2.7) holds for all $k \le n$, $n > 0$. To obtain (2.7) for the case $k = n+1$, we make the change of variables

$$\phi_j \to \phi_j + i\sum_\rho \zeta_\rho a_\rho^{n+1} \quad . \tag{2.10}$$

We must check that this doesn't carry us outside the analyticity domain of the functions $I_\beta(\cdot)$. This is guaranteed by the observation that if for some nearest neighbor bond $\langle i,j\rangle$ one has $\sum_\rho\sum_{m=0}^{n}[a_\rho^m(i) - a_\rho^m(j)] \neq 0$, then $\sum_\rho[a_\rho^{n+1}(i) - a_\rho^{n+1}(j)] = 0$. Thus no function $I_\beta(\cdot)$ is translated more than once. Since by (2.3), (2.6) and Proposition II.1, a single translation doesn't carry us outside the analyticity domain of the functions $I_\beta(\cdot)$, (2.10) is an allowed change of variables. Substituting (2.10) into (2.9) and using the definition of a_ρ^{n+1} leads to (2.7).

The effect of the translations on the $\tilde{V}_X(\cdot)$ functions and on the $I_\beta(\cdot)$ functions is straightforward. We examine how this

change affects the exponential factor

$$e^{i\zeta_\rho\phi(\rho)} \to e^{i\zeta_\rho\phi(\rho)} \cdot e^{-\zeta_\rho^2 \sum_i a_\rho^k(i)\cdot\rho(i)} .$$

By the definition of our translating functions, $|a_\rho^k(i)| = \gamma \ln_2\sqrt{2}$ on supp ρ. Also, $\zeta_\rho^2 = 1$ so that

$$\zeta_\rho^2 \sum_i a_\rho^k(i)\rho(i) = \sum_{s'\in S_k'(\rho)} \gamma(\mathrm{sgn}\, q_\rho^{k,s'}) \ln_2\sqrt{2} \sum_{i\in s'} \rho(i)$$

$$= \ln_2\sqrt{2}\,\gamma \sum_{s'\in S_k'(\rho)} |q_\rho^{k,s'}| . \tag{2.11}$$

We recall that $q_\rho^{k,s'} = \sum_{i\in s'} \rho(i)$. This leads immediately to the expression for $q_{k,\rho}$ in (2.8).

We now prove that no $I_\beta(\cdot)$ function is translated more than once. Assume that $a_\rho^m(i) - a_\rho^m(j) \neq 0$ for some $m \le n$ Eqs. (2.1)-(2.4) imply that $\langle i,j\rangle$ intersects the disk of radius $2\cdot 2^m$ centered on some square $s_1' \in S_m'(\rho_1)$ for some $\rho_1 \in \Gamma$. Now assume that there is some change density $\rho_2 \in \Gamma$ with $a_{\rho_2}^{n+1}(i) - a_{\rho_2}^{n+1}(j) \neq 0$. If $d_\Lambda(\rho_1) < 2^{n+1}$, then $\rho_1 \in \{E_{n+1,\ell}^{s_2'}\}$ for some ℓ, and $a_{\rho_1}^m(i) - a_{\rho_1}^m(j) = 0$, a contradiction. Similarly, if $d_\Lambda(\rho_1) \ge 2^{n+1}$, $\mathrm{dist}(\rho_1,\rho_2) \ge M\cdot 2^{(n+1)\alpha}$. Since $a_{\rho_2}^{n+1}(i) - a_{\rho_2}^{n+1}(j) \neq 0$, this implies $2^{n+1} \le \mathrm{dist}(\rho_2,\langle i,j\rangle) \le 2^{n+2}$. The triangle inequality implies

$$\mathrm{dist}(\rho_1,\langle i,j\rangle) \ge \mathrm{dist}(\rho_1,\rho_2) - \mathrm{dist}(\rho_2,\langle i,j\rangle)$$

$$\ge M\cdot 2^{(n+1)\alpha} - 2^{n+2} \ge 2^{n+2} \tag{2.12}$$

for $\alpha > 3/2$ and $M > 6$. This implies that $\langle i,j\rangle$ does not intersect the disk of radius $2\cdot 2^m$ centered on $s_1' \in S_m'(\rho_1)$ for any $m \le n$. This contradiction proves the stated result.

Continue the renormalization process up to a scale $k = \ln_2|\Lambda|^{1/2}$. This is greater than $\ln_2 d_\Lambda(\rho)$ for any $\rho \in N$, so $a_\rho^m = 0$ for all $\rho \in N$, $m > k$. Define $a_\rho = \sum_{m=0}^{\infty} a_\rho^m$.

<u>Corollary II.7</u>.

$$\sum_{\langle i,j\rangle\cap\Lambda} \frac{I_\beta(\phi_i-\phi_j+\sigma_i-\sigma_j+i\sum_\rho\zeta_\rho[a_\rho(i)-a_\rho(j)])}{I_\beta(\phi_i-\phi_j+\sigma_i-\sigma_j)}$$

$$= \prod_\rho \prod_{\langle i,j\rangle\cap\Lambda} \left(\frac{I_\beta(\phi_i-\phi_j+\sigma_i-\sigma_j+i\zeta_\rho[a_\rho(i)-a_\rho(j)])}{I_\beta(\phi_i-\phi_j+\sigma_i-\sigma_j)}\right) \equiv \prod_\rho i(\phi+\sigma+i\zeta_\rho a_\rho) . \tag{2.13}$$

<u>Proof</u>. Since no $I_\beta(\cdot)$ functions is translated more than once, the first equality is immediate. It will prove convenient later to have a more explicit form for $i(\phi+\sigma+i\zeta_\rho a_\rho)$.

Let

$$\begin{aligned} \Omega_\rho^0 &= \{\langle i,j\rangle : [a_\rho^0(i)-a_\rho^0(j)] \neq 0\} \\ &\vdots \\ \Omega_\rho^k &= \{\langle i,j\rangle : [a_\rho^k(i)-a_\rho^k(j)] \neq 0\} . \end{aligned} \tag{2.14}$$

The discussion above implies that $\Omega_\rho^{k_1} \cap \Omega_\rho^{k_2} = \phi$ unless $k_1 = k_2$. This

implies

$$\prod_{\rho}\prod_{<i,j>\cap\Lambda}\left(\frac{I_\beta(\phi_i-\phi_j+\sigma_i-\sigma_j+i\zeta_\rho[a_\rho(i)-a_\rho(j)])}{I_\beta(\phi_i-\phi_j+\sigma_i-\sigma_j)}\right)$$

$$=\prod_{\rho}\left(\prod_{m=0}^{\infty}\prod_{\substack{<i,j>:\\ <i,j>\in\Omega_\rho^m}}\frac{I_\beta(\phi_i-\phi_j+\sigma_i-\sigma_j+i\zeta_\rho[a_\rho(i)-a_\rho(j)])}{I_\beta(\phi_i-\phi_j+\sigma_i-\sigma_j)}\right) . \qquad (2.15)$$

Comparison with (2.13) implies that $i(\phi+\sigma+i\zeta_\rho a_\rho)$ is equal to the quantity in square brackets.

Now define

$$\Delta V_X(\phi:i\sum_\rho\zeta_\rho a_\rho) = \tilde{V}_X(\{\phi+\sigma+i\sum_\rho\zeta_\rho a_\rho\}|_X) - \tilde{V}_X(\{\phi+\sigma\}|_X) ,$$

and

$$d\tilde{\nu}_\Lambda^V(\phi) = e^{\sum_{X\subset\Lambda}\tilde{V}_X(\{\phi\}|_X)}\,d\mu_\Lambda(\phi) . \qquad (2.16)$$

We obtain, upon combining Lemma II.6 and Corollary II.7,

<u>Proposition II.8</u>.

$$Z_N(\sigma,\varepsilon) = \sum_\Gamma\int(\tfrac{1}{2})^{|\Gamma|}\sum_{\{\zeta\}}\Big(\prod_{\rho\in\Gamma}K(\rho)e^{i\zeta_\rho(\phi+\sigma)(\rho)}e^{-q_\rho}{}_{i(\phi+\sigma+i\zeta_\rho a_\rho)}$$

$$\times e^{i\varepsilon\zeta_\rho(a_\rho(0)-a_\rho(x))}\Big)e^{\sum_{X\subset\Lambda}\Delta V_X(\phi+\sigma:i\sum_\rho\zeta_\rho a_\rho)}\,d\tilde{\nu}_\Lambda^V(\phi) , \qquad (2.17)$$

where $q_\rho = q_{n(\rho),\rho}$. This is the form from which we shall start the Mayer expansion in the next section. We also note that since all our translating

functions were chosen independently of the observable, one may immediately write down an analogous formula for $Z_N(0,0)$.

<u>Proposition II.8'</u>.

$$Z_N(0,0) = \sum_\Gamma\sum_{\{\zeta\}}\int(\tfrac{1}{2})^{|\Gamma|}\Big(\prod_{\rho\in\Gamma}K(\rho)e^{i\zeta_\rho(\phi)(\rho)}e^{-q_\rho}{}_{i(\phi+i\zeta_\rho a_\rho)}$$

$$\times e^{\sum_{X\in\Lambda}\Delta V_X(\phi:i\sum_\rho\zeta_\rho a_\rho)}\,d\tilde{\nu}_\Lambda^V(\phi) . \qquad (2.18)$$

II.3. The Mayer Expansion

In this section we perform a Mayer expansion on expressions (2.17) and (2.18). This expansion is exponentiated yielding a positive measure.

The motivation for this expansion lies in the observation that if it were not for the non-local potentials, the integrand in (2.17) would factorize, yielding

$$Z_N(\sigma,\varepsilon) = \sum_{\Gamma}\int \Big(\prod_{\rho\in\Gamma} Q(\rho,\phi+\sigma,\varepsilon)\Big)\, d\tilde{\nu}^{V}_{\Lambda}(\phi) \quad . \tag{3.1}$$

Because of the exponential falloff of $\tilde{V}_X(\cdot)$ we expect corrections to (3.1) to be small, for widely separated charge densities ρ. The Mayer expansion allows us to write (2.17) as an exactly factorized part, plus corrections. One then expands a second time, this time isolating the correction term, and creating in the process new corrections. This inductive procedure eventually terminates and yields an expression of the form (3.1) but with functions $Q(\rho,\phi+\sigma,\varepsilon)$ which are much more complicated than those which occur in the $\tilde{V}_X(\cdot) \equiv 0$ case.

Begin by ordering the charge densities in N in an arbitrary fashion. Define inductively a sequence of interpolated potentials.

<u>Definition</u>. Let $\Gamma_k \subset N$, $k = 1,\ldots,n$, and let $\zeta a(\Gamma_k) \equiv \sum_{\rho\in\Gamma_k} \zeta_\rho a_\rho$. Define a series of convex sums of potentials by

$$\Delta V_X^{s_1}(\phi:ia(\Gamma_1);ia(\Gamma_2)) = s_1\Delta V_X(\phi:ia(\Gamma_1)+ia(\Gamma_2))$$
$$+ (1-s_1)\left[\Delta V_X(\phi:ia(\Gamma_1))+\Delta V_X(\phi:ia(\Gamma_2))\right] \quad , \tag{3.2}$$

and by induction

$$\Delta V_X^{s_1\cdots s_n}(\phi:ia(\Gamma_1)+\ldots+ia(\Gamma_n);ia(\Gamma_{n+1}))$$
$$= s_n\Delta V_X^{s_1\cdots s_{n-1}}(\phi:ia(\Gamma_1)+\ldots+ia(\Gamma_{n-1});ia(\Gamma_n)+ia(\Gamma_{n+1}))$$
$$+ (1-s_n)\Big[\Delta V_X^{s_1\cdots s_{n-1}}(\phi:ia(\Gamma_1)+\ldots+ia(\Gamma_{n-1});ia(\Gamma_n))$$
$$+ \Delta V_X(\phi:ia(\Gamma_{n+1}))\Big] \quad . \tag{3.3}$$

Armed with these definitions, we turn to a consideration of the N dependent part of the integrand of expression (2.17). For fixed Γ we define

$$R_\Gamma = \sum_{\{\zeta\}} \left(\tfrac{1}{2}\right)^{|\Gamma|}\Big(\prod_{\rho\in\Gamma} R(\rho,\phi+\sigma,\zeta_\rho,\varepsilon)\Big) e^{\sum_{X\subset\Lambda}\Delta V_X(\phi+\sigma:i\zeta a(\Gamma))} \quad , \tag{3.4}$$

with

$$R(\rho,\phi+\sigma,\zeta_\rho,\varepsilon) =$$
$$K(\rho)e^{i\zeta_\rho(\phi+\sigma)(\rho)}\, e^{i\varepsilon\zeta_\rho(a_\rho(0)-a_\rho(x))}\, e^{-\gamma q_\rho} \times i(\phi+\sigma+i\zeta_\rho a_\rho). \tag{3.5}$$

Let ρ' be the minimal charge density with respect to the above ordering. We decouple ρ' from the remainder of the system using the fundamental theorem of calculus. This yields

$$R_\Gamma = \sum_{\{\zeta\}} \left(\frac{1}{2}\right)^{|\Gamma|} \left[\prod_{\rho\in\Gamma} R(\rho,\phi+\sigma,\zeta_\rho,\varepsilon)\right] \times e^{\sum_{X\subset\Lambda}\left[\Delta V_X(\phi+\sigma:i\zeta_{\rho'},a_{\rho'})+\Delta V_X(\phi+\sigma:i\zeta a(\Gamma\backslash\rho'))\right]}$$

$$+ \sum_{\{\zeta\}} \left(\frac{1}{2}\right)^{|\Gamma|} \left[\prod_{\rho\in\Gamma} R(\rho,\phi+\sigma,\zeta_\rho,\varepsilon)\right] \int_0^1 ds_1 \left(\frac{d}{ds_1}\right) \times e^{\sum_{X\subset\Lambda} \Delta V_X^{s_1}(\phi+\sigma:i\zeta_{\rho'},a_{\rho'};i\zeta a(\Gamma\backslash\rho'))} . \tag{3.6}$$

Evaluate the derivative using (3.2) obtaining

$$\sum_{X_1\subset\Lambda}\Big[\Delta V_{X_1}(\phi+\sigma:i\zeta_{\rho'},a_{\rho'},+i\zeta a(\Gamma\backslash\rho'))-\Delta V_{X_1}(\phi+\sigma:i\zeta_{\rho'},a_{\rho'}) - \Delta V_{X_1}(\phi+\sigma:i\zeta a(\Gamma\backslash\rho'))\Big] e^{\sum_{X\subset\Lambda}\Delta V_X^{s_1}(\phi+\sigma:i\zeta_{\rho'},a_{\rho'};i\zeta a(\Gamma\backslash\rho'))} . \tag{3.7}$$

Notice that any term in the sum over X_1 vanishes unless X_1 intersects both supp $a_{\rho'}$ and supp $a(\Gamma\backslash\rho')$. This follows since if $X_1 \cap \text{supp } a_{\rho'} = \phi$, $\Delta V_X(\phi:i\zeta_{\rho'},a_{\rho'}) = 0$ and $\Delta V_{X_1}(\phi:i\zeta_{\rho'},a_{\rho'},+i\zeta a(\Gamma\backslash\rho')) = \Delta V_{X_1}(\phi:i\zeta a(\Gamma\backslash\rho'))$ so that the corresponding term in (3.7) vanishes. If X_1 intersects supp $a_{\rho'}$ but not supp $a(\Gamma\backslash\rho')$, $\Delta V_{X_1}(\phi:i\zeta a(\Gamma\backslash\rho')) = 0$ and $\Delta V_{X_1}(\phi:i\zeta_{\rho'},a_{\rho'},+i\zeta a(\Gamma\backslash\rho')) = \Delta V_{X_1}(\phi:i\zeta_{\rho'},a_{\rho'})$ and again the term in (3.7) vanishes. The general situation is covered by the following lemmas.

<u>Definition</u>

$$\Delta \mathcal{V}_X(\phi:i\zeta a(\Gamma_\ell)+\ldots+i\zeta a(\Gamma_m))$$

$$= \begin{cases} \Delta V_X(\phi:i\zeta a(\Gamma_\ell)+\ldots+i\zeta a(\Gamma_m))-\Delta V_X(\phi:i\zeta a(\Gamma_{\ell+1})+\ldots+i\zeta a(\Gamma_m)) \\ -\Delta V_X(\phi:i\zeta a(\Gamma_\ell)+\ldots+i\zeta a(\Gamma_{m-1}))+\Delta V_X(\phi:i\zeta a(\Gamma_{\ell+1})+\ldots+i\zeta a(\Gamma_{m-1})) & \text{if } \ell \le m-2, \\ \Delta V_X(\phi:i\zeta a(\Gamma_\ell)+i\zeta a(\Gamma_m))-\Delta V_X(\phi:i\zeta a(\Gamma_\ell)) \\ -\Delta V_X(\phi:i\zeta a(\Gamma_m)) & \text{if } \ell = m-1. \end{cases} \tag{3.8}$$

<u>Lemma II.9.</u> $\Delta \mathcal{V}_X(\phi:i\zeta a(\Gamma_\ell)+\ldots+i\zeta a(\Gamma_m)) = 0$ unless $X \cap \text{supp } a(\Gamma_\ell) \neq \phi$ <u>and</u> $X \cap \text{supp } a(\Gamma_m) \neq \phi$.

<u>Proof.</u> If $\ell = m-1$, the lemma follows from our discussion of $\left(\frac{d}{ds_1}\right) V_X^{s_1}$ above. If $\ell \le m-2$ assume that $X \cap \text{supp } a(\Gamma_\ell) = \phi$. Then $\Delta V_X(\phi:i\zeta a(\Gamma_\ell)+\ldots+i\zeta a(\Gamma_m)) = \Delta V_X(\phi:i\zeta a(\Gamma_{\ell+1})+\ldots+i\zeta a(\Gamma_m))$, and $\Delta V_X(\phi:i\zeta a(\Gamma_\ell)+\ldots+i\zeta a(\Gamma_{m-1})) = \Delta V_X(\phi:i\zeta a(\Gamma_{\ell+1})+\ldots+i\zeta a(\Gamma_{m-1}))$, and $\Delta \mathcal{V}_X(\cdot:\cdot)$ vanishes. If $X \cap \text{supp } a(\Gamma_m) = \phi$, $\Delta V_X(\phi:i\zeta a(\Gamma_\ell)+\ldots+i\zeta a(\Gamma_m)) = \Delta V_X(\phi:i\zeta a(\Gamma_\ell)+\ldots+i\zeta a(\Gamma_{m-1}))$ and $\Delta V_X(\phi:i\zeta a(\Gamma_{\ell+1})+\ldots+i\zeta a(\Gamma_m)) = \Delta V_X(\phi:i\zeta a(\Gamma_{\ell+1})+\ldots+i\zeta a(\Gamma_{m-1}))$ and once again the sum vanishes.

Lemma II.10. The interpolated potentials satisfy

$$\Delta V_X^{s_1\cdots s_n}(\phi{:}ia(\Gamma_1)+\ldots+ia(\Gamma_n);ia(\Gamma_{n+1})) = \sum_{j=1}^{n+1}\Delta V_X(\phi{:}ia(\Gamma_j))$$

$$+\sum_{k=1}^{n}\sum_{m=1}^{k} s_m\cdots s_k\cdot\Delta V_X(\phi{:}ia(\Gamma_m)+\ldots+ia(\Gamma_{k+1})) \quad . \tag{3.9}$$

Proof. The proof again is by induction. Assume it is true for all $k \le n-1$. (By (3.2) it is true for $n = 1$.) Then,

$$\Delta V_X^{s_1,\ldots,s_n}(\phi{:}ia(\Gamma_1)+\ldots+ia(\Gamma_n);ia(\Gamma_{n+1}))$$

$$= s_n\Delta V_X^{s_1\cdots s_{n-1}}(\phi{:}ia(\Gamma_1)+\ldots+ia(\Gamma_{n-1});ia(\Gamma_n\cup\Gamma_{n+1}))$$

$$+ (1-s_n)\Big[\Delta V_X^{s_1\cdots s_{n-1}}(\phi{:}ia(\Gamma_1)+\ldots+ia(\Gamma_{n-1});ia(\Gamma_n))$$

$$+ \Delta V_X(\phi{:}ia(\Gamma_{n+1}))\Big] \quad . \tag{3.10}$$

Applying the induction hypothesis gives

$$\Delta V_X^{s_1\cdots s_n}(\phi{:}ia(\Gamma_1)+\ldots+ia(\Gamma_n);ia(\Gamma_{n+1}))$$

$$= s_n\sum_{m=1}^{n-1} s_m\cdots s_{n-1}\Big[\Delta V_X(\phi{:}ia(\Gamma_m)+\ldots+ia(\Gamma_n\cup\Gamma_{n+1}))$$

$$-\Delta V_X(\phi{:}ia(\Gamma_m)+\ldots+ia(\Gamma_n))\Big]$$

$$+\sum_{k=1}^{n-1}\sum_{m=1}^{k} s_m\cdots s_k\Delta V_X(\phi{:}ia(\Gamma_m)+\ldots+ia(\Gamma_{k+1}))$$

Eq. (3.11) to be continued

$$+ s_n\Big[\Delta V_X(\phi{:}ia(\Gamma_n\ \Gamma_{n+1}))-\Delta V_X(\phi{:}ia(\Gamma_n))-\Delta V_X(\phi{:}ia(\Gamma_{n+1}))\Big]$$

$$+\sum_{j=1}^{n+1}\Delta V_X(\phi{:}ia(\Gamma_j)) \quad . \tag{3.11}$$

Note that

$$\Delta V_X(\phi{:}ia(\Gamma_n\cup\Gamma_{n+1}))-\Delta V_X(\phi{:}ia(\Gamma_n))-\Delta V_X(\phi{:}ia(\Gamma_{n+1}))$$

$$= \Delta V_X(\phi{:}ia(\Gamma_n)+ia(\Gamma_{n+1})) \quad . \tag{3.12}$$

Furthermore,

$$\Delta V_X(\phi{:}ia(\Gamma_m)+\ldots+ia(\Gamma_n\cup\Gamma_{n+1}))-\Delta V_X(\phi{:}ia(\Gamma_m)+\ldots+ia(\Gamma_n))$$

$$= \Delta V_X(\phi{:}ia(\Gamma_m)+\ldots+ia(\Gamma_{n+1})) \quad . \tag{3.13}$$

Substituting (3.12) and (3.13) into (3.11) completes the proof.

Corollary II.11.

$$\frac{d}{ds_n}\Delta V_X^{s_1\cdots s_n}(\phi{:}ia(\Gamma_1)+\ldots+ia(\Gamma_n);ia(\Gamma_{n+1}))$$

$$= \sum_{k=1}^{n}(s_k\cdots s_{n-1})\Delta V_X(\phi{:}ia(\Gamma_{\eta(m)})+\ldots+ia(\Gamma_{n+1})) \quad . \tag{3.14}$$

Proof. This follows by differentiating (3.9).

With this information in hand, let us return to the task of expanding R_Γ. Recall that by combining (3.6) and (3.7) we had demonstrated that

$$R_\Gamma = \sum_{\{\zeta\}} (\tfrac{1}{2})^{|\Gamma|} \Big(\prod_{\rho\in\Gamma} R(\rho,\phi+\sigma,\zeta_\rho,\varepsilon)\Big) e^{\sum_{X\subset\Lambda} \Delta V_X(\phi+\sigma;i\zeta_{\rho'},a_{\rho'})}$$

$$\times\ e^{\sum_{X\subset\Lambda}\Delta V_X(\phi+\sigma;i\zeta a(\Gamma\backslash\rho'))}$$

$$+ \sum_{\{\zeta\}} \sum_{X_1} (\tfrac{1}{2})^{|\Gamma|} \Big(\prod_{\rho\in\Gamma} R(\rho,\phi+\sigma,\zeta_\rho,\varepsilon)\Big) \int_0^1 ds_1 \Big(\Delta V_{X_1}(\phi+\sigma;i\zeta_{\rho'},a_{\rho'},+i\zeta a(\Gamma\backslash\rho'))$$

$$\times e^{\sum_{X\subset\Lambda}\Delta V_X^{s_1}(\phi+\sigma;i\zeta_{\rho'},a_{\rho'};i\zeta a(\Gamma\backslash\rho'))} \quad . \tag{3.15}$$

Let $\rho' = \Gamma_1$. Then choose $\Gamma_2 = \{\rho \in \Gamma\backslash\Gamma_1 : X \cap \operatorname{supp} a_\rho \neq \phi\}$, according to which term in the sum over X_1 we are considering. We noted that all terms with $\Gamma_2 = \phi$ vanish so we ignore this possibility. Now introduce a second decoupling parameter, s_2, and decouple $\Gamma_2 \cup \Gamma_1$ from $\Gamma\backslash(\Gamma_2\cup\Gamma_1)$. Continuing this process, we arrive at

<u>Proposition II.12</u>. For any $n \geq 2$ one may write

$$R_\Gamma = \sum_{k=1}^{n-1} \sum_{\{\zeta\}} \sum_{X_1} \cdots \sum_{X_{k-1}} \sum_\eta \int_0^1 ds_1 \ldots ds_{k-1} (\tfrac{1}{2})^{|\Gamma|}$$

$$\times \Big(\prod_\rho R(\rho,\phi+\sigma,\zeta_\rho,\varepsilon)\Big) \times \prod_{m=2}^{k} \Big(s_{\eta(m)} \cdots s_{m-2} \Delta V_{X_{m-1}}(\phi+\sigma;i\,\zeta\, a(\Gamma_{\eta(m)})+\ldots+i\zeta a(\Gamma_m))\Big)$$

$$\times e^{\sum_{X\subset\Lambda}\Big(\Delta V_X^{s_1\cdots s_{k-1}}(\phi+\sigma:i\zeta a(\Gamma_1)+\ldots+i\zeta a(\Gamma_{k-1});i\zeta a(\Gamma_k))}$$

$$+\Delta V_X(\phi+\sigma:i\zeta a(\Gamma\backslash\Gamma_k\backslash\ldots\backslash\Gamma_1)\Big)$$

Eq. (3.16) to be continued

$$+ \sum_{\{\zeta\}} \sum_{X_1} \cdots \sum_{X_{n-1}} \sum_\eta \int_0^1 ds_1 \ldots ds_{k-1} (\tfrac{1}{2})^{|\Gamma|} \Big(\prod_\rho R(\rho,\phi+\sigma,\zeta_\rho,\varepsilon)\Big)$$

$$\prod_{m=2}^{n} \Big(s_{\eta(m)} \cdots s_{m-2} \Delta V_{X_{m-1}}(\phi+\sigma:i\zeta a(\Gamma_{\eta(m)})+\ldots+i\zeta a(\Gamma_m))\Big)$$

$$\times\ e^{\sum_{X\subset\Lambda}\Delta V_X^{s_1\cdots s_{n-1}}(\phi+\sigma:i\zeta a(\Gamma_1)+\ldots+i\zeta a(\Gamma_{n-1});i\zeta a(\Gamma\backslash\Gamma_{n-1}\backslash\ldots\backslash\Gamma_1))} \quad . \tag{3.16}$$

Here η is a tree function, i.e., a function on the integers less than or equal to k satisfying $\eta(\ell) < \ell$. We adopt the convention that when $\eta(m) = m-1$ the product of s-factors is set equal to one. Finally, Γ_{k+1} is defined inductively as all those charge densities, ρ, in $\Gamma\backslash\Gamma_k\backslash\ldots\backslash\Gamma_1$ such that $X_k \cap \operatorname{supp} a_\rho \neq \phi$.

<u>Proof</u>. The proof is by induction. By (3.6) and (3.7) the lemma holds for $n = 2$. Assume that it is true for all k less than or equal to $n-1$. Then interpolate the exponential factor in the second term of (3.16), isolating Γ_n. Under this operation the exponential part of the second term in (3.16) becomes

$$e^{\sum_{X\subset\Lambda}\Big(\Delta V_X^{s_1\cdots s_{n-1}}(\phi+\sigma:i\zeta a(\Gamma_1)+\ldots+i\zeta a(\Gamma_{n-1});i\zeta a(\Gamma_n))\ +}$$

$$\Delta V_X(\phi+\sigma:i\zeta a(\Gamma\backslash\Gamma_n\backslash\ldots\backslash\Gamma_1)\Big)$$

$$+ \int_0^1 ds_n \Big(\sum_{X_n\subset\Lambda} \frac{d}{ds_n}(\Delta V_X^{s_1\cdots s_n}(\phi+\sigma;i\zeta a(\Gamma_1)+\ldots+i\zeta a(\Gamma_n);i\zeta a(\Gamma\backslash\Gamma_n\backslash\ldots\backslash\Gamma_1))\Big)$$

Eq. (3.17) to be continued

$$\times \left[e^{\sum_{X\subset\Lambda} \Delta V_X^{s_1\cdots s_n}(\phi+\sigma:i\zeta a(\Gamma_1)+\ldots+i\zeta a(\Gamma_n);i\zeta a(\Gamma\setminus\Gamma_n\setminus\ldots\setminus\Gamma_1))} \right] . \quad (3.17)$$

The first term of (3.17) becomes the $k = n$ part of the first term of (3.16) while the second, after application of Corollary II.11, yields the second term of (3.16) in the case $k = n$. This completes the proof.

One continues the decoupling process until Γ is depleted. This must happen after a finite number of steps since $|\Gamma|$ is finite and each Γ_k is non-empty. One then uses the fact that $\Delta V_X(\phi+\sigma;ia(\phi)) = 0$ to write the second term of (3.16) as the $k = n$ part of the first one. Defining all terms in the sum with higher values of k to be zero, one has

Corollary II.13.

$$R_\Gamma = \sum_{k=1}^{\infty} \sum_{\{\zeta\}} \sum_{X_1} \cdots \sum_{X_{k-1}} \sum_{\eta} \int_0^1 ds_1 \cdots ds_{k-1} \left(\tfrac{1}{2}\right)^{|\Gamma|}$$

$$\times \left(\prod_{\rho\in\Gamma} R(\rho,\phi+\sigma,\zeta_\rho,\varepsilon) \right)$$

$$\times \prod_{m=2}^{k} \left[s_{\eta(m)} \cdots s_{m-2} \Delta V_{X_{m-1}}(\phi+\sigma:i\zeta a(\Gamma_{\eta(m)})+\ldots+i\zeta a(\Gamma_m)) \right.$$

$$e^{\sum_{X\subset\Lambda} \Delta V_X^{s_1\cdots s_{k-1}}(\phi+\sigma:i\zeta a(\Gamma_1)+\ldots+i\zeta a(\Gamma_{k-1});i\zeta a(\Gamma_k))}$$

$$\times e^{\sum_{X\subset\Lambda} \Delta V_X(\phi+\sigma:i\zeta a(\Gamma\setminus\Gamma_k\setminus\ldots\setminus\Gamma_1))} . \quad (3.18)$$

II.4 The Cluster Formalism

Having completed the decoupling procedure we introduce the cluster formalism.

Definition. A cluster, Y, consists of

(a) an integer $k \geq 1$,

(b) a collection of $k-1$ connected sets of plaquettes in $(\mathbb{Z}^2)^* \cap \Lambda$, $\{X_1,\ldots,X_{k-1}\}$.

(c) a tree function η.

(d) for each integer ℓ, $1 \leq \ell \leq k$ a non-empty subset of Γ, Γ_ℓ, satisfying $\Gamma_\ell \cap \Gamma_{\ell'} = \emptyset$ for $\ell \neq \ell'$ and with $\Gamma_1 =$ minimal $\rho \in \cup_\ell \Gamma_\ell$ with respect to the ordering of N defined above. (4.1)

(e) for each integer ℓ, $1 \leq \ell < k$, an interpolation parameter $s_\ell \in [0,1]$.

Definition. We define a cluster function $\tilde{Q}(Y,\phi+\sigma,\varepsilon)$ as

$$\tilde{Q}(Y,\phi+\sigma,\varepsilon) = \sum_{\{\zeta\}_{\Gamma(Y)}} \left(\tfrac{1}{2}\right)^{|\Gamma(Y)|} \left[\prod_{\rho\in\Gamma(Y)} R(\rho,\phi+\sigma,\zeta_\rho,\varepsilon) \right]$$

$$\times \prod_{m=2}^{k} \left[s_{\eta(m)} \cdots s_{m-2} \Delta V_{X_{m-1}}(\phi+\sigma:i\zeta a(\Gamma_{\eta(m)}) + \cdots + i\zeta a(\Gamma_m)) \right]$$

$$\exp\left\{ \sum_{X\subset\Lambda} \Delta V_X^{s_1\cdots s_{k-1}}(\phi+\sigma:i\zeta a(\Gamma_1) + \cdots + i\zeta a(\Gamma_{k-1});i\zeta a(\Gamma_k)) \right\}. \quad (4.2)$$

Here, $\Gamma(Y)$ means $\cup_{\ell=1}^{k} \Gamma_\ell$, and the sum over $\{\zeta\}_{\Gamma(Y)}$ runs over $\zeta_\rho = \pm 1$ for all $\rho \in \Gamma(Y)$. Define $\Gamma_1(Y) = \Gamma_1 = \rho'$.

Using (4.2) we may rewrite (3.18) as

$$R_\Gamma = \sum_{\substack{Y:\Gamma(Y)\subset\Gamma \\ \Gamma_1(Y)=\rho'}} \tilde{Q}(Y,\phi+\sigma,\varepsilon) R_{\Gamma\setminus\Gamma(Y)} \quad , \tag{4.3}$$

where we have compressed our notation so that

$$\sum_{\substack{Y:\Gamma(Y)\subset\Gamma \\ \Gamma_1=\rho'}} = \sum_{k=1}^{\infty} \sum_{X_1} \cdots \sum_{X_{k-1}} \sum_{\eta} \int_0^1 ds_1,\ldots,ds_{k-1} \ . \tag{4.4}$$

One now applies the Mayer expansion to $R_{\Gamma\setminus\Gamma(Y)}$. This time we choose Γ_1 to be the minimal ρ in $\Gamma\setminus\Gamma(Y)$. The clusters generated in this expansion must satisfy a compatibility condition with respect to those generated in the first expansion, namely,

$$\Gamma(Y_1) \cap \Gamma(Y_2) = \emptyset \tag{4.5}$$

for all Y_2 generated in the expansion of $R_{\Gamma\setminus\Gamma(Y_1)}$. Repeating this procedure until Γ is exhausted we arrive at the expression

$$R_\Gamma = \sum_{\substack{(Y_1,\ldots,Y_m) \\ \text{all } Y_\ell\text{'s compatible} \\ \Gamma_1(Y_1)<\Gamma_1(Y_2)<\ldots<\Gamma_1(Y_m) \\ \cup_1^m \Gamma(Y_\ell)=\Gamma}} \prod_{\ell=1}^{m} \tilde{Q}(Y_\ell,\phi+\sigma,\varepsilon) \ . \tag{4.6}$$

We remove the compatibility conditions on the sum over clusters as in [4,7,16,25]. Define

$$U(Y_1,Y_2) = \begin{cases} 1 & \text{if } \Gamma(Y_1) \cap \Gamma(Y_2) = \emptyset \\ 0 & \text{otherwise} \end{cases} \ . \tag{4.7}$$

Remove the restriction that $\Gamma_1(Y_1) < \Gamma_1(Y_2) < \cdots < \Gamma_1(Y_m)$ by summing over ordered sets of m clusters and introduce a factor of $1/m!$ to compensate for overcounting. Remove the restriction that $\cup_1^m \Gamma(Y_\ell) = \Gamma$ by summing over all Γ. We then obtain,

$$\sum_\Gamma R_\Gamma = \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{(Y_1,\ldots,Y_m)} \left[\prod_{\ell=1}^{m} \tilde{Q}(Y_\ell,\phi+\sigma,\varepsilon)\right]\left[\prod_{(i,j)} U(Y_i,Y_j)\right]. \tag{4.8}$$

The product over (i,j) runs over all pairs of integers in $\{1,\ldots,m\}$. Define $A(Y_1,Y_2) = U(Y_1,Y_2) - 1$, and expand the product over pairs of clusters. The resulting expression may be exponentiated (see the references listed for details) and one finally obtains

<u>Proposition II.14</u>. There exist $0 < \beta_1 < \beta_2$ and $J_1(\beta_1) > 0$ such that for all $\beta \in (\beta_1,\beta_2)$ and $J > J_1(\beta_1)$ one has

$$\sum_\Gamma R_\Gamma = \exp\left\{\sum_{m=1}^{\infty} \frac{1}{m!} \sum_{(Y_1,\ldots,Y_m)} \left(\sum_{g\in G_c} \prod_{\ell\in g} A(\ell)\right) \prod_{\ell=1}^{m} \tilde{Q}(Y_\ell,\phi+\sigma,\varepsilon)\right\} . \tag{4.9}$$

Here, G_c is the set of connected graphs on vertices $Y_1,\ldots,Y_m$. The product over ℓ runs over all legs in a given graph, and $A(\ell) = A(Y_i,Y_j)$ where Y_i and Y_j are the vertices which terminate ℓ. The restrictions to small β and large J arise from the fact that if these conditions are not met, the sums in (4.9) may not be convergent, rendering the manipulations leading to this form meaningless. We shall see in the following section that these conditions insure absolute convergence of the sums over clusters Y_ℓ.

From (2.17) one has immediately

Corollary II.15. There exist $0<\beta_1<\beta_2$ and $J_1(\beta_1)>0$ such that for all $\beta\in(\beta_1,\beta_2)$ and $J>J_1(\beta_1)$ one has

$$Z_N(\sigma,\varepsilon) = \int e^{\varepsilon(\phi_o-\phi_x)} \exp\left\{\sum_{m=1}^{\infty}\frac{1}{m!}\times\right.$$
$$\left.\times\left[\sum_{(Y_1,\dots,Y_m)}\left(\sum_{g\in G_c}\prod_{\ell\in g}A(\ell)\right)\left(\prod_{\ell=1}^{m}\tilde{Q}(Y_\ell,\phi+\sigma,\varepsilon)\right)\right]\right\}d\tilde{\nu}_\Lambda^V(\phi+\sigma) \quad . \tag{4.10}$$

One may repeat the above procedure for $Z_N(0,0)$ and one obtains

Corollary II.15'. Under the conditions of the previous corollary

$$Z_N(0,0) = \int \exp\left\{\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{(Y_1,\dots,Y_m)}\left(\sum_{g\in G_c}\prod_{\ell\in g}A(\ell)\right)\left(\prod_{\ell=1}^{m}\tilde{Q}(Y_\ell,\phi,0)\right)\right\}d\tilde{\nu}_\Lambda^V(\phi). \tag{4.11}$$

We conclude this section with the following observation.

Lemma II.16. $\tilde{Q}(Y,\phi,\varepsilon)$ is a real function of ϕ and ε.

Proof. Consider a term in (4.2) with fixed $\{\zeta\}_{\Gamma(Y)}=\{\zeta'\}_{\Gamma(Y)}$, which we denote $T(\{\zeta'\}_{\Gamma(Y)})$. We shall show that

$$T^*(\{\zeta'\}_{\Gamma(Y)}) = T(-\{\zeta'\}_{\Gamma(Y)}) \quad . \tag{4.12}$$

Since $\tilde{Q}(Y,\phi,\varepsilon)$ may be rewritten as a sum of $2^{|\Gamma(Y)|-1}$ pairs of terms of the form

$$T(\{\zeta'\}_{\Gamma(Y)})+T(-\{\zeta'\}_{\Gamma(Y)}) = T(\{\zeta'\}_{\Gamma(Y)})+T^*(\{\zeta'\}_{\Gamma(Y)})$$
$$= 2\,\mathrm{Re}\,T(\{\zeta'\}_{\Gamma(Y)}) \quad . \tag{4.13}$$

it is a sum of real functions and hence real.

To prove (4.12), we show that $T(\{\zeta\}_{\Gamma(Y)})$ is a product of factors all of which possess the property that their complex conjugate is equal to the same function with $\{\zeta'\}_{\Gamma(Y)}$ changed to $-\{\zeta'\}_{\Gamma(Y)}$. By (2.15) and Proposition II.1(b) we have

$$i(\phi+i\zeta'_\rho a_\rho) = i^*(\phi-i\zeta'_\rho a_\rho) \ , \tag{4.14}$$

and thus

$$R(\rho,\phi,\zeta'_\rho,\varepsilon) = K(\rho)e^{-\gamma q_\rho}{}_{i(\phi+i\zeta'_\rho a_\rho)}\exp[i\zeta'_\rho\phi(\rho)+i\varepsilon(a_\rho(o)-a_\rho(x))]$$
$$= R^*(\rho,\phi,-\zeta'_\rho,\varepsilon) \quad . \tag{4.15}$$

Since $\tilde{V}_X(\cdot)$ is an analytic function which is real for real values of its arguments, and invariant under $\{\phi\}\to-\{\phi\}$ one has

$$\tilde{V}_X(\{\phi+i\zeta'_\rho a_\rho\}|_X) = \tilde{V}_X^*(\{\phi-i\zeta'_\rho a_\rho\}|_X) \quad . \tag{4.16}$$

The functions $\Delta V_X^{s_1\dots s_{k-1}}(\cdot;\cdot:\cdot)$ and $\Delta V_X(\cdot:\cdot)$ are sums of $\tilde{V}_X(\cdot)$ functions and hence inherit property (4.16). Thus each of the factors in $T(\{\zeta'\}_{\Gamma(Y)})$ possesses this property, and the proof is complete.

II.5 Jensen's Inequality and the Proof of Theorem II.5

In this section we apply Jensen's inequality (4.10) to derive the lower bound of Theorem II.5. The first step is embodied in

Proposition II.17. There exist $0<\beta_1<\beta_2$ and $J_1(\beta_1)>0$ such that for all $\beta\in(\beta_1,\beta_2)$ and $J>J_1(\beta_1)$, the measure

$$d\nu_\Lambda^N(\phi) = \exp\left\{\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{(Y_1,\dots,Y_m)}\left(\sum_{g\in G_c}\prod_{\ell\in g}A(\ell)\right)\left(\prod_{\ell=1}^{m}\tilde{Q}(Y_\ell,\phi,0)\right)\right\}d\tilde{\nu}_\Lambda^V(\phi) \tag{5.1}$$

is finite. Furthermore,

$$\exp\left\{\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{(Y_1,\dots,Y_m)}\left(\sum_{g\in G_c}\prod_{\ell\in g}A(\ell)\right)\left(\prod_{\ell=1}^{m}\tilde{Q}(Y_\ell,\phi,0)\right)\right\}\prod_{\langle i,j\rangle}I_\beta(\phi_i-\phi_j) \tag{5.2}$$

is invariant under the global change of variables $\{\phi\}\to-\{\phi\}$.

Proof. Finiteness of the measure comes from the following convergence lemma, whose proof is delayed until a later chapter.

Lemma II.18. There exist $0<\beta_1<\beta_2$ and $J_1(\beta_1)>0$ such that for $\beta\in(\beta_1,\beta_2)$ and $J>J_1(\beta_1)$ and every charge density $\rho_0\in\Gamma$, one has

$$\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{\substack{(Y_1,\dots,Y_m)\\ \rho_0\in\cup_1^m\Gamma(Y_j)}}\left|\left(\sum_{g\in G_c}\left(\prod_{\ell\in g}A(\ell)\right)\left(\prod_{\ell=1}^{m}\tilde{Q}(Y_\ell,\phi,0)\right)\right)\right|\leq 1 \quad . \tag{5.3}$$

By the lemma,

$$\left|\exp\left\{\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{(Y_1,\dots,Y_m)}\left(\sum_{g\in G_c}\prod_{\ell\in g}A(\ell)\right)\left(\prod_{\ell=1}^{m}\tilde{Q}(Y_\ell,\phi,0)\right)\right\}\right|$$

Equation (5.4) to be continued

$$\leq \exp\left\{\sum_{\rho\in N}\sum_{m=1}^{\infty}\frac{1}{m!}\left|\sum_{\substack{(Y_1,\dots,Y_m)\\ \rho\in\cup_1^m\Gamma(Y)}}\left(\sum_{g\in G_c}\prod_{\ell\in g}A(\ell)\right)\left(\prod_{\ell=1}^{m}\tilde{Q}(Y_\ell,\phi,0)\right)\right|\right\}$$

$$\leq \exp|N| \quad . \tag{5.4}$$

Similarly by (1.2),

$$\exp\left\{\sum_{X\subset\Lambda}\tilde{v}_X(\{\phi\}|_X)\right\}\leq\exp\left\{\sum_{i\in\mathbb{Z}^2\cap\Lambda}\sum_{\substack{X\subset\Lambda:\\ i\subset X}}\left|\tilde{v}_X(\{\phi\}|_X)\right|\right\}\leq e^{c'\cdot|\Lambda|} \quad , \tag{5.5}$$

where the last inequality follows by

$$\begin{aligned}\sum_{\substack{X\subset\Lambda:\\ i\subset X}}\left|\tilde{v}_X(\{\phi\}|_X)\right| &\leq \sum_{\substack{X\subset\Lambda:\\ i\subset X}} e^{-J|X|}\left|2\sum_{m=1}^{\infty}\frac{1}{m^2}\right|^{|X|}\\ &\leq \sum_{|X|=3}^{\infty} e^{-J|X|}\cdot c^{|X|}\cdot c'^{|X|}\\ &\leq c'' \quad ,\end{aligned} \tag{5.6}$$

if J is sufficiently large. The first inequality follows from (1.2) while the second is the standard Peierls estimate. Since $d\mu_\Lambda(\phi)$ is a finite measure, $\int d\nu_\Lambda^N(\phi)\leq e^{|N|+c''|\Lambda|}\int d\mu_\Lambda(\phi)<\infty$. Note that $I_\beta(\phi)=I_\beta(-\phi)$. Thus the invariance of (5.2) under $\{\phi\}\to-\{\phi\}$ follows from $\tilde{Q}(Y,\phi,0)=\tilde{Q}(Y,-\phi,0)$ which we now demonstrate.

Recall that

$$\begin{aligned}\tilde{Q}(Y,-\phi,0) &= \sum_{\{\zeta\}_{\Gamma(Y)}}\left(\frac{1}{2}\right)^{|\Gamma(Y)|}\left[\prod_{\rho\in\Gamma(Y)}R(\rho,-\phi,\zeta_\rho,0)\right]\\ &\times\prod_{m=2}^{k}\left[s_{\eta(m)}\cdots s_{m-2}\,\Delta V_{X_{m-1}}(-\phi:i\zeta a(\Gamma_{\eta(m)})+\cdots+i\zeta a(\Gamma_m))\right]\\ &\times\exp\left\{\sum_{X\subset\Lambda}\Delta v_X^{s_1\cdots s_{k-1}}(-\phi:i\zeta a(\Gamma_1)+\cdots+i\zeta a(\Gamma_{k-1});i\zeta a(\Gamma_k))\right\} .\end{aligned} \tag{5.7}$$

The definition of $R(\cdot,\cdot,\cdot,\cdot)$ implies

$$R(\rho,-\phi,\ \zeta_\rho,0) \ = \ R(\rho,\phi,-\zeta_\rho,0) \quad . \tag{5.8}$$

Similarly, the invariance of the potentials $\tilde{V}_X(\{\phi\}|_X)$ under $\{\phi\}\to-\{\phi\}$ implies

$$\Delta V_X(-\phi:i\zeta a(\Gamma_{\eta(m)})+\cdots+i\zeta a(\Gamma_m))$$
$$= \ \Delta V_X(\phi:-i\zeta a(\Gamma_{\eta(m)})-\cdots-i\zeta a(\Gamma_m)) \quad , \tag{5.9}$$

and

$$\Delta V_X^{s_1\cdots s_{k-1}}(-\phi:i\zeta a(\Gamma_1)+\cdots+i\zeta a(\Gamma_{k-1});i\zeta a(\Gamma_k))$$
$$= \ \Delta V_X^{s_1,\ldots,s_{k-1}}(\phi:-i\zeta a(\Gamma_1)-\cdots-i\zeta a(\Gamma_{k-1});-i\zeta a(\Gamma_k)) \ . \tag{5.10}$$

Substitute (5.8), (5.9), and (5.10) into (5.7). Now change variables in the sum over $\{\zeta\}_{\Gamma(Y)}$ to $\{\zeta'\}=-\{\zeta\}$. Summing over $\zeta'_\rho=\pm1$ is equivalent to summing over $\zeta_\rho=\pm1$. Thus the r.h.s. of (5.7) becomes

$$\sum_{\{\zeta'\}_{\Gamma(Y)}}\left(\tfrac{1}{2}\right)^{|\Gamma(Y)|}\left[\prod_{\rho\in\Gamma}R(\rho,\phi,\zeta'_\rho,0)\right]$$
$$\times\prod_{m=2}^{k}\ [s_{\eta(m)}\cdots s_{m-2}\ \Delta V_{X_{m-1}}(\phi:i\zeta' a(\Gamma_{\eta(m)})+\cdots+i\zeta' a(\Gamma_m))]$$
$$\exp\left\{\sum_{X\subset\Lambda}\ \Delta V_X^{s_1\cdots s_{k-1}}(\phi:i\zeta' a(\Gamma_1)+\cdots+i\zeta' a(\Gamma_{k-1});i\zeta' a(\Gamma_k))\right\}$$
$$= \ \tilde{Q}(Y,\phi,0) \quad . \tag{5.11}$$

This completes the proof of (5.2).

We take the following lemma from [15].

<u>Lemma II.19.</u> For β sufficiently small

$$\frac{I_\beta(\phi_i-\phi_j+\sigma_i-\sigma_j)}{I_\beta(\phi_i-\phi_j)} \geqslant \exp\left[\frac{I'_\beta(\phi_i-\phi_j)}{I_\beta(\phi_i-\phi_j)}\cdot(\sigma_i-\sigma_j)-c\beta(\sigma_i-\sigma_j)^2\right]. \tag{5.12}$$

We do not reproduce the proof of this lemma, but merely note that it follows easily from Proposition II.1(d) and Taylor's Theorem with remainder. Note also, that the invariance of $I_\beta(\phi)$ under $\{\phi\}\to-\{\phi\}$ insures that $I'_\beta(\phi)/I_\beta(\phi)$ is odd under this same transformation. Combining (5.1) and (4.10) we see that

$$Z_N(\sigma,\varepsilon) \ = \ \int e^{\varepsilon(\phi_o-\phi_x)}\exp\left\{\sum_{X\subset\Lambda}\ \tilde{V}_X(\{\phi+\sigma\}|_X)-\tilde{V}_X(\{\phi\}|_X)\right\}$$
$$\times\exp\left\{\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{(Y_1,\ldots,Y_m)}\left(\sum_{g\in G_c}\prod_{\ell\in g}A(\ell)\right)\left[\prod_{\ell=1}^{m}\tilde{Q}(Y_\ell,\phi+\sigma,\varepsilon)-\prod_{\ell=1}^{m}\tilde{Q}(Y_\ell,\phi,0)\right]\right\}$$
$$\times\left[\prod_{\langle i,j\rangle}\frac{I_\beta(\phi_i-\phi_j+\sigma_i-\sigma_j)}{I_\beta(\phi_i-\phi_j)}\right]dv_\Lambda^N(\phi) \quad . \tag{5.13}$$

The integrand of this expression is positive and thus we use Lemma II.18 to bound it from below by replacing the ratio of $I_\beta(\cdot)$ functions by

$$\exp\left\{\sum_{\langle i,j\rangle}\frac{I'_\beta(\phi_i-\phi_j)}{I_\beta(\phi_i-\phi_j)}\ (\sigma_i-\sigma_j)-c\beta\|\partial\sigma\|_2^2\right\} \tag{5.14}$$

where

$$\|\partial\sigma\|_2^2 \ = \ \sum_{\langle i,j\rangle}\ (\sigma_i-\sigma_j)^2 \quad .$$

Combining (5.1) and (3.9) we obtain $Z_N(0,0) = \int d\nu_\Lambda^N(\phi)$. Apply Jensen's inequality in the measure $d\nu_\Lambda^N(\phi)$ to obtain

<u>Lemma II.20.</u> For β sufficiently small and J sufficiently large one has

$$\frac{Z_N(\sigma,\varepsilon)}{Z_N(0,0)} \geq \exp\Bigg[\left(\frac{1}{d\nu_\Lambda^N(\phi)}\right)\cdot\int\Bigg\{\sum_{X\subset\Lambda}\left[\tilde{V}_X(\{\phi+\sigma\}|_X) - \tilde{V}_X(\{\phi\}|_X)\right]$$

$$+\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{(Y_1,\ldots,Y_m)}\left(\sum_{g\in G_c}\left(\prod_{\ell\in g}A(\ell)\right)\right)\left[\prod_{\ell=1}^{m}\tilde{Q}(Y_\ell,\phi+\sigma,\varepsilon) - \prod_{\ell=1}^{m}\tilde{Q}(Y_\ell,\phi,0)\right]$$

$$-c\beta\|\partial\sigma\|_2^2\Bigg\}\Bigg]\, d\nu_\Lambda^N(\phi) \quad . \tag{5.15}$$

This follows by noting that

$$\int(\phi_0-\phi_x)\,d\nu_\Lambda^N(\phi) = \sum_{\langle i,j\rangle}\int\frac{I_\beta'(\phi_i-\phi_j)}{I_\beta(\phi_i-\phi_j)}\cdot(\sigma_i-\sigma_j)\,d\nu_\Lambda^N(\phi) = 0$$

since the integrand is in both cases an odd function under $\{\phi\}\to-\{\phi\}$.

We now deal with the two remaining sums. Consider the simpler of the two first.

<u>Lemma II.21.</u> There exist $0<\beta_1<\beta_2$ and $J_1(\beta_1)>0$ such that for $\beta\in(\beta_1,\beta_2)$ and $J>J_1(\beta_1)$ there exists $c'(J)$ with

$$\frac{\left|\int\left[\sum_{X\subset\Lambda}\tilde{V}_X(\{\phi+\sigma\}|_X) - \tilde{V}_X(\{\phi\}|_X)\right]d\nu_\Lambda^N(\phi)\right|}{\int d\nu_\Lambda^N(\phi)} \leq c'(J)\|\partial\sigma\|_2^2 \tag{5.16}$$

where $c'(J)$ can be made arbitrarily small by taking J sufficiently

<u>Proof.</u> Consider the definition (1.2) of $\tilde{V}_X(\{\phi+\sigma\}|_X)$. By Taylor's Theorem with remainder we see that

$$\tilde{V}_X(\{\phi+\sigma\}|_X) = \sum_{\substack{\{h\}|_X:\\ h_i \text{ fixed}}} V_X(\{\partial h\}|_X)\prod_{\langle i,j\rangle\subset X}\Big\{f(\partial\phi_{ij}+\partial h_{ij})$$

$$+(\partial\sigma_{ij})f'(\partial\phi_{ij}+\partial h_{ij}) + \frac{\lambda_0}{2}(\partial\sigma_{ij})^2 f''(\partial\phi_{ij}+\partial h_{ij}+\lambda_0\partial\sigma_{ij})\Big\} \tag{5.17}$$

where $f(x) = \frac{\sin^2\pi x}{\pi^2x^2}$, $\partial\phi_{ij} = \phi_i - \phi_j$ and λ_0 is some ($\partial\sigma_{ij}$ dependent) constant in $[0,1]$. Note that there exists a constant θ such that

$$|f(\partial\phi_{ij}+\partial h_{ij})| \leq \theta C(\partial\phi_{ij}+\partial h_{ij}) \quad ,$$

$$|f'(\partial\phi_{ij}+\partial h_{ij})| \leq \theta C(\partial\phi_{ij}+\partial h_{ij}) \quad , \tag{5.18}$$

and

$$|f''(\partial\phi_{ij}+\partial h_{ij}+\lambda_0\partial\sigma_{ij})| \leq \theta C(\partial\phi_{ij}+\partial h_{ij}+\lambda_0\partial\sigma_{ij}) \quad ,$$

with

$$C(x) = \begin{cases} 1 & \text{if } |x|\leq 1 \\ \frac{1}{x^2} & \text{if } |x|>1 \end{cases} \quad . \tag{5.19}$$

The explicit expression for $f'(\partial\phi_{ij}+\partial h_{ij})$, coupled with the fact that $V_X(\{\partial h\}|_X) = V_X(-\{\partial h\}|_X)$ yields

$$\sum_{\{\partial h\}|_X} V_X(\{\partial h\}|_X) f'(\partial\phi_{ij}+\partial h_{ij}) \prod_{\substack{\langle i',j'\rangle\subset X\\ \langle i',j'\rangle\neq\langle i,j\rangle}} f(\partial\phi_{i'j'}+\partial h_{i'j'})$$

$$= -\sum_{\{\partial h\}|_X} V_X(\{\partial h\}|_X) f'(-\partial\phi_{ij}+\partial h_{ij}) \prod_{\substack{\langle i',j'\rangle\subset X\\ \langle i',j'\rangle\neq\langle i,j\rangle}} f(-\partial\phi_{ij}+\partial h_{ij}) \quad . \tag{5.20}$$

By the sum over $\{\partial h\}|_X$ we mean a sum over $\{h\}|_X$ with the height of some column in X fixed. By (5.20) one has

$$\sum_{\langle i,j\rangle\subset X} \int \left[\sum_{\{\partial h\}|_X} \tilde{V}_X(\{\partial h\}|_X)(\partial\sigma_{ij}) f'(\partial\phi_{ij}+\partial h_{ij}) \prod_{\substack{\langle i',j'\rangle\subset X\\ \langle i',j'\rangle\neq\langle i,j\rangle}} f(\partial\phi_{ij}+\partial h_{ij}) \right] d\nu_\Lambda^N(\phi) = 0 \quad . \tag{5.21}$$

Combining this with (5.17) we see that

$$\int [\tilde{V}_X(\{\phi+\sigma\}|_X) - \tilde{V}_X(\{\phi\}|_X)]\, d\nu_\Lambda^N(\phi) \tag{5.22}$$

consists of $3^{|\tilde{X}|} - |\tilde{X}| - 1$ terms, (where $|\tilde{X}|$ denotes the number of nearest neighbor pairs in X) each of which has at least two factors of $(\partial\sigma_{ij})$. Consider any such term, e.g.

$$\int \sum_{\{\partial h\}|_X} V_X(\{\partial h\}|_X) \left[\prod_{\langle i_1,j_1\rangle\in S_1} f(\partial\phi_{i_1j_1} + \partial h_{i_1j_1}) \right]$$

$$\times \left[\prod_{\langle i_2,j_2\rangle\in S_2} (\partial\sigma_{i_2j_2}) f'(\partial\phi_{i_2j_2}+\partial h_{i_2j_2}) \right] \left[\prod_{\langle i_3,j_3\rangle} \frac{\lambda_0}{2} (\partial\sigma_{i_3j_3})^2 f''(\partial\phi_{i_3j_3}+\partial h_{i_3j_3}+\lambda_0\partial\sigma_{i_3j_3}) \right] d\nu_\Lambda^N(\phi) \tag{5.23}$$

where $S_1 \cup S_2 \cup S_3 = \{$set of nearest neighbor pairs $\langle i,j\rangle$ in $X\}$, and $|S_3| > 1$ or $|S_2| \geq 2$. Note that $|\partial\sigma_{ij}| < \varepsilon|\ln\beta|\cdot c$, independent of i,j or x. This follows from the inequality $C_D \leq C_F$, where C_F is the covariance with free boundary conditions at infinity, and then standard estimates on the free covariance. Combining this observation with the fact that $f(\cdot)$ $f'(\cdot)$ are bounded by $\theta\cdot C(\partial\phi_{ij}+\partial h_{ij})$, we find that

$$\left| \sum_{\{\partial h\}|_X} V_X(\{\partial h\}|_X) \prod_{\langle i,j\rangle\subset X} \theta\cdot C(\partial\phi_{ij}+\partial h_{ij}) \right| \leq e^{-J|X|} \sum_{\{\partial h\}|_X} \prod_{\langle i,j\rangle\subset X} C(\partial\phi_{ij}+\partial h_{ij})\cdot\theta \tag{5.24}$$

(by (I.2.2)(b))

$$\leq c^{|X|} e^{-J|X|} \leq e^{-7/8\, J|X|} \quad ,$$

provided J is sufficiently large. This in turn implies that (5.23) is bounded by

$$\sup_{\langle i,j\rangle\subset X} (\partial\sigma_{ij})^2 (c|\ln\beta|\theta)^{|X|} e^{-\frac{7}{8}J|X|} \int d\nu_\Lambda^N(\phi) \quad . \tag{5.25}$$

Pick $J_1(\beta_1)$ sufficiently large that for $\beta\in(\beta_1,\beta_2)$ and $J > J_1(\beta_1)$ one has,

$$3^{|X|} (c|\ln\beta|\theta)^{|X|} e^{-\frac{1}{8}J|X|} \leq 1 \quad . \tag{5.26}$$

The factor of $3^{|X|}$ comes from bounding the number of terms like (5.23) in (5.22). Then

$$\left|\sum_{X\subset\Lambda}\int[\tilde{v}_X(\{\phi+\sigma\}|_X)-\tilde{v}_X(\{\phi\}|_X)]\,d\nu_\Lambda^N(\phi)\right|$$

$$\leq \sum_{\langle i,j\rangle\subset\Lambda}\left\{\sum_{\substack{X:\\ \sup|\partial\sigma_{i'j'}|=|\partial\sigma_{ij}|\\ \langle i',j'\rangle\subset X}}\left|\int[\tilde{v}_X(\{\phi+\sigma\}|_X)-\tilde{v}_X(\{\phi\}|_X)]\,d\nu_\Lambda^N(\phi)\right|\right\}$$

$$\leq \sum_{\langle i,j\rangle\subset\Lambda}\sum_{\substack{X:\\ \langle i,j\rangle\subset X}}(\partial\sigma_{ij})^2\,e^{-\frac{3}{4}J|X|}\int d\nu_\Lambda^N(\phi)$$

$$\leq e^{-\frac{1}{2}J}\,\|\partial\sigma\|_2^2\int d\nu_\Lambda^N(\phi) \qquad . \tag{5.27}$$

In the second inequality we applied (5.23) and (5.26) and in the last step we used the Peierls estimate to bound the sum over X's containing a given bond. This completes the proof of Lemma II.21.

The more difficult part of (5.13) is bounded by

<u>Proposition II.22</u>. There exist $0<\beta_1<\beta_2$ and $J_1(\beta_1)>0$ such that for all $\beta\in(\beta_1,\beta_2)$ and $J>J_1(\beta_1)$, there exists a constant $D>0$ such that

$$\left|\int\left\{\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{(Y_1,\ldots,Y_m)}\left(\sum_{g\in G_c}\left(\prod_{\ell\in g}A(\ell)\right)\right)\left(\prod_{\ell=1}^{m}\tilde{Q}(Y_\ell,\phi+\sigma,\varepsilon)-\prod_{\ell=1}^{m}\tilde{Q}(Y_\ell,\phi,0)\right)\right\}d\nu_\Lambda^N(\phi)\right|$$

$$\leq\left[\left(e^{-J/8}+2e^{-\frac{D}{64}|\ln\beta|}\right)\|\partial\sigma\|_2^2+2\varepsilon^2\right]\int d\nu_\Lambda^N(\phi) \qquad . \tag{5.28}$$

The proof of this lemma is postponed until Chapter V.

Note that Proposition II.22, Proposition II.20 and Lemma II.21 combine to give

$$\frac{Z_N(\sigma,\varepsilon)}{Z_N(0,0)}\geq e^{-\tilde{c}(\beta,J)\|\partial\sigma\|_2^2}\cdot e^{-2\varepsilon^2} \qquad , \tag{5.29}$$

with

$$\tilde{c}(\beta,J) = (c\cdot\beta+e^{-J/8}+e^{-D/64\,|\ln\beta|}+c'(J)) \qquad .$$

The proof of Theorem II.5 is completed by noting that

$$\|\partial\sigma\|_2^2\leq\varepsilon^2(\ln\beta)^2c''\cdot\ln(1+|x|) \qquad . \tag{5.30}$$

This yields

$$\frac{Z_N(\sigma,\varepsilon)}{Z_N(0,0)}\geq e^{-\varepsilon^2k(\beta,J)\cdot\ln(1+|x|)}\,e^{-2\varepsilon^2} \qquad , \tag{5.31}$$

with $k(\beta,J)=c''(\ln\beta)^2\cdot\tilde{c}(\beta,J)$, which is precisely the bound of (1.15). Equation (4.30) is proved by noting that

$$\|\partial\sigma\|_2^2 = \sum_{\langle i,j\rangle}(\sigma_i-\sigma_j)^2 = \langle\sigma,(-\Delta_\Lambda^D)\sigma\rangle$$

$$= \varepsilon^2(\ln\beta)^2\langle C_Df,f\rangle\leq\varepsilon^2(\ln\beta)^2\langle C_Ff,f\rangle \ . \tag{5.32}$$

Here C_F is the Green's function of minus the two-dimensional lattice Laplacian with free boundary conditions at infinity. For a further discussion of the Dirichlet covariance see Appendix I. By $\langle\cdot,\cdot\rangle$ we mean the ordinary L^2 inner product for functions on $\mathbb{Z}^2\cap\Lambda$. Finally,

$$\langle C_Ff,f\rangle = 2[C_F(0,0)-C_F(0,x)]\leq 2c'''\cdot\ln(1+|x|) \tag{5.33}$$

by standard estimates. This completes the proof of (5.31).

CHAPTER III

AN UPPER BOUND ON CORRELATIONS IN THE SOLID ON SOLID MODEL

In this chapter we prove logarithmic bounds on the fluctuations of the interface of the solid on solid model. The proof combines a duality transformation [15, 31] with the complex translation technique of McBryan and Spencer [30].

Consider the SOS model on a two-dimensional square lattice, whose finite volume partition function is

$$Z_\Lambda = \sum_{\{h\}} \prod_{\langle i,j\rangle} e^{-\beta|h_i-h_j|} \quad . \tag{1.1}$$

We take boundary conditions with $h_i \equiv 0$ for $i \notin \Lambda$. The product runs over all nearest neighbor pairs in $\mathbb{Z}^2$. We assume that Λ is a square centered at the origin, and by $|\Lambda|$ we mean the number of points in $\mathbb{Z}^2 \cap \Lambda$. The main result is the bound of Corollary I.4. There exists a constant $K_u(\beta)$ such that

$$\langle (h_o - h_x)^2 \rangle_\Lambda \leq K_u(\beta) \ln(1 + |x|) \quad . \tag{I.3.5}$$

This follows from the more general bound of

<u>Theorem III.1</u>. For every $\beta > 0$, and $x = (n,0) \in \mathbb{Z}^2 \cap \Lambda$ with $0 < n < |\Lambda|^{1/8}$, there exists a constant $K_u(\beta) > 0$ and a constant $\varepsilon_0(\beta) > 0$ such that for all $0 < \varepsilon < \varepsilon_0(\beta)$ one has

$$\langle e^{\varepsilon(h_o-h_x)} \rangle_\Lambda \leq e^{\varepsilon^2 K_u(\beta) \ln(1+|x|)} \quad . \tag{1.2}$$

Corollary I.4 follows immediately from this bound by subtracting one from both sides of (1.2) and expanding to third order in ε. The term on the l.h.s. of (1.2) which is first order in ε vanishes due to the symmetry of the SOS model under $\{h\} \to -\{h\}$, and the bound (I.3.5) follows upon dividing by ε^2 and taking the limit $\varepsilon \to 0$.

<u>Proof</u>. The proof requires a change to dual variables. Consider the unnormalized expectation

$$Z_\Lambda \langle e^{\varepsilon(h_o-h_x)} \rangle_\Lambda = \sum_{\{h\}} e^{\varepsilon(h_o-h_x)} \prod_{\langle i,j\rangle} e^{-\beta|h_i-h_j|} \quad . \tag{1.3}$$

Let γ be the path running from 0 to x. We also let γ denote the set of lattice bonds in γ. It will be clear from the context which meaning is desired. Rewrite the unnormalized expectation as

$$Z_\Lambda \langle e^{\varepsilon(h_o-h_x)} \rangle_\Lambda = \sum_{\{h\}} \prod_{\langle m,k\rangle \in \gamma} e^{\varepsilon(h_m-h_k)} \prod_{\langle i,j\rangle} e^{-\beta|h_i-h_j|} \quad . \tag{1.4}$$

We adopt the convention that the bonds $\langle m,k\rangle$ are directed, with the positive direction either up or to the right.

The duality transformation consists of exchanging our "site" variables for "bond" variables $n_{ij} = h_i - h_j$. The bond variables n_{ij} are not independent. They must satisfy the constraint that their sum around any plaquette be equal to zero. The origin of this restriction becomes obvious when we consider that n_{ij} is just the lattice gradient of the $\{h\}$ field so that $\oint \vec{n} \cdot d\vec{\ell} = \oint \vec{\nabla} h \cdot d\vec{\ell} = 0$.

Let $I(i^*, \langle i,j\rangle)$ be the incidence function which is $+1$ when the bond $\langle i,j\rangle$ is contained in the boundary of the plaquette associated with the site i^* in the dual lattice and is oriented in the positive direction, -1 if it is oriented in the negative direction, and 0 if

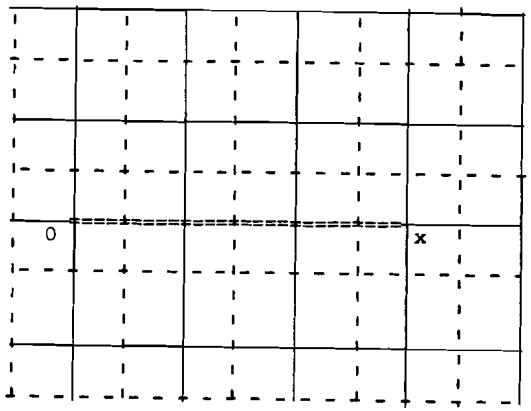


Figure III.1. The original lattice, — , the dual lattice, -- , and the bonds γ, == .

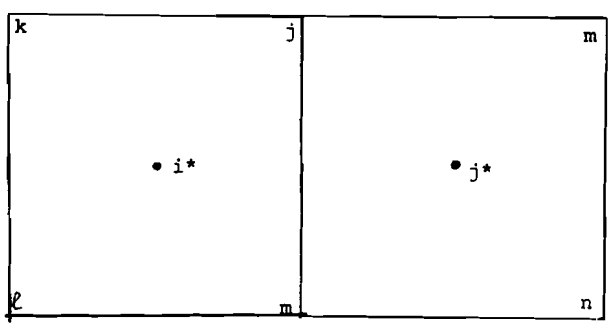


Figure III.2. Representative sites in the original and dual lattices.

if $\langle i,j\rangle$ does not lie in the boundary of i^*.

The coboundary operator ∇, defines a function (on sites of the dual lattice)

$$(\nabla n)(i^*) \equiv \sum_{\langle i,j\rangle} I(i^*,\langle i,j\rangle)n_{ij} \quad .$$

(See [1, 22] for a concise explication of these lattice operations.)

For the site i^* in Figure 2 one just has

$$(\nabla n)(i^*) = n_{ij} - n_{kj} - n_{\ell k} + n_{\ell i} = h_i - h_j + h_j - h_k + h_k - h_\ell + h_\ell - h_i = 0 \ .$$

We may sum over the bond variables as if they were independent provided we enforce the constraints on their sums by introducing Kronecker δ-functions for each plaquette which are non-zero only when $(\nabla n)(i^*) = 0$.

$$Z_\Lambda \langle e^{\varepsilon(h_o - h_x)}\rangle_\Lambda = \sum_{\{n\}} \prod_{\langle k,\ell\rangle\in\gamma} e^{\varepsilon n_{k\ell}} \prod_{\langle i,j\rangle} e^{-\beta|n_{ij}|} \prod_{i^*\in\Lambda^*} (\delta_{0,(\nabla n)(i^*)}) . \tag{1.5}$$

Λ^* is the set of all sites in the dual lattice $\mathbb{Z}^{2^*} = (\mathbb{Z}+1/2)^2$ such that the plaquette associated with this site touches Λ. Note that our boundary conditions translate into the restriction that n_{ij} vanish for all bonds $\langle i,j\rangle \subset \Lambda^c$. Throughout this chapter and the next "starred" quantities (e.g., i^*, $\langle i^*,j^*\rangle$) refer to the dual lattice and unstarred quantities refer to the original lattice. Rewrite (1.5) using

$$\delta_{0,n} = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{in\theta} \quad . \tag{1.6}$$

One obtains

$$Z_\Lambda \langle e^{\varepsilon(h_o - h_x)} \rangle_\Lambda = \sum_{\{n\}} \left[\prod_{i^* \in \Lambda} \int_0^{2\pi} \frac{d\theta_{i^*}}{2\pi} \right] \prod_{\langle k,\ell \rangle \in \gamma} e^{\varepsilon n_{k\ell}} \prod_{\langle i,j \rangle} e^{-\beta |n_{ij}|}$$

$$\times \prod_{i^* \in \Lambda^*} e^{i\theta_{i^*}(\nabla n)(i^*)} \quad . \tag{1.7}$$

From Figure 2 we see that each n_{ij} appears twice in $\prod_{i^* \in \Lambda^*} e^{i\theta_{i^*}(\nabla n)(i^*)}$, once with a positive sign when i^* is the site of the dual lattice immediately above or to the left of $\langle i,j \rangle$, and once with a negative sign when i^* is the site in the dual lattice immediately below or to the right of $\langle i,j \rangle$. Denote $\prod_{i^* \in \Lambda^*} \int_0^{2\pi} \frac{d\theta_{i^*}}{2\pi} = \int D\theta$ and rewrite (1.7) as

$$Z_\Lambda \langle e^{\varepsilon(h_o - h_x)} \rangle_\Lambda = \int D\theta \sum_{\{n\}} \prod_{\langle k,\ell \rangle \in \gamma} e^{\varepsilon n_{k\ell}} \prod_{\langle i,j \rangle} \left[e^{-\beta |n_{ij}|} e^{i n_{ij}(\theta_{i^*} - \theta_{j^*})} \right] . \tag{1.8}$$

Here i^* is the site in the dual lattice to the left or above $\langle i,j \rangle$ and j^* is the site to the right or below that bond. Perform the sum over $\{n\}$ leading to

$$Z_\Lambda \langle e^{\varepsilon(h_o - h_x)} \rangle_\Lambda = \int D\theta \prod_{\langle i^*,j^* \rangle} \tilde{I}_\beta(\theta_{i^*} - \theta_{j^*} + i\,\varepsilon\,\delta_{\langle i^*,j^* \rangle,\gamma}) \quad . \tag{1.9}$$

$$\tilde{I}_\beta(\theta) = \frac{1 - e^{-2\beta}}{1 + e^{-2\beta} - 2e^{-\beta}\cos\theta} \quad , \tag{1.10}$$

and

$$\delta_{\langle i^*,j^* \rangle,\gamma} = \begin{cases} 1 & \text{if } \langle i^*,j^* \rangle \cap \gamma \\ 0 & \text{otherwise} \end{cases} \quad . \tag{1.11}$$

Note that there exists a constant $\lambda(\beta) > 0$ such that $\tilde{I}_\beta(\theta) \geqslant \lambda(\beta)$ for all θ.

We now use the periodicity of the functions $\tilde{I}_\beta(\theta)$ to change variables in (1.9) according to

$$\theta_{i^*} \to \theta_{i^*} + i a_{i^*} \quad . \tag{1.12}$$

If a_{i^*} is properly chosen (we discuss this choice below) there exists a constant $c'(\beta)$ with

$$\left| \frac{\tilde{I}_\beta(\theta_{i^*} - \theta_{j^*} + i a_{i^*} - i a_{j^*} + i\,\varepsilon\,\delta_{\langle i^*,j^* \rangle,\gamma})}{\tilde{I}_\beta(\theta_{i^*} - \theta_{j^*})} \right| \leqslant e^{c'(\beta)(a_{i^*} - a_{j^*} + \varepsilon\delta_{\langle i^*,j^* \rangle,\gamma})^2} . \tag{1.13}$$

$$\frac{\tilde{I}_\beta(\theta + ia)}{\tilde{I}_\beta(\theta)} = \left\{ 1 - \frac{2e^{-\beta}[\cos\theta(\cosh a - 1) + i\sin\theta\sinh a]}{1 + e^{-2\beta} - 2e^{-\beta}\cos\theta} \right\}^{-1} . \tag{1.14}$$

Suppose there exists some constant K such that $|a| < K\varepsilon$. Since

$$|2e^{-\beta}[\cos\theta(\cosh a - 1) + i\sin\theta\sinh a]| \leqslant 2K\varepsilon \ ,$$

for ε sufficiently small, there exists a constant $\varepsilon_0(\beta)$ such that for all $0 < \varepsilon < \varepsilon_0(\beta)$ one has

$$\left| \frac{\tilde{I}_\beta(\theta + ia)}{\tilde{I}_\beta(\theta)} \right| = \left| e^{-\ell n \left\{ 1 - \frac{2e^{-\beta}[\cos\theta(\cosh a - 1) + i\sin\theta\sinh a]}{1 + e^{-2\beta} - 2e^{-\beta}\cos\theta} \right\}} \right|$$

$$\leqslant e^{c(\beta)\cdot a^2} \quad , \tag{1.15}$$

with $c(\beta) > 0$. (In fact for β small, $c(\beta) \sim c \cdot \beta^{-4}$.) This estimate used Taylor's theorem to expand the logarithm to second order and then used the bounds

$$|\cosh(a) - 1| \leqslant a^2, \qquad \sinh a \leqslant 3/2a \ , \tag{1.16}$$

for a small. (This may necessitate taking $\varepsilon_0(\beta)$ somewhat smaller than it was previously.) Equation (1.15) yields (1.13) if we can prove

$$|a_{i^*} - a_{j^*} + \varepsilon\delta_{\langle i^*,j^*\rangle,\gamma}| \leqslant K\cdot\varepsilon \quad . \tag{1.17}$$

Define γ^* as the set of bonds in the dual lattice which intersects γ and define γ_L^* as sites in the dual lattice which are the upper end of some bond in γ^*. $\chi(\cdot)$ is the function on $\mathbb{Z}^{2^*}$ defined by

$$\chi(i^*) = \begin{cases} 1 & \text{if } i^* \in \gamma_L^* \\ 0 & \text{otherwise} \end{cases} \quad . \tag{1.18}$$

Finally, let ∂_2 be the y component of the lattice gradient, i.e., if f is some lattice function, $(\partial_2 f)(i) \equiv f(i+\hat{e}_y) - f(i)$. We pick our translating function a_{i^*} to be

$$a_{i^*} = \varepsilon \sum_{j^*} C_D(i^*,j^*)(\partial_2\chi)(j^*). \tag{1.19}$$

Here C_D is the Green's function, or covariance of the negative of the two-dimensional lattice Laplacian with Dirichlet b.c.'s on $\partial\Lambda^*$.

We prove in Appendix I that there exists a constant c, such that for i^*,j^* nearest neighbor sites, $|a_{i^*} - a_{j^*}| < c\cdot\varepsilon$. If we take $K = c+1$,

$$|a_{i^*} - a_{j^*} + \varepsilon\delta_{\langle i^*,j^*\rangle,\gamma}| < K\cdot\varepsilon \quad ,$$

verifying (1.17). Use (1.13) to rewrite (1.9) as:

$$Z_\Lambda \langle e^{\varepsilon(h_o - h_x)}\rangle_\Lambda = \int D\theta \prod_{\langle i^*,j^*\rangle\subset\Lambda^*} \tilde{I}_\beta(\theta_{i^*} - \theta_{j^*} + ia_{i^*} - ia_{j^*} + i\varepsilon\delta_{\langle i^*,j^*\rangle,\gamma})$$

$$= \int D\theta \prod_{\langle i^*,j^*\rangle\subset\Lambda^*} \left\{ \frac{\tilde{I}_\beta(\theta_{i^*} - \theta_{j^*} + ia_{i^*} - ia_{j^*} + i\varepsilon\delta_{\langle i^*,j^*\rangle,\gamma})}{\tilde{I}_\beta(\theta_{i^*} - \theta_{j^*})} \right\} \{\tilde{I}_\beta(\theta_{i^*} - \theta_{j^*})\}$$

$$\leqslant e^{c'(\beta) \sum_{\langle i^*,j^*\rangle} (a_{i^*} - a_{j^*} + \varepsilon\delta_{\langle i^*,j^*\rangle,\gamma})^2}$$

$$\times \int D\theta \prod_{\langle i^*,j^*\rangle\subset\Lambda^*} \tilde{I}_\beta(\theta_{i^*} - \theta_{j^*}) \quad . \tag{1.20}$$

But

$$Z_\Lambda = \int D\theta \prod_{\langle i^*,j^*\rangle\subset\Lambda^*} \tilde{I}_\beta(\theta_{i^*} - \theta_{j^*}) \quad .$$

Hence

$$\langle e^{\varepsilon(h_o - h_x)}\rangle_\Lambda \leqslant \exp\left\{ c'(\beta) \sum_{\langle i^*,j^*\rangle} (a_{i^*} - a_{j^*} + \varepsilon\delta_{\langle i^*,j^*\rangle,\gamma})^2 \right\} . \tag{1.21}$$

Now bound the exponent of this expression.

$$\sum_{\langle i^*,j^*\rangle} (a_{i^*} - a_{j^*} + \varepsilon\delta_{\langle i^*,j^*\rangle,\gamma})^2$$

$$= \sum_{\langle i^*,j^*\rangle} [(a_{i^*} - a_{j^*})^2 + 2\varepsilon(a_{i^*} - a_{j^*})\delta_{\langle i^*,j^*\rangle,\gamma} + \varepsilon^2\delta^2_{\langle i^*,j^*\rangle,\gamma}] \tag{1.22}$$

The definition of $\delta_{\langle i^*,j^*\rangle,\gamma}$ implies that

$$\sum_{\langle i^*,j^*\rangle} \delta^2_{\langle i^*,j^*\rangle,\gamma} = |\gamma| \ . \tag{1.23}$$

Using the definition of $\chi(\cdot)$ we write

$$\varepsilon \sum_{\langle i^*,j^*\rangle} (a_{i^*}-a_{j^*})\delta_{\langle i^*,j^*\rangle,\gamma} = -\varepsilon \sum_{i^*} a_{i^*}(\partial_2\chi)(i^*)$$

$$\equiv -\varepsilon\langle a,\partial_2\chi\rangle, \tag{1.24}$$

where $\langle\cdot,\cdot\rangle$ is the usual L^2 inner product on the space of functions on $\mathbb{Z}^2$ which vanish in Λ^c. The definition of Dirichlet boundary conditions for the lattice Laplacian (see Appendix I) implies that

$$\sum_{\langle i^*,j^*\rangle} (a_{i^*}-a_{j^*})^2 = \langle a,(-\Delta^D_\Lambda)a\rangle$$

$$= \varepsilon^2\langle c_D\partial_2\chi,(-\Delta^D_\Lambda)c_D\partial_2\chi\rangle \tag{1.25}$$

$$= \varepsilon^2\langle c_D\partial_2\chi,\partial_2\chi\rangle = \varepsilon\langle a,\partial_2\chi\rangle \ . \tag{1.26}$$

Combining (1.24) and (1.26) gives

$$\sum_{\langle i^*,j^*\rangle} [(a_{i^*}-a_{j^*})^2 + 2\varepsilon(a_{i^*}-a_{j^*})\delta_{\langle i^*,j^*\rangle,\gamma}]$$

$$= -\varepsilon^2 \langle c_D\partial_2\chi,\partial_2\chi\rangle \ . \tag{1.27}$$

<u>Lemma III.2</u>. There exists a constant $c_u(\beta)$ such that

$$|c'(\beta)\{-\varepsilon^2\langle c_D\partial_2\chi,\partial_2\chi\rangle + \varepsilon^2|\gamma|\}| \leq c_u(\beta)\cdot\varepsilon^2\cdot\ln(1+|x|) \ . \tag{1.28}$$

This lemma, combined with (1.21), (1.22), (1.23) and (1.27) yields Theorem III.1. The lemma is proved by integration by parts, but the proof is left to Appendix I where the Dirichlet covariance is discussed.

CHAPTER IV

AN UPPER BOUND FOR MODELS WITH NON-LOCAL POTENTIALS

In this chapter we extend the methods used to prove the upper bound in the SOS model to the models with non-local potentials. This requires dual transforming as in the SOS case. Because the interaction potentials are non-local, however, the dual transformed model does not factorize into nearest neighbor interactions. A more manageable form of the model is obtained after a Mayer expansion which exhibits a partial factorization. We prove the upper bound on the fluctuations given in Theorem I.1,

$$\langle(h_o-h_x)^2\rangle_\Lambda \leq c_u(\beta,J)\ln(1+|x|) \ . \tag{I.3.1}$$

This bound was shown to follow from

<u>Theorem I.3</u>. For every $\beta>0$ there exist constants $J_2(\beta)>0$ and $\varepsilon_0(\beta)>0$ such that for all $J>J_2(\beta)$ and $0<\varepsilon<\varepsilon_0(\beta)$ there exists a constant $c_u(\beta,J)>0$ such that for $x=(n,0)\in\mathbb{Z}^2\cap\Lambda$, with $0<n<|\Lambda|^{1/8}$ one has:

$$\langle e^{\varepsilon(h_0-h_x)}\rangle_\Lambda \leq e^{\varepsilon^2 c_u(\beta,J)\ln(1+|x|)} \ . \tag{0.1}$$

Furthermore, $c_u(\beta,J)$ may be uniformly bounded by some constant $K(\beta)$ for all $J>J_2(\beta)$.

IV.1 The Mayer Expansion for Upper Bounds

The models studied have an unnormalized expectation given by

$$Z_\Lambda \langle e^{\varepsilon(h_o - h_x)} \rangle_\Lambda = \sum_{\{h\}} \prod_{\langle k,\ell\rangle \in \gamma} e^{\varepsilon(h_k - h_\ell)} \prod_{\langle i,j\rangle} I_\beta(h_i - h_j)\, e^{\sum_{X\subset\Lambda} V_X(\{h_i - h_j\}|_X)} . \tag{1.1}$$

As in the previous chapter, γ is the set of lattice bonds connecting 0 to x, $I_\beta(h) = e^{-\beta|h|}$, and the potentials $V_X(\cdot)$ obey the conditions (I.2.2). Introduce "bond" variables $n_{ij} = h_i - h_j$ as before, and implement the constraints on these variables via Kronecker δ-functions, which are rewritten as integrals over variables θ_{i*}, living on the dual lattice. The manipulations are the same as those in the previous chapter and yield

$$Z_\Lambda \langle e^{\varepsilon(h_o - h_x)} \rangle_\Lambda = \int D\theta \sum_{\{n\}} \left(\prod_{\langle i,j\rangle} I_\beta(n_{ij})\, e^{i n_{ij}(\theta_{i*} - \theta_{j*} + i\varepsilon\delta_{\langle i*,j*\rangle,\gamma})} \times e^{\sum_{X\subset\Lambda} V_X(\{n\}|_X)} \right) . \tag{1.2}$$

At this point we see where the problems arise--the sum over $\{n\}$ does not factorize as was previously the case. To circumvent this difficulty we Mayer expand (1.2). As a first step in this process define an arbitrary order for nearest neighbor pairs (bonds) which intersect the region Λ. Denote the first bond (with respect to this ordering) $\langle i_1, j_1\rangle$, and define the interpolated potential

$$V_X^{s_1}(\{n\}|_X; \{\langle i_1,j_1\rangle\}) = \begin{cases} s_1 V_X(\{n\}|_X) & \text{if } \langle i_1,j_1\rangle \subset X \\ V_X(\{n\}|_X) & \text{otherwise.} \end{cases} \tag{1.3}$$

Then rewrite

$$\sum_{\{n\}} \left(\prod_{\langle i,j\rangle} I_\beta(n_{ij})\, e^{i n_{ij}(\theta_{i*} - \theta_{j*} + i\varepsilon\delta_{\langle i*,j*\rangle,\gamma})} \right) e^{\sum_{X\subset\Lambda} V_X(\{n\}|_X)} , \tag{1.4}$$

using the fundamental theorem of calculus as

$$\sum_{\{n\}} \left(\prod_{\langle i,j\rangle} I_\beta(n_{ij})\, e^{i n_{ij}(\theta_{i*} - \theta_{j*} + i\varepsilon\delta_{\langle i*,j*\rangle,\gamma})} \right) e^{\sum_{\substack{X\subset\Lambda \\ \langle i_1,j_1\rangle \not\subset X}} V_X(\{n\}|_X)}$$

$$+ \int_0^1 ds_1 \sum_{\{n\}} \left(\prod_{\langle i,j\rangle} I_\beta(n_{ij})\, e^{i n_{ij}(\theta_{i*} - \theta_{j*} + i\varepsilon\delta_{\langle i*,j*\rangle,\gamma})} \right)$$

$$\times \sum_{\substack{X_1\subset\Lambda \\ X_1 \supset \langle i_1,j_1\rangle}} V_{X_1}(\{n\}|_{X_1}) \cdot e^{\sum_{X\subset\Lambda} V_X^{s_1}(\{n\}|_X; \langle i_1,j_1\rangle)} . \tag{1.5}$$

The purpose of this interpolation is to decouple the bond $\langle i_1, j_1\rangle$ from all the non-local potentials in the first term. Now continue to decouple the second term via the following inductive procedure.

Let $\Gamma_1, \ldots, \Gamma_n$ be connected collections of lattice bonds such that for all $i \in \{1,\ldots,n\}$ there exists $j \in \{1,\ldots,i-1\}$ with $\Gamma_i \cap \Gamma_j \neq \emptyset$. Then define

$$V_X^{s_1\cdots s_n}(\{n\}|_X;\Gamma_1\cup\cdots\cup\Gamma_n)$$

$$= \begin{cases} s_n V_X^{s_1\cdots s_{n-1}}(\{n\}|_X;\Gamma_1\cup\cdots\cup\Gamma_{n-1}) & \text{if there exists} \\ \quad \langle i,j\rangle\subset X \text{ s.t. } \langle i,j\rangle\in\Gamma_1\cup\cdots\cup\Gamma_n \text{ and there exists} \\ \quad \langle i',j'\rangle\subset X \text{ s.t. } \langle i',j'\rangle\notin\Gamma_1\cup\cdots\cup\Gamma_n \\ V_X^{s_1\cdots s_{n-1}}(\{n\}|_X;\Gamma_1\cup\cdots\cup\Gamma_{n-1}) & \text{otherwise} \end{cases} \quad . \tag{1.6}$$

Now study the second term in (1.5).

Let $\Gamma_1=\{\langle i_1,j_1\rangle\}$ and choose $\Gamma_2=\{$set of lattice bonds contained in $X_1\}\equiv\Gamma(X_1)$, according to which term in the sum over X_1 we are considering. Decouple using $V_X^{s_1s_2}(\{n\}|_X;\Gamma_1\cup\Gamma_2)$ to obtain

$$\sum_{X_1\subset\Lambda}\sum_{X_2\subset\Lambda}\sum_{\{n\}}\int_0^1 ds_1ds_2\chi_{X_1}(\Gamma_1)\chi_{X_2}(\Gamma_1\cup\Gamma_2)V_{X_1}(\{n\}|_{X_1})V_{X_2}^{s_1}(\{n\}|_{X_2};\Gamma_1)$$

$$\times\prod_{\langle i,j\rangle}(I_\beta(n_{ij})e^{in_{ij}(\theta_{i*}-\theta_{j*}+i\varepsilon\delta_{\langle i*,j*\rangle},\gamma)})e^{\sum_{X\subset\Lambda}V_X^{s_1s_2}(\{n\}|_X;\Gamma_1\cup\Gamma_2)}$$

$$+\sum_{X_1\subset\Lambda}\sum_{\{n\}}\int_0^1 ds_1\chi_{X_1}(\Gamma_1)V_{X_1}(\{n\}|_{X_1})\prod_{\langle i,j\rangle}(I_\beta(n_{ij})e^{in_{ij}(\theta_{i*}-\theta_{j*}+i\varepsilon\delta_{\langle i*,j*\rangle},\gamma)})$$

$$\times e^{\sum_{X\subset X_1}V_X^{s_1}(\{n\}|_X;\Gamma_1)}\,e^{\sum_{\substack{X\subset\Lambda\\ \Gamma(X)\cap(\Gamma_1\cup\Gamma_2)=\emptyset}}V_X(\{n\}|_X)} \quad . \tag{1.7}$$

The characteristic functions are defined by

$$\chi_X(\Gamma) = \begin{cases} 1 & \text{if there exists } \langle i,j\rangle\in\Gamma \text{ with } \langle i,j\rangle\subset X \text{ and} \\ & \text{there exists } \langle i',j'\rangle\notin\Gamma \text{ with } \langle i',j'\rangle\subset X \\ 0 & \text{otherwise.} \end{cases}$$

We have used the facts that $V_X^{s_1s_2=0}(\{n\}|_X;\Gamma_1\cup\Gamma_2)=0$ for any X with $\Gamma(X)\cap\Gamma(X_1)\neq\emptyset$ and $\Gamma(X)\cap(\Gamma_\Lambda\backslash\Gamma(X_1))\neq\emptyset$ ($\Gamma_\Lambda=\{$set of all bonds intersecting $\Lambda\}$), $V_X^{s_1}(\{n\}|_X;\Gamma_1)=V_X(\{n\}|_X)$ for $\Gamma(X)\cap\Gamma_1=\emptyset$, and $V_X^{s_1,s_2=0}(\{n\}|_X;\Gamma_1\cup\Gamma_2)=V_X^{s_1}(\{n\}|_X;\Gamma_1)$ if $X\subset X_1$, all of which follow from the definition of the interpolated potentials. (In general, as in the special case above, $\Gamma(X)=\{$set of lattice bonds contained in $X\}$.)

Rearrange the second term of (1.7) as

$$\sum_{X_1}\left\{\int_0^1 ds_1\sum_{\{n\}|_{\Gamma_1\cup\Gamma_2}}\chi_{X_1}(\Gamma_0)V_{X_1}(\{n\}|_{X_1})\prod_{\langle i,j\rangle\subset X_1}\left(I_\beta(n_{ij})\times e^{in_{ij}(\theta_{i*}-\theta_{j*}+i\varepsilon\delta_{\langle i*,j*\rangle},\gamma)}\right)e^{\sum_{X\subset X_1}V_X^{s_1}(\{n\}|_X;\Gamma_1)}\right\}$$

$$\times\left\{\sum_{\{n\}|_{\Gamma_\Lambda\backslash(\Gamma_1\cup\Gamma_2)}}\prod_{\substack{\langle i,j\rangle:\\ \langle i,j\rangle\not\subset X_1}}\left(I_\beta(n_{ij})e^{in_{ij}(\theta_{i*}-\theta_{j*}+i\varepsilon\delta_{\langle i*,j*\rangle},\gamma)}\right)\times e^{\sum_{\substack{X\subset\Lambda\\ \Gamma(X)\cap(\Gamma_1\cup\Gamma_2)=\emptyset}}V_X(\{n\}|_X)}\right\} \quad . \tag{1.8}$$

Now return to (1.7) to decouple X_1 and X_2 further. The general result is the following lemma. First, given $\Gamma_1,\ldots,\Gamma_k$ and X_k, define for $m<k$

$$\tilde{\chi}_{X_k}(\Gamma_1 \cup \dots \cup \Gamma_{m-1};\Gamma_m) = \begin{cases} 1 & \text{if there exists } \langle i,j\rangle \subset X_k \text{ s.t. } \langle i,j\rangle \in \Gamma_m, \\ & \langle i',j'\rangle \subset X_k \text{ s.t. } \langle i',j'\rangle \notin \Gamma_1 \cup \dots \cup \Gamma_m \text{ and} \\ & \text{no } \langle i'',j''\rangle \subset X_k \text{ s.t. } \langle i'',j''\rangle \in \Gamma_1 \cup \dots \cup \Gamma_{m-1} \\ 0 & \text{otherwise} \end{cases} . \tag{1.9}$$

<u>Lemma IV.1.</u> For any positive integer m

$$\sum_{\{n\}} \left(\prod_{\langle i,j\rangle} I_\beta(n_{ij}) e^{in_{ij}(\theta_{i*}-\theta_{j*}+i\varepsilon\delta_{\langle i*,j*\rangle},\gamma)} \right) e^{\sum_{X\subset\Lambda} V_X(\{n\}|_X)} =$$

$$\sum_{k=1}^{m} \sum_{X_1} \dots \sum_{X_{k-1}} \sum_{\eta} \int_0^1 ds_1 \dots ds_{k-1}$$

$$\times \left\{ \sum_{\{n\}|_{\Gamma_1\cup\dots\cup\Gamma_k}} \prod_{\substack{\langle i,j\rangle: \\ \langle i,j\rangle\in\Gamma_1\cup\dots\cup\Gamma_k}} I_\beta(n_{ij}) e^{in_{ij}(\theta_{i*}-\theta_{j*}+i\varepsilon\delta_{\langle i*,j*\rangle},\gamma)} \right.$$

$$\times\ e^{\sum_{X\subset X_1\cup\dots\cup X_{k-1}}} V_{X_{\ell-1}}^{s_1\dots s_{k-1}}(\{n\}|_X;\Gamma_1\cup\dots\cup\Gamma_{k-1})$$

$$\left. \times \prod_{\ell=2}^{k} \left[s_{\eta(\ell)}\dots s_{\ell-2} \tilde{\chi}_{X_{\ell-1}}(\Gamma_1\cup\dots\cup\Gamma_{\eta(\ell)-1};\Gamma_{\eta(\ell)}) V_{X_{\ell-1}}(\{n\}|_{X_{\ell-1}}) \right] \right\}$$

$$\times \left\{ \sum_{\{n\}|_{\Gamma_\Lambda\setminus(\Gamma_1\cup\dots\cup\Gamma_k)}} \prod_{\substack{\langle i,j\rangle: \\ \langle i,j\rangle\notin\Gamma_1\cup\dots\cup\Gamma_k}} I_\beta(n_{ij}) e^{in_{ij}(\theta_{i*}-\theta_{j*}+i\varepsilon\delta_{\langle i*,j*\rangle},\gamma)} \right.$$

$$\left. e^{\sum_{\substack{X\subset\Lambda \\ \Gamma(X)\cap(\Gamma_1\cup\dots\cup\Gamma_k)=\emptyset}} V_X(\{n\}|_X)} \right\}$$

Eq. (1.10) to be continued

$$+ \sum_{X_1} \dots \sum_{X_m} \sum_{\eta} \int_0^1 ds_1 \dots ds_m \sum_{\{n\}} \left[\prod_{\ell=2}^{m+1} |s_{\eta(\ell)}\dots s_{\ell-2}| \right.$$

$$\left. \times\ \chi_{X_{\ell-1}}(\Gamma_\ell \cup \dots \cup \Gamma_{\eta(\ell)-1},\Gamma_{\eta(\ell)}) V_{X_{\ell-1}}\left(\{n\}\middle|X_{\ell-1}\right) \right]$$

$$\prod_{\langle i,j\rangle} I_\beta(n_{ij}) e^{in_{ij}(\theta_{i*}-\theta_{j*}+i\varepsilon\delta_{\langle i*,j*\rangle},\gamma)} e^{\sum_{X\subset\Lambda} V_X^{s_1\dots s_m}(\{n\}|_X;\Gamma_1\cup\dots\cup\Gamma_m)} \tag{1.10}$$

We define Γ_k inductively as $\Gamma_k = \Gamma(X_{k-1})$. The sum over η in the first term runs over all tree functions on k vertices while that in the second term runs over all tree functions on $m+1$ vertices. The proof is also by induction. Begin by proving the following lemma about the interpolated potentials.

<u>Lemma IV.2.</u> For any integer $k \geq 1$,

$$\sum_X \chi_X(\Gamma_1\cup\dots\cup\Gamma_k) V_X^{s_1\dots s_{k-1}}(\{n\}|_X;\Gamma_1\cup\dots\cup\Gamma_{k-1})$$

$$= \sum_X \sum_{j=1}^{k} (s_j\dots s_{k-1}) \tilde{\chi}_X(\Gamma_1\cup\dots\cup\Gamma_{j-1};\Gamma_j) V_X(\{n\}|_X) . \tag{1.11}$$

<u>Proof</u>. This is true for $k=1$. Assume that it is true for $k<n$. From the definition of $\chi_X(\Gamma_1\cup\dots\cup\Gamma_k)$ we can rewrite the l.h.s. of (1.11) when $k=n$ as

$$\sum_X [\chi_X(\Gamma_1 \cup \ldots \cup \Gamma_{n-1}) + \tilde{\chi}_X(\Gamma_1 \cup \ldots \cup \Gamma_{n-1};\Gamma_n)]\, V_X^{s_1\ldots s_{n-1}}(\{n\}|_X;\Gamma_1 \cup \ldots \cup \Gamma_{n-1})$$

$$= s_{n-1} \sum_X \sum_{j=1}^{n-1} (s_j \ldots s_{n-2})\tilde{\chi}_X(\Gamma_1 \cup \ldots \cup \Gamma_{n-2};\Gamma_{n-1}) V_X(\{n\}|_X)$$

$$+ \sum_X \tilde{\chi}_X(\Gamma_1 \cup \ldots \cup \Gamma_{n-1};\Gamma_n) V_X^{s_1\ldots s_{n-1}}(\{n\}|_X;\Gamma_1 \cup \ldots \cup \Gamma_{n-1}) \,. \qquad (1.12)$$

We have applied the definition of the interpolated potentials and then the induction hypothesis to the first term on the r.h.s. of (1.12). The second term is handled by the observation that if $\Gamma(X) \cap \{\Gamma_1 \cup \ldots \cup \Gamma_{n-1}\} = \emptyset$, then $V_X^{s_1\ldots s_{n-1}}(\{n\}|_X;\Gamma_1 \cup \ldots \cup \Gamma_{n-1}) = V_X(\{n\}|_X)$. (This follows from the definition of the interpolated potentials.) With this observation the second term in (1.12) becomes

$$\sum_X \tilde{\chi}_X(\Gamma_1 \cup \ldots \cup \Gamma_{n-1};\Gamma_n) V_X(\{n\}|_X) \,,$$

which when combined with the first term completes the proof.

<u>Proof (of Lemma IV.1)</u>. By (1.5), the lemma holds for $k=1$. Assume that it holds for $m < m_0$. Consider the second term in (1.10). Introducing a new decoupling parameter s_{m_0}, expand

$$e^{\sum_{X\subset\Lambda} V_X^{s_1\ldots s_{m_0-1}}(\{n\}|_X;\Gamma_1\cup\ldots\cup\Gamma_{m_0-1})} = e^{\sum_{X\subset X_1\cup\ldots\cup X_{m_0-1}} V_X^{s_1\ldots s_{m_0-1}}(\{n\}|_X;\Gamma_1\cup\ldots\cup\Gamma_{m_0-1})}$$

$$\times\; e^{\sum_{\substack{X\subset\Lambda \\ \Gamma(X)\cap(\Gamma_1\cup\ldots\cup\Gamma_{m_0})=\emptyset}} V_X(\{n\}|_X)}$$

Eq. (1.13) to be continued

$$+ \int_0^1 ds_{m_0} \left(\sum_{X_{m_0}} V_{X_{m_0}}^{s_1\ldots s_{m_0-1}}(\{n\}|_{X_{m_0}};\Gamma_1\cup\ldots\cup\Gamma_{m_0-1}) \chi_{X_{m_0}}(\Gamma_1\cup\ldots\cup\Gamma_{m_0}) \right)$$

$$e^{\sum_{X\subset\Lambda} V_X^{s_1\ldots s_{m_0}}(\{n\}|_X;\Gamma_1\cup\ldots\cup\Gamma_{m_0})} \,. \qquad (1.13)$$

In the first term we have used the fact that

$$V_X^{s_1\ldots s_{m-1}}(\{n\}|_X;\Gamma_1\cup\ldots\cup\Gamma_{m-1}) = V_X(\{n\}|_X)$$

if $\Gamma(X) \cap \{\Gamma_1 \cup \ldots \cup \Gamma_{m-1}\} = \emptyset$. The first of these two terms becomes the $k = m_0$ part of the first term in (1.10). Rewrite the second term with the aid of Lemma IV.2 as

$$\int_0^1 ds_{m_0} \left(\sum_{X_{m_0}} \sum_{j=1}^{m_0} (s_j \ldots s_{m_0-1}) \tilde{\chi}_{X_{m_0}}(\Gamma_1\cup\ldots\cup\Gamma_{j-1};\Gamma_j) V_{X_{m_0}}(\{n\}|_{X_{m_0}}) \right)$$

$$e^{\sum_{X\subset\Lambda} V_X^{s_1\ldots s_{m_0}}(\{n\}|_X;\Gamma_1\cup\ldots\cup\Gamma_{m_0})} \,.$$

Putting this back into (1.10) yields the second term in (1.10) in the case when $m = m_0$. This completes the proof.

The expansion process must terminate after a finite number of steps, because $\Gamma_1 \cup \ldots \cup \Gamma_k \subset \Gamma_1 \cup \ldots \cup \Gamma_{k+1}$ and the lattice is of finite size. When $\Gamma_1 \cup \ldots \cup \Gamma_k = \Gamma_\Lambda$,

$$V_X^{s_1\ldots s_{k+1}}(\{n\}|_X;\Gamma_1\cup\ldots\cup\Gamma_{k+1}) = V_X^{s_1\ldots s_k}(\{n\}|_X;\Gamma_1\cup\ldots\cup\Gamma_k)$$

and

$$\frac{d}{ds_{k+1}} V_X^{s_1\cdots s_{k+1}}(\{n\}|_X;\Gamma\cup\ldots\cup\Gamma_{k+1}) = 0 .$$

Thus the second half of (1.10) will vanish. If we adopt the convention that for all k greater than that k_0 for which $\Gamma_1\cup\ldots\cup\Gamma_{k_0}$ exhausts all the bonds on the lattice, the sums in (1.10) are set equal to zero we obtain

Corollary IV.3.

$$\sum_{\{n\}}\left(\prod_{\langle i,j\rangle} I_\beta(n_{ij})e^{in_{ij}(\theta_{i*}-\theta_{j*}+i\varepsilon\delta_{\langle i*,j*\rangle},\gamma)}\right) e^{\sum_{X\subset\Lambda} V_X(\{n\}|_X)}$$

$$= \sum_{k=1}^{\infty}\sum_{X_1}\cdots\sum_{X_{k-1}}\sum_{\eta}\int_0^1 ds_1\ldots ds_{k-1}$$

$$\times\left\{\sum_{\{n\}|_{\Gamma_1\cup\ldots\cup\Gamma_k}}\prod_{\substack{\langle i,j\rangle:\\ \langle i,j\rangle\in\Gamma_1\cup\ldots\cup\Gamma_k}} I_\beta(n_{ij})e^{in_{ij}(\theta_{i*}-\theta_{j*}+i\varepsilon\delta_{\langle i*,j*\rangle},\gamma)}\right.$$

$$\times\prod_{\ell=2}^{k}\left[s_{\eta(\ell)}\cdots s_{\ell-2}\tilde{X}_{X_{\ell-1}}\left(\Gamma_1\cup\ldots\cup\Gamma_{\eta(\ell)-1};\Gamma_{\eta(\ell)}\right)V_{X_{\ell-1}}(\{n\}|_{X_{\ell-1}})\right]$$

$$\left. e^{\sum_{X\subset X_1\cup\ldots\cup X_{k-1}} V_X^{s_1\cdots s_{k-1}}(\{n\}|_X;\Gamma_1\cup\ldots\cup\Gamma_{k-1})}\right\}$$

$$\times\left\{\sum_{\{n\}|_{\Gamma_\Lambda\setminus(\Gamma_1\cup\ldots\cup\Gamma_k)}}\prod_{\substack{\langle i,j\rangle:\\ \langle i,j\rangle\notin\Gamma_1\cup\ldots\cup\Gamma_k}} I_\beta(n_{ij})e^{in_{ij}(\theta_{i*}-\theta_{j*}+i\varepsilon\delta_{\langle i*,j*\rangle},\gamma)}\right.$$

$$\left.\times\; e^{\sum_{\substack{X\subset\Lambda\\ \Gamma(X)\cap(\Gamma_1\cup\ldots\cup\Gamma_k)=\emptyset}} V_X(\{n\}|_X)}\right\} \qquad (1.14)$$

IV.2 The Cluster Formalism

We now introduce the cluster formalism discussed in Chapter 2.

Definition: A cluster Y consists of

(a) An integer $k\geq 1$

(b) A collection of connected subsets of Λ, $\{X_1,\ldots,X_{k-1}\}$

(c) A collection of sets of lattice bonds

$$\{\{\langle i_1,j_1\rangle\},\ \Gamma(X_1),\ldots,\Gamma(X_{k-1})\}\equiv\{\Gamma_1,\Gamma_2,\ldots,\Gamma_k\} \text{ with} \qquad (2.1)$$

$\langle i_1,j_1\rangle$, the minimal bond in $\cup_i^k\Gamma_j$.

(d) A tree function η on k vertices.

(e) An interpolation parameter s_m for $m=1,\ldots,k-1$.

Definition: Given a cluster Y we define the cluster function:

$$Q(Y,\theta,\varepsilon) = \sum_{\{n\}|_{\tilde{\Gamma}(Y)}}\left\{\prod_{\substack{\langle i,j\rangle:\\ \langle i,j\rangle\in\Gamma(Y)}} I_\beta(n_{ij})e^{in_{ij}(\theta_{i*}-\theta_{j*}+i\varepsilon\delta_{\langle i*,j*\rangle},\gamma)}\right.$$

$$\times\left[\prod_{\ell=2}^{k} s_{\eta(\ell)}\cdots s_{\ell-2}\tilde{X}_{X_{\ell-1}}(\Gamma_1\cup\ldots\cup\Gamma_{\eta(\ell)-1};\Gamma_{\eta(\ell)})V(\{n\}|_{X_{\ell-1}})\right]$$

$$\left.\times\; e^{\sum_{X\subset X_1\cup\ldots\cup X_{k-1}} V_X^{s_1\cdots s_{k-1}}(\{n\}|_X;\Gamma_1\cup\ldots\cup\Gamma_{k-1})}\right\} \qquad (2.2)$$

where by $\tilde{\Gamma}(Y)$ we mean $\Gamma_1\cup\ldots\cup\Gamma_k$.

With these two definitions we may write

$$\sum_{\{n\}}\left(\prod_{\langle i,j\rangle} I_\beta(n_{ij})e^{in_{ij}(\theta_{i*}-\theta_{j*}+i\varepsilon\delta_{\langle i*,j*\rangle},\gamma)}e^{\sum_{X\subset\Lambda} V_X(\{n\}|X)}\right)$$

$$= \sum_{\substack{Y:\Gamma_1=\{\langle i_1,j_1\rangle\}\\ \tilde\Gamma(Y)\subset\Lambda}} Q(Y,\theta,\varepsilon)\left[\sum_{\{n\}|\tilde\Gamma_\Lambda\setminus\Gamma(Y)}\prod_{\substack{\langle i,j\rangle:\\ \langle i,j\rangle\in\Gamma_\Lambda\setminus\tilde\Gamma(Y)}} e^{in_{ij}(\theta_{i*}-\theta_{j*}+i\varepsilon\delta_{\langle i*,j*\rangle},\gamma)}\right.$$

$$\left.\times e^{\sum_{\substack{X\subset\Lambda\\ \Gamma(X)\cap\tilde\Gamma(Y)=\emptyset}} V_X(\{n\}|X)}\right] \quad . \qquad (2.3)$$

Where we have compressed our notation so that

$$\sum_{\substack{Y:\Gamma_1=\{\langle i_1,j_1\rangle\}\\ \tilde\Gamma(Y)\subset\Gamma_\Lambda}} = \sum_{k=1}^{\infty}\sum_{X_1}\cdots\sum_{X_{k-1}}\sum_{\eta}\int_0^1 ds_0\ldots ds_{k-1} \quad .$$

Repeat the expansion process, this time applying it to the bracketed quantity on the r.h.s. of (2.3). We choose $\langle i_1',j_1'\rangle$ so that Γ_1 is the smallest bond (with respect to the previously defined ordering) in $\Gamma_\Lambda\setminus\tilde\Gamma(Y)$. The clusters generated in this expansion must satisfy a compatibility condition with respect to those generated in the first expansion. Specifically, we require that $\tilde\Gamma(Y_1)\cap\tilde\Gamma(Y_2)=\emptyset$. Continuing this process until Γ_Λ is exhausted we arrive finally at the expression,

<u>Lemma IV.4.</u>

$$\sum_{\{n\}}\left(\prod_{\langle i,j\rangle} I_\beta(n_{ij})\, e^{in_{ij}(\theta_{i*}-\theta_{j*}+i\varepsilon\delta_{\langle i*,j*\rangle},\gamma)}\, e^{\sum_{X\subset\Lambda} V_X(\{n\}|X)}\right)$$

$$\sum_{\substack{\{Y_1,\ldots,Y_k\}:\\ \tilde\Gamma(Y_m)\cap\tilde\Gamma(Y_j)=\emptyset;\, j\neq m\\ \Gamma_1(Y_1)<\ldots<\Gamma_1(Y_k)\\ \cup_1^k\tilde\Gamma(Y_m)=\Gamma_\Lambda}} \prod_{m=1}^{k} Q(Y_m,\theta,\varepsilon) \quad . \qquad (2.4)$$

By $\Gamma_1(Y_j)$ we mean Γ_1 for the cluster Y_j. Now remove the various restrictions on the allowed sets of clusters. First remove the requirement that $\cup_1^k\tilde\Gamma(Y_m)=\Gamma_\Lambda$. Note, that by picking ε sufficiently small we can insure that $|\tilde I_\beta(\theta_{i*}-\theta_{j*}-i\varepsilon\delta_{\langle i*,j*\rangle},\gamma)|\geq\lambda(\beta,\varepsilon)$, for some constant $\lambda(\beta,\varepsilon)>0$. Recall that

$$\tilde I_\beta(\theta) = \frac{1-e^{-2\beta}}{1+e^{-2\beta}-2e^{-\beta}\cos\theta} \quad .$$

Thus we may rewrite (1.18) as

$$\sum_{\substack{\{Y_1,\ldots,Y_k\}\\ \tilde\Gamma(Y_m)\cap\tilde\Gamma(Y_j)=\emptyset;\, m\neq j\\ \Gamma_1(Y_1)<\ldots<\Gamma_1(Y_k)\\ \cup_1^k\tilde\Gamma(Y_k)=\Gamma_\Lambda}}\left(\prod_{\langle i,j\rangle\in\Gamma_\Lambda}\tilde I_\beta(\theta_{i*}-\theta_{j*}+i\varepsilon\delta_{\langle i*,j*\rangle},\gamma)\right)$$

$$\times\prod_{m=1}^{k}\left[\frac{Q(Y_m,\theta,\varepsilon)}{\prod_{\langle i,j\rangle\in\tilde\Gamma(Y_m)}\tilde I_\beta(\theta_{i*}-\theta_{j*}+i\varepsilon\delta_{\langle i*,j*\rangle},\gamma)}\right]. \qquad (2.5)$$

Note that for any cluster Y, with $k=1$, $\tilde\Gamma(Y)=\{\langle i,j\rangle\}$ for some $\langle i,j\rangle$. We can restrict our sum over clusters to run only over those with $k>1$. For those clusters we define

$$\hat Q(Y_m,\theta,\varepsilon) = \frac{Q(Y_m,\theta,\varepsilon)}{\prod_{\langle i,j\rangle\in\tilde\Gamma(Y_m)}\tilde I_\beta(\theta_{i*}-\theta_{j*}+i\varepsilon\delta_{\langle i*,j*\rangle},\gamma)} \qquad (2.6)$$

Rewrite (2.4) as

$$\sum_{\substack{\{Y_1,\ldots,Y_k\}\\ \tilde\Gamma(Y_m)\cap\tilde\Gamma(Y_j)=\emptyset;\, m\neq j\\ \Gamma_1(Y_1)<\ldots<\Gamma_1(Y_k)}}{}' \left[\prod_{\langle i,j\rangle\in\Gamma_\Lambda}\tilde I_\beta(\theta_{i*}-\theta_{j*}+i\varepsilon\delta_{\langle i*,j*\rangle},\gamma)\right]\left[\prod_1^k\hat Q(Y_m,\theta,\varepsilon)\right]. \qquad (2.7)$$

$\sum'$ just reminds us that we sum over clusters with $k > 1$. Given a term in (2.7), associate with each $\langle i,j\rangle \in \Gamma_\Lambda \backslash \cup_1^m \tilde{\Gamma}(Y_j)$ the cluster $Y_{m'}$ with $\tilde{\Gamma}(Y_{m'})$. For such a cluster $Q(Y_{m'},\theta,\varepsilon) = \tilde{I}_\beta(\theta_{i*}-\theta_{j*}+i\varepsilon\delta_{\langle i*,j*\rangle,\gamma})$. Thus

$$\left[\prod_{\langle i,j\rangle\in\Gamma_\Lambda} \tilde{I}_\beta(\theta_{i*}-\theta_{j*}+i\varepsilon\delta_{\langle i*,j*\rangle,\gamma})\right]\left[\prod_i^k \hat{Q}(Y_m,\theta,\varepsilon)\right]$$

$$= \left[\prod_{m'} Q(Y_{m'},\theta,\varepsilon)\right]\left[\prod_{\substack{\langle i,j\rangle\varepsilon \\ \Gamma_\Lambda\backslash(\cup_{m'}\tilde{\Gamma}(Y_{m'}))}} \tilde{I}_\beta(\theta_{i*}-\theta_{j*}+i\varepsilon\delta_{\langle i*,j*\rangle,\gamma})\right]\prod_1^k \hat{Q}(Y_m,\theta,\varepsilon)$$

$$= \prod_{m'} Q(Y_{m'},\theta,\varepsilon) \prod_{m=1}^k Q(Y_m,\theta,\varepsilon) \quad ,$$

as required by (2.4)

One eliminates the requirement that $\Gamma_1(Y_1) < \ldots < \Gamma_1(Y_k)$ by summing over all ordered sets of clusters $(Y_1,\ldots,Y_k)$ and dividing by $1/k!$ to cancel the overcounting. Finally remove the restriction that $\tilde{\Gamma}(Y_m) \cap \tilde{\Gamma}(Y_j) = \emptyset$, by introducing the familiar $U(Y_i,Y_j)$ functions defined by:

$$U(Y_i,Y_j) = \begin{cases} 1 & \text{if } \tilde{\Gamma}(Y_i)\cap\tilde{\Gamma}(Y_j) = \emptyset \\ 0 & \text{otherwise} \end{cases} \quad . \tag{2.8}$$

With these modifications (2.4) becomes

$$\prod_{\langle i,j\rangle} \tilde{I}_\beta(\theta_{i*}-\theta_{j*}+i\varepsilon\delta_{\langle i*,j*\rangle,\gamma}) \sum_{m=0}^\infty \frac{1}{m!} \sum_{(Y_1,\ldots,Y_m)}' \prod_1^m \hat{Q}(Y_m,\theta,\varepsilon) \prod_{1\le i<j\le m} U(Y_i,Y_j) \quad . \tag{2.9}$$

From here one follows the procedure used in the proof of the lower bound. Introduce functions $A = U-1$, expand the products over A's, and then exponentiate the resulting sum, finally arriving at

<u>Lemma IV.5.</u> There exists $\varepsilon_0(\beta)$ and $J_2(\beta) > 0$ such that for all $0 < \varepsilon < \varepsilon_0(\beta)$ and all $J > J_2(\beta)$ one has

$$Z_\Lambda\langle e^{\varepsilon(h_o-h_x)}\rangle_\Lambda = \int D\theta\left(\prod_{\langle i*,j*\rangle} \tilde{I}_\beta(\theta_{i*}-\theta_{j*}+i\varepsilon\delta_{\langle i*,j*\rangle,\gamma})\right)$$

$$\times \exp\left[\sum_{m=1}^\infty \frac{1}{m!} \sum_{(Y_1,\ldots,Y_m)}' \left(\sum_{g\in G_c(Y_1,\ldots,Y_m)} \prod_{\ell\in g} A(\ell)\right) \prod_{j=1}^m \hat{Q}(Y_j,\theta,\varepsilon)\right] . \tag{2.10}$$

Recall that $G_c(Y_1,\ldots,Y_m)$ is the set of connected graphs on the vertices $(Y_1,\ldots,Y_m)$. The restriction on J just insures that the sum in the exponent is absolutely convergent, and hence that the manipulations leading to (2.10) are justified.

<u>Corollary IV.6.</u> Under the conditions of Lemma IV.5

$$Z_\Lambda = \int D\theta\left(\prod_{\langle i*,j*\rangle} \tilde{I}_\beta(\theta_{i*}-\theta_{j*})\right)$$

$$\times \exp\left[\sum_{m=1}^\infty \frac{1}{m!} \sum_{(Y_1,\ldots,Y_m)}' \left(\sum_{g\in G_c(Y_1,\ldots,Y_m)} \prod_{\ell\in g} A(\ell)\right) \prod_{j=1}^m \hat{Q}(Y_j,\theta,\varepsilon=0)\right]. \tag{2.11}$$

We close this section with

<u>Lemma IV.7.</u> For any cluster Y, $\hat{Q}(Y,\theta,\varepsilon=0)$ is real.

Proof.

$$\hat{Q}(Y,\theta,\varepsilon=0) = \left[\frac{1}{\prod_{\langle i,j\rangle\in\tilde{\Gamma}(Y)} \tilde{I}_\beta(\theta_{i*}-\theta_{j*})}\right]\cdot Q(Y,\theta,\varepsilon=0) \quad .$$

Since $\tilde{I}_\beta(\theta_{i*}-\theta_{j*})$ is real, so we may concentrate on $Q(Y,\theta,\varepsilon=0)$. Suppose we make the change of variables $\{n'\} = \{-n\}$ in the sum defining $Q(Y,\theta,\varepsilon=0)$. Because of the invariance of $I_\beta(n)$, $V_X(\{n\}|_X)$ and hence of $V_X^{s_1\cdots s_{k-1}}(\{n\}|_X;\Gamma_1\cup\ldots\cup\Gamma_{k-1})$ under this change of variables we see that

$$\begin{aligned} Q(Y,\theta,\varepsilon=0) &= \sum_{\{n'\}|_{\tilde{\Gamma}(Y)}} \left\{ \prod_{\substack{\langle i,j\rangle: \\ \langle i,j\rangle\in\tilde{\Gamma}(Y)}} I_\beta(n'_{ij})e^{-in'_{ij}(\theta_{i*}-\theta_{j*})} \right. \\ &\times \prod_{\ell=2}^{k}\left[s_{\eta(\ell)}\cdots s_{\ell-2}\tilde{\chi}_{X_{\ell-1}}(\Gamma_1\cup\ldots\cup\Gamma_{\eta(\ell)-1};\Gamma_{\eta(\ell)})\, V_{X_{\ell-1}}(\{n'\}|_{X_{\ell-1}})\right] \\ &\times \left. e^{\sum_{X\subset X_1\cup\ldots\cup X_{k-1}} V_X^{s_1\cdots s_{k-1}}(\{n'\}|_X;\Gamma_1\cup\ldots\cup\Gamma_{k-1})} \right\} \\ &= Q^*(Y,\theta,\varepsilon=0). \end{aligned} \tag{2.12}$$

(Here $Q^*(\cdot,\cdot,\cdot)$ refers to the complex conjugate, not the dual, of $Q(\cdot,\cdot,\cdot)$). This implies that $Q(Y,\theta,\varepsilon=0)$ is real, and completes the proof of Lemma IV.7.

IV.3 Complex Translations and an Upper Bound

In this section we change variables and derive bounds yielding Theorem I.3. As a first step use the periodicity of the integrand in expression (2.10) to translate

$$\theta_{i*}\to\theta_{i*}+ia_{i*} \quad .$$

As in Chapter III we choose our translating function a_{i*} to be

$$a_{i*} = \varepsilon\sum_{j*} C_D(i*,j*)(\delta_2\chi)(j*) \quad .$$

Recall that $C_D(i*,j*)$ is the covariance of the negative of the 2-dim Laplacian with Dirichlet b.c.'s and $\chi(\cdot)$ is the function defined in (III.1.18). Under this change of variables,

$$\begin{aligned} Z_\Lambda\langle e^{\varepsilon(h_o-h_x)}\rangle_\Lambda &= \int D\theta\left(\prod_{\langle i*,j*\rangle}\tilde{I}_\beta(\theta_{i*}-\theta_{j*}+ia_{i*}-ia_{j*}+i\varepsilon\delta_{\langle i*,j*\rangle,\gamma})\right) \\ &\times \exp\left[\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{(Y_1,\ldots,Y_m)}{}'\left(\sum_{g\in G_c(Y_1,\ldots,Y_m)}\right)\left[\prod_{\ell\in g}A(\ell)\right]\right. \\ &\qquad\left.\times\prod_{j=1}^{m}\hat{Q}(Y_j,\theta+ia,\varepsilon)\right], \\ &= \int D\theta\prod_{\langle i*,j*\rangle}\tilde{I}_\beta(\theta_{i*}-\theta_{j*})\exp\left[\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{(Y_1,\ldots,Y_m)}{}'\left(\sum_{g\in G_c(Y_1,\ldots,Y_m)}\right)\left[\prod_{\ell\in g}A(\ell)\right]\right. \\ &\qquad\left.\times\prod_{j=1}^{m}\hat{Q}(Y_j,\theta,\varepsilon=0)\right] \end{aligned}$$

Eq. (3.1) to be continued

$$\left[\frac{\prod_{\langle i^*,j^*\rangle} \tilde{I}_\beta(\theta_{i^*}-\theta_{j^*}+ia_{i^*}-ia_{j^*}+i\varepsilon\delta_{\langle i^*,j^*\rangle},\gamma)}{\prod_{\langle i^*,j^*\rangle} \tilde{I}_\beta(\theta_{i^*}-\theta_{j^*})}\right]$$

$$\cdot \exp\left[\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{(Y_1,\dots,Y_m)}{}'\left(\sum_{g\in G_c}\prod_{\ell\in g}A(\ell)\right)\left\{\prod_{j=1}^{m}\hat{Q}(Y_j,\theta+ia,\varepsilon)-\prod_{j=1}^{m}\hat{Q}(Y_j,\theta,\varepsilon=0)\right\}\right],$$

$$\leqslant \sup_{\{\theta\}}\left\{\left|\frac{\prod_{\langle i^*,j^*\rangle} \tilde{I}_\beta(\theta_{i^*}-\theta_{j^*}+ia_{i^*}-ia_{j^*}+i\varepsilon\delta_{\langle i^*,j^*\rangle},\gamma)}{\prod_{\langle i^*,j^*\rangle} \tilde{I}_\beta(\theta_{i^*}-\theta_{j^*})}\right|\right.$$

$$\left.\left|\exp\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{(Y_1,\dots,Y_m)}{}'\left(\sum_{g\in G_c}\left[\prod_{\ell\in g}A(\ell)\right]\right)\left[\prod_{j=1}^{m}\hat{Q}(Y_j,\theta+ia,\varepsilon)-\prod_{j=1}^{m}\hat{Q}(Y_j,\theta,\varepsilon=0)\right]\right|\right\}$$

$$\times\int D\theta\left[\prod_{\langle i^*,j^*\rangle}\tilde{I}_\beta(\theta_{i^*}-\theta_{j^*})\right]\exp\left[\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{(Y_1,\dots,Y_m)}{}'\left(\sum_{g\in G_c}\left[\prod_{\ell\in g}A(\ell)\right]\right)\prod_{j=1}^{m}\hat{Q}(Y_j,\theta,\varepsilon=0)\right]. \tag{3.1}$$

In this step we use the fact that the reality of $\hat{Q}(Y_j,\theta,\varepsilon=0)$ implies that the integrand is positive. Note that this integral is eactly the partition function, so that (3.1) insures

$$\langle e^{\varepsilon(h_o-h_x)}\rangle_\Lambda \leqslant \sup_{\{\theta\}}\left\{\left|\frac{\prod_{\langle i^*,j^*\rangle} \tilde{I}_\beta(\theta_{i^*}-\theta_{j^*}+ia_{i^*}-ia_{j^*}+i\varepsilon\delta_{\langle i^*,j^*\rangle},\gamma)}{\prod_{\langle i^*,j^*\rangle} \tilde{I}_\beta(\theta_{i^*}-\theta_{j^*})}\right|\right.$$

Eq. (3.2) to be continued

$$\left.\times\left|\exp\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{(Y_1,\dots,Y_m)}\left(\sum_{g\in G_c}{}'\left[\prod_{\ell\in g}A(\ell)\right]\right)\left[\prod_{j=1}^{m}\hat{Q}(Y_j,\theta+ia,\varepsilon)-\prod_{j=1}^{m}\hat{Q}(Y_j,\theta,\varepsilon=0)\right]\right|\right\} \tag{3.2}$$

In the previous chapter (see III.1.15) we saw that for ε sufficiently small there exists a constant $c'(\beta)$ such that

$$\sup_{\{\theta\}}\left\{\left|\frac{\prod_{\langle i^*,j^*\rangle} \tilde{I}_\beta(\theta_{i^*}-\theta_{j^*}+ia_{i^*}-ia_{j^*}+i\varepsilon\delta_{\langle i^*,j^*\rangle},\gamma)}{\prod_{\langle i^*,j^*\rangle} \tilde{I}_\beta(\theta_{i^*}-\theta_{j^*})}\right|\right\}$$

$$\leqslant e^{c'(\beta)\sum_{\langle i^*,j^*\rangle}(a_{i^*}-a_{j^*}+\varepsilon\delta_{\langle i^*,j^*\rangle},\gamma)^2}. \tag{3.3}$$

The remainder of the expression is controlled by

<u>Proposition IV.8</u>. For every $\beta>0$, and $x=(n,0)\in\mathbf{Z}^2\cap\Lambda$ there exists $J_2(\beta)>0$ and $\varepsilon_0(\beta)>0$ such that for $J>J_2(\beta)$ and $0<\varepsilon<\varepsilon_0(\beta)$, there exists a constant $c''(\beta,J)$ such that

$$\sup_{\{\theta\}}\left|\exp\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{(Y_1,\dots,Y_m)}{}'\left(\sum_{g\in G_c}\left[\prod_{\ell\in g}A(\ell)\right]\right)\left[\prod_{j=1}^{m}\hat{Q}(Y_j,\theta+ia,\varepsilon)-\prod_{j=1}^{m}\hat{Q}(Y_j,\theta,\varepsilon=0)\right]\right|$$

$$\leqslant \exp c''(\beta,J)\sum_{\langle i^*,j^*\rangle}(a_{i^*}-a_{j^*}+\varepsilon\delta_{\langle i^*,j^*\rangle},\gamma)^2. \tag{3.4}$$

Furthermore, $c''(\beta,J)$ can be made arbitrarily small by choosing J sufficiently large.

Equation (III.1.28) implies the existence of a constant K with

$$\sum_{\langle i^*,j^*\rangle} (a_{i^*}-a_{j^*}+\varepsilon\delta_{\langle i^*,j^*\rangle,\gamma})^2 \leqslant K\,\ell n(1+|x|) \ .$$

Thus, if we take $c_u(\beta,J) = K(c'(\beta,J)+c''(\beta,J))$, we obtain, by combining (2.2), (2.3) and (2.4), the estimate

$$\langle e^{\varepsilon(h_o-h_x)}\rangle_\Lambda \leqslant \exp\left\{(c'(\beta,J)+c''(\beta,J)) \sum_{\langle i^*,j^*\rangle} (a_{i^*}-a_{j^*}+\varepsilon\delta_{\langle i^*,j^*\rangle,\gamma})^2\right\}$$

$$\leqslant \exp\{c_u(\beta,J)\,\ell n(1+|x|)\} \quad . \tag{3.5}$$

This completes the proof of Theorem I.3. The proof of Proposition IV.8 is delayed until Chapter VI.

CHAPTER V

CONVERGENCE ESTIMATES FOR THE LOWER BOUND

In this chapter we prove the estimates which insure convergence of the expansion of Chapter II. The proof is divided into four sections. In the first section Proposition II.21 is shown to follow from the convergence of a comparison series involving new cluster functions $\tilde{S}_n(Y)$. In the second section the convergence of this comparison series is proven. In the third section, ratios of the cluster functions $Q(Y,\phi,\varepsilon)$ and $\tilde{S}_n(Y)$ are estimated, and these estimates are used to prove Lemma II.17. In the final section differences of cluster functions like $Q(Y,\phi+\theta,\varepsilon) - Q(Y,\phi,0)$ are bounded.

V.1 The Proof of Proposition II.22.

Define a set of new cluster functions $\tilde{s}_n(Y)$, $n=1,\dots,64$ by

$$\tilde{s}_n(Y) = \Big(\prod_{\rho\in\Gamma(Y)} z_\rho^{n/64}\Big)\Big(\prod_{i=2}^{k} s_{\eta(i)}\cdots s_{i-2}\chi_{X_{i-1}}(\Gamma_{\eta(i)},\Gamma_i)\, e^{-\frac{nJ}{64}|X_{i-1}|}\Big) \tag{1.1}$$

Here

$$\chi_X(\Gamma_i,\Gamma_j) = \begin{cases} 1 & \text{if } X\cap \operatorname{supp} a_\rho \neq \emptyset \text{ for all } \rho\in\Gamma_j \text{ and} \\ & \quad X\cap \operatorname{supp} a(\Gamma_i)\neq\emptyset \\ 0 & \text{otherwise} \end{cases}, \tag{1.2}$$

and

$$z_\rho = \exp\Big(-\frac{D}{2}\,|\ln\beta|A(\rho)\Big), \tag{1.3}$$

where D is a (β and J independent) constant determined by the combinatorial estimates of [15] (see Appendix II), and $A(\rho)$ is the function defined in Theorem II.3.

These new cluster functions are related to the functions $Q(Y,\phi,\varepsilon)$ by the following lemmas.

Lemma V.1. There exist $0<\beta_1<\beta_2$ and $J_1(\beta_1)>0$ such that for $\beta\in(\beta_1,\beta_2)$ and $J>J_1(\beta_1)$ one has

$$\Big|\int\Big[\prod_{\ell=1}^{m}\tilde{Q}(Y_\ell,\phi+\sigma,\varepsilon)-\prod_{\ell=1}^{m}\tilde{Q}(Y_\ell,\phi,0)\Big]\,d\nu_\Lambda^N(\phi)\Big|$$

$$\leq \varepsilon^2\prod_{\ell=1}^{m}\tilde{s}_n(Y_\ell)\int d\nu_\Lambda^N(\phi), \tag{1.4}$$

for $n=1,\dots,39$.

Lemma V.2. Let $\{Y_\ell\}$ be a set of clusters which satisfy $\{0,x\}\cap \operatorname{supp} a(\Gamma(Y_\ell))=\emptyset$ for $\ell=1,\dots,m$. Then there exists $0<\beta_1<\beta_2$ and $J_1(\beta_1)>0$ such that for all $J>J_1(\beta_1)$ and $\beta\in(\beta_1,\beta_2)$ one has

$$\Big|\int\Big[\prod_{\ell=1}^{m}\tilde{Q}(Y_\ell,\phi+\sigma,\varepsilon)-\prod_{\ell=1}^{m}\tilde{Q}(Y_\ell,\phi,0)\Big]\,d\nu_\Lambda^N(\phi)\Big|$$

$$\leq\Big\{\sup_{\ell=1,\dots,m}\Big(\sup_{\substack{\rho\in\Gamma(Y_\ell)\\ \langle i,j\rangle\cap D(\rho)}}|\partial\sigma_{ij}|^2+\sup_{\substack{X\cap\operatorname{supp} a_\rho\\ \rho\in\Gamma(Y_\ell)\\ \langle i,j\rangle\subset X}}|\partial\sigma_{ij}|^2\,e^{-\frac{1}{4}J|X|}\Big)\Big\}$$

$$\times\prod_{\ell=1}^{m}\tilde{s}_n(Y_\ell)\int d\nu_\Lambda^N(\phi), \tag{1.5}$$

for $n=1,\dots,39$.

The functions $\tilde{s}_n(Y)$ are in turn controlled by:

Lemma V.3. There exists $0<\beta_1<\beta_2$ and $J_1(\beta_1)>0$ such that for $\beta\in(\beta_1,\beta_2)$ and $J>J_1(\beta_1)$ one has

$$\sum_{\substack{(Y_1,\dots,Y_m)\\ \rho_0\in\cup_i\Gamma(Y_i)}}\Big|\sum_{g\in G_c}\Big(\prod_{\ell\in g}A(\lambda)\Big)\prod_{\ell=1}^{m}\tilde{s}_n(Y_\ell)\Big|\leq m!\,e^{-m}(z_{\rho_0})^{\left(\frac{n-7}{64}\right)}, \tag{1.6}$$

for $m=10,\dots,64$.

We defer the proofs of these theorems until later in the chapter, and conclude this section by showing how these lemmas lead to a proof of Proposition II.22. Note first that

$$\int \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{(Y_1,\ldots,Y_m)} \left(\sum_{g\in G_c} \prod_{\ell\in g} A(\ell) \right) \left[\prod_{\ell=1}^{m} \tilde{Q}(Y_\ell,\phi+\sigma,\varepsilon) - \prod_{\ell=1}^{m} \tilde{Q}(Y_\ell,\phi,0) \right] d\nu_\Lambda^N(\phi) \Bigg|$$

$$\leqslant \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{(Y_1,\ldots,Y_m)} \left| \left(\sum_{g\in G_c} \prod_{\ell\in g} A(\ell) \right) \prod_{k=1}^{m} \tilde{S}_{21}(Y_k) \right|$$

$$\times \frac{\left| \int \left[\prod_{1}^{m} \tilde{Q}(Y_\ell,\phi+\sigma,\varepsilon) - \prod_{1}^{m} \tilde{Q}(Y_\ell,\phi,0) \right] d\nu_\Lambda^N(\phi) \right|}{\prod_{1}^{m} \tilde{S}_{21}(Y_k)} \tag{1.7}$$

$$\leqslant \Bigg\{ \sum_{\substack{\rho\in N \\ \operatorname{supp} a_\rho \cap \{0,x\}\neq\emptyset}} \left[\varepsilon^2 \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{\substack{(Y_1,\ldots,Y_m) \\ \rho\in \cup_1^m \Gamma(Y_k)}} \left| \left(\sum_{g\in G_c} \prod_{\ell} A(\ell) \right) \prod_{1}^{m} \tilde{S}_{21}(Y_k) \right| \right]$$

$$+ \sum_{\langle i,j\rangle\subset\Lambda} \sum_{\substack{\rho\in N: \\ \sup|\partial\sigma_{i'j'}|=|\partial\sigma_{ij}| \\ \langle i',j'\rangle\cap D(\phi)}} |\partial\sigma_{ij}|^2 \left[\sum_{m=1}^{\infty} \frac{1}{m!} \sum_{\substack{(Y_1,\ldots,Y_m) \\ \rho\in \cup_i^m \Gamma(Y_k)}} \right.$$

$$\left. \times \left| \left(\sum_{g\in G_c} \prod_{\ell} A(\ell) \right) \left(\prod_{k=1}^{m} \tilde{S}_{21}(Y_k) \right) \right| \right]$$

$$+ \sum_{\langle i,j\rangle\subset\Lambda} \sum_{\rho\in N} \sum_{\substack{X: X\supset\langle i,j\rangle \\ X\cap \operatorname{supp} a_\rho}} |\partial\sigma_{ij}|^2 e^{-\frac{1}{4}J|X|}$$

$$\times \left[\sum_{m=1}^{\infty} \frac{1}{m!} \sum_{\substack{(Y_1,\ldots,Y_m) \\ \rho\in \cup_1^m \Gamma(Y_k)}} \left| \left(\sum_{g\in G_c} \prod_{\ell} A(\ell) \right) \prod_{k=1}^{m} \tilde{S}_{21}(Y_k) \right| \right] \Bigg\} \, d\nu_\Lambda^N(\phi) \tag{1.8}$$

The second inequality results from applying either Lemma V.1 or Lemma V.2 to each term in the sum over $(Y_1,\ldots,Y_m)$ in the first inequality depending on whether or not $\operatorname{supp} a_\rho \cap \{0,x\} \neq \emptyset$ for some $\rho \in \cup_1^m \Gamma(Y_k)$.

The first term in brackets in (1.8) is controlled by noting that Theorem II.3, (II.2.3) and (II.2.6) imply that there are at most two charge densities, ρ, satisfying $2^n < d_\Lambda(\rho) \leqslant 2^{n+1}$ and $\operatorname{supp} a_\rho \cap \{0,x\} \neq \emptyset$. Applying Lemma V.3 we bound this term by

$$\varepsilon^2 \sum_{\substack{\rho\in N: \\ \operatorname{supp} a_\rho\cap\{0,x\}\neq\emptyset}} \sum_{m=1}^{\infty} \frac{1}{m!} \cdot m! \cdot e^{-m} z_\rho^{7/32}$$

$$\leqslant \varepsilon^2 \sum_{n=1}^{\infty} \sum_{\substack{\rho\in N: \\ 2^N<d_\Lambda(\rho)\leqslant 2^{n+1} \\ \operatorname{supp} a_\rho\cap\{0,x\}}} z_\rho^{7/32}$$

$$\leqslant 2\varepsilon^2 \sum_{n=1}^{\infty} e^{-c|\ln \beta|\cdot n}$$

$$\leqslant 2\varepsilon^2 \quad , \tag{1.9}$$

for β sufficiently small. In the next to last inequality we have used the fact that $A(\rho) \geqslant c' \ln_2 d_\Lambda(\rho)$ (see Appendix II) to bound $z_\rho^{7/32}$.

The second term in (1.8) is bounded in an analogous fashion. Applying Lemma V.3 one bounds that term by

$$\sum_{\langle i,j\rangle\subset\Lambda} \sum_{\substack{\rho\in N \\ \sup|\partial\sigma_{i'j'}|=|\partial\sigma_{ij}| \\ \langle i',j'\rangle\cap D(\rho)}} |\partial\sigma_{ij}|^2 z_\rho^{7/32} \cdot \sum_{m=1}^{\infty} \frac{1}{m!} \cdot m! \cdot e^{-m} . \tag{1.10}$$

Once again, there is at most ore charge density $\rho \in N$ with $2^n < d_\Lambda(\rho) \leq 2^{n+1}$, and with $\langle i,j\rangle \cap (D(\rho) \cap \Lambda)$. Combining this with the fact that $z_\rho^{1/32} \leq \exp(-D/64\ |\ell n\ \beta|)$ for any $\rho \in N$ one bounds (1.10) by

$$e^{-\frac{D}{64}|\ell n\ \beta|} \sum_{\langle i,j\rangle \subset \Lambda} \sum_{\substack{\rho \in N: \\ \sup|\partial\sigma_{i'j'}|=|\partial\sigma_{ij}| \\ \langle i',j'\rangle \cap D(\rho)}} z_\rho^{6/32} |\partial\sigma_{ij}|^2$$

$$\leq e^{-\frac{D}{64}|\ell n\ \beta|} \sum_{\langle i,j\rangle} \sum_{n=1}^{\infty} e^{-c''\cdot|\ell n\ \beta|\cdot n} \cdot |\partial\sigma_{ij}|^2$$

$$\leq 2e^{-\frac{D}{64}|\ell n\ \beta|} \|\partial\sigma\|_2^2 \qquad , \tag{1.11}$$

for β sufficiently small.

Finally we turn to the last of the three terms of (1.8). Applying Lemma V.3 to control the sum over clusters this is bounded by

$$\sum_{\langle i,j\rangle} (\partial\sigma_{ij})^2 \sum_{\rho \in N} \sum_{\substack{X \supset\!\!\!< \langle i,j\rangle \\ X \cap \mathrm{supp}\ a_\rho}} e^{-\frac{1}{4}J|X|} z_\rho^{7/32}$$

$$= \sum_{\langle i,j\rangle} (\partial\sigma_{ij})^2 \sum_{\rho \in N} (1+\mathrm{dist}(i,\rho))^{-|\ell n\ \beta|^{1/2}}$$

$$\times \sum_{\substack{X \supset\!\!\!< \langle i,j\rangle \\ X \cap \mathrm{supp}\ a_\rho}} (1+\mathrm{dist}(i,\rho))^{|\ell n\ \beta|^{1/2}} e^{-\frac{1}{2}J|X|} z_\rho^{7/32} \quad . \tag{1.12}$$

From the definition of a_ρ and the fact that $\langle i,j\rangle \subset X$ and $X \cap \mathrm{supp}\ a_\rho$, one has $\mathrm{dist}(i,\rho) \leq |X| + d(\mathrm{supp}\ a_\rho) \leq |X| + 3d_\Lambda(\rho)$. For J sufficiently large, and β sufficiently small,

$$(1+\mathrm{dist}(i,\rho))^{|\ell n\ \beta|^{1/2}} e^{-\frac{1}{32}J|X|} z_\rho^{7/32}$$

$$\leq (1+|X|+3d_\Lambda(\rho))^{|\ell n\ \beta|^{1/2}} e^{-\frac{J}{32}|X|} e^{-\frac{7D}{32}|\ell n\beta|\ell n_2 d_\Lambda(\rho)}$$

$$\leq 1. \tag{1.13}$$

One may bound the number of charge densities a distance n from the point i by $O(1)\cdot n$. This follows by noting that there are $O(1)\cdot n$ points in $\mathbb{Z}^2$ a distance n from i, and any charge density a distance n from i must contain one of these points. Thus (1.12) is bounded by

$$\sum_{\langle i,j\rangle} (\partial\sigma_{ij})^2 \sum_{\rho \in N} (1+\mathrm{dist}(i,\rho))^{-|\ell n\ \beta|^{1/2}} \sum_{X \supset\!\!\!< \langle i,j\rangle} e^{-\frac{7}{32}J|X|}$$

$$\leq e^{-\frac{3}{16}J} \sum_{\langle i,j\rangle} (\partial\sigma_{ij})^2 \sum_{n=1}^{\infty} \sum_{\substack{\rho \in N: \\ \mathrm{dist}(i,\rho)=n}} (1+n)^{-|\ell n\ \beta|^{1/2}}$$

$$\leq e^{-\frac{3}{16}J} \sum_{\langle i,j\rangle} (\partial\sigma_{ij})^2 \sum_{n=1}^{\infty} O(1)\cdot n(1+n)^{-|\ell n\ \beta|^{1/2}}$$

$$\leq c'\ e^{-\frac{3}{16}J} \|\partial\sigma\|_2^2 \leq e^{-\frac{1}{8}J} \|\partial\sigma\|_2^2 \qquad , \tag{1.14}$$

for J sufficiently large. In the first of these inequalities we have used the Peierls argument to bound the sum over X with $\langle i,j\rangle \subset X$.

Combining (1.8), (1.9), (1.11), and (1.14) we obtain the bound of Proposition II.22.

V.2 The Convergence of the Comparison Series for the Lower Bound

In this section we prove a series of estimates which yield Lemma V.3. To control the sums over $\tilde{S}_n(Y)$ we take advantage both of the convergence coming from the number and size of the charge densities $\rho \in \Gamma(Y)$, and the exponential decay of the factors $\exp((-nJ/64)|X_i|)$.

The first step is to control sums over a single cluster function.

Lemma V.4. There exist $0 < \beta_1 < \beta_2$ and $J_1(\beta) > 0$ such that for all $J > J_1(\beta_1)$ and $\beta \in (\beta_1, \beta_2)$

$$\sum_{Y:\rho_0 \in \Gamma(Y)} \tilde{S}_n(Y) \leqslant (z_{\rho_0})^{\frac{n-6}{64}} \tag{2.1}$$

provided $n = 6,\ldots,64$.

Proof.

$$\begin{aligned}\sum_{Y:\rho_0 \in \Gamma(Y)} \tilde{S}_n(Y) &= \sum_{\rho \in N} \sum_{\substack{Y:\rho = \Gamma_1(Y)\\ \rho_0 \in \Gamma(Y)}} \tilde{S}_n(Y) \\ &= \sum_{\rho \in N} z_{\rho_0}^{(\frac{n-5}{64})} (1+\mathrm{dist}(\rho,\rho_0))^{-|\ln \beta|^{1/2}} \\ &\quad \times \sum_{\substack{Y:\rho = \Gamma_1(Y)\\ \rho_0 \in \Gamma(Y)}} (1+\mathrm{dist}(\rho,\rho_0))^{|\ln \beta|^{1/2}} z_{\rho_0}^{-\frac{(n-5)}{64}} \tilde{S}_n(Y) .\end{aligned} \tag{2.2}$$

This is bounded in two steps. We first claim that for any cluster Y, with $\tilde{\rho} = \Gamma_1(Y)$, $\rho_0 \in \Gamma_i$, $i \in \{1,\ldots,k\}$, and for β, J as specified by the lemma.

$$\begin{aligned}(1+\mathrm{dist}(\tilde{\rho},\rho_0))^{|\ln \beta|^{1/2}} &\leqslant \left[\prod_{j=1}^{i-1} \Big(\prod_{\rho \in \Gamma_j} z_\rho^{-3/64}\Big) e^{\frac{1}{32} J|X_{j-1}|}\right] \\ &\quad \Big(\prod_{\rho \in \Gamma_i} z_\rho^{-1/64}\Big) e^{\frac{1}{32} J|X_{i-1}|} .\end{aligned} \tag{2.3}$$

For $i = 1$ the statement is obvious since the l.h.s. is equal to 1, and $z_\rho < 1$ for all ρ provided β is sufficiently small. Assume it is true for $i \leqslant k$. Let $\rho_0 \in \Gamma_{k+1}$. Pick some $\rho_k \in \Gamma_k$ such that $X_k \cap \mathrm{supp}\, a_{\rho_k} \neq \emptyset$. By the triangle inequality one has

$$\begin{aligned}(1+\mathrm{dist}(\tilde{\rho},\rho_0))^{|\ln \beta|^{1/2}} &\leqslant (1+\mathrm{dist}(\tilde{\rho},\rho_k)+\mathrm{dist}(\rho_k,\rho_0)+d(\rho_k))^{|\ln \beta|^{1/2}} \\ &\leqslant (1+\mathrm{dist}(\tilde{\rho},\rho_k))^{|\ln \beta|^{1/2}} (1+\mathrm{dist}(\rho_k,\rho_0))^{|\ln \beta|^{1/2}} (d(\rho_k)-1)^{|\ln \beta|^{1/2}} .\end{aligned} \tag{2.4}$$

In the second inequality we have used the fact that if $k \geqslant 2$ $(1+\mathrm{dist}(\tilde{\rho},\rho_k))$, $(1+\mathrm{dist}(\rho_k,\rho_0))$, and $(d(\rho_k)-1)$ are all larger than 2 (if the constant M in Theorem II.3 is chosen larger than 4), and the easy estimate $(A+B)^k \leqslant A^k B^k$ for A,B greater than two. By the induction hypothesis,

$$\begin{aligned}(1+\mathrm{dist}(\tilde{\rho},\rho_k))^{|\ln \beta|^{1/2}} &\leqslant \left[\prod_{j=1}^{k-1} \Big(\prod_{\rho \in \Gamma_j} z_\rho^{-3/64}\Big) e^{\frac{1}{32} J|X_{j-1}|}\right] \\ &\quad \times \prod_{\rho \in \Gamma_k} (z_\rho^{-1/64}) e^{\frac{1}{32} J|X_{k-1}|} .\end{aligned} \tag{2.5}$$

We now demonstrate that

$$(1+\mathrm{dist}(\tilde{\rho}_k,\rho_0))^{|\ln\beta|^{1/2}} \leqslant \Big(\prod_{\rho\in\Gamma_{k+1}\cup\Gamma_k} z_\rho^{-\frac{1}{64}}\Big)\, e^{\frac{1}{32}J|X_k|} \quad . \tag{2.6}$$

We know that $X_k \cap \mathrm{supp}\, a_{\rho_k} \neq \emptyset$ and $X_k \cap \mathrm{supp}\, a_{\rho_0} \neq \emptyset$ since by definition X_k intersects all $\rho \in \Gamma_{k+1}$. By the definition of our translating functions a_ρ, no point in $\mathrm{supp}\, a_\rho$ is farther than $d_\Lambda(\rho)$ from $\mathrm{supp}\, \rho$. Thus, $\mathrm{dist}(\rho_k,\rho_0) \leqslant d_\Lambda(\rho_k) + d_\Lambda(\rho_0) + |X_k|$. Thus

$$\begin{aligned}(1+\mathrm{dist}(\rho_k,\rho_0))^{|\ln\beta|^{1/2}} &\leqslant (1+d_\Lambda(\rho_k)+d_\Lambda(\rho_0)+|X_k|)^{|\ln\beta|^{1/2}} \\ &\leqslant (d_\Lambda(\rho_k))^{|\ln\beta|^{1/2}}\cdot(d_\Lambda(\rho_0))^{|\ln\beta|^{1/2}}\cdot(1+|X_k|)^{|\ln\beta|^{1/2}} \\ &\leqslant z_{\rho_k}^{-1/64}\cdot z_{\rho_0}^{-1/64}\cdot e^{\frac{1}{32}J|X_k|} \quad , \end{aligned} \tag{2.7}$$

for β sufficiently small and J sufficiently large,

$$\leqslant \Big(\prod_{\rho\in\Gamma_{k+1}\cup\Gamma_k} (z_\rho^{-1/64})\Big)\, e^{\frac{1}{32}J|X_k|} \quad .$$

Combining (2.7), (2.5), and the observation that $(d(\rho_k)-1)^{|\ln\beta|^{1/2}} \leqslant z_{\rho_k}^{-1/64}$ for β sufficiently large completes the induction and proves (2.3).

Combining the definition of $\tilde{S}_n(Y)$ with (2.3) gives

$$(1+\mathrm{dist}(\rho,\rho_0))^{|\ln\beta|^{1/2}} (z_{\rho_0})^{-\frac{(n-5)}{64}}\, \tilde{S}_n(Y) \leqslant \tilde{S}_2(Y) \quad , \tag{2.8}$$

for $n = 6,\ldots,64$, and hence

$$\sum_{\substack{Y:\rho=\Gamma_1(Y)\\ \rho_0\in\Gamma(Y)}} (1+\mathrm{dist}(\rho,\rho_0))^{|\ln\beta|^{1/2}} (z_{\rho_0})^{-\frac{(n-5)}{64}}\, \tilde{S}_n(Y) \leqslant \sum_{Y:\rho=\Gamma_1(Y)} \tilde{S}_2(Y)$$

$$= \sum_{k=1}^{\infty} \sum_{X_1} \cdots \sum_{X_{k-1}} \sum_{\eta} \int_0^1 ds_1\ldots ds_{k-1}$$

$$\Big(\prod_{\rho\in\Gamma(Y)} z_\rho^{1/32}\Big) \prod_{i=2}^{k} [s_{\eta(i)}\cdots s_{i-2}\chi_{X_{i-1}}(\Gamma_{\eta(i)},\Gamma_i)\, e^{-\frac{J}{32}|X_{i-1}|}]. \tag{2.9}$$

In this last expression $\Gamma_1(Y) = \rho$ is fixed. We now demonstrate that

$$\Big(\prod_{\rho\in\Gamma_k} z_\rho^{1/64}\Big)\Big(\sum_{X_{k-1}} \chi_{X_{k-1}}(\Gamma_{\eta(k)},\Gamma_k)\, e^{-\frac{J}{32}|X_{k-1}|}\Big) \leqslant 1 \quad . \tag{2.10}$$

Note that if X_{k-1} makes a non-vanishing contribution to the sum, it must intersect $\mathrm{supp}\, a_\rho$ for all $\rho \in \Gamma_k$. In particular, it must intersect $\mathrm{supp}\, a_{\hat{\rho}}$ for $\hat{\rho}$ that charge density in Γ_k which is minimal with respect to the ordering defined in Chapter II. Thus we find

$$\begin{aligned}\sum_{X_{k-1}} \chi_{X_{k-1}}(\Gamma_{\eta(k)},\Gamma_k)\exp(-1/32\, J|X_{k-1}|) &\leqslant \sum_{i\in\mathrm{supp}\, a_{\hat{\rho}}} \Big(\sum_{X:i\in X} \exp(-1/32\, J|X|)\Big) \\ &\leqslant 8\pi[d_\Lambda(\hat{\rho})]^2 \exp(-1/64\, J) \quad . \end{aligned} \tag{2.11}$$

In the last inequality we have used the Peierls argument to bound the sum over X with $i \in X$, and the fact that the diameter of $\mathrm{supp}\, a_{\hat{\rho}}$ is less than or equal to $3d_\Lambda(\hat{\rho})$ and hence $|\mathrm{supp}\, a_{\hat{\rho}}| \leqslant 8\pi[d_\Lambda(\hat{\rho})]^2$. For β sufficiently large one has

$$8\pi[d_\Lambda(\hat{\rho})]^2 \cdot z_{\hat{\rho}}^{1/64} < 1 \quad . \tag{2.12}$$

Combining (2.12) with (2.11) yields (2.10).

Since for any charge density ρ, $A(\rho) \geq 1$ there exists some constant c' such that $z_\rho^{1/64} < \exp(-c'|\ln \beta|^{1/2})$ for all ρ. Since $\Gamma_i \neq \emptyset$ for $i = 1,\ldots,k$,

$$\prod_{\rho\in\Gamma(Y)} z_\rho^{1/64} < e^{-c'k|\ln \beta|^{1/2}} \quad . \tag{2.13}$$

Applying (2.13) to the r.h.s. of (2.9) gives a bound of

$$\sum_{k=1}^{\infty} e^{-c'k|\ln \beta|^{1/2}} \sum_{X_1} \cdots \sum_{X_{k-1}} \sum_{\eta} \int_0^1 ds_1 \ldots ds_{k-1}$$

$$\left(\prod_{\rho\in\Gamma(Y)} z_\rho^{1/64}\right) \prod_{i=2}^{k} [s_{\eta(i)} \cdots s_{i-2} \chi_{X_{i-1}}(\Gamma_{\eta(i)}, \Gamma_i) e^{-\frac{J}{32}|X_{i-1}|}]. \tag{2.14}$$

Now use (2.10) to estimate each of the sums over X_j, starting with $j = k-1$ and working backwards. This yields

$$\sum_{\substack{Y:\rho=\Gamma_1(Y)\\ \rho_0\in\Gamma(Y)}} (1+\mathrm{dist}(\rho,\rho_0))^{|\ln \beta|^{1/2}} \left(z_{\rho_0}^{-(\frac{n-5}{64})}\right) \tilde{s}_n(Y)$$

$$\leq \sum_{k=1}^{\infty} e^{-c'k|\ln \beta|^{1/2}} \sum_{\eta} \int_0^1 ds_1 \ldots ds_{k-1} \prod_{i=2}^{k} s_{\eta(i)} \cdots s_{i-2}$$

$$\leq \sum_{k=1}^{\infty} e^{-c'k|\ln \beta|^{1/2}} \cdot e^k \leq 1 \quad , \tag{2.15}$$

for β sufficiently small.

In the next to last inequality we have used the standard estimate (see [9])

$$\sum_{\eta} \int_0^1 ds_1,\ldots,ds_{k-1} \prod_{i=2}^{k} s_{\eta(i)} \cdots s_{i-2} \leq e^k \tag{2.16}$$

When (2.15) is combined with (2.2) it yields

$$\sum_{Y:\rho_0\in\Gamma(Y)} \tilde{s}_n(Y) \leq \sum_{\rho\in N} z_{\rho_0}^{(\frac{n-5}{64})} (1+\mathrm{dist}(\rho,\rho_0))^{-|\ln \beta|^{1/2}}$$

$$= \sum_{m=1}^{\infty} z_{\rho_0}^{(\frac{n-5}{64})} \sum_{\substack{\rho\in N:\\ \mathrm{dist}(\rho,\rho_0)=m}} (1+m)^{-|\ln \beta|^{1/2}} \quad . \tag{2.17}$$

There are at most $O(1)\cdot(d(\rho_0)+m)^2 d^2(\rho_0)$ points x in $\mathbb{Z}^2$ satisfying $\mathrm{dist}(x,\rho_0) = m$. To see this suppose that $\mathrm{supp}\,\rho_0$ has k connected components. Define ρ^α, $\alpha = 1,\ldots,k$, to be the restriction of ρ_0 to the component α. For a connected region one has no more than $O(1)(d(\rho^\alpha)+m)^2$ points in $\mathbb{Z}^2$ a distance m from $\mathrm{supp}\,\rho^\alpha$. Therefore the number of points in $\mathbb{Z}^2$ a distance m from $\mathrm{supp}\,\rho_0$ is bounded by

$$\sum_{\alpha=1}^{k} O(1)(d(\rho^\alpha) + m)^2 \quad . \tag{2.18}$$

If we draw a circle of radius $2d(\rho_0)$ about any point in $\mathrm{supp}\,\rho$, all ρ^α's must be contained in this circle. Thus $k \leq O(1)d^2(\rho_0)$. Clearly, $d(\rho_0) \leq d(\rho^\alpha)$ for all α. These two facts allow us to bound (2.18) by $O(1)d^2(\rho_0)(d(\rho_0)+m)^2$ as claimed. Since any charge density a distance m from ρ_0 contains a point a distance m from ρ_0. This estimate extends to bound the number of charge densities $\rho \in N$ satisfying $\mathrm{dist}(\rho,\rho_0) = m$.

Combining this with (2.17) gives

$$\sum_{Y:\rho_0\in\Gamma(Y)} \tilde{S}_n(Y) \leqslant \sum_{m=1}^{\infty} z_{\rho_0}^{(\frac{n-5}{64})}\, O(1)\cdot d^2(\rho_0)\,(d(\rho_0)+m)^2\,(1+m)^{-|\ln\beta|^{1/2}}$$

$$\leqslant z_{\rho_0}^{(\frac{n-5}{64})}\cdot O(1)\cdot d^4(\rho_0)$$

$$\leqslant z_{\rho_0}^{(\frac{n-6}{64})} \qquad , \tag{2.19}$$

provided β is sufficiently small. This completes the proof of Lemma V.4.

We must now control sums over more than one cluster. Following [4,7,16,25] we define

$$\Phi^n(Y_1,\ldots,Y_r;Z_1,\ldots,Z_k) = \sum_{g\in G_c(Y_1,\ldots,Y_r;Z_1,\ldots,Z_k)} \Big(\prod_{\ell\in g} A(\ell)\Big) \prod_{s=1}^{k} \tilde{S}_n(Z_s) \ .$$

Here $G_c(Y_1,\ldots,Y_r;Z_1,\ldots,Z_k)$ = the set of graphs connecting each Z_j cluster (directly or indirectly) to a Y_m. Sums of such functions are controlled by

<u>Lemma V.5</u>. There exist $0<\beta_1<\beta_2$ and $J_1(\beta_1)>0$ such that for $\beta\in(\beta_1,\beta_2)$ and $J>J_1(\beta_1)$ one has

$$\sum_{\substack{(Z_1,\ldots,Z_k):\\ \Sigma_{j=1}^k|\Gamma(Z_j)|=M}} |\Phi^n(Y_1,\ldots,Y_r;Z_1,\ldots,Z_k)| \leqslant k!\ \hat{z}^{\frac{(n-8)}{64}M} \exp\Big\{\sum_{i=1}^{r}|\Gamma(Y_i)|\Big\} \ , \tag{2.20}$$

for $n=8,\ldots,64$. Here, $\hat{z}=\max_{\rho\in N} z_\rho$, and satisfies the bound $0<\hat{z}<\exp(-c|\ln\beta|)$ for some c.

<u>Proof</u>. The proof is by induction on $r+k$. Begin by noting that $\Phi^n(Y_1,\ldots,Y_r;\phi)=1$ and $\Phi^n(\phi,Z_1,\ldots,Z_k)=0$. If we now define, for $g\in G_c$, $\Omega=\{s:\ell(Y_1,Z_s)\in g\}$ we find

$$\Phi^n(Y_1,\ldots,Y_r;Z_1,\ldots,Z_r) = \sum_{\Omega}\Big[\prod_{s\in\Omega}\tilde{S}_n(Z_s)\Big]\Big[\prod_{s\in\Omega}A(Y_1,Z_s)\Big]\Big[\prod_{\substack{j=2\\ s\in\Omega}}^{r}U(Y_j,Z_s)\Big]$$
$$\times\Big[\prod_{\substack{s_1<s_2\\ s_i\in\Omega}}U(Y_{s_1},Y_{s_2})\Big]\Phi^n(Y_2,\ldots,Y_r,(Z_s)_{s\in\Omega};(Z_s)_{s\notin\Omega}) \ . \tag{2.21}$$

For details of the derivation of this equation see the references above. Inserting (2.21) into the l.h.s. of (2.20) we estimate the sum which results by first fixing the cardinality of Ω, and the total number, $N\leqslant M$, of charge densities in $\cup_{s\in\Omega}\Gamma(Z_s)$. Using $|U(Y_j,Z_s)|\leqslant 1$ and isolating the terms with $N=0$ and $N=M$ one obtains

$$\sum_{\substack{(Z_1,\ldots,Z_s):\\ \Sigma_{s=1}^k|\Gamma(Z_s)|=N}} |\Phi^n(Y_1,\ldots,Y_r;Z_1,\ldots,Z_k)| \leqslant \sum_{\substack{(Z_1,\ldots,Z_k)\\ \Sigma|\Gamma(Z_s)|=M}} |\Phi^n(Y_2,\ldots,Y_r;Z_1,\ldots,Z_k)|$$

$$+\ k!\sum_{|\Omega|=1}^{k-1}\frac{1}{|\Omega|!}\sum_{N=|\Omega|}^{M-1}\sum_{\substack{(Z_1',\ldots,Z_{|\Omega|}')\\ \Sigma|\Gamma(Z_s')|=N}}\Big[\prod_{s\in\Omega}|A(Y_1,Z_s')|\,|\tilde{S}_n(Z_s')|\Big]$$

$$\times\frac{1}{(k-|\Omega|)!}\sum_{\substack{(Z_1'',\ldots,Z_{k-|\Omega|}'')\\ \Sigma|\Gamma(Z_s'')|=M-N}}|\Phi^n(Y_2,\ldots,Y_r,Z_1',\ldots,Z_{|\Omega|}';Z_1'',\ldots,Z_{k-|\Omega|}'')|$$

$$+\sum_{\substack{(Z_1,\ldots,Z_k)\\ \Sigma|\Gamma(Z_s)|=M}}\prod_{1}^{k}[|A(Y_1,Z_s)|\,|\tilde{S}_n(Z_s)|] \ . \tag{2.22}$$

The first term in this expression is bounded by the induction hypothesis. The last term is controlled by noting that $|A(Y_1,Z_s)| = 0$ unless $\Gamma(Z_s) \cap \Gamma(Y_1) \neq \emptyset$, and then applying the bound of Lemma V.6 below. Finally, the second term is estimated by using the induction hypothesis and the bound of Lemma V.6. Combining these extimates gives

$$\sum_{\substack{(Z_1,\dots,Z_s):\\ \Sigma_{s=1}^{k}|\Gamma(Z_s)|=M}} |\phi^n(Y_1,\dots,Y_r;Z_1,\dots,Z_k)|$$

$$\leqslant k!\ \hat{z}^{(\frac{n-8}{64})M} \exp\Big\{ \sum_{i=2}^{r} |\Gamma(Y_i)| \Big\} \Big\{ 1 + \frac{|\Gamma(Y_1)|^k}{k!} + \sum_{|\Omega|=1}^{k-1} \frac{1}{|\Omega|!} \sum_{N=|\Omega|}^{M-1} \hat{z}^{\frac{N}{64}} \cdot e^N |\Gamma(Y_1)|^{|\Omega|} \Big\} . \tag{2.23}$$

If β is sufficiently small the quantity in curly brackets is bounded by $\exp(|\Gamma(Y_1)|)$ which completes the proof of Lemma V.5.

<u>Lemma V.6</u>. There exist $0 < \beta_1 < \beta_2$ and $J_1(\beta_1) > 0$ such that for all $\beta \in (\beta_1,\beta_2)$ and $J > J_1(\beta_1)$ one has

$$\sum_{\substack{(Z_1,\dots,Z_m)\\ \Gamma(Z_s)\cap\Gamma(Y_1)\neq\emptyset\\ s=1,\dots,m\\ \Sigma|\Gamma(Z_s)| = M}} \prod_{s=1}^{m} \tilde{s}_n(Z_s) \leqslant |\Gamma(Y_1)|^m\ \hat{z}^{\frac{M}{64}(n-7)} , \tag{2.24}$$

for $n = 7,\dots,64$.

<u>Proof</u>. There are at most 2^M ways of dividing the M charge densities in $\Sigma\ |\Gamma(Z_s)|$ into m piles with M_s charge densities in pile s, $s = 1,\dots,m$. Thus, (2.24) may be bounded by

$$2^M \sup_{\substack{\{M_1,\dots,M_m\}\\ \Sigma_s M_s = M}} \sum_{\substack{(Z_1,\dots,Z_m)\\ \Gamma(Z_s)\cap\Gamma(Y_1)\neq\emptyset\\ |\Gamma(Z_s)|=M_s\\ s=1,\dots,m}} \prod_{s=1}^{m} \tilde{s}_n(Z_s)$$

$$\leqslant 2^M \sup_{\substack{\{M_1\dots M_m\}\\ \Sigma M_s = M}} \Big[\prod_{s=1}^{m} \Big(\sum_{\rho\in\Gamma(Y_1)} \sum_{\substack{Z_s:\\ \rho\in\Gamma(Z_s)\\ |\Gamma(Z_s)|=M}} \tilde{s}_n(Z_s) \Big) \Big]$$

$$\leqslant 2^M \sup_{\substack{\{M_1,\dots,M_m\}\\ \Sigma M_s = M}} \Big[\prod_{s=1}^{m} \Big(\hat{z}^{(\frac{n-6}{64})M_s} \Big) \Big(\sum_{\rho\in\Gamma(Y_1)} \sum_{\substack{Z_s:\\ \rho\in\Gamma(Z_s)\\ |\Gamma(Z_s)|=M_s}} \tilde{s}_6(Z_s) \Big) \Big] . \tag{2.25}$$

In the last inequality we have just used the observation

$$\tilde{s}_n(Z_s) \leqslant \Big(\prod_{\rho\in\Gamma(Z_s)} z_\rho^{\frac{n-6}{64}} \Big) \tilde{s}_6(Z_s) . \tag{2.26}$$

Estimate the sums over Z_s in (2.25) by Lemma V.4, and then use the fact that there are $|\Gamma(Y_1)|$ terms in the sum over $\rho \in \Gamma(Y_1)$ to bound (2.25) by

$$2^M \sup_{\substack{\{M_1,\dots,M_m\}\\ \Sigma M_s = M}} \Big[\prod_{s=1}^{m} \Big(\hat{z}^{\frac{(n-6)}{64}M_s} \Big) |\Gamma(Y_1)| \Big] = 2^M\ \hat{z}^{\frac{(n-6)M}{64}} |\Gamma(Y_1)|^m$$

$$\leqslant \hat{z}^{\frac{(n-7)M}{64}} |\Gamma(Y_1)|^m . \tag{2.27}$$

This completes the proof of Lemma V.6.

We conclude this section by using Lemma V.5 to prove Lemma V.3. First note that

$$\sum_{\substack{(Y_1,\dots,Y_m)\\ \rho_0\in\cup_i\Gamma(Y_i)}} \left| \Big(\sum_{g\in G_c} \prod_{\ell} A(\ell)\Big) \prod_{k=1}^{m} \tilde{S}_n(Y_k)\right|$$

$$\leqslant m \sum_{Y_1:\rho_0\in Y_1} \tilde{S}_n(Y_1) \sum_{(Y_2,\dots,Y_m)} |\Phi^n(Y_1;Y_2,\dots,Y_m)| \ . \tag{2.28}$$

We have compensated for choosing $\rho_0\in Y_1$ by multiplying by m. The r.h.s. of (2.28) is rewritten as

$$m \sum_{Y_1:\rho_0\in Y_1} \tilde{S}_n(Y_1) \sum_{N=m}^{\infty} \sum_{\substack{(Y_2,\dots,Y_m)\\ \Sigma_{s=2}^{m}|\Gamma(Y_s)|=N}} |\Phi^n(Y_1;Y_2,\dots,Y_m)|$$

$$\leqslant m \sum_{Y_1:\rho_0\in Y_1} \tilde{S}_n(Y_1) \sum_{N=m}^{\infty} (m-1)!\ \hat{z}^{\frac{(n-8)}{64}N}\ e^{|\Gamma(Y_1)|}$$

$$\leqslant m! \sum_{Y_1:\rho_0\in Y_1} \hat{z}^{\frac{(n-9)}{64}m}\ e^{|\Gamma(Y_1)|}\ \tilde{S}_n(Y_1) \tag{2.29}$$

provided β is sufficiently small. Now use the fact that

$$\Big(\prod_{\rho\in\Gamma(Y_1)} z_\rho^{1/64}\Big) e^{|\Gamma(Y_1)|} \leqslant 1 \ , \tag{2.30}$$

provided β is sufficiently small to bound (2.29) by

$$m!\ \hat{z}^{\frac{(n-9)}{64}m} \sum_{Y_1:\rho_0\in Y_1} \tilde{S}_{n-1}(Y_1) \leqslant m!\ \hat{z}^{\frac{(n-9)}{64}m} (z_{\rho_0})^{\frac{(n-7)}{64}} \ . \tag{2.31}$$

In the last inequality we used Lemma V.4 to bound the sum over Y_1. If β is sufficiently small that $\hat{z}^{1/64}\leqslant e^{-1}$, (2.31) is bounded by

$$m!\ e^{-m}\cdot(z_{\rho_0})^{\frac{(n-7)}{64}} \ . \tag{2.32}$$

This completes the proof of Lemma V.3.

V.3 Ratios of Cluster Functions

In this section we prove estimates relating the cluster functions $\tilde{Q}(Y,\phi,\varepsilon)$ and $\tilde{S}_n(Y)$. These estimates lead to a proof of Lemma II.18. Estimates on the factors $K(\rho)$, $e^{-\gamma q_\rho}$, and $i(\phi+i\zeta_\rho a_\rho)$ follow from [15] and are deferred until Appendix II. We now estimate the contributions of the non-local potentials.

Lemma V.7. There exists $0<\beta_1<\beta_2$ and $J_1(\beta)>0$ such that for all $J>J_1(\beta)$ and $\beta\in(\beta_1,\beta_2)$ one has

$$\left|\exp\left\{\sum_{X\subset\Lambda}\Delta V_X^{s_1\cdots s_{k-1}}(\phi:i\zeta a(\Gamma_1)+\ldots+i\zeta a(\Gamma_{k-1});i\zeta a(\Gamma_k))\right\}\right|$$

$$\leqslant \exp\left\{c''\gamma^2 e^{-3/4\,J}\sum_{j=1}^{k}\sum_{\rho\in\Gamma_j}\sum_{m=0}^{\infty}|S_m'(\rho)|\right\} \tag{3.1}$$

for some constant c. Recall that $S_m'(\rho)$ is defined by (II.2.2).

Proof. The proof breaks into several parts. First we demonstrate that for any set of translating functions, $a(\Gamma_\ell),\ldots,a(\Gamma_k)$, one has

$$|\exp\Delta V_X(\phi:i\zeta a(\Gamma_\ell)+\ldots+i\zeta a(\Gamma_k))|$$

$$\leqslant \exp\left\{e^{-\frac{15}{16}J|X|}\left[\sup_{\substack{j=\ell,\ldots,k\\ \rho\in\Gamma_j}}\left(\sup_{\langle i',j'\rangle\subset X}(a_\rho(i')-a_\rho(j'))^2\right)\right]\right\} \tag{3.2}$$

These bounds are then extended to the interpolated potentials, and finally sums of the interpolated potentials are treated.

Step 1. Define $\tilde{a}=\Sigma_{j=\ell}^{k}\zeta a(\Gamma_j)$. By (II.1.2) one has

$$\Delta V_X(\phi:i\lambda\zeta a(\Gamma_\ell)+\ldots+i\lambda\zeta a(\Gamma_k))$$

$$=\sum_{\{\partial h\}|_X}V_X(\{\partial h\}|_X)\left\{\prod_{\langle i,j\rangle\subset X}f(\partial\phi_{ij}+\partial h_{ij}+i\lambda\partial\tilde{a}_{ij})-\prod_{\langle i,j\rangle\subset X}f(\partial\phi_{ij}+\partial h_{ij})\right\}. \tag{3.3}$$

Recall that $f(z)=(\sin^2\pi z)/\pi^2z^2$, $\partial\phi_{ij}=\phi_i-\phi_j$, the sum over $\{\partial h\}|_X$ is a sum over $\{h\}|_X$, with $h_i=\text{constant}$, for some $i\subset X$, and $\lambda\in[0,1]$. By Taylor's theorem with remainder one has

$$\Delta V_X(\phi:i\zeta a(\Gamma_\ell)+\ldots+i\zeta a(\Gamma_k))$$

$$=\sum_{\langle i,j\rangle\subset X}\left\{\sum_{\{\partial h\}|_X}V_X(\{\partial h\}|_X)\frac{df}{d\lambda}(\partial\phi_{ij}+\partial h_{ij}+i\lambda\partial\tilde{a}_{ij})\Big|_{\lambda=0}\times\left[\prod_{\substack{\langle i',j'\rangle\subset X\\ \langle i',j'\rangle\neq\langle i,j\rangle}}f(\partial\phi_{ij}+\partial h_{ij})\right]\right\}$$

$$+\frac{1}{2}\sum_{\{\partial h\}|_X}V_X(\{\partial h\}|_X)\frac{d^2}{d\lambda^2}\prod_{\langle i,j\rangle\subset X}f(\partial\phi_{ij}+\partial h_{ij}+i\lambda\partial\tilde{a}_{ij})\Big|_{\lambda=\lambda_0}, \tag{3.4}$$

for some $\lambda_0\in[0,1]$, and we have used the fact that $\Delta V_X(\phi:0)=0$. By explicit computation of the derivative of $f(\cdot)$ one finds that the first term is pure imaginary.

Now consider the terms in which two derivatives act.

$$\frac{d^2}{d\lambda^2}\left[\prod_{\langle i,j\rangle\subset X} f(\partial\phi_{ij}+\partial h_{ij}+i\lambda\partial\tilde{a}_{ij})\right]\Big|_{\lambda=\lambda_0}$$

$$= -\sum_{\langle i_1,j_1\rangle\subset X}(\partial\tilde{a}_{i_1j_1})^2 f''(\partial\phi_{i_1j_1}+\partial h_{i_1j_1}+i\lambda_0\partial\tilde{a}_{i_1j_1})$$

$$\times\left[\prod_{\substack{\langle i,j\rangle\subset X\\ \langle i,j\rangle\neq\langle i_1,j_1\rangle}} f(\partial\phi_{ij}+\partial h_{ij}+i\lambda_0\partial\tilde{a}_{ij})\right]$$

$$-\sum_{\langle i_1,j_1\rangle\subset X}\sum_{\langle i_2,j_2\rangle\subset X}(\partial\tilde{a}_{i_1j_1})(\partial\tilde{a}_{i_2j_2})f'(\partial\phi_{i_1j_1}+\partial h_{i_1j_1}+i\lambda_0\partial\tilde{a}_{i_1j_1})$$

$$\times f'(\partial\phi_{i_2j_2}+\partial h_{i_2j_2}+i\lambda_0\partial\tilde{a}_{i_2j_2})\left[\prod_{\substack{\langle i,j\rangle\subset X\\ \langle i,j\rangle\neq\langle i_1,j_1\rangle\\ \langle i,j\rangle\neq\langle i_2,j_2\rangle}} f(\partial\phi_{ij}+\partial h_{ij}+i\lambda_0\partial\tilde{a}_{ij})\right] \tag{3.5}$$

We saw in Chapter II that for any nearest neighbor pair $\langle i,j\rangle$, $(a_\rho(i)-a_\rho(j))$ was non-zero for at most one $\rho\in N$. Thus,

$$|\partial\tilde{a}_{ij}|\leqslant 2\gamma \quad , \tag{3.6}$$

for γ the constant in the definition of our translating functions. Making use of the explicit expression for the function $f(z)$ it is easy to check that there exists a constant $K>0$ such that

$$\begin{aligned}
|f(\partial\phi_{ij}+\partial h_{ij}+i\lambda_0\partial\tilde{a}_{ij})| &\leqslant Ke^{4\gamma\pi}C(\partial\phi_{ij}+\partial h_{ij}) \ ,\\
|f'(\partial\phi_{ij}+\partial h_{ij}+i\lambda_0\partial\tilde{a}_{ij})| &\leqslant Ke^{4\gamma\pi}C(\partial\phi_{ij}+\partial h_{ij}) \ ,\\
|f''(\partial\phi_{ij}+\partial h_{ij}+i\lambda_0\partial\tilde{a}_{ij})| &\leqslant Ke^{4\gamma\pi}C(\partial\phi_{ij}+\partial h_{ij}) \ ,
\end{aligned} \tag{3.7}$$

where

$$C(\phi) = \begin{cases} 1 & \text{if } |\phi|\leqslant 1\\ \phi^{-2} & \text{if } |\phi|>1\end{cases} \quad . \tag{3.8}$$

Combining (3.4), (3.5), and (3.7) we see that

$$\begin{aligned}
&|\exp\{\Delta V_X(\phi:i\zeta a(\Gamma_\ell)+\ldots+i\zeta a(\Gamma_k))\}|\\
&\leqslant \exp\left|\sum_{\{\partial h\}|_X} V_X(\{\partial h\}|_X)\frac{d^2}{d\lambda^2}\prod_{\langle i,j\rangle\subset X} f(\partial\phi_{ij}+\partial h_{ij}+i\lambda_0\partial\tilde{a}_{ij})\Big|_{\lambda=\lambda_0}\right|\\
&\leqslant \exp\left\{\sup_{\langle i,j\rangle\subset X}(\partial\tilde{a}_{ij})^2(|X|+|X|^2)e^{-J|X|}\sum_{\{\partial h\}|_X}\prod_{\langle i,j\rangle} Ke^{4\pi\gamma}C(\partial\phi_{ij}+\partial h_{ij})\right\}\\
&\leqslant \exp\left\{\sup_{\langle i,j\rangle\subset X}(\partial\tilde{a}_{ij})^2(|X|+|X|^2)(Kc'e^{4\pi\gamma})^{|X|}e^{-J|X|}\right\}\\
&\leqslant \exp\left\{\sup_{\langle i,j\rangle\subset X}(\partial\tilde{a}_{ij})^2\, e^{-\frac{7}{8}J|X|}\right\} \ ,
\end{aligned} \tag{3.9}$$

provided J is sufficiently large. In the first of these inequalities we have dropped the exponential of those terms in (3.4) in which only one derivative acted because it was of magnitude one. We then used (3.5) and (3.7) to control the terms with two derivatives. The number of terms in which both derivatives act on a single function, $f(\cdot)$, is bounded by $|X|$, while the number of terms in which the derivatives act on different functions is bounded by $|X|^2$. In the third inequality c' just bounds the sum over ∂h_{ij} of $C(\partial\phi_{ij}+\partial h_{ij})$ which is uniformly bounded independent of $\partial\phi_{ij}$.

Step 2. The bound of (3.9) is extended to give a bound on the interpolated potentials of the form

$$\left|\exp\{\Delta V_X^{s_1\cdots s_{k-1}}(\phi:i\zeta a(\Gamma_1)+\ldots+i\zeta a(\Gamma_{k-1});i\zeta a(\Gamma_k))\}\right|$$

$$\leqslant \exp\left\{e^{-\frac{7}{8}J|X|}\left[\sum_{\ell=1}^{k}\sup_{\langle i,j\rangle\subset X}(\partial a_{ij}(\Gamma_\ell))^2\right]\right\}. \tag{3.10}$$

Equation (3.9) insures that this bound holds for the case $k=1$ when we recall from Chapter II, that for any bond $\langle i,j\rangle$, $a_\rho(i)-a_\rho(j)$ is non-zero for at most one $\rho\in N$. Thus, $(\partial\tilde{a}_{ij})^2=\Sigma_{m=\ell}^{k}(\partial a_{ij}(\Gamma_m))^2$. Assume (3.10) holds for $k<n$. Then, combining the definition of the interpolated potentials with the induction hypothesis, yields

$$\left|\exp\{\Delta V_X^{s_1\cdots s_n}(\phi:i\zeta a(\Gamma_1)+\ldots+i\zeta a(\Gamma_n);i\zeta a(\Gamma_{n+1}))\}\right|$$

$$\leqslant \exp\left\{s_n\left[e^{-\frac{7}{8}J|X|}\sum_{\ell=1}^{n}\sup_{\langle i,j\rangle\subset X}(\partial a_{ij}(\hat{\Gamma}_\ell))^2\right]\right.$$

$$\left.+(1-s_n)\left[e^{-\frac{7}{8}J|X|}\sum_{\ell=1}^{n}\sup_{\langle i,j\rangle\subset X}(\partial a_{ij}(\Gamma_\ell))^2+e^{-\frac{7}{8}J|X|}\sup_{\langle i,j\rangle\subset X}(\partial a_{ij}(\Gamma_{n+1}))^2\right]\right\} \tag{3.11}$$

Here

$$\hat{\Gamma}_\ell=\begin{cases}\Gamma_\ell & \text{if } 1\leqslant\ell\leqslant n-1\\ \Gamma_n\cup\Gamma_{n+1} & \text{if } \ell=n\end{cases} \tag{3.12}$$

Our previous comment on the non-overlapping support of $\partial a_{ij}(\Gamma_\ell)$ insures that

$$(\partial a_{ij}(\hat{\Gamma}_n))^2\leqslant(\partial a_{ij}(\Gamma_n))^2+(\partial a_{ij}(\Gamma_{n+1}))^2. \tag{3.13}$$

Combining this observation with (3.11) yields (3.10).

Step 3. The proof of Lemma V.7. By (3.10)

$$\left|\exp\left\{\sum_{X\subset\Lambda}\Delta V_X^{s_1,\ldots,s_{k-1}}(\phi:i\zeta a(\Gamma_1)+\ldots+i\zeta a(\Gamma_{k-1});i\zeta a(\Gamma_k))\right\}\right|$$

$$\leqslant\exp\left\{\sum_{X\subset\Gamma}e^{-\frac{7}{8}J|X|}\left[\sum_{\ell=1}^{k}\sup_{\langle i,j\rangle\subset X}(\partial a_{ij}(\Gamma_\ell))^2\right]\right\}. \tag{3.14}$$

Rewrite

$$\sum_{X\subset\Lambda}\sup_{\langle i,j\rangle\subset X}(\partial a_{ij}(\Gamma_\ell))^2=\sum_{\langle i,j\rangle\subset\Lambda}\sum_{\substack{X:\\ \sup_{\langle i',j'\rangle\subset X}(\partial a_{i'j'}(\Gamma_\ell))^2=(\partial a_{ij}(\Gamma_\ell))^2}}(\partial a_{ij}(\Gamma_\ell))^2. \tag{3.15}$$

Applying the Peierls argument we have

$$\sum_{\substack{X:\\ \sup_{\langle i',j'\rangle\subset X}(\partial a_{i'j'}(\Gamma_\ell))^2=(\partial a_{ij}(\Gamma_\ell))^2}}e^{-\frac{7}{8}J|X|}(\partial a_{ij}(\Gamma_\ell))^2\leqslant e^{-3/4\,J}(\partial a_{ij}(\Gamma_\ell))^2. \tag{3.16}$$

If we again use the fact that for any given $\langle i,j\rangle$, $(a_\rho(i)-a_\rho(j))$ is non-zero for at most one ρ we may rewrite $(\partial a_{ij}(\Gamma_\ell))^2$ as $\Sigma_{\rho\in\Gamma_\ell}(a_\rho(i)-a_\rho(j))^2$. This, coupled with (3.15) and (3.16) allows us to bound (3.14) by

$$\exp\left\{ e^{-3/4\, J} \sum_{\ell=1}^{k} \sum_{\rho\in\Gamma_\ell} \sum_{\langle i,j\rangle\subset\Lambda} (a_\rho(i) - a_\rho(j))^2 \right\}$$

$$\leqslant \exp\left\{ e^{-3/4\, J}\, \gamma^2 \ell n_2 \sqrt{2}\, (c') \sum_{\ell=1}^{k} \sum_{\rho\in\Gamma_\ell} \sum_{m=1}^{\infty} |S'_m(\rho)| \right\} . \tag{3.17}$$

In this last inequality we have used Lemma A.II.2 of Appendix II to control the sum over $\langle i,j\rangle$. Expression (3.17) is precisely the bound of Lemma V.7.

<u>Corollary V.8</u>. There exist constants $0<\beta_1<\beta_2$ and $J_1(\beta_1)>0$ such that for all $\beta\in(\beta_1,\beta_2)$ and $J>J_1(\beta_1)$ one has

$$\left| \prod_{m=2}^{k} [s_{\eta(m)}\cdots s_{m-2} \Delta V_{X_{m-1}}(\phi : i\zeta a(\Gamma_{\eta(m)}) + \ldots + i\zeta a(\Gamma_m))] \right|$$

$$\leqslant \prod_{m=2}^{k} \left| s_{\eta(m)}\cdots s_{m-2} \chi_{X_{m-1}}(\Gamma_{\eta(m)},\Gamma_m)\, e^{-\frac{15}{16} J |X_{m-1}|} \right| . \tag{3.18}$$

<u>Proof</u>. By Lemma II.8 $\Delta V_{X_{m-1}}(\phi : i\zeta a(\Gamma_{\eta(m)}) + \ldots + i\zeta a(\Gamma_m))$ vanishes unless $X_{m-1} \cap \operatorname{supp} a(\Gamma_{\eta(m)}) \neq \emptyset$. By the definition of Γ_m, $X_{m-1} \cap \operatorname{supp} a_\rho \neq \emptyset$ for all $\rho\in\Gamma_m$. Hence

$$\Delta V_{X_{m-1}}(\phi : i\zeta a(\Gamma_{\eta(m)}) + \ldots + i\zeta a(\Gamma_m))$$

$$= \chi_{X_{m-1}}(\Gamma_{\eta(m)},\Gamma_m)\, \Delta V_{X_{m-1}}(\phi : i\zeta a(\Gamma_{\eta(m)}) + \ldots + i\zeta a(\Gamma_m)) . \tag{3.19}$$

From the definition of $\Delta V_X(\cdot:\cdot)$ and the bounds on the translated potentials in the proof of the previous lemma, one finds

$$|\Delta V_{X_{m-1}}(\phi : i\zeta a(\Gamma_{\eta(m)}) + \ldots + i\zeta a(\Gamma_m))|$$

$$\leqslant 8\, e^{-J|X_{m-1}|} (Kc' e^{4\pi\gamma})^{|X_{m-1}|} \tag{3.20}$$

Pick $J_1(\beta_1)$ sufficiently large that for all $J>J_1(\beta_1)$

$$e^{-\frac{1}{16} J |X_{m-1}|} (Kc' e^{4\pi\gamma})^{|X_{m-1}|} \leqslant 1.$$

Then (3.19) and (3.20) yield (3.18).

We now turn to the proof of Lemma II.17. The estimates of Appendix II yield

$$|K(\rho)\, i(\phi + i\zeta_\rho a_\rho) e^{-\gamma q_\rho}| \leqslant \exp\left\{ (c+\beta^{1/6}\gamma^2 c' - \gamma D) A(\rho) \right\} . \tag{3.21}$$

From the definition of $A(\rho)$, one has

$$\sum_{m=1}^{\infty} |S'_m(\rho)| \leqslant A(\rho) . \tag{3.22}$$

This observation, when combined with Lemma V.7, Corollary V.8 and (3.21) yields

$$|\tilde{Q}(Y,\phi,0)| \leqslant \sum_{\{\zeta\}_{\Gamma(Y)}} \left(\tfrac{1}{2}\right)^{|\Gamma(Y)|} \prod_{m=2}^{k} \left| s_{\eta(m)}\cdots s_{m-2} \chi_{X_{m-1}}(\Gamma_{\eta(m)},\Gamma_m) e^{-\frac{15}{16} J |X_{m-1}|} \right|$$

$$\times \exp\left\{ \sum_{\rho\in\Gamma(Y)} (c+\beta^{1/6}\gamma^2 c' + c''\gamma^2 e^{-3/4\, J} - \gamma D) A(\rho) \right\} . \tag{3.23}$$

The r.h.s. of (3.23) is now independent of $\{\zeta\}$, so we drop the sum over $\{\zeta\}_{\Gamma(Y)}$ and compensate by multiplying by $2^{|\Gamma(Y)|}$. Choose β

sufficiently small and J sufficiently large that

$$(c+\beta^{1/6}\gamma^2 c'+c''\gamma^2 e^{-3/4J}-\gamma D) < -\gamma\,\frac{3D}{4} \quad . \tag{3.24}$$

Recall that we chose $\gamma = c'''|\ln\beta|$, for some small constant c'''. These two comments imply that

$$|\tilde{Q}(Y,\phi,0)| \leqslant \tilde{S}_n(Y) \ , \tag{3.25}$$

for β sufficiently small and $n=1,\ldots,60$. In analagous fashion one has

$$|\tilde{Q}(Y,\phi+\sigma,\varepsilon)| \leqslant \tilde{S}_n(Y) \ , \tag{3.26}$$

for $n=1,\ldots,60$, independent of σ and ε. Now note that

$$\sum_{m=1}^{\infty}\frac{1}{m!}\left|\sum_{\substack{(Y_1\ldots Y_m)\\ \rho_0\subset\cup_1^m\Gamma(Y_k)}}\Big(\sum_{g\in G_c}\prod_{\ell}A(\ell)\Big)\prod_{k=1}^{m}\tilde{Q}(Y_k,\phi,0)\right|$$

$$\leqslant \sum_{m=1}^{\infty}\frac{1}{m!}\sum_{\substack{(Y_1,\ldots,Y_m)\\ \rho_0\in\cup_1^m\Gamma(Y_k)}}\left|\Big(\sum_{g\in G_c}\prod_{\ell\in g}A(\ell)\Big)\prod_{k=1}^{m}\tilde{S}_{60}(Y_m)\right|$$

$$\leqslant \sum_{m=1}^{\infty}\frac{1}{m!}\cdot m!\,e^{-m}z_{\rho_0}^{1/2} \leqslant z_{\rho_0}^{1/2} \leqslant 1 \quad .$$

In the first of these inequalities we used (3.24) while the second applied Lemma V.3. This completes the proof of Lemma II.18.

V.4. Differences of Cluster Functions

In this section we turn to the more difficult task of bounding differences of products of cluster functions. Define

$$F_\lambda(\phi) = \prod_{\ell=1}^{m}\tilde{Q}(Y_\ell,\phi+\lambda\sigma,\lambda\varepsilon) - \prod_{\ell=1}^{m}\tilde{Q}(Y_\ell,\phi,0) \quad . \tag{4.1}$$

We wish to study $F_{\lambda=1}(\phi)$ which we write, using Taylor's theorem with remainder, as

$$F_{\lambda=1}(\phi) = F_{\lambda=0}(\phi) + \frac{dF_\lambda}{d\lambda}(\phi)\Big|_{\lambda=0} + \tfrac{1}{2}\Big(\frac{d^2F_\lambda}{d\lambda^2}(\phi)\Big)\Big|_{\lambda=\lambda_0} \ ,$$

for some $\lambda_0 \in (0,1)$. We see from (4.1) that $F_{\lambda=0}(\phi) = 0$. The remainder of the section is devoted to obtaining bounds on the derivatives of $F_\lambda(\phi)$.

$$\frac{dF_\lambda}{d\lambda}(\phi)\Big|_{\lambda=0} = \sum_{\ell=1}^{m}\Big(\frac{d\tilde{Q}}{d\lambda}(Y_\ell,\phi+\lambda\sigma,\lambda\varepsilon)\Big)\Big|_{\lambda=0}\times\prod_{\substack{\ell'=1\\ \ell'\neq\ell}}^{m}\tilde{Q}(Y_{\ell'},\phi,0) \quad . \tag{4.2}$$

We now show that

$$\tilde{Q}(Y,\phi-\lambda\sigma,-\lambda\varepsilon) = \tilde{Q}(Y,-\phi+\lambda\sigma,\lambda\varepsilon) \quad . \tag{4.3}$$

By definition

$$\tilde{Q}(Y,\phi-\lambda\sigma,-\lambda\varepsilon) = \sum_{\{\zeta\}_{\Gamma(Y)}}(\tfrac{1}{2})^{|\Gamma(Y)|}\prod_{\rho\in\Gamma(Y)}R(\rho,\phi-\lambda\sigma,\zeta_\rho,-\lambda\varepsilon)$$

Eq. (4.4) to be continued

$$\times \prod_{m=2}^{k} [s_{\eta(m)} \cdots s_{m-2} \Delta V_{X_{m-1}} (\phi-\lambda\sigma; i\zeta a(\Gamma_{\eta(m)}) + \ldots + i\zeta a(\Gamma_m))]$$

$$\times \exp\Big\{ \sum_{X\subset\Lambda} \Delta V_X^{s_1 \cdots s_{k-1}} (\phi-\lambda\sigma : i\zeta a(\Gamma_1) + \ldots + i\zeta a(\Gamma_{k-1}); i\zeta a(\Gamma_k)) \Big\} \,. \tag{4.4}$$

Use the facts that

$$R(\rho,\phi-\lambda\sigma,\zeta_\rho,-\lambda\varepsilon) = R(\rho,-\phi+\lambda\sigma,-\zeta_\rho,\lambda\varepsilon) \quad ,$$

$$\Delta V_{X_{m-1}} (\phi-\lambda\sigma; i\zeta a(\Gamma_{\eta(m)}) + \ldots + i\zeta a(\Gamma_m))$$

$$= \Delta V_{X_{m-1}} (-\phi+\lambda\sigma; -i\zeta a(\Gamma_{\eta(m)}) - \ldots - i\zeta a(\Gamma_m)) \quad ,$$

and

$$\Delta V_X (\phi-\lambda\sigma : i\zeta a(\Gamma_1) + \ldots + i\zeta a(\Gamma_{k-1}); i\zeta a(\Gamma_k))$$

$$= \Delta V_X (-\phi+\lambda\sigma : -i\zeta a(\Gamma_1) - \ldots - i\zeta a(\Gamma_{k-1}); -i\zeta a(\Gamma_k)) \quad . \tag{4.5}$$

Inserting these expressions into (4.4), and then changing variables in the sum over $\{\zeta\}_{\Gamma(Y)}$ to $\zeta'_\rho = -\zeta_\rho$, (4.4) becomes $\tilde{Q}(Y,-\phi+\lambda\sigma,-\lambda\varepsilon)$ as claimed.

From (4.3) we see that $\tilde{Q}(Y,\phi,0) = \tilde{Q}(Y,-\phi,0)$ and that $\frac{d\tilde{Q}}{d\lambda}(Y,\phi+\lambda\sigma,\lambda\varepsilon)\Big|_{\lambda=0} = -\frac{d\tilde{Q}}{d\lambda}(Y,-\phi+\lambda\sigma,\lambda\varepsilon)\Big|_{\lambda=0}$. These observations, combined with (4.2), imply that

$$\frac{dF_\lambda}{d\lambda}(\phi)\Big|_{\lambda=0} = -\frac{dF_\lambda}{d\lambda}(-\phi)\Big|_{\lambda=0} \quad , \tag{4.6}$$

and hence

$$\int \frac{dF_\lambda}{d\lambda}(\phi)\Big|_{\lambda=0} d\nu_\Lambda^N(\phi) = 0 \quad . \tag{4.7}$$

Now turn to a consideration of

$$\frac{d^2F_\lambda}{d\lambda^2} = \sum_{\ell=1}^{m} \Big(\frac{d^2\tilde{Q}}{d\lambda^2}(Y_\ell,\phi+\lambda\sigma,\lambda\varepsilon)\Big) \prod_{\substack{\ell'=1\\ \ell\neq\ell'}}^{m} \tilde{Q}(Y_{\ell'},\phi+\lambda\sigma,\lambda\varepsilon)$$

$$+ 2 \sum_{1\le\ell_1<\ell_2\le m} \left[\frac{d\tilde{Q}(Y_{\ell_1},\phi+\lambda\sigma,\lambda\varepsilon)}{d\lambda}\right]\left[\frac{d\tilde{Q}(Y_{\ell_2},\phi+\lambda\sigma,\lambda\varepsilon)}{d\lambda}\right]$$

$$\times \prod_{\substack{\ell'=1\\ \ell'\neq\ell_1,\ell'\neq\ell_2}}^{m} \tilde{Q}(Y_{\ell'},\phi+\lambda\sigma,\lambda\varepsilon) \quad . \tag{4.8}$$

These sums are controlled by the following lemmas.

<u>Lemma V.9</u>. There exist $0 < \beta_1 < \beta_2$ and $J_1(\beta_1) > 0$ such that for all $\beta \in (\beta_1,\beta_2)$ and $J > J_1(\beta_1)$, one has

$$\left|\frac{d\tilde{Q}}{d\lambda}(Y,\phi+\lambda\sigma,\lambda\varepsilon\mu)\Big|_{\lambda_0}\right| \le \Big\{ \sup_{\substack{\rho\in\Gamma(Y)\\ \langle i,j\rangle\cap D(\rho)}} |\partial\sigma_{ij}| + \sup_{\substack{X\cap\mathrm{supp}\, a_\rho\\ \rho\in\Gamma(Y)\\ \langle i,j\rangle\subset X}} (|\partial\sigma_{ij}| e^{-\frac{1}{4}J|X|})$$

$$+ \varepsilon\mu \Big\} S_{40}(Y) \quad . \tag{4.9}$$

<u>Lemma V.10</u>. There exists $0 < \beta_1 < \beta_2$ and $J_1(\beta_1) > 0$ such that for all $\beta \in (\beta_1,\beta_2)$ and $J > J_1(\beta_1)$, one has

$$\left|\frac{d^2\tilde{Q}(Y,\phi+\lambda\sigma,\lambda\varepsilon\mu)}{d\lambda^2}\Big|_{\lambda_o}\right| \le \Bigg\{ \sup_{\substack{\rho\in\Gamma(Y)\\ \langle i,j\rangle\cap D(\rho)}} |\partial\sigma_{ij}|^2$$

$$+ \sup_{\substack{X\cap \mathrm{supp}\, a_\rho\\ \rho\in\Gamma(Y)\\ \langle i,j\rangle\subset X}} \left(|\partial\sigma_{ij}|^2 e^{-\frac{1}{4}J|X|}\right) + \varepsilon^2(\mu^2+\mu)\Bigg\} S_{40}(Y) \quad . \tag{4.10}$$

Lemma V.1 follows from these two lemmas. We first recall that because of the definition of σ_i, and standard estimates on the 2-dimensional covariance, $|\partial\sigma_{ij}| \le c\cdot|\ln\beta|\varepsilon$, for some constant c. Furthermore, we have already shown (see (3.24)) that

$$|\tilde{Q}(Y,\phi,0)| \le S_{60}(Y) \quad . \tag{4.11}$$

Thus

$$\left|\frac{d^2F_\lambda}{d\lambda^2}(\phi)\Big|_{\lambda_o}\right| \le \varepsilon^2\Big\{ m(2c|\ln\beta|^2+2) + m^2\cdot 2(2c|\ln\beta|+1)^2\Big\}\prod_{\ell=1}^m S_{40}(Y_\ell)$$

$$\le \varepsilon^2 \prod_{\ell=1}^m S_{39}(Y_\ell) \quad . \tag{4.12}$$

In the second inequality we have used the fact that for any cluster, Y,

$$|S_1(Y)| \le e^{-\frac{D}{128}|\ln\beta|} \quad , \tag{4.13}$$

so that for β sufficiently small,

$$\{m(2c|\ln\beta|^2+2) + 2m^2(2c|\ln\beta|+1)^2\}\prod_{\ell=1}^m S_1(Y_\ell) \le 1 \quad . \tag{4.14}$$

Combining (4.12) and (4.7) yields Lemma V.1.

Lemma V.2 also follows from these results. Assume that $\mathrm{supp}\, a_\rho \cap \{0,x\} = \phi$ for all $\rho \in \Gamma(Y)$. Then from the definition of our cluster functions, we find $\tilde{Q}(Y,\phi+\lambda\sigma,\lambda\varepsilon) = \tilde{Q}(Y,\phi+\lambda\sigma,0)$ for such a cluster.

Define

$$F^0_\lambda(\phi) = \prod_{\ell=1}^m \tilde{Q}(Y_\ell,\phi+\lambda\sigma,0) - \prod_{\ell=1}^m \tilde{Q}(Y_\ell,\phi,0) \quad . \tag{4.15}$$

Then by precisely the same argument as before, one has

$$F^0_{\lambda=1}(\phi) = F^0_{\lambda=0}(\phi) + \frac{dF^0_\lambda}{d\lambda}(\phi)\Big|_{\lambda=0} + \frac{1}{2}\left(\frac{d^2F^0_\lambda}{d\lambda^2}(\phi)\right)\Big|_{\lambda=\lambda_o} \quad , \tag{4.16}$$

and

$$\int \left(\frac{dF^0_\lambda}{d\lambda}(\phi)\right)\Big|_{\lambda=0} d\nu^N_\Lambda(\phi) = 0 \quad . \tag{4.17}$$

Furthermore, writing out an expression for the second derivative of $F^0_\lambda(\phi)$, analogous to (4.8) and using (4.9) and (4.10), with $\mu = 0$, leads to a bound

$$\left|\frac{d^2F^0_\lambda}{d\lambda^2}(\phi)\Big|_{\lambda_o}\right| \le \Bigg\{ \sup_{\substack{\rho\in\Gamma(Y_\ell)\\ \ell=1,\dots,m\\ \langle i,j\rangle\cap D(\rho)}} |\partial\sigma_{ij}|^2 + \sup_{\substack{X\cap \mathrm{supp}\, a_\ell\\ \rho\in\Gamma(Y_\ell)\\ \ell=1,\dots,m\\ \langle i,j\rangle\subset X}} \left(|\partial\sigma_{ij}|^2 e^{-\frac{1}{4}J|X|}\right)\Bigg\} \times \prod_{\ell=1}^m S_{39}(Y\ ). \tag{4.18}$$

Equations (4.15), (4.16), (4.17) and inequality (4.18) combine to prove Lemma V.2.

We must now prove Lemmas V.9 and V.10. Begin by proving bounds on derivatives of parts of the cluster functions. We first note that by (II.3.5),

$$\left|\frac{dR}{d\lambda}(\rho,\phi+\lambda\sigma,\zeta_\rho,\lambda\mu\varepsilon)\Big|_{\lambda_o}\right|$$

$$\le K(\rho)e^{-\gamma q_\rho}|i(\phi+\lambda_o\sigma+i\zeta a_\rho)|\Big\{|i\sigma(\rho)+i\varepsilon\mu(a_\rho(0)-a_\rho(X))$$

$$+\sum_{\langle i,j\rangle\in\bigcup_{m=0}^{n(\rho)}\Omega_\rho^m}\frac{d}{d\lambda}\ln\left(\frac{I_\beta(\partial\phi_{ij}+i\lambda\partial\sigma_{ij}+i\zeta_\rho(a_\rho(i)-a_\rho(j)))}{I_\beta(\partial\phi_{ij}+i\lambda\partial\sigma_{ij})}\right)\Big|_{\lambda_o}\Big|\Big\}\ . \tag{4.19}$$

The first three factors on the r.h.s. of (4.19) are bounded by the estimates of Appendix II which we used in section V.3 to bound $Q(Y,\phi,0)$. (See (3.21).) The last term in curly brackets is bounded by

$$\sup_{\langle i,j\rangle\cap D(\rho)} c'\cdot|\partial\sigma_{ij}|\cdot\beta^{\frac{1}{3}}(g(a_\rho(i)-a_\rho(j))+1)\cdot d_\Lambda^2(\rho)\ . \tag{4.20}$$

Here, $g(\cdot)$ is the function defined in Theorem II.1, and the factors $\beta^{\frac{1}{3}}(g(a_\rho(i)-a_\rho(j))+1)$ in this bound come from bounding the derivative of the logarithm of the $I_\beta(\cdot)$ functions by means of (II.1.1)(d). The factor of $d_\Lambda^2(\rho)$ comes from bounding the number of terms in the sum over $\langle i,j\rangle$. This follows because any $\langle i,j\rangle\in\bigcup_{m=0}^{n(\rho)}\Omega_\rho^m$ is contained in

supp a_ρ, and $|\text{supp } a_\rho|\le O(1)\cdot d_\Lambda^2(\rho)$. The second term in curly brackets is bounded by noting that for any point i, $|a_\rho(i)|\le\gamma\cdot(n(\rho)+1)$. This follows from the observation that at any given length scale 2^m, there is at most one $s'\in S_m'(\rho)$ with $i\in\text{supp } a_{\rho,s'}^m$. Since $|a_{\rho,s'}^m(i)|\le\gamma$, $|a_\rho(i)|\le\sum_{m=0}^{n(\rho)}\sum_{s'\in S_m'(\rho)}|a_{\rho,s'}^n(i)|\le\sum_{m=0}^{n(\rho)}\gamma$. Thus

$$|i\varepsilon\mu(a_\rho(0)-a_\rho(x))|\le 2\gamma\varepsilon\mu n(\rho)\ . \tag{4.21}$$

The first term in curly brackets in (4.19) is bounded by

$$|\sigma(\rho)|\le 6d_\Lambda^2(\rho)(d_\Lambda(\rho)+1)\sup_{\langle i,j\rangle\cap D(\rho)}|\partial\sigma_{ij}|\ . \tag{4.22}$$

If ρ is a neutral charge density, this follows immediately from equation (5.23) of [15]. If ρ is charged, let i_R denote the site in $\mathbb{Z}^2\cap\Lambda^c$ closest to i. Let τ be the set of lattice bonds connecting i to i_R of minimal length. Then

$$|\sigma(i)|=|\sigma(i)-\sigma(i_R)|\le\sum_{\langle m,n\rangle\in\tau}|\sigma(m)-\sigma(n)|$$

$$\le 2(d_\Lambda(\rho)+1)\sup_{\langle i,j\rangle\cap D(\rho)}|\partial\sigma_{ij}|\ . \tag{4.23}$$

We have used here the fact that no point in supp ρ is farther than $2d_\Lambda(\rho)$ from the boundary of Λ. If we note finally that there are no more than $3d_\Lambda^2(\rho)$ points in supp ρ, we arrive at (4.22).

Combining (4.19), (4.20), (4.21), (4.22), we have

$$\left|\frac{dR}{d\lambda}(\rho,\phi+\lambda\sigma,\zeta_\rho,\lambda\mu\varepsilon)\Big|_{\lambda_o}\right|$$

(Eq. (4.24) to be continued)

$$\leq z_\rho \left\{ \left[6d^2(\rho)(d_\Lambda(\rho)+1) + c''\beta^{\frac{1}{3}} e^{4\pi\gamma} \cdot d_\Lambda^2(\rho) \right] \sup_{<i,j>\cap D(\rho)} |\partial\sigma_{ij}| \right.$$

$$\left. + 2\gamma\epsilon\mu n(\rho) \right\} . \tag{4.24}$$

We have used the bound on the $g(\cdot)$ functions coming from Theorem II.1 in the second term inside the square brackets. Recall that $\gamma = c|\ln\beta|$. For the constant c sufficiently small, $c''\beta^{\frac{1}{3}}e^{4\pi\gamma} < \beta^{\frac{1}{4}}$ for β sufficiently small. Also, for β sufficiently small,

$$z_\rho^{\frac{1}{8}}\{6d_\Lambda^2(\rho)(d_\Lambda(\rho)+1) + d_\Lambda^2(\rho) + 2\gamma n(\rho)\} \leq 1 . \tag{4.25}$$

These two facts imply

$$\left| \frac{dR}{d\lambda}(\rho, \phi+\lambda\sigma, \zeta_\rho, \lambda\mu\epsilon) \Big|_{\lambda_o} \right| \leq z_\rho^{\frac{7}{8}} \left(\sup_{<i,j>\cap D(\rho)} |\partial\sigma_{ij}| + \epsilon\mu \right) .$$

Now derive bounds on second derivatives of $R(\cdot,\cdot,\cdot,\cdot)$. Again, using the explicit form of (II.3.5) we find

$$\left| \frac{d^2R}{d\lambda^2}(\rho, \phi+\lambda\sigma, \zeta_\rho, \lambda\mu\epsilon) \Big|_{\lambda_o} \right|$$

$$\leq K(\rho) e^{-\gamma q_\rho} |i(\phi+\lambda_o\sigma+i\zeta a_\rho)| \left\{ \left[|\sigma(\rho)| + \epsilon\mu(a_\rho(0)-a_\rho(x)) \right]^2 \right.$$

$$+ \sum_{<i,j>\subset \bigcup_0^{n(\rho)} \Omega_\rho^m} \frac{d^2}{d\lambda^2} \ln\left(\frac{I_\beta(\partial\phi_{ij}+i\lambda\partial\sigma_{ij}+i\zeta_\rho(a_\rho(i)-a_\rho(j)))}{I_\beta(\partial\phi_{ij}+i\lambda\partial\sigma_{ij})} \right) \Big|_{\lambda_o}$$

Eq. (4.26) to be continued

$$+ \sum_{\substack{<i,j>\subset \bigcup_0^{n(\rho)} \Omega_\rho^m \\ <i',j'>\subset \bigcup_0^{n(\rho)} \Omega_\rho^m \\ <i',j'>\neq<i,j>}} \left[\frac{d}{d\lambda} \ln\left(\frac{I_\beta(\partial\phi_{ij}+i\lambda\partial\sigma_{ij}+i\zeta_\rho(a_\rho(i)-a_\rho(j)))}{I_\beta(\partial\phi_{ij}+i\lambda\partial\sigma_{ij})} \right) \Big|_{\lambda_o} \right]$$

$$\left. \times \left[\frac{d}{d\lambda} \ln\left(\frac{I_\beta(\partial\phi_{i'j'}+i\lambda\partial\sigma_{i'j'}+i\zeta_\rho(a_\rho(i')-a_\rho(j')))}{I_\beta(\partial\sigma_{i'j'}+i\lambda\partial\sigma_{i'j'})} \right) \Big|_{\lambda_o} \right] \right\} . \tag{4.26}$$

The only term in (4.26) which is not controlled by our previous estimates is the second one inside the curly brackets. This is bounded by (II.1.1)(d), and we find, for β sufficiently small, that

$$\left| \frac{d^2R}{d\lambda^2}(\rho, \phi+\lambda\sigma, \zeta_\rho, \lambda\mu\epsilon) \Big|_{\lambda_o} \right| \leq z_\rho^{\frac{7}{8}} \left(\sup_{<i,j>\cap D(\rho)} |\partial\sigma_{ij}|^2 + \epsilon^2(\mu^2+\mu) \right) . \tag{4.27}$$

We have used the fact that $|\sigma(i)| \leq c\cdot|\ln\beta|\cdot\epsilon$ to absorb the term proportional to $|\sigma(i)|\cdot\epsilon\mu$ into the $\epsilon^2\mu$ term on the r.h.s. of (4.27).

We now estimate derivatives of the interpolated potentials. If we once again denote $\tilde{a} = \sum_{k=\ell}^{m} \zeta a(\Gamma_k)$, we find

$$\frac{d}{d\lambda} \Delta V_X(\phi+\lambda\sigma : i\zeta a(\Gamma_\ell) + \ldots + i\zeta a(\Gamma_m)) \Big|_{\lambda_o}$$

$$= \sum_{<i,j>\subset X} \left\{ \sum_{\{\partial h\}|_X} V_X(\{\partial h\}|_X)(\partial\sigma_{ij}) \left[f'(\partial\sigma_{ij}+\lambda_o\partial\sigma_{ij}+\partial h_{ij}+i\partial\tilde{a}_{ij}) \right.\right.$$

$$\times \prod_{\substack{<i',j'>\subset X \\ <i',j'>\neq<i,j>}} f(\partial\phi_{i'j'}+\lambda_o\partial\sigma_{i'j'}+\partial h_{i'j'}+i\partial\tilde{a}_{i'j'})$$

Eq. (4.28) to be continued

$$- f'(\partial\phi_{ij}+\lambda_o\partial\sigma_{ij}+\partial h_{ij}) \times \prod_{\substack{<i',j'>\subset X \\ <i',j'>\neq<i,j>}} f(\partial\phi_{ij}+\lambda_o\partial\sigma_{ij}+\partial h_{ij})\Big]\Big\} . \tag{4.28}$$

Applying the bounds of (3.7) to control the factors of $f(\cdot)$ (recall that $f(z) = \frac{\sin^2\pi z}{\pi^2 z^2}$), we find

$$\left|\frac{d\Delta V_X}{d\lambda}(\phi+\lambda\sigma:i\zeta a(\Gamma_\ell) +\ldots+ i\zeta a(\Gamma_m))\Big|_{\lambda_o}\right|$$

$$\leq 2\sum_{<i,j>\subset X}|\partial\sigma_{ij}|\sum_{\{\partial h\}|_X}|V_X(\{\partial h\}|_X)|(\kappa e^{\gamma\pi})^{|X|}\prod_{<i,j>\subset X}C(\partial\phi_{ij}+\lambda_o\partial\sigma_{ij}+\partial h_{ij})$$

$$\leq \sup_{<i,j>\subset X}|\partial\sigma_{ij}|\cdot e^{-J|X|}\cdot 2|X|(c\kappa e^{\gamma\pi})^{|X|}$$

$$\leq \sup_{<i,j>\subset X}|\partial\sigma_{ij}|e^{-\frac{7}{8}J|X|} , \tag{4.29}$$

for J sufficiently large. Here we have used the fact that $\sum_{n=-\infty}^{\infty} C(\phi+n) < c$, in the second inequality, and also bounded the number of terms in the sum over $<i,j>$ by $|X|$. The estimates of (3.7) also yield, in an identical fashion,

$$\left|\frac{d^2}{d\lambda^2}\Delta V_X(\phi+\lambda\sigma:i\zeta a(\Gamma_\ell) +\ldots+ i\zeta a(\Gamma_m))\Big|_{\lambda_o}\right| \leq \sup_{<i,j>\subset X}|\partial\sigma_{ij}|^2 e^{-\frac{7}{8}J|X|} . \tag{4.30}$$

Now extend these bounds to the interpolated potentials. We claim that

$$\left|\frac{d^m}{d\lambda^m}\Delta V_X{}^{s_1\cdots s_{k-1}}(\phi+\lambda\sigma:i\zeta a(\Gamma_1)+\ldots+i\zeta a(\Gamma_{k-1});i\zeta a(\Gamma_k))\Big|_{\lambda_o}\right|$$

$$\leq k\cdot\sup_{<i,j>\subset X}|\partial\sigma_{ij}|^m e^{-\frac{7}{8}J|X|} , \tag{4.31}$$

for $m = 1,2$. By (4.29) and (4.30), this bound holds for $k = 1$. Assume that it holds for $k < n$. Then

$$\left|\frac{d^m}{d\lambda^m}\Delta V_X{}^{s_1\cdots s_{n-1}}(\phi+\lambda\sigma:i\zeta a(\Gamma_1)+\ldots+i\zeta a(\Gamma_{n-1});i\zeta a(\Gamma_n))\Big|_{\lambda_o}\right|$$

$$\leq s_{n-1}\left|\frac{d^m}{d\lambda^m}\Delta V_X{}^{s_1\cdots s_{n-1}}(\phi+\lambda\sigma:i\zeta a(\Gamma_1)+\ldots+i\zeta a(\Gamma_{n-2});i\zeta a(\Gamma_{n-1}\cup\Gamma_n))\Big|_{\lambda_o}\right|$$

$$+ (1-s_{n-1})\left|\frac{d^m}{d\lambda^m}\Delta V_X{}^{s_1\cdots s_{n-1}}(\phi+\lambda\sigma:i\zeta a(\Gamma_1)+\ldots+i\zeta a(\Gamma_{n-2});i\zeta a(\Gamma_{n-1}))\Big|_{\lambda_o}\right.$$

$$\left. + \frac{d^m}{d\lambda^m}\Delta V_X(\phi+\lambda\sigma:i\zeta a(\Gamma_n))\Big|_{\lambda_o}\right|$$

$$\leq \left\{s_{n-1}(n-1)+(1-s_{n-1})[(n-1)+1]\right\}\sup_{<i,j>\subset X}|\partial\sigma_{ij}|^m e^{-\frac{7}{8}J|X|}$$

$$\leq n\sup_{<i,j>\subset X}|\partial\sigma_{ij}|^m e^{-\frac{7}{8}J|X|} . \tag{4.32}$$

This completes the proof. In the second inequality, we have applied the induction hypothesis.

We are now ready to prove Lemmas V.9 and V.10. Using the defini-

tion of our cluster functions, (II.4.2), we have

$$\left|\frac{d\tilde{Q}}{d\lambda}(Y,\phi+\lambda\varepsilon,\lambda\mu\varepsilon)\Big|_{\lambda_o}\right| \le$$

$$\sum_{\{\zeta\}_{\Gamma(Y)}} (\tfrac{1}{2})^{|\Gamma(Y)|}\left|\exp\sum_{X\subset\Lambda}\Delta V_X^{s_1\cdots s_{k-1}}(\phi:i\zeta a(\Gamma_1)+\ldots+i\zeta a(\Gamma_{k-1});i\zeta a(\Gamma_k))\right|$$

$$\Bigg\{\Big(\sum_{\rho\in\Gamma(Y)}\frac{dR}{d\lambda}(\rho,\phi+\lambda\sigma,\zeta_\rho,\lambda\mu\varepsilon)\Big|_{\lambda_o}\Big|\prod_{\substack{\rho'\in\Gamma(Y)\\ \rho'\neq\rho}}\left|R(\rho',\phi+\lambda_o\sigma,\zeta_\rho,\lambda_o\mu\varepsilon)\right|\Big)$$

$$\Big(\prod_{m=2}^{k} s_{\eta(m)}\cdots s_{m-2}\left|\Delta V_{X_{m-1}}(\phi+\lambda_o\sigma:i\zeta a(\Gamma_{\eta(m)})+\ldots+i\zeta a(\Gamma_m))\right|\Big)$$

$$+\Big(\sum_{m=2}^{k}(s_{\eta(m)}\cdots s_{m-2})\left|\frac{d\Delta V_{X_{m-1}}}{d\lambda}(\phi+\lambda\sigma:i\zeta a(\Gamma_{\eta(m)})+\ldots+i\zeta a(\Gamma_m))\right|_{\lambda_o}\Big|$$

$$\times\prod_{\substack{m'=2\\ m'\neq m}}^{k} s_{\eta(m)}\cdots s_{m-2}\left|\Delta V_{X_{m-1}}(\phi+\lambda_o\sigma:i\zeta a(\Gamma_{\eta(m)})+\ldots+i\zeta a(\Gamma_m))\right|\Big)$$

$$\Big(\prod_{\rho\in\Gamma(Y)}\left|R(\rho,\phi+\lambda_o\sigma,\zeta_\rho,\lambda_o\mu\varepsilon)\right|\Big)$$

$$+\Big(\prod_{\rho\in\Gamma(Y)}|R(\rho,\phi+\lambda_o\sigma,\zeta_\rho,\lambda_o\mu\varepsilon)|\Big)$$

$$\Big(\prod_{m=2}^{k}[s_{\eta(m)}\cdots s_{m-2}]\times\left|\Delta V_{X_{m-1}}(\phi+\lambda_o\sigma:i\zeta a(\Gamma_{\eta(m)})+\ldots+i\zeta a(\Gamma_m))\right|\Big)$$

$$\times\Big(\sum_{\substack{X\subset\Lambda\\ X\cap\operatorname{supp} a(\Gamma_\ell)\\ \ell=1,\ldots,k}}\frac{d\Delta V_X^{s_1\cdots s_{k-1}}}{d\lambda}(\phi+\lambda\sigma;i\zeta a(\Gamma_1)+\ldots+i\zeta a(\Gamma_{k-1});i\zeta a(\Gamma_k))\Big|_{\lambda_o}\Big|\Big)\Bigg\}\,. \tag{4.33}$$

In the last term in the curly brackets, we have used the fact that $\Delta V_X^{s_1\cdots s_{k-1}}(\phi+\lambda\sigma:i\zeta a(\Gamma_1)+\ldots+i\zeta a(\Gamma_{k-1});i\zeta a(\Gamma_k))$ vanishes unless X intersects $\operatorname{supp} a(\Gamma_j)$ for some j, to restrict the sum over X. In (4.33) factors of $|R(.,.,.,.)|$ are controlled by (3.21), the factor $|\exp\sum_{X\subset\Lambda}\Delta V_X^{s_1\cdots s_{k-1}}(.:.;.)|$ is controlled by Lemma V.7, and factors of $\Delta V_X(.:.)$ by Corollary V.8. The factors on which derivatives act are bounded by (4.24) or (4.31). Since by (II.3.8), $\Delta V_{X_m}(\cdot:\cdot)$ is made up of at most four terms of $\Delta V_X(\cdot:\cdot)$, derivatives of $\Delta V_{X_m}(\cdot:\cdot)$ may also be bounded by (4.31). Combining all these estimates yields

$$\left|\frac{d\tilde{Q}}{d\lambda}(Y,\phi+\lambda\varepsilon,\lambda\mu\varepsilon)\Big|_{\lambda_o}\right| \le$$

$$\sum_{\{\zeta\}_{\Gamma(Y)}}(\tfrac{1}{2})^{|\Gamma(Y)|}\cdot\exp\Big(c''\gamma^2 e^{-\frac{3}{4}J}\sum_{\rho\in\Gamma(Y)}\sum_{m=0}^{\infty}\left|S_m'(\rho)\right|\Big)$$

$$\times\Big(\prod_{\rho\in\Gamma(Y)}z_\rho^{\frac{7}{8}}\Big)\Big(\prod_{m=2}^{k}[s_{\eta(m)}\cdots s_m]\chi_{X_{m-1}}(\Gamma_{\eta(m)},\Gamma_m)e^{-\frac{5}{8}J|X_{m-1}|}\Big)$$

$$\times\Bigg\{|\Gamma(Y)|\Big(\sup_{\langle i,j\rangle\cap D(\rho)}|\partial\sigma_{ij}|+\epsilon\mu\Big)+4k\sup_{\substack{\langle i,j\rangle\subset X\\ X\cap\operatorname{supp} a_\rho\\ \rho\in\Gamma(Y)}}|\partial\sigma_{ij}|e^{-\frac{1}{4}J|X|}$$

$$+\sum_{\substack{X\subset\Lambda\\ X\cap\operatorname{supp} a_\rho\\ \rho\in\Gamma(Y)}}\Big(k\cdot e^{-\frac{7}{8}J|X|}\sup_{\langle i,j\rangle\subset X}|\partial\sigma_{ij}|\Big)\Bigg\}\,. \tag{4.34}$$

We have inserted the characteristic functions on the r.h.s. of (4.34)

because the functions $\Delta V_{X_{m-1}}(\cdot:\cdot)$ in (4.33) vanish unless X_{m-1} intersects both $\Gamma_{\eta(m)}$ and Γ_m. Note that nothing on the r.h.s. of (4.34) depends on $\{\zeta\}$, so we can drop the sum over $\{\zeta\}$ and compensate by multiplying by $2^{|\Gamma(Y)|}$. The sum over X on the r.h.s. of (4.34) is bounded by

$$k\cdot \sup_{\substack{X\cap \mathrm{supp}\, a_\rho \\ \rho\in\Gamma(Y) \\ \langle i,j\rangle\subset X}} |\partial\sigma_{ij}| e^{-\frac{1}{4}J|X|} \sum_{\substack{X\subset\Lambda \\ X\cap \mathrm{supp}\, a_\rho \\ \rho\in\Gamma(Y)}} e^{-\frac{5}{8}J|X|}$$

$$\le k\cdot \sup_{\substack{X\cap \mathrm{supp}\, a_\rho \\ \rho\in\Gamma(Y) \\ \langle i,j\rangle\subset X}} |\partial\sigma_{ij}| e^{-\frac{1}{4}J|X|} \cdot \sum_{\substack{i\in \mathrm{supp}\, a_\rho \\ \rho\in\Gamma(Y)}} \sum_{\substack{X\subset\Lambda \\ i\subset X}} e^{-\frac{7}{8}J|X|}$$

$$\le k\cdot \sup_{\substack{X\cap \mathrm{supp}\, a_\rho \\ \rho\in\Gamma(Y) \\ \langle i,j\rangle\subset X}} |\partial\sigma_{ij}| e^{-\frac{1}{4}J|X|} \Big(\sum_{\rho\in\Gamma(Y)} c' d_\Lambda^2(\rho)\Big) \quad . \tag{4.35}$$

In the last inequality we have used the Peierl's argument to control the sum over X containing i, and then bounded the number of points in $\mathrm{supp}\, a_\rho$ by $c' d_\Lambda^2(\rho)$. Finally, pick β sufficiently small and J sufficiently large that

$$\Big(\prod_{\rho\in\Gamma(Y)} z_\rho^{\frac{1}{4}}\Big) \exp\Big(c'' \gamma^2 e^{-\frac{3}{4}J} \sum_{\rho\in\Gamma(Y)} A(\rho)\Big) \times \Big\{|\Gamma(Y)| + 4k + k\cdot c'\cdot \sum_{\rho\in\Gamma(Y)} d_\Lambda^2(\rho)\Big\} \le 1 \quad . \tag{4.36}$$

If we use the fact that $\sum_{m=0}^{\infty} |S'_m(\rho)| < A(\rho)$, (4.36) combined with (4.34) yields

$$\left|\frac{d\tilde{Q}}{d\lambda}(Y,\phi+\lambda\varepsilon,\lambda\mu\varepsilon)\right|_{\lambda_0} \le \Big(\sup_{\langle i,j\rangle\cap D(\rho)} |\partial\sigma_{ij}| + \sup_{\substack{\langle i,j\rangle\subset X \\ X\cap \mathrm{supp}\, a_\rho \\ \rho\in\Gamma(Y)}} |\partial\sigma_{ij}| e^{-\frac{1}{4}J|X|} + \varepsilon\mu\Big)$$

$$\times \Big(\prod_{\rho\in\Gamma(Y)} z_\rho^{\frac{5}{8}}\Big) \left[\prod_{m=2}^{k} (s_{\eta(m)}\cdots s_{m-2}) \chi_{X_{m-1}}(\Gamma_{\eta(m)},\Gamma_m) e^{-\frac{5}{8}J|X|}\right] \quad . \tag{4.37}$$

This is precisely the bound of (4.9) and it completes the proof of Lemma V.9.

The proof of Lemma V.10 is very similar. The second derivative of the cluster function is bounded by

$$\left|\frac{d^2\tilde{Q}}{d\lambda^2}(Y,\phi+\lambda\sigma,\zeta,\lambda\varepsilon\mu)\right|_{\lambda_0} \le$$

$$\sum_{\{\zeta\}_{\Gamma(Y)}} \Big(\frac{1}{2}\Big)^{|\Gamma(Y)|} \left|\exp \sum_{X\subset\Lambda} \Delta V_X^{s_1\cdots s_{k-1}}(\phi:i\zeta a(\Gamma_1)+\ldots+i\zeta a(\Gamma_{k-1});i\zeta a(\Gamma_k))\right|$$

$$\Bigg\{ \prod_{m=2}^{k} (s_{\eta(m)}\cdots s_{m-2}) |\Delta V_{X_{m-1}}| \times \left|\frac{d^2}{d\lambda^2}\Big(\prod_{\rho\in\Gamma(Y)} R_\rho\Big)\right|$$

$$+ \prod_{\rho\in\Gamma(Y)} |R_\rho| \Big(\prod_{m=2}^{k} (s_{\eta(m)}\cdots s_{m-2}) |\Delta V_{X_{m-1}}|\Big) \Bigg\{ \sum_{\substack{X\subset\Lambda \\ X\cap \mathrm{supp}\, a_\rho \\ \rho\in\Gamma(Y)}} \left|\frac{d^2}{d\lambda^2} \Delta V_X^{s_1\cdots s_{k-1}}\right|$$

$$+ \Big(\sum_{\substack{X\subset\Lambda \\ X\cap \mathrm{supp}\, a_\rho \\ \rho\in\Gamma(Y)}} \left|\frac{d}{d\lambda} \Delta V_X^{s_1\cdots s_{k-1}}\right|\Big)^2 \Bigg\} + \Big(\prod_{\rho\in\Gamma(Y)} |R_\rho|\Big) \left|\frac{d^2}{d\lambda^2} \prod_{m=2}^{k} (s_{\eta(m)}\cdots s_{m-2}) |\Delta V_{X_{m-1}}|\right|$$

Eq. (4.38) to be continued

$$+ 2\Big(\sum_{\rho\in\Gamma(Y)} \Big|\frac{dR}{d\lambda}\Big| \prod_{\substack{\rho'\in\Gamma(Y)\\ \rho'\neq\rho}} |R_\rho|\Big)\Big(\sum_{m=2}^{k} (s_{\eta(m)}\cdots s_{m-2}) \Big|\frac{d\Delta V_{X_{m-1}}}{d\lambda}\Big|$$

$$\times \prod_{\substack{m'=2\\ m\neq m'}}^{k} (s_{\eta(m')}\cdots s_{m'-2}|\Delta V_{X_{m-1}}|)$$

$$+ 2\Big(\sum_{\substack{X\subset\Lambda\\ X\cap \operatorname{supp} a_\rho\\ \rho\in\Gamma(Y)}} \Big|\frac{d\Delta V_X^{s_1\cdots s_{k-1}}}{d\lambda}\Big|\Big)\Big(\sum_{\rho\in\Gamma(Y)} \Big|\frac{dR_\rho}{d\lambda}\Big| \prod_{\substack{\rho'\in\Gamma(Y)\\ \rho'\neq\rho}} |R_\rho|\Big)$$

$$\times \Big(\prod_{m=2}^{k} (s_{\eta(m)}\cdots s_{m-2}) |\Delta V_{X'_{m-1}}|\Big)$$

$$+ 2\Big(\sum_{\substack{X\subset\Lambda\\ X\cap \operatorname{supp} a_\rho\\ \rho\in\Gamma(Y)}} \Big|\frac{d\Delta V_X}{d\lambda}\Big|\Big)\Big(\prod_{\rho\in\Gamma(Y)} |R_\rho|\Big)\Big(\sum_{m=2}^{k} (s_{\eta(m)}\cdots s_{m-2}) \Big|\frac{d\Delta V_{X_{m-1}}}{d\lambda}\Big|$$

$$\times \prod_{\substack{m'=2\\ m'\neq m}}^{k} (s_{\eta(m')}\cdots s_{m'-2}) |\Delta V_{X_{m'-1}}|\Big)\Bigg\}. \tag{4.38}$$

We have omitted the arguments of the functions here in an effort to save space. They are in each case identical to those of the corresponding function in (4.33). The last three of these terms are controlled using the estimates from the proof of Lemma V.9. The first three use in addition (4.27), (4.30) and (4.31) to control the factors with second derivatives with respect to λ. As was the case in Lemma V.9, the sums over $X \cap \operatorname{supp} a_\rho$ are controlled by the Peierls argument. Applying all of the above bounds to (4.38) yields

$$\Big|\frac{d^2\tilde{Q}}{d\lambda^2}(Y,\phi+\lambda\sigma,\zeta,\lambda\varepsilon\mu)\Big|_{\lambda_0}\Big|$$

$$\le \exp\Big(c''\gamma^2 e^{-\frac{3}{4}J} \sum_{\rho\in\Gamma(Y)} A(\rho)\Big)\Big(\prod_{\rho\in\Gamma(Y)} z_\rho^{\frac{7}{8}}\Big)$$

$$\times \Big(\prod_{m=2}^{k} [s_{\eta(m)}\cdots s_{m-2}]\chi_{X_{m-1}}(\Gamma_{\eta(m)},\Gamma_m) e^{-\frac{5}{8}J|X_{m-1}|}\Big)$$

$$\times\Bigg\{ |\Gamma(Y)|\Big(\sup_{\langle i,j\rangle\cap D(\rho)} (|\partial\sigma_{ij}|^2+\varepsilon^2(\mu^2+\mu))\Big)$$

$$+ \Big(4k+k\sum_{\rho\in\Gamma(Y)} c' d_\Lambda^2(\rho)\Big)\Big(\sup_{\substack{\langle i,j\rangle\subset X\\ X\cap \operatorname{supp} a_\rho\\ \rho\in\Gamma(Y)}} |\partial\sigma_{ij}|^2 e^{-\frac{1}{4}J|X|}\Big)$$

$$+ \Bigg(|\Gamma(Y)|\Big(\sup_{\langle i,j\rangle\cap D(\rho)} |\partial\sigma_{ij}|+\varepsilon\mu\Big) + \Big(4k+k\sum_{\rho\in\Gamma(Y)} c' d_\Lambda^2(\rho)\Big)$$

$$\times \Big(\sup_{\substack{\langle i,j\rangle\subset X\\ X \operatorname{supp} a_\rho\\ \rho\in\Gamma(Y)}} |\partial\sigma_{ij}| e^{-\frac{1}{4}J|X|}\Big)\Bigg)^2\Bigg\}. \tag{4.39}$$

Since we can pick β sufficiently small, and J sufficiently large that

$$\Big(\prod_{\rho\in\Gamma(Y)} z_\rho^{\frac{1}{8}}\Big)\Big(|\Gamma(Y)|+4k+kc'\sum_{\rho\in\Gamma(Y)} d_\Lambda^2(\rho)\Big)^2 \le 1 \quad ,$$

there exist $0 < \beta_1 < \beta_2$ and $J_1(\beta_1) > 0$ such that for all $\beta \in (\beta_1,\beta_2)$ and $J > J_1(\beta_1)$, (4.39) is bounded by

$$\left(\prod_{\rho\in\Gamma(Y)} z_\rho^{\frac{5}{8}}\right)\left(\prod_{m=2}^{k}[s_{\eta(m)}\cdots s_{m-2}]\chi_{X_{m-1}}(\Gamma_{\eta(m)},\Gamma_m)e^{-\frac{5}{8}J|X_{m-1}|}\right)$$

$$\left(\sup_{\substack{\rho\in\Gamma(Y)\\ \langle i,j\rangle\subset D(\rho)}}|\partial\sigma_{ij}|^2+\sup_{\substack{\langle i,j\rangle\subset X\\ X\cap\operatorname{supp} a_\rho\\ \rho\in\Gamma(Y)}}|\partial\sigma_{ij}|^2e^{-\frac{1}{4}J|X|}+\varepsilon^2(\mu^2+\mu)\right) .$$

This completes the proof of Lemma V.10.

CHAPTER VI

CONVERGENCE ESTIMATES FOR THE UPPER BOUNDS

VI.1. Bounds on the Cluster Functions

In this chapter we prove Proposition IV.8 which yields convergence of the expansion for the upper bound. As a first step, consider differences of products of cluster functions.

Lemma VI.1. There exists $\varepsilon_o(\beta) > 0$ such that for all $0 < \varepsilon < \varepsilon_o(\beta)$

$$\left(\prod_{j=1}^{m}\hat{Q}(Y_j,\theta+ia,\varepsilon) - \prod_{j=1}^{m}\hat{Q}(Y_j,\theta,\varepsilon=0)\right)$$

$$= iQ_1(Y_1,\ldots,Y_m,\theta,a,\varepsilon) + Q_2(Y_1,\ldots,Y_m,\theta,a,\varepsilon) \quad , \qquad (1.1)$$

where $Q_1(Y_1,\ldots,Y_m,\theta,a,\varepsilon)$ is a real function for all values of its arguments. There exists $J_2(\beta) > 0$ such that for all $J > J_2(\beta)$,

$$|Q_2(Y_1,\ldots,Y_m,\theta,a,\varepsilon)| \le \sup_{\langle i^*,j^*\rangle\in\bigcup_1^m\Gamma(Y_k)}(a_{i^*}-a_{j^*}+\varepsilon\delta_{\langle i^*,j^*\rangle},\gamma)^2\prod_{k=1}^{m}S(Y_k) \quad , \qquad (1.2)$$

with

$$S(Y) = \left(\prod_{j=2}^{k}[s_{\eta(j)}\cdots s_{j-2}]\tilde{\chi}_{X_{j-1}}(\Gamma_1\cup\cdots\cup\Gamma_{\eta(j)-1};\Gamma_{\eta(j)})e^{\frac{3}{4}J|X_{j-1}|}\right) . \qquad (1.3)$$

<u>Proof</u>. Define $G_\lambda(\theta,a,\varepsilon) = \prod_{j=1}^{m} \hat{Q}(Y_j,\theta+i\lambda a,\lambda\varepsilon) - \prod_{j=1}^{m} \hat{Q}(Y_j,\theta,\varepsilon=0)$.

By Taylor's theorem,

$$G_{\lambda=1}(\theta,a,\varepsilon) = G_{\lambda=0}(\theta,a,\varepsilon) + \frac{dG_\lambda}{d\lambda}\bigg|_{\lambda=0} + \frac{1}{2}\frac{d^2G_\lambda}{d\lambda^2}\bigg|_{\lambda=\lambda_o} , \quad (1.4)$$

for some $\lambda_o \in [0,1]$.

Define $Q_1(Y_1,\ldots,Y_m,\theta,a,\varepsilon) = (-i)\frac{dG_\lambda}{d\lambda}\Big|_{\lambda=0}$. We prove that $(-i)\frac{d\hat{Q}}{d\lambda}\Big|_{\lambda=0}$ is real. Since we have already shown that $\hat{Q}(Y_j,\theta,0)$ is real, this suffices to prove that $(-i)\frac{dG_\lambda}{d\lambda}\Big|_{\lambda=0}$ and hence $Q_1(Y_1,\ldots,Y_m,\theta,a,\varepsilon)$ is real.

$$\frac{d\hat{Q}}{d\lambda}(Y,\theta+i\lambda a,\lambda\varepsilon)\Big|_{\lambda=0} = \Bigg(\prod_{\substack{<i,j>\\ <i,j>\in\tilde{\Gamma}(Y)}} \tilde{I}_\beta(\theta_{i*}-\theta_{j*})\Bigg)^{-1}\hat{Q}(Y,\theta,0)$$

$$\left[\sum_{\substack{<i,j>\\ <i,j>\in\Gamma(Y)}} \frac{-\frac{dI_\beta}{d\lambda}(\theta_{i*}-\theta_{j*}+i\lambda a_{i*}-i\lambda a_{j*}+i\lambda\varepsilon\delta_{<i*,j*>,\gamma})\Big|_{\lambda=0}}{\tilde{I}_\beta(\theta_{i*}-\theta_{j*})}\right]$$

$$+\left[\frac{1}{\prod_{\substack{<i,j>\\ <i,j>\in\Gamma(Y)}} \tilde{I}_\beta(\theta_{i*}-\theta_{j*})}\right]$$

$$\left\{\sum_{\{n\}|_{\tilde{\Gamma}(Y)}} \Big(\sum_{\substack{<i,j>\in\\ \tilde{\Gamma}(Y)}} I_\beta(n_{ij})(-n_{ij})(a_{i*}-a_{j*}+\varepsilon\delta_{<k*,j*>,\gamma})e^{in_{ij}(\theta_{i*}-\theta_{i*})}\right\}$$

Eq. (1.6) to be continued

$$\times\left(\prod_{\substack{<i',j'>\in\tilde{\Gamma}(Y)\\ <i',j'>\neq<i,j>}} I_\beta(n_{i'j'})e^{in_{i'j'}(\theta_{i*}-\theta_{j*})}\right)$$

$$\times\prod_{\ell=2}^{k}\left(s_{\eta(\ell)}\cdots s_{\ell-2}\tilde{\chi}_{X_{\ell-1}}(\Gamma_1\cup\cdots\cup\Gamma_{\eta(\ell)-1};\Gamma_{\eta(\ell)}\cap X_{\ell-1}(\{n\})|_{X_{\ell-1}})\right)$$

$$\times e^{\sum_{X\subset X_1\cup\cdots\cup X_{k-1}} V_X^{s_1\cdots s_{k-1}}(\{n\}|_X;\Gamma_1\ \cdots\ \Gamma_{k-1})} . \quad (1.6)$$

The first term is pure imaginary. This follows from the fact that $Q(Y_j,\theta,\varepsilon=0)$ and $\tilde{I}_\beta(\theta_{i*}-\theta_{j*})$ are real, while $\frac{d\tilde{I}_\beta}{d\lambda}(\theta_{i*}-\theta_{j*}+i\lambda a_{i*}-i\lambda a_{j*}+i\lambda\,\delta_{<i*,j*>,\gamma})|_{\lambda=0}$ is pure imaginary, by explicit computation. In the second term we may ignore the factors of $\tilde{I}_\beta(\theta_{i*}-\theta_{j*})$ since they are real. Change variables in the sum over $\{n\}|_{\tilde{\Gamma}(Y)}$ to $\{m\} = \{-n\}$. Using the invarance of $I_\beta(n)$ and $V_X(\{n\}|_X)$ under this change of variables we find

$$I \equiv \sum_{\{n\}|_{\tilde{\Gamma}(Y)}} \Big(\sum_{<i,j>\in\tilde{\Gamma}(Y)} I_\beta(n_{ij})(-n_{ij})(a_{i*}-a_{j*}+\varepsilon\delta_{<i*,j*>,\gamma})e^{in_{ij}(\theta_{i*}-\theta_{j*})}$$

$$\times\prod_{\substack{<i',j'>\in\tilde{\Gamma}(Y)\\ <i',j'>\neq<i,j>}} I_\beta(n_{i'j'})e^{in_{i'j'}(\theta_{i'*}-\theta_{j'*})}$$

$$= \sum_{\{m\}|_{\tilde{\Gamma}(Y)}} \Big(\sum_{<i,j>\in\tilde{\Gamma}(Y)} I_\beta(m_{ij})(m_{ij})(a_{i*}-a_{j*}+\varepsilon\delta_{<i*,j*>,\gamma})e^{-im_{ij}(\theta_{i*}-\theta_{j*})}$$

Eq. (1.7) to be continued

$$\times \left[\prod_{\substack{<i',j'>\in\tilde{\Gamma}(Y) \\ <i',j'>\neq<i,j>}} I_\beta(m_{i'j'}) e^{-im_{i'j'}(\theta_{i'*}-\theta_{j'*})} \right]$$

$$= -I^* \quad . \tag{1.7}$$

We have ignored the factors with $V_X(\cdot)$ in them since these are all invariant under $\{n\} \to -\{n\}$. Equation (1.7) proves that the second term in (1.6) and hence $\frac{d\tilde{Q}}{d\lambda}(Y,\theta+i\lambda a,\lambda\varepsilon)\big|_{\lambda=0}$ is pure imaginary. Thus $Q_1(Y_1,\ldots,Y_m,\theta,a,\varepsilon)$ is a real function as claimed in the first half of Lemma VI.1.

Now define

$$Q_2(Y_1,\ldots,Y_m,\theta,a,\varepsilon) \equiv \frac{1}{2}\frac{d^2G_\lambda}{d\lambda^2}\bigg|_{\lambda=\lambda_o} \quad . \tag{1.8}$$

We first prove some bounds on $Q(Y_\ell,\theta+i\lambda_o a,\lambda_o\varepsilon)$ and its derivatives.

$$|V_X^{s_1\cdots s_{k-1}}(\{n\}|_X;\Gamma_1\cup\cdots\cup\Gamma_{k-1})| \le e^{-J|X|} \quad . \tag{1.9}$$

This is true by the assumption for $k = 0$, and since the interpolation procedure either leaves $V_X^{s_1\cdots s_k}(\cdot;\cdot)$ unchanged, or multiplies it by some real constant of magnitude less than one, it holds for all larger k as well. By the standard Peierl's argument, there exists $J_2(\beta) > 0$ such that for all $J > J_2(\beta)$ one has

$$\left| \sum_{X:x_o\in X} V_X^{s_1\cdots s_{k-1}}(\{n\}|_X;\Gamma_1\cup\cdots\cup\Gamma_{k-1}) \right| \le \sum_{|X|=3}^{\infty} e^{-J|X|}\cdot c^{|X|} \le e^{-\frac{3}{4}J} \quad . \tag{1.10}$$

Thus,

$$\left| \sum_{X\subset X_1\cup\cdots\cup X_{k-1}} V_X^{s_1\cdots s_{k-1}}(\{n\}|_X;\Gamma_1\cup\cdots\cup\Gamma_{k-1}) \right|$$

$$\le \sum_{x\in X_1\cup\cdots\cup X_{k-1}} \left| \sum_{X:x\in X} V_X^{s_1\cdots s_{k-1}}(\{n\}|_X;\Gamma_1\cup\cdots\cup\Gamma_{k-1}) \right|$$

$$\le e^{-\frac{3}{4}J} \left| \bigcup_{m=1}^{k-1} X_m \right| \quad . \tag{1.11}$$

Recall from Chapter III that $|a_{i*}-a_{j*}| \le K\cdot\varepsilon$. Pick $\varepsilon_o(\beta)$ so that $\varepsilon(K+1) \le \frac{1}{2}\beta$ for all $0 < \varepsilon < \varepsilon_o(\beta)$. Then for $\lambda_o \in [0,1]$,

$$\left| \sum_n I_\beta(n) e^{in\theta} e^{-\lambda_o(a_{i*}-a_{j*}+\varepsilon\delta_{<i*,j*>,\gamma})n} \right| \le 2\sum_{n=0}^{\infty} e^{-\frac{1}{2}\beta n} = 2(1-e^{-\frac{1}{2}\beta})^{-1}. \tag{1.12}$$

Combining (1.9), (1.11), (1.12), and the definition of $Q(Y,\theta,\varepsilon)$ from (IV.2.2) yields

$$\left| Q(Y,\theta+i\lambda_o\varepsilon,\lambda_o\varepsilon) \right| \le \left(\prod_{\ell=2}^{k} s_{\eta(\ell)}\cdots s_{\ell-2}\, \chi_{X_{\ell-1}}(\Gamma_1\cup\cdots\cup\Gamma_{n(\ell)-1};\Gamma_{n(\ell)}) e^{-J|X_{\ell-1}|} \right)$$

$$\times \exp\left(e^{-\frac{3}{4}J} \left| \bigcup_{m=1}^{k-1} X_m \right|\right) \{2(1-e^{-\frac{1}{2}\beta})\}^{|\tilde{\Gamma}(Y)|} \tag{1.13}$$

Now bound derivatives of $Q(Y,\theta+i\lambda a,\lambda\varepsilon)$.

$$\left|\frac{dQ}{d\lambda}(Y,\theta+i\lambda a,\lambda\varepsilon)\right|_{\lambda_o}\Bigg| \le$$

$$e^{e^{-\frac{3}{4}J\left|\bigcup_1^{k-1} X_m\right|}}\left(\prod_{\ell=2}^{k} s_{\eta(\ell)}\cdots s_{\ell-2}\tilde{\chi}_{X_{\ell-1}}(\Gamma_1\cup\ldots\cup\Gamma_{\eta(\ell)-1};\Gamma_{\eta(\ell)})e^{-J|X_{\ell-1}|}\right)$$

$$\times\Bigg|\sum_{\{n\}|_{\tilde{\Gamma}(Y)}}\Big(\sum_{<i,j>\in\tilde{\Gamma}(Y)} I_\beta(n_{ij})(-n_{ij})(a_{i*}-a_{j*}+\varepsilon\delta_{<i*,j*>},Y)$$

$$\times e^{in_{ij}(\theta_{i*}-\theta_{j*})}e^{-n_{ij}\lambda_o(a_{i*}-a_{j*}+\varepsilon\delta_{<i*,j*>},Y)}\Big)$$

$$\prod_{\substack{<i',j'>\in\tilde{\Gamma}(Y)\\ <i',j'>\neq<i,j>}} I_\beta(n_{ij})e^{in_{i'j'}(\theta_{i'*}-\theta_{j'*})}e^{\lambda_o(a_{i'*}-a_{j'*}+\varepsilon\delta_{<i'*,j'*>},Y)n_{i'j'}}\Bigg|. \tag{1.14}$$

The first line in this inequality results from applying (1.9) and (1.11). The sums over $\{n\}$ are bounded by (1.12) and

$$\sum_{n=0}^{\infty} ne^{-\frac{1}{2}\beta n} \le 2e^{-\frac{1}{2}\beta}(1-e^{-\frac{1}{2}\beta})^{-2} . \tag{1.15}$$

This yields

$$\frac{dQ}{d\lambda}(Y,\theta+i\lambda a,\lambda\varepsilon) \le 2\Big(\sup_{<i*,j*>\in\tilde{\Gamma}(Y)}|a_{i*}-a_{j*}+\varepsilon\delta_{<k*,j*>},Y|\Big)|\tilde{\Gamma}(Y)|$$

$$\times 2^{|\tilde{\Gamma}(Y)|}(1-e^{-\frac{1}{2}\beta})^{-2-|\tilde{\Gamma}(Y)|}\times\exp\{e^{-\frac{3}{4}J\left|\bigcup_1^{k-1}X_m\right|}\}\times e^{-\frac{\beta}{2}}$$

Eq. (1.16) to be continued

$$\times\left(\prod_{\ell=2}^{k}(s_{\eta(\ell)}\cdots s_{\ell-2})\chi_{X_{\ell-1}}(\Gamma_1\cup\ldots\cup\Gamma_{\eta(\ell)-1};\Gamma_{\eta(\ell)})e^{-J|X_{\ell-1}|}\right) . \tag{1.16}$$

The factor of $|\tilde{\Gamma}(Y)|$ just comes from estimating the number of terms in $\sum_{<i,j>\in\tilde{\Gamma}(Y)}$.

Finally we bound second derivatives of cluster functions. The only additional estimate required is

$$\sum_{n=0}^{\infty} n^2e^{-\frac{1}{2}\beta n} \le 2e^{-\frac{\beta}{2}}(1-e^{-\frac{\beta}{2}})^{-3} . \tag{1.17}$$

This is used to bound those cases in which both derivatives act on a single exponential factor. Applying these estimates, one obtains

$$\left|\frac{d^2Q}{d\lambda^2}(Y,\theta+i\lambda a,\varepsilon\lambda)\right|_{\lambda=\lambda_o}\Bigg| \le \sup_{<i*,j*>\in\tilde{\Gamma}(Y)}(a_{i*}-a_{j*}+\varepsilon\delta_{<i*,j*>},Y)^2$$

$$\times\{|\tilde{\Gamma}(Y)|2^{|\tilde{\Gamma}(Y)|+1}e^{-\frac{\beta}{2}}(1-e^{-\frac{1}{2}\beta})^{-|\tilde{\Gamma}(Y)|-2}$$

$$+|\tilde{\Gamma}(Y)|^2 2^{|\tilde{\Gamma}(Y)|+2}e^{-\beta}(1-e^{-\frac{\beta}{2}})^{-|\tilde{\Gamma}(Y)|-4}\}$$

$$\times\exp\{e^{-\frac{3}{4}J\left|\bigcup_1^m X_m\right|}\}\Big(\prod_{\ell=2}^{k}[s_{\eta(\ell)}\cdots s_{\ell-2}]\tilde{\chi}_{X_{\ell-1}}(\Gamma_1\cup\ldots\cup\Gamma_{\eta(\ell)-1};\Gamma_{\eta(\ell)})e^{-J|X_{\ell-1}|}\Big)\}. \tag{1.18}$$

In this formula the factor of $|\tilde{\Gamma}(Y)|$ comes from estimating the number of terms in which both derivatives act on a single exponential, and the factor $|\tilde{\Gamma}(Y)|^2$ comes from estimating the number of terms in which the

derivatives act on different exponential factors, in $Q(Y,\theta+i\lambda a,\lambda\varepsilon)$.

Now consider

$$\frac{1}{2}\frac{d^2G}{d\lambda^2}\Big|_{\lambda=\lambda_o} = \frac{1}{2}\sum_{j=1}^{m}\{\frac{d^2\hat{Q}}{d\lambda^2}(Y_j,\theta+i\lambda a,\lambda\varepsilon)\Big|_{\lambda=\lambda_o}\prod_{\substack{\ell\neq j\\ \ell=1}}^{m}\hat{Q}(Y_j,\theta+i\lambda_o a,\lambda_o\varepsilon)\}$$

$$+\sum_{1\le j<k\le m}\{\frac{d\hat{Q}}{d\lambda}(Y_j,\theta+i\lambda a,\lambda\varepsilon)\Big|_{\lambda=\lambda_o}\cdot\frac{d\hat{Q}}{d\lambda}(Y_k,\theta+i\lambda a,\lambda\varepsilon)\Big|_{\lambda=\lambda_o}$$

$$\times\prod_{\substack{\ell=1\\ \ell\neq j;\,\ell\neq k}}^{m}\hat{Q}(Y_\ell,\theta+i\lambda_o a,\lambda_o\varepsilon)\}\ . \tag{1.19}$$

Explicitly

$$\frac{d\hat{Q}}{d\lambda}(Y,\theta+i\lambda a,\lambda\varepsilon)\Big|_{\lambda=\lambda_o}=\{\prod_{<i^*,j^*>\in\tilde{\Gamma}(Y)}\tilde{I}_\beta(\ldots)\}^{-1}\times\{\frac{dQ}{d\lambda}(Y,\theta+i\lambda a,\lambda\varepsilon)\Big|_{\lambda_o}\}$$

$$+Q(Y,\theta+i\lambda_o a,\lambda_o\varepsilon)\{\prod_{<i^*,j^*>\in\tilde{\Gamma}(Y)}\tilde{I}_\beta(\ldots)\}^{-1}$$

$$\times\{\sum_{<i^*,j^*>\in\tilde{\Gamma}(Y)}\frac{dI_\beta}{d\lambda}(\ldots)[I_\beta(\ldots)]^{-1}\}\ . \tag{1.20}$$

The arguments of the $I_\beta(\cdot)$ functions have been omitted to save space. They are in each case equal to $[\theta_{i^*}-\theta_{j^*}+i\lambda_o(a_{i^*}-a_{j^*}+\varepsilon\delta_{<i^*,j^*>,\gamma})]$. The definition of $I_\beta(\cdot)$ allows one to calculate the derivatives, which gives

$$\Big|\sum_{<i^*,j^*>\in\tilde{\Gamma}(Y)}[I_\beta(\ldots)]^{-1}\Big(\frac{dI_\beta}{d\lambda}(\ldots)\Big)\Big|\le 4|\tilde{\Gamma}(Y)|\,(1-e^{-\frac{1}{2}\beta})^{-2}$$

$$\times\sup_{<i^*,j^*>\in\tilde{\Gamma}(Y)}|a_{i^*}-a_{j^*}+\varepsilon\delta_{<i^*,j^*>,\gamma}|\ . \tag{1.21}$$

Combining (1.21) with (1.20) and (1.16) gives

$$\Big|\frac{d\hat{Q}}{d\lambda}(Y,\theta+i\lambda a,\lambda\varepsilon)\Big|_{\lambda=\lambda_o}\Big|\le\sup_{<i^*,j^*>\in\tilde{\Gamma}(Y)}|a_{i^*}-a_{j^*}+\varepsilon\delta_{<i^*,j^*>,\gamma}|$$

$$\times\Big[\prod_{\ell=2}^{k}(s_{\eta(\ell)}\cdots s_{\ell-2})\tilde{X}_{X_{\ell-1}}(\Gamma_1\cup\ldots\cup\Gamma_{\eta(\ell)-1};\Gamma_{\eta(\ell)})\ e^{-J|X_{\ell-1}|}\Big]$$

$$\times 2|\tilde{\Gamma}(Y)|2^{|\tilde{\Gamma}(Y)|}(1-e^{-\frac{\beta}{2}})^{-|\tilde{\Gamma}(Y)|}\exp(e^{-\frac{3}{4}J}\Big|\bigcup_1^{k-1}X_m\Big|)$$

$$\times\{4(1-e^{-\frac{1}{2}\beta})^{-2}+2|\tilde{\Gamma}(Y)|e^{-\frac{\beta}{2}}(1-e^{-\frac{\beta}{2}})^{-2-|\tilde{\Gamma}(Y)|}\}\ . \tag{1.22}$$

To bound the second derivative of $\hat{Q}(Y,\theta+i\lambda a,\lambda\varepsilon)$, we use (1.13), (1.18), (1.21) and

$$\Big|\sum_{<i^*,j^*>\in\tilde{\Gamma}(Y)}\Big(\frac{d^2\tilde{I}_\beta}{d\lambda^2}\Big|_{\lambda=\lambda_o}\Big)\times\Big(I_\beta\Big)^{-1}\Big|$$

$$\le|\tilde{\Gamma}(Y)|\Big(\frac{8}{(1-e^{-\frac{1}{2}\beta})^3}\Big)\sup_{<i^*,j^*>\in\tilde{\Gamma}(Y)}(a_{i^*}-a_{j^*}+\varepsilon\delta_{<i^*,j^*>,\gamma})^2\ , \tag{1.23}$$

which follows again from the explicit form of $\tilde{I}_\beta(\cdot)$. (See (III.1.10).) Once again the arguments of the functions on the l.h.s. of (1.23) are just $(\theta_{i*}-\theta_{j*}+\lambda_o(a_{i*}-a_{j*}+\varepsilon\delta_{<i*,j*>},\gamma)$. One then obtains

$$\left|\frac{d^2\hat{Q}}{d\lambda^2}(Y,\theta+i\lambda a,\lambda\varepsilon)\right|_{\lambda=\lambda_o}\Bigg| \le \sup_{<i*,j*>\in\tilde{\Gamma}(Y)} (a_{i*}-a_{j*}+\varepsilon\delta_{<i*,j*>},\gamma)^2$$

$$\times\left[\prod_{\ell=2}^{k}(s_{\eta(\ell)}\cdots s_{\ell-2})\tilde{\chi}_{X_{\ell-1}}(\Gamma_1\cup\ldots\cup\Gamma_{\eta(\ell)-1};\Gamma_{\eta(\ell)})e^{-J|X_{\ell-1}|}\right]$$

$$\times\left\{\left[\exp(e^{-\frac{3}{4}J\left|\bigcup_{m=1}^{k-1}X_m\right|})\right](4)^{|\tilde{\Gamma}(Y)|}(1-e^{-\frac{1}{2}\beta})^{-|\tilde{\Gamma}(Y)|}(|\tilde{\Gamma}(Y)|)\right\}$$

$$\times\Big\{(16)(1-e^{-\frac{1}{2}\beta})^{-|\tilde{\Gamma}(Y)|-2}+4|\tilde{\Gamma}(Y)|(1-e^{-\frac{1}{2}\beta})^{-|\tilde{\Gamma}(Y)|-4}$$

$$+(16)(1-e^{-\frac{1}{2}\beta})^{-|\tilde{\Gamma}(Y)|-2}+|\tilde{\Gamma}(Y)|(1-e^{-\frac{1}{2}\beta})^{-4}+8(1-e^{-\frac{1}{2}\beta})^{-3}\Big\}\ . \tag{1.24}$$

In the last curly bracket, the first two terms come from estimating the case in which both derivatives act on the function $Q(.,.,.)$, the third term comes from the case when one derivative acts on the function $Q(.,.,.)$ and one on the product of $\tilde{I}_\beta(\cdot)$ functions, and the last two terms come from the case in which both derivatives act on the product of $\tilde{I}_\beta(\cdot)$ functions.

The important point is that there exist constants $K_1(\beta)$ and $K_2(\beta)$ such that the quantities in curly brackets in (1.23) and (1.24) may be bounded from above by $\exp(K_1(\beta)|\tilde{\Gamma}(Y)|)$ and $\exp(K_2(\beta)|\tilde{\Gamma}(Y)|)$ respectively. (We use the fact that because of the definition of Γ_k one has $\left|\bigcup_1^{k-1}X_m\right| \le c|\tilde{\Gamma}(Y)|$.) Using (1.13) and our bounds on $I_\beta(\cdot)$, one has

$$|\hat{Q}(Y,\theta+i\lambda_o a,\lambda_o\varepsilon)| \le e^{K_3(\beta)|\tilde{\Gamma}(Y)|}$$

$$\times\prod_{\ell=2}^{k}[s_{\eta(\ell)}\cdots s_{\ell-2}]\tilde{\chi}_{X_{\ell-1}}(\Gamma_1\cup\ldots\cup\Gamma_{\eta(\ell)-1};\Gamma_{\eta(\ell)})e^{-J|X_{\ell-1}|}\ . \tag{1.25}$$

Pick $J_2(\beta) > 8\cdot\max(K_1(\beta),K_2(\beta),K_3(\beta))$ and use the fact that $\prod_{\ell=2}^{k-1}\exp(-J|X_\ell|) \le \exp(-J|\tilde{\Gamma}(Y)|)$. Then

$$\left|\frac{1}{2}\frac{d^2G_\lambda}{d\lambda^2}\right|_{\lambda=\lambda_o}\Bigg| \le \sup_{\substack{<i*,j*>\in\tilde{\Gamma}(Y_\ell)\\ \ell=1,\ldots,m}} (a_{i*}-a_{j\&}+\varepsilon\delta_{<i*,j*>},\gamma)^2(m+2m^2)$$

$$\times\prod_{\ell=1}^{m}\Big(\prod_{j=2}^{k_\ell}[s_{\eta(j)}\cdots s_{j-2}]\tilde{\chi}_{X_{j-1}(Y_\ell)}(\Gamma_1(Y_\ell)\cup\ldots\cup\Gamma_{\eta(j)-1}(Y_\ell);\Gamma_{\eta(j)}(Y_\ell))$$

$$\times e^{-\frac{7J}{8}|X_{j-1}(Y_\ell)|}\Big)\ . \tag{1.26}$$

By k_ℓ we mean the integer k associated with the cluster Y_ℓ. Similarly $\Gamma_j(Y_\ell)$ and $X_j(Y_\ell)$ are the sets Γ_j and X_j associated with Y_ℓ. Since (possibly by enlarging $J_2(\beta)$) one has

$$(m+2m^2)\prod_{\ell=1}^{m}\Big(\prod_{j=1}^{k_\ell}e^{-\frac{1}{8}J|X_{j-1}(Y_\ell)|}\Big) \le 1\ , \tag{1.27}$$

for all $J > J_2(\beta)$, eqn. (1.26) leads immediately to (1.2).

VI.2. Bounds on Sums of Cluster Functions

In this section we prove estimates on sums of the new cluster functions $S(Y)$, defined in (1.3). The principle result is

Lemma VI.2. For every $\beta > 0$ there exists $J_2(\beta) > 0$ such that for all $J > J_2(\beta)$,

$$\sum_{\substack{(Y_1,\ldots,Y_m):\\ <i^*,j^*>\in \bigcup_1^m \tilde{\Gamma}(Y_k)}}{}' \left| \sum_{g\in G_c} \left(\prod_{\ell\in g} A(\ell) \right) \prod_{k=1}^m S(Y_k) \right| \le m!\, e^{-\frac{1}{8}Jm} . \tag{2.1}$$

Recall that $\sum'$ just means that we sum only over clusters with $k \ge 2$.

This lemma implies Proposition IV.8. By Lemma VI.1,

$$\left| \exp\left\{ \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{(Y_1,\ldots,Y_m)}{}' \left(\sum_{g\in G_c} [\prod_{\ell} A(\ell)] \right) \left(\prod_{j=1}^m \hat{Q}(Y_j,\theta+ia,\varepsilon) - \prod_{j=1}^m \hat{Q}(Y,\theta,0) \right) \right\} \right|$$

$$= \left| \exp \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{(Y_1,\ldots,Y_m)}{}' \left(\sum_{g\in G_c} [\prod_{\ell} A(\ell)] \right) Q_2(Y_1,\ldots,Y_m,\theta,a,\varepsilon) \right|$$

$$\le \exp \left| \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{(Y_1,\ldots,Y_m)}{}' \left(\sum_{g\in G_c} [\prod_{\ell} A(\ell)] \right) \prod_{k=1}^m S(Y_k) \cdot \frac{Q_2(Y_1,\ldots,Y_m,\theta,a)}{\prod_{k=1}^m S(Y_k)} \right|$$

$$\le \exp \left\{ \sum_{<i^*,j^*>} \left(\sum_{m=1}^{\infty} \frac{1}{m} \sum_{\substack{(Y_1,\ldots,Y_m):\\ <i^*,j^*>\in \bigcup_1^m \tilde{\Gamma}(Y_k)}}{}' \left| \sum_{g\in G_c} [\prod_{\ell} A(\ell)] \prod_{k=1}^m S(Y_k) \right| \right) \times (a_{i^*}-a_{j^*}+\varepsilon\delta_{<i^*,j^*>,\gamma})^2 \right\}. \tag{2.2}$$

In the last inequality we have applied (1.3) and insured that the $\sup(a_{i^*}-a_{j^*}+\varepsilon\delta_{<i^*,j^*>,\gamma})^2$ is attained by summing over all $<i^*,j^*>$ in $\bigcup_1^m \tilde{\Gamma}(Y_k)$. Lemma VI.2 bounds this expression by

$$\exp\left(\sum_{m=1}^{\infty} e^{-\frac{1}{8}Jm} \right) \left(\sum_{<i^*,j^*>} (a_{i^*}-a_{j^*}+\varepsilon\delta_{<i^*,j^*>,\gamma})^2 \right) . \tag{2.3}$$

This is the bound of Proposition IV.8 if we make the identification $c''(\beta,J) = \sum_{m=1}^{\infty} e^{-\frac{1}{8}Jm}$.

Note that Lemma VI.2 combined with (1.25), also insures that the sums over $(Y_1,\ldots,Y_m)$ in (IV.2.10) are absolutely convergent and hence that the manipulations leading to the exponentiated form of the expansion were justified.

We now prove Lemma VI.2. The first step is

Lemma VI.3. There exists $J_2(\beta) > 0$ such that for $J > J_2(\beta)$

$$\sum_{\substack{(Z_1,\ldots,Z_m)\\ \tilde{\Gamma}(Z_s)\cap\tilde{\Gamma}(Y_1)\neq\phi\\ s=1,\ldots,m\\ \sum_s|\tilde{\Gamma}(Z_s)|=M}} \prod_{s=1}^m S(Z_s) \le |\tilde{\Gamma}(Y_1)|^m \cdot e^{-\frac{3}{16}JM} , \tag{2.4}$$

for any cluster Y_1.

Proof. There exists $J_2(\beta) > 0$ such that $J > J_2(\beta)$ implies

$$\sum_{\substack{Y:\langle i^*,j^*\rangle\in\tilde{\Gamma}(Y)\\ |\tilde{\Gamma}(Y)|=N}}' S(Y) \le e^{-\frac{1}{4}JN} . \tag{2.5}$$

The l.h.s. of this expression may be rewritten as

$$\sum_{k=2}^{\infty} \sum_{\substack{X_1\ldots X_{k-1}:\\ |\Gamma(Y)|=N;\langle i^*,j^*\rangle\in\tilde{\Gamma}(Y)}} \sum_{\eta} \int_0^1 ds_1\ldots ds_{k-1}$$

$$\times \left(\prod_{j=2}^{k} [s_{\eta(j)}\cdots s_{j-2}]\tilde{\chi}_{X_{j-1}}(\Gamma_1\cup\ldots\cup\Gamma_{\eta(j)-1};\Gamma_{\eta(j)})e^{-\frac{3}{4}J|X_{j-1}|}\right) . \tag{2.6}$$

Now fix $|\Gamma_2| = n_2,\ldots,|\Gamma_k| = n_k$. (One has $|\Gamma_1| = 1$ always.) There are at most 2^N ways of partitioning the Γ_k's. Expression (2.6) is then bounded by

$$2^N\cdot \sum_{k=2}^{\infty}(k)e^{-\frac{5}{16}JN} \sup_{\{n_1,\ldots,n_k\}} \sum_{\substack{X_1\ldots X_{k-1}:\\ \langle i^*,j^*\rangle\in\Gamma(X_1)\\ |\Gamma_2|=n_2,\ldots,|\Gamma_k|=n_k}} \sum_{\eta} \int_0^1 ds_1,\ldots,ds_{k-1}$$

$$\times\left\{\prod_{j=2}^{k}(s_{\eta(j)}\cdots s_{j-2})\tilde{\chi}_{X_j}(\Gamma_1\cup\ldots\cup\Gamma_{\eta(j)-1};\Gamma_{\eta(j)})e^{-\frac{1}{8}J|X_{j-1}|}\right\} . \tag{2.7}$$

The last step assumed that $\langle i^*,j^*\rangle \in \Gamma(X_1)$ and compensated by multiplying by k. We now claim that

$$\sum_{X_1}\cdots\sum_{\substack{X_{k-1}:\\ \langle i^*,j^*\rangle\in\Gamma(X_1)\\ |\Gamma_2|=n_2,\ldots,|\Gamma_k|=n_k}} \left(\prod_{j=2}^{k}[s_{\eta(j)},\ldots,s_{j-2}]\tilde{\chi}_{X_{j-1}}(\Gamma_1\cup\ldots\cup\Gamma_{\eta(j)-1};\Gamma_{\eta(j)})e^{-\frac{1}{8}J|X_{j-1}|}\right)$$

$$\le \prod_{j=2}^{k-1}\{[s_{\eta(j)}\cdots s_{j-2}]n_{\eta(j)}\}\, e^{-\frac{kJ}{32}} . \tag{2.8}$$

For $k = 2$, we need to show that

$$\sum_{\substack{X_1\\ \langle i^*,j^*\rangle\in\Gamma(X_1)\\ |\Gamma_2|=n_2}} e^{-\frac{1}{8}J|X_1|} \le e^{-\frac{J}{16}} . \tag{2.9}$$

This follows from the standard Peierls argument. Assume that (2.8) holds for $k \le m-1$.

$$\sum_{X_{m-1}}(s_{\eta(m)}\cdots s_{m-2})\tilde{\chi}_{X_{m-1}}(\Gamma_1\cup\ldots\cup\Gamma_{\eta(m)-1};\Gamma_{\eta(m)})e^{-\frac{J}{8}|X_{m-1}|}$$

$$\le \sum_{\langle i^*,j^*\rangle\in\Gamma_{\eta(m)}} \sum_{\substack{X:\\ \langle i^*,j^*\rangle\in\Gamma(X)}} (s_{\eta(m)}\cdots s_{m-2})e^{-\frac{J}{8}|X|}$$

$$\le (s_{\eta(m)}\cdots s_{m-2}) \times |\Gamma_{\eta(m)}|e^{-\frac{J}{32}} . \tag{2.10}$$

The first inequality follows because all terms in the first sum with

$\Gamma(X_{m-1}) \cap \Gamma_{\eta(m)} = \phi$ vanish. In the second inequality, the Peierls argument is used to bound the sum over X with $\langle i^*,j^*\rangle \in \Gamma(X)$. Equation (2.10) implies

$$\sum_{\substack{X_1 \\ \langle i^*,j^*\rangle \in \Gamma(X_1) \\ |\Gamma_2| = n_2,\ldots,|\Gamma_m|=n_m}} \cdots \sum_{X_{m-1}} \left(\prod_{j=2}^{m} s_{\eta(j)} \cdots s_{j-2} \tilde{\chi}_{X_{j-1}} (\Gamma_1 \cup \cdots \cup \Gamma_{\eta(j)-1}; \Gamma_{\eta(j)}) e^{-\frac{1}{8}J|X_{j-1}|} \right)$$

$$\leq \sum_{\substack{X_1 \\ \langle i^*,j^*\rangle \in \Gamma(X_1) \\ |\Gamma_2|=n_2,\ldots,|\Gamma_{m-1}|=n_{m-1}}} \cdots \sum_{X_{m-2}} \left(\prod_{j=2}^{m-1} s_{\eta(j)} \cdots s_{j-2} \tilde{\chi}_{X_{j-1}} (\Gamma_1 \cup \cdots \cup \Gamma_{\eta(j)-1}; \Gamma_{\eta(j)}) e^{-\frac{1}{8}J|X_{j-1}|} \right) \times (s_{\eta(m)} \cdots s_{m-2} n_{\eta(m)}) e^{-\frac{J}{16}} \quad . \tag{2.11}$$

Bound the remaining sums by the induction hypothesis, thereby obtaining (2.8). By standard estimates (see [9])

$$\sum_{\eta} \int_0^1 ds_1 \cdots ds_{k-1} \left(\prod_{j=1}^{k} (s_{\eta(j)} \cdots s_{j-2} n_{\eta(j)}) \right) \leq \exp\left(\sum_1^k n_j\right) = e^N \quad . \tag{2.12}$$

Combining (2.5), (2.7), (2.8) and (2.12) yields

$$\sum_{\substack{Y:\langle i^*,j^*\rangle \in \tilde{\Gamma}(Y) \\ |\tilde{\Gamma}(Y)|=N}}' S(Y) \leq 2^N \sum_{k=2}^{\infty} (k) e^{-\frac{5}{16}JN} \cdot e^N e^{-\frac{kJ}{32}}$$

$$\leq e^{-\frac{1}{4}JN} \quad , \tag{2.13}$$

for J sufficiently large.

Now use (2.5) to prove (2.4).

$$\sum_{\substack{(Z_1,\ldots,Z_m) \\ \tilde{\Gamma}(Z_s)\cap\tilde{\Gamma}(Y_1)\neq\phi \\ s=1,\ldots,m \\ \sum_s |\tilde{\Gamma}(Z_s)|=M}}' \prod_{s=1}^{m} S(Z_s) \leq 2^M \cdot \sup_{\substack{\{n_1,\ldots,n_m\} \\ \Sigma n_k = M}} \sum_{\substack{(Z_1,\ldots,Z_m) \\ \tilde{\Gamma}(Z_s)\cap\tilde{\Gamma}(Y_1)\neq\phi \\ s=1,\ldots,m \\ |\tilde{\Gamma}(Z_1)|=n_1,\ldots,|\tilde{\Gamma}(Z_m)|=n_m}}' \prod_{s=1}^{m} S(Z_s) \tag{2.14}$$

The factor of 2^M bounds the number of possibilities for splitting the total number of bonds in $\Sigma|\tilde{\Gamma}(Z_s)|$ into subsets $n_1,\ldots,n_m$. Since each $\tilde{\Gamma}(Z_s)$ intersects $\tilde{\Gamma}(Y_1)$, the r.h.s. of (2.14) is bounded by

$$2^M \cdot \sup_{\substack{\{n_1,\ldots,n_m\} \\ \Sigma n_k = M}} \prod_{s=1}^{m} \left(\sum_{\langle i^*,j^*\rangle \in \tilde{\Gamma}(Y_1)} \sum_{\substack{Z_s:\langle i^*,j^*\rangle \in \tilde{\Gamma}(Z_s) \\ |\Gamma(Z_s)|=n_s}}' S(Z_s) \right) \cdot$$

$$\leq 2^M \cdot \sup_{\substack{\{n_1,\ldots,n_m\} \\ \Sigma n_k = M}} \prod_{s=1}^{m} \left(|\tilde{\Gamma}(Y_1)| e^{-\frac{1}{4}Jn_s} \right)$$

$$\leq 2^M |\tilde{\Gamma}(Y_1)|^m \cdot e^{-\frac{1}{4}JM} \leq |\tilde{\Gamma}(Y_1)|^m e^{-\frac{3}{16}JM} \quad , \tag{2.15}$$

for J sufficiently large. In the first inequality, we have used (2.5) to bound the sum over Z_s and then bounded the number of terms in the sum over $\langle i^*,j^*\rangle$ by $|\tilde{\Gamma}(Y_1)|$. This completes the proof of Lemma VI.3.

In analogy with the proof of the lower bounds in Chapter V, define

$$\Psi(Y_1,\ldots,Y_n;Z_1,\ldots,Z_k) = \sum_{g\in G_c(Y_1,\ldots,Y_r;Z_1,\ldots,Z_k)} \Big(\prod_{\ell\in g} A(\ell)\Big) \prod_{j=1}^{k} S(Z_j) \quad . \tag{2.16}$$

Recall that $G_c(Y_1,\ldots,Y_r;Z_1,\ldots,Z_k)$ refers to the set of graphs in which each Z_j vertex is connected (directly or indirectly) to a Y_k vertex. One then has

<u>Lemma VI.4</u>. There exists $J_2(\beta) > 0$ such that for all $J > J_2(\beta)$ one has

$$\sum_{\substack{(Z_1,\ldots,Z_k)\\ \sum_1^k|\tilde\Gamma(Z_s)|=N}}{}' |\Psi(Y_1,\ldots,Y_r;Z_1,\ldots,Z_k)| \le k!\,e^{-\frac{J}{8}N}\cdot\exp\Big(\sum_1^r |\tilde\Gamma(Y_j)|\Big) \quad . \tag{2.17}$$

<u>Proof</u>. First note that $\Psi(\phi;Z_1,\ldots,Z_k) \equiv 0$ and $\Psi(Y_1,\ldots,Y_r;\phi) \equiv 1$. We then rewrite $\Psi(Y_1,\ldots,Y_r;Z_1,\ldots,Z_k)$ in terms of Ψ functions with less than $r+k$ vertices, via the manipulations detailed in [4, 7, 12, 25]. This leads to

$$\Psi(Y_1,\ldots,Y_r;Z_1,\ldots,Z_k) = \sum_{\Omega}\Big(\prod_{s\in\Omega} S(Z_s)\Big) \prod_{s\in\Omega} A(Y_1,Z_s) \prod_{\substack{j=2\\ s\in\Omega}}^{r} U(Y_1,Z_s)$$

$$\times\Big|\prod_{\substack{s_1<s_2\\ s_i\in\Omega}} U(Z_{s_1},Z_{s_2})\,\Psi(Y_2,\ldots,Y_r,(Z_s)_{s\in\Omega};(Z_s)_{s\notin\Omega})\Big| \quad . \tag{2.18}$$

Recall that given a graph $g \in G_c$, $\Omega \equiv \{s:\ell(Y_1,Z_s)\in g\}$. This leads to

$$\sum_{\substack{(Z_1,\ldots,Z_k)\\ \sum_1^k|\tilde\Gamma(Z_s)|=N}}{}' |\Psi(Y_1,\ldots,Y_r;Z_1,\ldots,Z_k)| \le \sum_{\substack{(Z_1,\ldots,Z_k)\\ \sum|\tilde\Gamma(Z_s)|=N}}{}' \prod_{s=1}^{k}\big[|A(Y_1,Z_s)|\,S(Z_s)\big]$$

$$+ k!\sum_{|\Omega|=1}^{k-1}\frac{1}{|\Omega|!}\sum_{M=|\Omega|}^{N-1}\sum_{\substack{(Z_1',\ldots,Z'_{|\Omega|})\\ \sum|\tilde\Gamma(Z_s')|=M}}{}' \Big|\prod_{s\in\Omega} A(Y_1,Z_s)\Big|\Big(\prod_{s\in\Omega} S(Z_s')\Big)$$

$$\times\frac{1}{(k-|\Omega|)!}\sum_{\substack{(Z_1'',\ldots,Z''_{z-|\Omega|})\\ \sum|\tilde\Gamma(Z_s'')|=N-M}}{}' |\Psi(Y_2,\ldots,Y_r,Z_s',\ldots,Z'_{|\Omega|};Z_1'',\ldots,Z''_{k-|\Omega|})|$$

$$+ \sum_{\substack{(Z_1,\ldots,Z_k)\\ \sum|\tilde\Gamma(Z_s)|=N}} |\Psi(Y_2,\ldots,Y_r;Z_1,\ldots,Z_k)| \quad . \tag{2.19}$$

The first term is handled by noting that $A(Y_1,Z_s)$ vanishes unless $\tilde\Gamma(Z_s) \cap \tilde\Gamma(Y_1) \neq \phi$, and then applying Lemma VI.3. The third term is handled by the induction hypothesis and the second combines the induction hypothesis and Lemma VI.3. This bounds (2.19) by

$$k!\,e^{-\frac{J}{8}N}\cdot\exp\Big(\sum_2^r|\tilde\Gamma(Y_j)|\Big)$$

$$\times\Big\{1+\frac{1}{k!}|\tilde\Gamma(Y_1)|^k e^{-\frac{1}{8}JN} + \sum_{|\Omega|=1}^{k-1}\frac{1}{|\Omega|!}e^{M}e^{\frac{1}{8}JM}|\Gamma(Y_1)|^{|\Omega|}e^{-\frac{1}{4}JM}\Big\} \quad . \tag{2.20}$$

If J is sufficiently large $\exp(\frac{1}{8}J + 1 - \frac{1}{4}J)M \le 1$, and the quantity in curly brackets is bounded by $\exp(|\tilde\Gamma(Y_1)|)$ which completes the

$\Gamma(X_{m-1}) \cap \Gamma_{\eta(m)} = \phi$ vanish. In the second inequality, the Peierls argument is used to bound the sum over X with $\langle i^*,j^*\rangle \in \Gamma(X)$. Equation (2.10) implies

$$\sum_{\substack{X_1 \\ \langle i^*,j^*\rangle \in \Gamma(X_1) \\ |\Gamma_2| = n_2,\ldots,|\Gamma_m|=n_m}} \cdots \sum_{X_{m-1}} \left(\prod_{j=2}^{m} s_{\eta(j)} \cdots s_{j-2} \tilde{\chi}_{X_{j-1}} (\Gamma_1 \cup \cdots \cup \Gamma_{\eta(j)-1}; \Gamma_{\eta(j)}) e^{-\frac{1}{8}J|X_{j-1}|} \right)$$

$$\leq \sum_{\substack{X_1 \\ \langle i^*,j^*\rangle \in \Gamma(X_1) \\ |\Gamma_2|=n_2,\ldots,|\Gamma_{m-1}|=n_{m-1}}} \cdots \sum_{X_{m-2}} \left(\prod_{j=2}^{m-1} s_{\eta(j)} \cdots s_{j-2} \tilde{\chi}_{X_{j-1}} (\Gamma_1 \cup \cdots \cup \Gamma_{\eta(j)-1}; \Gamma_{\eta(j)}) e^{-\frac{1}{8}J|X_{j-1}|} \right) \times (s_{\eta(m)} \cdots s_{m-2}^{n_{\eta(m)}}) e^{\frac{J}{16}} \quad . \tag{2.11}$$

Bound the remaining sums by the induction hypothesis, thereby obtaining (2.8). By standard estimates (see [9])

$$\sum_{\eta} \int_0^1 ds_1 \cdots ds_{k-1} \left(\prod_{j=1}^{k} (s_{\eta(j)} \cdots s_{j-2}^{n_{\eta(j)}}) \right) \leq \exp\left(\sum_1^k n_j\right) = e^N \quad . \tag{2.12}$$

Combining (2.5), (2.7), (2.8) and (2.12) yields

$$\sum_{\substack{Y:\langle i^*,j^*\rangle \in \tilde{\Gamma}(Y) \\ |\tilde{\Gamma}(Y)|=N}}' S(Y) \leq 2^N \sum_{k=2}^{\infty} (k) e^{-\frac{5}{16}JN} \cdot e^N e^{-\frac{kJ}{32}}$$

$$\leq e^{-\frac{1}{4}JN} \quad , \tag{2.13}$$

for J sufficiently large.

Now use (2.5) to prove (2.4).

$$\sum_{\substack{(Z_1,\ldots,Z_m) \\ \tilde{\Gamma}(Z_s)\cap\tilde{\Gamma}(Y_1)\neq\phi \\ s=1,\ldots,m \\ \sum_s |\tilde{\Gamma}(Z_s)|=M}}' \prod_{s=1}^{m} S(Z_s) \leq 2^M \cdot \sup_{\substack{\{n_1,\ldots,n_m\} \\ \Sigma n_k = M}} \sum_{\substack{(Z_1,\ldots,Z_m) \\ \tilde{\Gamma}(Z_s)\cap\tilde{\Gamma}(Y_1)\neq\phi \\ s=1,\ldots,m \\ |\tilde{\Gamma}(Z_1)|=n_1,\ldots,|\tilde{\Gamma}(Z_m)|=n_m}}' \prod_{s=1}^{m} S(Z_s) \tag{2.14}$$

The factor of 2^M bounds the number of possibilities for splitting the total number of bonds in $\Sigma|\tilde{\Gamma}(Z_s)|$ into subsets $n_1,\ldots,n_m$. Since each $\tilde{\Gamma}(Z_s)$ intersects $\tilde{\Gamma}(Y_1)$, the r.h.s. of (2.14) is bounded by

$$2^M \cdot \sup_{\substack{\{n_1,\ldots,n_m\} \\ \Sigma n_k = M}} \prod_{s=1}^{m} \left(\sum_{\langle i^*,j^*\rangle \in \tilde{\Gamma}(Y_1)} \sum_{\substack{Z_s:\langle i^*,j^*\rangle \in \tilde{\Gamma}(Z_s) \\ |\Gamma(Z_s)|=n_s}}' S(Z_s) \right) \cdot$$

$$\leq 2^M \cdot \sup_{\substack{\{n_1,\ldots,n_m\} \\ \Sigma n_k = M}} \prod_{s=1}^{m} \left(|\tilde{\Gamma}(Y_1)| e^{-\frac{1}{4}Jn_s} \right)$$

$$\leq 2^M |\tilde{\Gamma}(Y_1)|^m \cdot e^{-\frac{1}{4}JM} \leq |\tilde{\Gamma}(Y_1)|^m e^{-\frac{3}{16}JM} \quad , \tag{2.15}$$

for J sufficiently large. In the first inequality, we have used (2.5) to bound the sum over Z_s and then bounded the number of terms in the sum over $\langle i^*,j^*\rangle$ by $|\tilde{\Gamma}(Y_1)|$. This completes the proof of Lemma VI.3.

In analogy with the proof of the lower bounds in Chapter V, define

$$\Psi(Y_1,\ldots,Y_n;Z_1,\ldots,Z_k) = \sum_{g\in G_c(Y_1,\ldots,Y_r;Z_1,\ldots,Z_k)} \Big(\prod_{\ell\in g} A(\ell)\Big) \prod_{j=1}^{k} S(Z_j) \quad . \tag{2.16}$$

Recall that $G_c(Y_1,\ldots,Y_r;Z_1,\ldots,Z_k)$ refers to the set of graphs in which each Z_j vertex is connected (directly or indirectly) to a Y_k vertex. One then has

<u>Lemma VI.4</u>. There exists $J_2(\beta) > 0$ such that for all $J > J_2(\beta)$ one has

$$\sum_{\substack{(Z_1,\ldots,Z_k)\\ \sum_1^k|\tilde\Gamma(Z_s)|=N}}' |\Psi(Y_1,\ldots,Y_r;Z_1,\ldots,Z_k)| \le k!e^{-\frac{J}{8}N}\cdot\exp\Big(\sum_1^r|\tilde\Gamma(Y_j)|\Big) \quad . \tag{2.17}$$

<u>Proof</u>. First note that $\Psi(\phi;Z_1,\ldots,Z_k) \equiv 0$ and $\Psi(Y_1,\ldots,Y_r;\phi) \equiv 1$. We then rewrite $\Psi(Y_1,\ldots,Y_r;Z_1,\ldots,Z_k)$ in terms of Ψ functions with less than $r+k$ vertices, via the manipulations detailed in [4, 7, 12, 25]. This leads to

$$\Psi(Y_1,\ldots,Y_r;Z_1,\ldots,Z_k) = \sum_{\Omega}\Big(\prod_{s\in\Omega} S(Z_s)\Big) \prod_{s\in\Omega} A(Y_1,Z_s) \prod_{\substack{j=2\\ s\in\Omega}}^{r} U(Y_1,Z_s)$$

$$\times\Big|\prod_{\substack{s_1<s_2\\ s_i\in\Omega}} U(Z_{s_1},Z_{s_2})\Psi(Y_2,\ldots,Y_r,(Z_s)_{s\in\Omega};(Z_s)_{s\notin\Omega})\Big| \quad . \tag{2.18}$$

Recall that given a graph $g \in G_c$, $\Omega \equiv \{s:\ell(Y_1,Z_s)\in g\}$. This leads to

$$\sum_{\substack{(Z_1,\ldots,Z_k)\\ \sum_1^k|\tilde\Gamma(Z_s)|=N}}' |\Psi(Y_1,\ldots,Y_r;Z_1,\ldots,Z_k)| \le \sum_{\substack{(Z_1,\ldots,Z_k)\\ \sum|\tilde\Gamma(Z_s)|=N}}' \prod_{s=1}^{k}\big[|A(Y_1,Z_s)|S(Z_s)\big]$$

$$+ k!\sum_{|\Omega|=1}^{k-1}\frac{1}{|\Omega|!}\sum_{M=|\Omega|}^{N-1}\sum_{\substack{(Z'_1,\ldots,Z'_{|\Omega|})\\ \Sigma|\tilde\Gamma(Z'_s)|=M}}' \Big|\prod_{s\in\Omega}A(Y_1,Z_s)\Big|\Big(\prod_{s\in\Omega}S(Z'_s)\Big)$$

$$\times\frac{1}{(k-|\Omega|)!}\sum_{\substack{(Z''_1,\ldots,Z''_{z-|\Omega|})\\ \Sigma|\tilde\Gamma(Z''_s)|=N-M}}' |\Psi(Y_2,\ldots,Y_r,Z'_s,\ldots,Z'_{|\Omega|};Z''_1,\ldots,Z''_{k-|\Omega|})|$$

$$+\sum_{\substack{(Z_1,\ldots,Z_k)\\ \Sigma|\tilde\Gamma(Z_s)|=N}} |\Psi(Y_2,\ldots,Y_r;Z_1,\ldots,Z_k)| \quad . \tag{2.19}$$

The first term is handled by noting that $A(Y_1,Z_s)$ vanishes unless $\tilde\Gamma(Z_s) \cap \tilde\Gamma(Y_1) \neq \phi$, and then applying Lemma VI.3. The third term is handled by the induction hypothesis and the second combines the induction hypothesis and Lemma VI.3. This bounds (2.19) by

$$k!e^{-\frac{J}{8}N}\cdot\exp\Big(\sum_2^r|\tilde\Gamma(Y_j)|\Big)$$

$$\times\Big\{1+\frac{1}{k!}|\tilde\Gamma(Y_1)|^k e^{-\frac{1}{8}JN} + \sum_{|\Omega|=1}^{k-1}\frac{1}{|\Omega|!}e^{M}e^{\frac{1}{8}JM}|\Gamma(Y_1)|^{|\Omega|}e^{-\frac{1}{4}JM}\Big\} \quad . \tag{2.20}$$

If J is sufficiently large $\exp(\frac{1}{8}J + 1 - \frac{1}{4}J)M \le 1$, and the quantity in curly brackets is bounded by $\exp(|\tilde\Gamma(Y_1)|)$ which completes the

induction.

Finally, we use Lemma VI.4 to prove Lemma VI.2.

$$\sum_{\substack{(Y_1,\ldots,Y_m)\\ \langle i^*,j^*\rangle\in\bigcup_1^m\tilde{\Gamma}(Y_k)}}{}' \left|\sum_{g\in G_c}(\prod_\ell A(\ell))\prod_{k=1}^m S(Y_k)\right|$$

$$\leq m\cdot\sum_{\substack{(Y_1,\ldots,Y_m)\\ \langle i^*,j^*\rangle\in\tilde{\Gamma}(Y_1)}}{}' \left|\sum_{g\in G_c}(\prod_\ell A(\ell))\prod_{k=1}^m S(Y_k)\right| , \qquad (2.21)$$

where we assumed $\langle i^*,j^*\rangle \in \tilde{\Gamma}(Y_1)$ and compensated by introducing a factor of m. Using (2.18) this may be bounded by

$$m\cdot\sum_{\substack{Y_1:\\ \langle i^*,j^*\rangle\in\tilde{\Gamma}(Y_1)}}{}' \Big(\sum_{N=m}^{\infty}\sum_{\substack{(Y_2,\ldots,Y_m)\\ \Sigma|\tilde{\Gamma}(Y_s)|=N}}{}' |\Psi(Y_1;Y_2,\ldots,Y_m)|\Big)S(Y_1)$$

$$\leq \sum_{\substack{Y_1:\\ \langle i^*,j^*\rangle\in\tilde{\Gamma}(Y_1)}}{}' \Big(\sum_{N=m}^{\infty} m\cdot(m-1)!e^{-\frac{J}{8}N}\Big)\cdot e^{|\tilde{\Gamma}(Y_1)|}S(Y_1) , \qquad (2.22)$$

by Lemma VI.4. This may be rewritten as

$$m!e^{-\frac{Jm}{8}}(1-e^{-\frac{J}{8}})^{-1}\cdot\sum_{n=1}^{\infty}\sum_{\substack{Y_1:\\ \langle i^*,j^*\rangle\in\tilde{\Gamma}(Y_1)\\ |\tilde{\Gamma}(Y_1)|=n}}{}' e^{|\tilde{\Gamma}(Y_1)|}S(Y_1)$$

Eq. (2.23) to be continued

$$\leq m!e^{-\frac{Jm}{8}}(1-e^{-\frac{J}{4}})^{-1}\sum_{m=1}^{\infty}e^n\cdot e^{-\frac{1}{4}Jn}$$

$$\leq m!e^{-\frac{Jm}{8}} , \qquad (2.23)$$

provided J is sufficiently large. In the next to last inequality, we have applied (2.5). This completes the proof of Lemma VI.2.

APPENDIX I

THE DIRICHLET COVARIANCE

In this appendix we prove various estimates on the Dirichlet covariance. Let Λ be some square region in $\mathbb{R}^2$, with sides of length $(2m+1)$ for $m \in \mathbb{Z}$. Define $L_\Lambda = \{f : f$ is a function on $\mathbb{Z}^2$ which vanishes on $\mathbb{Z}^2 \cap \Lambda^c\}$. Then for $f \in L_\Lambda$, define Δ_Λ^D, the lattice Laplacian with Dirichlet boundary conditions, on $\partial\Lambda$ by its associated quadratic form,

$$\langle f,(-\Delta_\Lambda^D)f\rangle = \sum_{\langle i,j\rangle} |f_i - f_j|^2 \quad . \tag{A.I.1}$$

If Δ is the lattice Laplacian with free boundary conditions, a straightforward calculation shows that this is equivalent to

$$\Delta_\Lambda^D f = \chi_\Lambda \Delta f \quad , \tag{A.I.2}$$

where χ_Λ is the characteristic function of the region Λ. We first prove a logarithmic lower bound on the Dirichlet covariance, given the restriction that $|x| \le |\Lambda|^{1/8}$. Recall that we wish to prove the existence of a constant $K'' > 0$ such that

$$C_D(0,0) - C_D(0,x) \ge K'' \ln(1+x) \quad . \tag{A.I.3}$$

This bound is trivial for $x = 0$. To prove it for $|x| \ge 1$ we rewrite the Dirichlet covariance in terms of the covariance with free boundary conditions. Let Ξ be the set of periodic lines, separated by a

distance $(2m+2)$ in the x and y directions, which divides R^2 into $(2m+2) \times (2m+2)$ squares Δ_j, one of which, Δ_0, is centered at the origin (and hence contains Λ). For any point $x \in \Lambda$, define a set of points, $\{x_j\}$, invariant under reflection in any line belonging to Ξ, and with $x \in \{x_j\}$. One then has following [23].

$$C_D(i,0) - C_D(i,x) = \sum_{j=0}^{\infty} (-1)^{\varepsilon_j} [C_F(i-0_j) - C_F(i-x_j)] \quad . \tag{A.I.4}$$

where $C_F(x-y)$ is the covariance with free boundary conditions at infinity. Note that $0_0 = 0$ and $x_0 = x$ since $\Lambda \subset \Delta_0$. Standard estimates on the free covariance imply $C_F(0) - C_F(x) \geqslant K \cdot \ln(1+|x|)$ so our task is to estimate the contribution to (A.I.4) from all squares other than Δ_0. Define

$$F(i) = \sum_{j=1}^{\infty} (-1)^{\varepsilon_j} [C_F(i-0_j) - C_F(i-x_j)] \quad . \tag{A.I.4}$$

We see that $F(\cdot)$ is a solution to Laplace's equation throughout Δ_0. Thus by the maximum-minimum principle for elliptic difference operators it must attain both its maximum and minimum values on $\partial\Delta_0$, the boundary of Δ_0. However, we know that for any site $i \in \Delta_0$,

$$C_D(i,0) - C_D(i,x) = 0 = C_F(i-0) - C_F(i-x) + F(i) \quad . \tag{A.I.5}$$

Thus for any site $i \in \mathbb{Z}^2 \cap \Delta_0$

$$|F(i)| \leqslant \sup_{j \in \partial\Delta_0 \cap \mathbb{Z}^2} |C_F(j-0) - C_F(j-x)| \quad . \tag{A.I.6}$$

Standard estimates on the two-dimensional dipole potential show that

$$|C_F(j-0)-C_F(j-x)| < \frac{c\cdot|x|}{\min(|j-x|,|j|)} \tag{A.I.7}$$

$$< c''|\Lambda|^{-7/8} \quad ,$$

provided $|\Lambda|>1$, where in the second inequality we have used our bounds on $|x|$. Hence

$$C_D(0,0)-C_D(0,x) > C_F(0)-C_F(x)-|F(0)|$$

$$> K\cdot\ln(1+|x|) - c''|\Lambda|^{-7/8}. \tag{A.I.8}$$

Choosing Λ_0 such that for $|\Lambda| > \Lambda_0$, $c''|\Lambda|^{-7/8} < K/2$, one obtains the bound of (A.I.3) with $K'' = K/2$.

We now prove the upper bounds of Chapter III.

We first prove that for the function a_{i*} defined by (III.1.9) and any pair of nearest neighbor sites $\langle i^*,j^*\rangle$ one has $|a_{i*}-a_{j*}| < c\cdot\varepsilon$ for some constant c. Heuristically, $a_{i*}-a_{j*}$ is just the electric field (lattice gradient of the potential) of a line of dipoles stretching from 0 to x, which elementary electrostatic estimates show is uniformly bounded. Note that the definition of $X(\cdot)$ implies

$$a_{i*}-a_{j*} = \varepsilon\sum_{\ell^*\in\gamma_{L^*}} [C_D(i^*,\ell^*-\hat{e}_y)-C_D(i^*,\ell^*)-C_D(j^*,\ell^*-\hat{e}_y)+C_D(j^*,\ell^*)]$$

$$= \varepsilon\sum_{k=0}^{\infty}(-1)^{\varepsilon_k}\sum_{\ell^*\in\gamma_{L^*}}\{C_F(i^*,(\ell^*-\hat{e}_y)_k)-C_F(i^*,\ell_k^*)$$

$$- C_F(j^*,(\ell^*-\hat{e}_y)_k)+C_F(j^*,\ell_k^*)\} \quad . \tag{A.I.9}$$

In the second equality we again used the representation of the Dirichlet covariance in terms of the free covariance. We shall study just the contribution of the cube Δ_0 which coincides with Λ. The contribution of the other cubes is handled in a fashion very similar to that discussed in the lower bound. In the cube Δ_0 we must estimate

$$\sum_{\ell^*\in\gamma_{L^*}} \{C_F(i^*,\ell^*-\hat{e}_y) - C_F(i^*,\ell^*) - C_F(j^*,\ell^*-\hat{e}_y) + C_F(j^*,\ell^*)\} \ . \tag{A.I.10}$$

Since i^* and j^* are nearest neighbor sites, if $i^*=(x_1,x_2)$, j^* is either $(x_1\pm 1,x_2)$ or $(x_1,x_2\pm 1)$. Assume that it is (x_1+1,x_2). All the other cases are estimated in analogous fashion. Similarly, $\ell^*\in\gamma_{L^*}$ has coordinates $(m,0)$, $0\leqslant m\leqslant|x|$. (We have moved the origin of our coordinate system to the origin of the dual lattice.)

Using the translational invariance of the free covariance we find

$$|C_F(i^*,\ell-\hat{e}_y) - C_F(i^*,\ell^*) - C_F(j^*,\ell^*-\hat{e}_y) + C_F(j^*,\ell^*)|$$

$$= |C_F(i^*,\ell^*-\hat{e}_y) - C_F(i^*,\ell^*) - C_F(i^*,\ell^*-\hat{e}_y-\hat{e}_x) + C_F(i^*,\ell^*-\hat{e}_x)|$$

$$\leqslant \frac{|K\cdot x_2|}{1+x_1^2+x_2^2+n^2} \quad , \tag{A.I.11}$$

where we have used the fact that the second line in (A.I.11) is just the two-dimensional quadrupole potential, and then applied standard estimates to this quantity. Hence

$$\sum_{\ell^*\in\gamma_{L^*}} |C_F(i^*,\ell^*-\hat{e}_y) - C_F(i^*,\ell^*) - C_F(j^*,\ell^*-\hat{e}_y) + C_F(j^*,\ell^*)|$$

$$\leqslant \sum_{n=0}^{|x|} \frac{Kx_2}{1+x_1^2+x_2^2+n^2} \leqslant K' \ . \tag{A.I.12}$$

Combining this with the bound on contributions to (A.I.9) coming from squares other than Δ_o, we find

$$|a_{i*}-a_{j*}| \le K''\cdot\varepsilon \quad , \tag{A.I.13}$$

as claimed in (III.1.17).

Finally we turn to the bound of Lemma III.3. The basic task is to compute $\langle C_D\partial_2\chi,\partial_2\chi\rangle$. This is complicated by the fact that $C_D(i^*,j^*)$ is not translation invariant. First note that

$$\begin{aligned} C_D\partial_2\chi &= C_D(\chi_\Lambda\partial_2)(\chi_\Lambda\Delta)C_D\chi \\ &= \chi_\Lambda\partial_2 C_D\chi + C_D[(\chi_\Lambda\partial_2),(\chi_\Lambda\Delta)]C_D\chi \quad , \end{aligned} \tag{A.I.14}$$

where $[\cdot,\cdot]$ just denotes the usual operator commutator. Thus

$$\langle C_D\partial_2\chi,\partial_2\chi\rangle = \langle\chi_\Lambda\partial_2 C_D\chi,\partial_2\chi\rangle + \langle C_D[(\chi_\Lambda\partial_2),(\chi_\Lambda\Delta)]C_D\chi,\partial_2\chi\rangle \ . \tag{A.I.15}$$

If we denote vertical bonds (i.e., $j^* = i^*+\hat{e}_y$) by $\langle i^*,j^*\rangle_v$, and horizontal bonds by $\langle i^*,j^*\rangle_h$, we may rewrite the first term on the r.h.s. of (A.I.16) as

$$\begin{aligned} \langle\chi_\Lambda\partial_2 C_D\chi,\partial_2\chi\rangle &= \sum_{\langle i^*,j^*\rangle_v\subset\Lambda^*}[(C_D\chi)(j^*)-(C_D\chi)(i^*)][\chi(j^*)-\chi(i^*)] \\ &= \sum_{\langle i^*,j^*\rangle\subset\Lambda^*}[(C_D\chi)(j^*)-(C_D\chi)(i^*)][\chi(j^*)-\chi(i^*)] \\ &\quad - \sum_{\langle i^*,j^*\rangle_h\subset\Lambda^*}[(C_D\chi)(i^*)-(C_D\chi)(i^*)][\chi(j^*)-\chi(i^*)] \\ &= \langle(-\Delta_\Lambda^D)C_D\chi,\chi\rangle - \langle\chi_\Lambda\partial_1 C_D\chi,\partial_1\chi\rangle \\ &= \langle\chi,\chi\rangle - \langle\chi_\Lambda\partial_1 C_D\chi,\partial_1\chi\rangle \ . \end{aligned} \tag{A.I.16}$$

If we now combine the observations that

$$\langle\chi,\chi\rangle = |\gamma|$$

and

$$\langle\chi_\Lambda\partial_1 C_D\chi,\partial_1\chi\rangle = \langle C_D\partial_1\chi,\partial_1\chi\rangle + \langle C_D[\chi_\Lambda\Delta,\chi_\Lambda\partial_1]C_D\chi,\partial_1\chi\rangle \quad ,$$

with we find

$$\begin{aligned} |\langle C_D\partial_2\chi,\partial_2\chi\rangle-|\gamma|| &\le |\langle C_D\partial_1\chi,\partial_1\chi\rangle| \\ &\quad + |\langle C_D[(\chi_\Lambda\Delta),(\chi_\Lambda\partial_1)]C_D\chi,\partial_1\chi\rangle \\ &\quad + \langle C_D[(\chi_\Lambda\partial_2),(\chi_\Lambda\Delta)]C_D\chi,\partial_2\chi\rangle| \ . \end{aligned} \tag{A.I.17}$$

The first term in (A.I.18) is bounded using the inequality $C_D \le C_F$, from which we see

$$|\langle C_D\partial_1\chi,\partial_1\chi\rangle| \le |\langle C_F\partial_1\chi,\partial_1\chi\rangle| \le 2|C_F(0^*)-C_F(x^*)| \le K\,\ell n(1+x) \ . \tag{A.I.18}$$

Here, 0^* and x^* are the points at which $\partial_1\chi$ is non-vanishing. (See Figure A.1.)

Define the functions $\Psi = \chi_\Lambda\Delta\chi_\Lambda$, and $\zeta = \chi_\Lambda\partial_2\chi_\Lambda$. Then a straightforward calculation shows

$$\langle C_D[(\chi_\Lambda\Delta),(\chi_\Lambda\partial_2)]C_D\chi,\partial_2\chi\rangle = \langle C_D\Psi\partial_2 C_D,\partial_2\chi\rangle - \langle C_D\zeta\Delta C_D\chi,\partial_2\chi\rangle \tag{A.I.19}$$

Note that Ψ is -1 on the sites in $\mathbb{Z}^2\cap\Lambda$ closest to $\partial\Lambda$ and zero elsewhere. Similarly, ζ is -1 on the "top" row of sites in Λ (i.e., those sites i in $\mathbb{Z}^2\cap\Lambda$ such that $i+\hat{e}_y\in\mathbb{Z}^2\cap\Lambda^c$) and zero elsewhere. Using this fact, one may rewrite

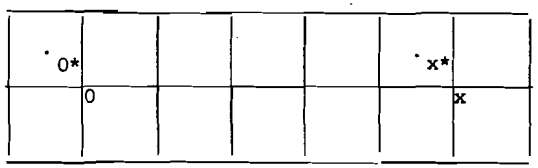


Figure A.1. The points 0* and x*.

$$\langle C_D \Psi \partial_2 C_D, \partial_2 \chi \rangle = \sum_{\substack{i^* \in \gamma_L^* \\ k^* \in \gamma_L^* \\ j^* \in \operatorname{supp} \Psi}} \{ [C_D(i^*-\hat{e}_y, j^*) - C_D(i^*, j^*)] \times [C_D(j^*-\hat{e}_y, k^*) - C_D(j^*, k^*)] \} \ . \quad \text{(A.I.20)}$$

Standard estimates imply $|C_D(i^*-\hat{e}_y, j^*) - C_D(i^*, j^*)| \le K_1 |i^*-j^*|^{-1}$. Because of the restrictions on the position of x, if $L = |\Lambda|^{\frac{1}{2}}$, one has $|i^*-j^*| > (2m+1)-(2m+1)^{\frac{1}{4}}$ and $|j^*-k^*| > (2m+1)-(2m+1)^{\frac{1}{4}}$. Also, there are at most $|x|$ terms in the sums over $i^* \in \gamma_L^*$ and $k^* \in \gamma_L^*$, and $4(2m+1)$ terms in the sum over $j^* \in \operatorname{supp} \Psi$, so that

$$|\langle C_D \Psi \partial_2 C_D \ , \partial_2 \chi \rangle| \le 4(2m+1) \cdot |x|^2 \cdot \frac{K_1^2}{((2m+1)-(2m+1)^{1/4})^2}$$

$$\le \frac{4(2m+1)^{3/2} \cdot K_1^2}{(2m+1)^2 - 2(2m+1)^{5/4} + (2m+1)^{5/4}}$$

$$\le K_2 \ . \quad \text{(A.I.21)}$$

In the second inequality, we have used the fact that $|x| < (2m+1)^{1/4}$. In analogous fashion, $|\langle C_D \zeta \Delta C_D \chi, \partial_2 \chi \rangle|$ and $|\langle C_D [(\chi_\Lambda \partial_2), (\chi_\Lambda \Delta)] C_D \chi, \partial_2 \chi \rangle|$ are both bounded by constants, which when combined with (A.I.24) and (A.I.25), yields the bound of Lemma III.3.

APPENDIX II

THE MULTIPOLE EXPANSION OF FRÖHLICH AND SPENCER

In this appendix we discuss the proof of Theorem II.3 and a number of related results from [15]. The idea of the expansion may be observed through the following approximate argument. Ignore for a moment the non-local potentials, and approximate the sum over $\{m\}$ in the expression (II.1.5) by the terms with $\{m\} \equiv 0$, and a term with one $m_i = +1$ and one $m_j = -1$. For the term with $\{m\} = 0$, the logarithmic lower bound on the correlations can be proved by elementary means. The term with non-zero $\{m\}$ is treated by noting that if the functions $I_\beta(\cdot)$ are replaced by Gaussian functions, the integrals over $\{\phi\}$ can be exactly performed and we obtain the electrostatic energy of a two-dimensional dipole with endpoints at i and j. Such a dipole (with one endpoint fixed) has an entropy proportional to $e^{c\ln|i-j|}$. Its energy behaves like $e^{-c'\beta^{-1}\ln|i-j|}$. Thus for small β, the energy will dominate the entropy, and the dipole term will contribute only a small correction to the $\{m\} = 0$ term. In making this argument rigorous, the two principal problems which must be overcome are first to discover a way to include the effects of all the rest of the terms in the sum over $\{m\}$, and second to estimate the electrostatic energy of charge configurations with the functions $I_\beta(\cdot)$ and all the non-local potentials present instead of just Gaussian functions.

Theorem II.3 provides the solution to the first of these problems.

We discuss some of the principle ideas of the proof. For full details see [15]. First pick a positive number, b, such that $b\sum_{n=1}^{\infty} n^{-4} = 1$. Then

$$(1+2\sum_{m=1}^{\infty}\cos(2\pi m\phi)) = \sum_{m=1}^{b} bm^{-4}(1+2b^{-1}m^4\cos(2\pi m\phi)) \quad . \qquad \text{(A.II.1)}$$

Thus

$$\prod_{i\in\Lambda\cap\mathbb{Z}^2}(1+2\sum_{m=1}^{\infty}\cos(2\pi m_i\phi_i))$$

$$= \sum_{\{m\}}\prod_{i\in\Lambda\cap\mathbb{Z}^2}(bm_i^{-4})\prod_{i\in\Lambda\cap\mathbb{Z}^2}(1+2b^{-1}m^4\cos(2\pi m_i\phi_i)) \quad . \qquad \text{(A.II.2)}$$

One now repeatedly applies the identity

$$[1+K_1\cos\phi(\rho_1)][1+K_2\cos\phi(\rho_2)] = \tfrac{1}{3}[1+3K_1\cos\phi(\rho_1)]$$
$$+ \tfrac{1}{3}[1+3K_2\cos\phi(\rho_2)] + \tfrac{1}{6}[1+3K_1K_2\cos\phi(\rho_1-\rho_2)]$$
$$+ \tfrac{1}{6}[1+3K_1K_2\cos\phi(\rho_1+\rho_2)] \quad . \qquad \text{(A.II.3)}$$

In the initial step of the induction, we just take $\phi(\rho_i) = 2\pi m_i\phi_i$. One then applies (A.II.3) to nearest neighbor pairs of sites, generating a set of charge densities. The constants c_N for this set of ensembles will just be the factors bm_i^{-4} multiplied by factors of $(\frac{1}{3})$ and $(\frac{1}{6})$, according to the various terms in (A.II.3) which contribute to a particular ensemble. If the ensembles generated by this procedure satisfy the conditions of Theorem II.3, we stop. If not, one reapplies (A.II.3) to these new charge densities. Some care is necessary to insure that one does not build up exceedingly large entropy

factors for some of the charge densities. Roughly speaking, one divides the charge densities generated after n-steps of this inductive procedure into two groups -- those which obey conditions (a) and (b) of Theorem II.3, and those which do not. To those which do not, one applies (A.II.3), while the rest are left untouched. Since there are only a finite number of charge densities to begin with, this inductive procedure must eventually terminate. One then checks to see that the entropy estimates of Theorem II.3(c) hold.

Suppose we have a set of charge densities, N_n^*, all of which are at least a distance 2^n from each other, and all obeying Theorem II.3(c). Apply (A.II.3) to all pairs of charge densities within a distance 2^{n+1} of one another, until an ensemble N_{n+1}^* is generated with all $\rho' \in N_{n+1}$ separated by at least 2^{n+1}. We show that

$$K(\rho') \le e^{cA_n(\rho')} \prod_{\rho\subset\rho'} K(\rho) \quad . \tag{A.II.4}$$

By $\rho \subset \rho'$ we mean those elements of N_n^* which are continued in ρ'. This, combined with the assumption that the $K(\rho)$'s all obey (II.1.7)(c) and an induction argument on n, proves that all charge densities generated in the first half of Theorem II.3 obey the entropy estimates.

First note that

$$K(\rho') = 3^{m(\rho')} \prod_{\rho\subset\rho'} K(\rho) \tag{A.II.5}$$

where $m(\rho')$ is the number of applications of (A.II.3) necessary to produce ρ' from N_n^*. We claim that $m(\rho') \le c'A_n(\rho')$ which will verify (A.II.4). Since we applied (A.II.3) only to charge densities

which were closer than 2^{n+1} to each other, the number of applications necessary to obtain ρ' is bounded by the number of charge densities in N_n^* within a distance of 2^{n+1} of ρ'. This follows from the fact that (A.II.3) takes two such charge densities, and either combines them or deletes one. A single $2^n\times 2^n$ square can intersect at most three charge densities in N_n^*. Hence $A_n(\rho') \ge c'' \sum_{\rho\subset\rho'} A_n(\rho)$. Furthermore, for a given ρ, a simple geometrical argument shows that there are at most $c''' A_n(\rho)$ charge densities in N_n^*, within a distance of 2^{n+1} of $\rho \in N_n^*$. (This follows from the requirement that all $\rho \in N_n^*$ be at least a distance 2^n from all other elements of N_n^*.) Hence

$$\begin{aligned} &|\#\text{ elements of } N_n^* \text{ within a distance } 2^{n+1} \text{ of } \rho'| \\ &\le c''' \sum_{\rho\subset\rho'} A_n(\rho) \le \left(\frac{c'''}{c''}\right) A_n(\rho') \quad , \end{aligned} \tag{A.II.6}$$

which proves (A.II.4).

Having succeeded in rewriting the sum over $\{m\}$ in (II.1.12) to include the effects of all terms, we proceed to the renormalization itself. Start from the observation that the integral

$$\int e^{ibx} e^{-ax^2} dx$$

can be evaluated by making the change of variables $x \to y - \frac{ib}{2a}$. One then "guesses" that the complicated integrals of (II.1.12) are "close" to Gaussian, makes the changes of variables detailed in Section II.2, and bounds the error which results. The bounds on errors result from combining the bounds of Section V.4 with bounds on the functions $i(\phi+i\zeta_\rho a_\rho)$ discussed below. Implicit in estimate (V.3.21) is the

bound of

<u>Lemma A.II.1.</u>

$$|A'(\rho)| \geq 4D|A(\rho)| \quad . \tag{A.II.7}$$

Recall that $A(\rho) = \sum_{m=0}^{n(\rho)} |S_k(\rho)|$, and $A'(\rho) = \sum_{m=0}^{n(\rho)} |S'_k(\rho)|$, with $S_k(\rho)$, the minimal set of $2^k \times 2^k$ squares which cover supp ρ, and $S'_k(\rho)$ the subset of $S_k(\rho)$ defined in (II.2.2). This is one of the few places where our choice of Dirichlet rather than free boundary conditions produces any significant difference from the detailed proof of [15]. The difference arises due to the fact that for fixed $S_k(\rho)$, $S'_k(\rho)$ may be smaller in the Dirichlet boundary condition case than in the free boundary condition case, because of the second half of condition (II.2.2) which insures that we don't translate so close to the edge of Λ that we violate our boundary conditions. We must check that, nonetheless, some inequality like (3.10) of [15] holds. Assume that $|S_{\gamma(k)}(\rho)| \geq 2$, where $\gamma(k) = [\alpha^{-1}(k-b-2)]$, where $b = \ln_2\sqrt{2}\cdot 2M$, and M and α are the constants appearing in Theorem II.3. (Note that if $|S_{\gamma(k)}(\rho)| < 2$, $\sqrt{2}\cdot 2^{\gamma(k)} > d(\rho)$ and $k > \ln[4M\cdot\ln_2 d(\rho)]$ so that $|S_{\gamma(k)}(\rho)| = 0$ if ρ is neutral, and $|S_{\gamma(k)}(\rho)| = |S'_{\gamma(k)}(\rho)|$ if ρ is charged, and in either case $A_k(\rho) = A'_k(\rho)$.) We divide $S_k(\rho)\backslash S'_k(\rho)$ into two sets, $S^n_k(\rho)$ and $S^f_k(\rho)$. If $s_1 \in S^f_k(\rho)$, $\mathrm{dist}(s_1,\Lambda^c) \leq 2^{k+2}$, and for every other $s_2 \in S_k(\rho)\backslash S'_k(\rho)$, $\mathrm{dist}(s_1,s_2) \geq \sqrt{2}\cdot 2M\cdot 2^{\alpha k}$. Define $S^n_k(\rho) = (S_k(\rho)\backslash S'_k(\rho))\backslash S^f_k(\rho)$. By the reasoning that leads to (3.8) of [15], one has

$$A_k(\rho) \leq \tfrac{1}{2}A_{\gamma(k)}(\rho) + |S'_{\gamma(k)}(\rho)| + |S^f_{\gamma(k)}(\rho)| \quad . \tag{A.II.8}$$

This results from the fact that since the squares in $S^n_{\gamma(k)}(\rho)$ are "close" to one another on the scale $2^{\gamma(k)}$, more than one of them may be covered by a single $2^k\times 2^k$ square. Iterating this inequality, one arrives at

$$A_k(\rho) \leq \sum_{m=0}^{\ell-1} 2^{-m}[|S'_{\gamma^{m+1}(k)}(\rho)| + |S^f_{\gamma^{m+1}(k)}(\rho)|] + 2^{-\ell}A_o(\rho) \quad . \tag{A.II.9}$$

This is analogous to Eq. (3.10) of [15]. Here, ℓ is the maximal integer such that $\gamma^{\ell}(k) \geq 0$. Then, following Fröhlich and Spencer, one shows that this implies

$$A(\rho) \leq EA'(\rho) + F\sum_{m=0}^{n(\rho)} |S^f_m(\rho)| \quad . \tag{A.II.10}$$

We now show that

$$\sum_{m=0}^{n(\rho)} |S^f_m(\rho)| \leq 2A_o(\rho) \tag{A.II.11}$$

which when combined with (A.II.10) yields the bound of Lemma A.II.1. To prove (A.II.11) we show that if $s_1 \in |S^f_m(\rho)|$, then the squares, s', which cover $\rho|_{s_1}$ on all scales k, less than or equal to $m-2$, can be arranged so that $\mathrm{dist}(s',\Lambda^c) \geq 2^m$, and hence, are never elements of $S^f_k(\rho)$. This implies that for $s_1 \in S^f_m(\rho)$ and $s_2 \in S^f_{m-2}(\rho)$, $s_1 \cap s_2 = \phi$. Thus,

$$\sum_{m=0}^{[n(\rho)/2]} |S^f_{2m}(\rho)| = |\bigcup_{m=0}^{[n(\rho)/2]} S^f_{2m}(\rho)| \leq A_o(\rho)$$

and

$$\sum_{m=0}^{[n(\rho)/2]} |S^f_{2m+1}(\rho)| \leq A_o(\rho)$$

which proves (A.II.11). ($[n(\rho)]$ is the integer part of $n(\rho)$.)

Consider $s_1 \in S_m^f(\rho)$. Defining $\rho_1 = \rho|_{s_1}$ we see that

$$\mathrm{dist}(\rho_1, \rho-\rho_1) \geq 2^{3/2} M 2^{\alpha m} \quad . \tag{A.II.12}$$

By (II.1.7)(b) this implies

$$\mathrm{dist}(\rho_1, \Lambda^c) \geq [(2M)^{-1}\mathrm{dist}(\rho_1, \rho-\rho_1)]^{1/\alpha} \geq (2)^{3/2\alpha} \cdot 2^m \quad . \tag{A.II.13}$$

Now pick the covering on the scale 2^k, $k < m-2$, such that no square, s, in the covering is closer to Λ^c than ρ_1 is. Then $\mathrm{dist}(s, \Lambda^c) \geq 2^{3/2\alpha} 2^m > 2^{k+2}$ so $s \notin S_k^f(\rho)$ as claimed.

Finally we turn to the task of bounding the factors of $i(\phi + i\zeta_\rho a_\rho)$. We need the following lemma.

<u>Lemma A.II.2</u>. For the translating functions $a_\rho^{m,s}$ defined in Chapter II,

$$\sum_{<i,j>} |a_\rho^{m,s}(i) - a_\rho^{m,s}(j)|^2 \leq \gamma^2 \cdot c' \tag{A.II.14}$$

for all $\rho, m, s \in S_m'(\rho)$.

This lemma follows from (4.42)-(4.59) of [15]. From (II.1.1)(c) we have

$$\left| \frac{I_\beta(\partial\phi_{ij} + i\zeta_\rho a_\rho(i) - i\zeta_\rho a_\rho(j))}{I_\beta(\partial\phi_{ij})} \right| \leq \exp\left[\beta^{\frac{1}{3}}(g(a_\rho^m(i) - a_\rho^m(j)))\right] \quad . \tag{A.II.15}$$

Because of the bounds on $g(\cdot)$, this is less than

$$\exp\left(\beta^{\frac{1}{3}} e^{2\pi\gamma^2} [a_\rho^m(i) - a_\rho(j)]^2\right) \leq \exp\left[\beta^{\frac{1}{6}}[a_\rho^m(i) - a_\rho^m(j)]\right]^2 \quad , \tag{A.II.15}$$

where we have used the fact that $\gamma = c|\ln\beta|$ and assumed the constant c is small. By (II.2.15) this implies

$$|i(\phi + i\zeta_\rho a_\rho)| \leq \exp\left\{\sum_{m=0}^{n(\rho)} \sum_{<i,j> \in \Omega_\rho^m} \beta^{\frac{1}{6}}[a_\rho^m(i) - a_\rho^m(j)]^2\right\} \quad . \tag{A.II.16}$$

Recalling from Chapter II that no bond $<i,j>$ is ever translated more than once, we may rewrite

$$\sum_{<i,j> \in \Omega_\rho^m} [a_\rho^m(i) - a_\rho^m(j)]^2 = \sum_{s \in S_m'(\rho)} [a_\rho^{m,s}(i) - a_\rho^{m,s}(j)]^2 \leq c'\gamma^2 |S_m'(\rho)| \quad , \tag{A.II.17}$$

for $m > 0$, and

$$\sum_{<i,j> \in \Omega_\rho^o} [a_\rho^o(i) - a_\rho^o(j)]^2 \leq c'\gamma^2 A_o(\rho) \quad , \tag{A.II.17}$$

using Lemma A.II.2. This, combined with (A.II.16), bounds $|i(\phi + i\zeta_\rho a_\rho)|$ by

$$|i(\phi + i\zeta_\rho a_\rho)| \leq \exp\left\{\sum_{m=1}^{n(\rho)} c'\gamma^2 |S_m'(\rho)| + c'\gamma^2 A_o(\rho)\right\}$$

$$= \exp\{c'\gamma^2 A'(\rho)\} \quad . \tag{A.II.18}$$

Combining Theorem II.3(c), (II.2.8) and (A.II.18), we obtain

$$|K(\rho)e^{-\gamma q_\rho}{}_{i(\phi+i\zeta_\rho a_\rho)}|$$

$$\leq \left(\prod_{i\in \mathrm{supp}\,\rho} (4\rho(i))^4\right)\exp\left\{cA(\rho)+c'\gamma^2\beta^{\frac{1}{6}}A'(\rho)-\gamma\sum_{i\in T_o^\rho}|\rho(i)|\right.$$

$$\left.-\gamma\ln_2\sqrt{2}\sum_{m=1}^{\infty}\sum_{s'\in S'_m(\rho)}|q_\rho^{m,s'}|\right\} . \qquad \text{(A.II.19)}$$

Since $\sum_{i\in T_o^\rho}|\rho(i)| \geq \frac{1}{4}\Omega(\rho) + \frac{1}{4}A_o(\rho)$, we see that for β sufficiently small,

$$\left|\prod_{i\in \mathrm{supp}\,\rho} (4\rho(i))^4\right|\exp\left\{-\frac{\gamma}{2}\sum_{i\in T_o^\rho}|\rho(i)|\right\}\leq 1 .$$

Since $|q_\rho^{m,s'}| \geq 1$ by our definition of $S'_m(\rho)$,

$$|K(\rho)e^{-\gamma q_\rho}{}_{i(\phi+i\zeta_\rho a_\rho)}|$$

$$\leq \exp\left\{(c+c'\gamma^2\beta^{\frac{1}{6}})A(\rho)-\frac{\gamma}{4}A_o(\rho)-\gamma\ln_2\sqrt{2}\sum_{m=1}^{\infty}|S'_m(\rho)|\right\}$$

$$\leq \exp\left|(c+c'\gamma^2\beta^{\frac{1}{6}}-\gamma D)A(\rho)\right| ,$$

where in the last step we used Lemma A.II.1. This is the estimate of (V.3.21). We note in passing that since $|A_n(\rho)| \geq 1$ for all $n \leq [\ln_2(Md(\rho)^\alpha)] + 1$, one clearly has $A(\rho) \geq \ln_2 Md(\rho)^\alpha > \ln_2 d_\Lambda(\rho)$. This completes our review of the results of [15].

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