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OVERLAPPING RESONANCES

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Abstract

We deal with an example of overlapping resonances; namely, a system of three particles in which one of the particles can form a two-particle resonance with either of the other two. There have been claims that overlapping two-particle resonances can cause an enhancement in the three-particle mass spectrum, and part of the motivation for this work was to investigate such a possibility. We show that when the calculations are done correctly and the results given the proper interpretation, the enhancements of previous models do not appear. We conclude, therefore, that enhancements in the three-particle mass can not be caused by overlapping two-particle resonances.

In Section I we describe the experimental procedure of making resonance cuts, and we discuss various choices for the definition of cross sections for resonance production. We treat both the case of a single final-state resonance and the case of overlapping final-state resonances.

In Section II we discuss overlapping resonances in the Lee model and in other static models. The Lee model is of particular interest because it presents the overlapping resonance problem within the context of a completely soluble field theory, and the static kinematics are useful for gaining understanding of some aspects of the problem.

In Section III we develop a nonstatic isobar model for describing overlapping final-state resonances, and we give two examples: a 3Π system in which two of the Π's are identical, and a $K\Pi\Pi$ system containing no identical particles and two overlapping resonances. We compare our model in which the resonances are treated coherently with the simpler model (from a calculational point of view) in which the resonances are treated incoherently. We find that there is not much difference between the two models in their predictions for the three-particle mass spectrum, but there is a considerable difference in their predictions for some differential cross sections - particularly for some angular distributions (in the incoherent model the presence of the second resonance is ignored in these differential cross sections). We feel that coherence should be taken into account when fitting the data, and that our form of the coherent model is reasonably simple to work with when performing calculations.

For all of our models we consider resonance projections which treat the resonances as though they were stable particles, and we get reasonably good approximations to the coherent model for the three-particle mass spectrum.

Amado has suggested that a product form for the scattering amplitude is somehow more fundamental than the linear form upon which our isobaric model and our approximations to the Lee model are based. We consider this point and our conclusion is that so long as the two-particle interactions can be described as resonances, an isobar model is probably adequate for treating overlapping two-particle interactions.

what's isobar model

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INTRODUCTION

In this work we will be dealing with an example of overlapping resonances; namely, a system of three pseudoscalar mesons in which one of the particles can form a two-particle, spin-one resonance with either of the other two. Experimentally such systems are usually found as a subsystem for a final state in a meson-nucleon production-reaction, and experimental data are becoming available with good enough statistics to allow a detailed study of such systems. Two cases of particular interest are the 3Π system with two identical Π's either of which can form a ρ with the third Π, and the charged K-Π-Π system which contains the appropriate quantum numbers for one ρ and (at least) one K^*. Much of the interest in these particular systems comes from the fact that there are experimentally observed enhancements in the three-particle mass spectrum for both cases near the overlap threshold, and one of the motivations for this work was to investigate the possibility that such enhancements could be caused merely by the resonance overlap.

In theoretical models final state resonances are often treated as though they were stable particles. If there are overlapping resonances in the final state, the resonance→stable particle projection is made for each resonance and the different amplitudes are added incoherently. In some cases the dynamics are such that one resonance term is considerably enhanced relative to another, and in other cases the presence of identical particles or some symmetry such as isotopic spin or SU(3)

i.e. take process $\langle Kp \to K^*\pi p \to K\pi\pi p \rangle = A$

$\langle Kp \to K\rho p \to K\pi\pi p \rangle = B$

and calculate cross as $A^2 + B^2$ not $(A+B)^2$?

can be used to fix the relative magnitude and phase of the different resonance terms in the amplitude. Reducing the number of particles in the final state through resonance projections simplifies the final state kinematics considerably so that the calculation of cross sections is often made trivial, and treating different resonance terms in the amplitude as incoherent means that symmetry predictions for the parts of the amplitude lead to symmetry predictions for the cross sections as well. In Section III below we develop a model for such an overlapping resonance system which can be compared directly with data, and within the context of this model we discuss the validity of resonance approximations and projections.

Resonances such as the ρ and the K^* can not be observed directly but can only be inferred from enhancements in the appropriate two-particle mass spectra and from various angular distributions and selection rules. In Section I we discuss the resonance-cut procedure used by experimentalists which involves selecting an event only if it has a two-particle mass near the resonance mass. The reason for such cuts is to suppress non-resonance events relative to resonance events. We discuss the relationship between cross sections based on these resonance cuts and what might be termed resonance cross sections. This section is primarily a background for the later sections.

In Section II we discuss a form of the Lee model which includes a three-particle final state with two overlapping resonances. We consider this model for the following three reasons: it is an exactly soluble field theory containing the problem

of interest, Chen-Cheung and Sommerfield[1] (C&S) found that either making a resonance projection or using a resonance cut led to an anomalous enhancement in the three-particle mass spectrum (see Fig. 1a), and the model is one of the cases cited by Amado[2] in which overlapping resonances appear more naturally in the amplitude as a product rather than as a sum (as shown in Appendix B resonance projections make sense when the amplitude is written as a linear sum of the resonance terms). We find that C&S made some errors in their derivation of the resonance projection and in their interpretation of the cross section based on their resonance cut, and that when these errors are corrected no enhancements occur. We show that the production process is well approximated by a Born approximation[3] and we discuss Amado's conjecture with respect to this result. We end this section by introducing a model which is a simplification of the Born approximation in that the production process is replaced by a point vertex and all information concerning the initial state is removed except for the total energy so that we are left with a model which only contains final state interactions which arise from the coherent sum of the two resonance terms. We calculate effective cross sections for this model and we find that the resonance-projection cross section and the resonance-cut cross section have essentially the same structure relative to the total cross section as they had in the Lee model and its Born approximation. We interpret this result as an indication that the mode of production of the three-particle system is not important when one is interested in studying the validity of

a resonance projection or the interpretation of a resonance cut.

In Section IIIA we derive a nonstatic model which is essentially a relativistic generalization of the final model of Section II. Here the only evidence of the origin of the system is the three-particle spin and parity (J^{η}) and the three-particle mass (M). We develop the model using the helicity framework of Wick[4] because it lends itself quite well to the treatment of overlapping resonances. Since dynamical approximations usually involve orbital angular momenta, we use McKerrell[5] to relate the helicity framework to the fully relativistic "canonical" framework given by MacFarlane.[6] We write down amplitudes for three-particle final states with either two or three overlapping two-particle resonances, and we put the corresponding effective cross sections into a form in which there remain only two nontrivial integrations with the variables of integration given by the usual Dalitz-plot variables. We also write down a resonance approximation within the framework of our model.

In Section IIIB we consider a system with overlapping resonances caused by a particle forming a resonance with either of two identical particles; specifically we consider the 3π system referred to above. We first consider a simplified version in which the ρ is taken to have spin $j = 0$ and the J^{η} is taken to be 0^-. We consider this case because our model then reduces to that of Chang[7] who claimed to have found an enhancement in the 3π mass in the $A_1(1080)$ region. We repeat his calculations and find no enhancement and we conclude that his calculation was in error. Next we give the ρ its correct spin-

parity , J^{η} is chosen to be 1^+, and each relative ρ-Π system is taken to be in a relative s-wave. These are the input for the model treated by Gleeson and Meggs[8,9] who also claimed to have found a 3Π enhancement near the A_1. We find no enhancement for the total cross section, and we show that their enhancement in one case[8] was caused by an invalid approximation and in the other[9] by an incorrect interpretation of the resonance-cut cross section.

In Section IIIC we consider an example of a system with overlapping resonances but no identical particles, the K-Π-Π system referred to above. For convenience we simplify on reality slightly by taking the ρ and the K^* spin-parity to be given by 0^+. Here the relative magnitude and phase of the two resonance terms in the amplitude are arbitrary. We calculate cross sections for various choices of these parameters including those predicted by SU(3) assuming the three-particle system has the internal quantum numbers of a K.[10]

In our examples of Sections IIIB and IIIC we find that the differential cross sections for the two-particle mass and one of the observable angular distributions for the model are quite different from the same quantities calculated for a model in which the effects of the second resonance are ignored. We also find that the resonance projection gives a reasonably good approximation to the total cross section. Our answer, therefore, to the question as to whether or not resonances can be treated as though they were particles is that it depends on the particular application.

In Section IIID we rewrite our amplitude in terms of directly observable quantities, and in so doing we arrive at the form of the amplitude used by two groups of experimentalists[11,12] for studying the $K\pi\pi$ system near the overlap threshold. The form of the amplitude derived in this section is much more cumbersome to work with in calculating cross sections than the form we have used in the previous sections. It is hoped that the demonstrated equivalence of the amplitude written in terms of experimental observables to a form which is much better suited for calculations will prove useful in future experimental analyses.

Lovelace[13] has suggested that the Veneziano model[14] treats the resonance overlap differently from the way it is treated in isobaric models such as the one we use. However, Boguta[49] has reproduced all of Lovelace's results using an isobaric model, and we find that near the overlap, Lovelace's amplitude can be written as an isobaric model amplitude times a slowly varying form factor.

Our conclusions are that overlapping two-particle resonances can not cause enhancements in the three-particle mass spectrum, and that although our models are far too simple to provide a complete description of a three-particle system containing overlapping resonances, the models at least provide a reasonable reference.

SECTION I: Resonance Cuts and Resonance Cross Sections

We start out this section by considering a situation in which we have a three-particle system dominated by a single two-particle resonance. One makes a resonance cut for such a system by accepting (rejecting) an event if the appropriate two-particle mass is inside (outside) an interval $[m^*-\kappa\Gamma, m^*+\kappa'\Gamma]$ where m^* is the mass corresponding to the peak of the resonance distribution, Γ is (approximately) the full width at half maximum of the distribution, and κ and κ' are to be determined. There are conflicting conditions to be met in choosing κ and κ': if the selection region is small the total number of events will be small so that the statistics will suffer but the probability that the event is actually a resonance event will be large since the ratio of resonance to background will be high, while if the selection region is larger more events will be included but the ratio of resonance to background will decrease. To illustrate this, consider that the background is incoherent to the resonance production and constant at about 10% of the maximum of the resonance, and assume that $\Gamma/m^* \ll 1$ and that the resonance has $j = 0$ so that the mass spectrum for the resonance is well approximated by a Lorentzian in the mass, and finally, assume that the kinematics are such that in performing the final integral over the resonance mass there is a negligible effect from letting the limits go to $\pm\infty$. Then, for $\kappa = \kappa' = 1/2$ we will be including 1/2 of all the resonance events, and the ratio of resonance to background is 5:1 at the end points and about 7.7:1 for the cross sections; for $\kappa = \kappa' = 1$ we will be including about 2/3 of all resonance events, and

the ratio of resonance to background will be 2:1 at the end points and the ratio of the cross sections will be about 5:1. As Γ/m^* increases and for $j \neq 0$, the resonance mass spectrum will no longer be a symmetric distribution in the mass so that κ and κ' will no longer be taken as equal. Generally one uses a selection region with a width of about 2Γ.

A similar procedure to the above can be used to suppress resonance events; for example, N* cuts are often made when studying the final state meson systems in meson-nucleon production reactions. Another method used to suppress N* production is to make a cut on the momentum transfer to the nucleon since there is an observed suppression of N* production for small momentum transfers.

why?

When there are overlapping resonances, resonance cuts such as those described above can be made and may be useful in suppressing nonresonant background, but the interpretation of the results of such cuts is more model dependent than for the single resonance case. If the final state contains two identical overlapping two-particle resonances, the cut is most naturally defined as selecting an event if either of the appropriate two-particle masses lie in some designated region. As we will see in Section III, the fact that the second resonance is coherent to the first distorts the shape of the two-particle mass distribution relative to the single resonance case. There is also a shape-distortion in the two-particle mass spectrum relative to the single resonance case when there are nonidentical overlapping resonances.

One of the applications for two-particle resonance cuts in three-particle states such as we have been considering is to look at the three-particle mass spectrum in which only the selected events are included. If there is a three-particle resonance whose decay is dominated by the two-particle resonance, the three-particle resonance peak is generally more pronounced for the selected events than for the uncut set of events. Care must be taken when interpreting such curves when there are overlapping two-particle resonances since we will see that a resonance cut can introduce structure which might be interpreted as a three-particle resonance. In the example we discuss, the structure does not occur until the width of the selection region is decreased to far below that which any experimentalist would choose, but there is at least one case on record in which a resonance was discredited partly because of arguments concerning structure created by resonance cuts.[15] The case involved the overlap of three ρ bands in a 3π state. This three-band overlap is analogous to the two-band overlap in the static models treated in Section II, and the structure introduced is more extreme than in the cases we will be discussing.

A cross section (or a three-particle mass distribution) calculated on the basis of the events in the selection region is not really a resonance-production cross section since it includes some nonresonant background and it does not include some resonance events. In correcting for these effects when there is only a single resonance, one uses some model for the background and a modified Lorentzian for the resonance mass distri-

bution assuming the resonance and background can be taken as incoherent. When there are overlapping resonances the procedure is not so well defined. If the resonances are not identical and if one wants a resonance-production cross section for only one of the resonances, then at some stage one must treat the two resonances as though they were incoherent. One possibility is to treat the two resonances as incoherent from the beginning to the extent that the data is fit using incoherent individual resonance terms with the interference term included somehow in the background. One reason for doing this may be that the models experimentalists have used which treat the different resonances coherently are very cumbersome to work with. We feel that in fitting the data the resonances should be kept coherent, and one of the motivations of this thesis is to provide models for doing this which are easy to calculate with. If one wants a number for the resonance-production cross section for only one of the resonances, we feel that the best way to calculate this is first to calculate the total resonance-production cross section keeping the two terms coherent. Then one calculates the fraction of this total which represents the incoherent production of the desired resonance using a model like the one in Section III. When the two resonances are identical one proceeds much as before in that one first calculates a total resonance-production cross section using a model which best fits the data. There is an ambiguity as to how one should proceed at this point: one could call this cross section the resonance cross section, but

by analogy with the nonidentical resonance case it would seem that we should subtract out a proportion associated with the interference of the two resonances.

The correct definition of the resonance-production cross section depends upon the reason one wants such a number. One reason might be in order to make some comparisons with a model in which the two-particle resonance is treated as though it were a stable particle. Then the appropriate choice is a theoretical resonance approximation such as that discussed in Section III and Appendix B which is essentially the lowest order term in Γ/m^* for either of the above definitions. This definition can also be used for the nonidentical resonance case. For the examples we work out there is very little difference in the numerical results for the various choices for the definition of the resonance cross section. The important point is that there is a difference between any of these definitions and the cross section based only on events in a resonance selection region.

SECTION II: Overlapping Resonances in Static Models

A. Background on Static Models

A static model is one in which some of the particles obey relativistic kinematics while others are treated in their static limit. In terms of a particle's three-momentum $\vec{p}$ and its mass m the particle's energy p° is given by

$$p^\circ = [\vec{p}.\vec{p} + m^2]^{1/2} = m + \vec{p}.\vec{p}/(2m) + \ldots \quad . \tag{1}$$

The static limit is obtained by keeping only the first term on the far right hand side of (1) so that the energy becomes decoupled from the three-momentum and is its rest mass and is constant. The applicability of the static limit in physical processes in particle physics is limited to a few special cases such as low energy pion-nucleon scattering where taking the nucleon as static is equivalent to ignoring the nucleon recoil. However, many of the effects of interest to us here occur in the static models as well as in the more physical nonstatic models, and because the static kinematics lead to some simplifications in calculations, the static models will prove useful for gaining some understanding of the causes of some of the effects.

We will now give some examples to illustrate simplifications which occur when working with static kinematics. For a three-particle state with all particles treated relativistically we obtain the following well-known relation among the two-particle invariant masses $[s_{ij}]^{1/2}$, the individual particle masses m_j, and the three-particle mass M

$s_{ij} = p_i p_j = m_i^2 + m_j^2 + 2p_i \cdot p_j$

$$s_{12} + s_{32} + s_{31} = M^2 + m_1^2 + m_2^2 + m_3^2 \quad . \tag{2}$$

It follows from (2) that two of the three invariant masses are independent. If the three-particle state contains the possibility of two-particle resonances either between #1 and #2 or between #3 and #2, the transition amplitude to that state will have a nontrivial dependence on both s_{12} and s_{32} so that in calculating a cross section the nontrivial part of the integral will be at least two dimensional. Also, the range in M for which s_{12} and s_{32} can simultaneously be at their resonance values (the overlap region in M) may be quite large: for example, for a 3Π state with two ρ-resonances, the overlap region is approximately 1080 MeV $\leq$ M $\leq$ 3900MeV. Thus if one is looking for structure due to the overlap, effects may be spread out over a 2BeV range in M.

Now we take as static both #2 and the two-particle resonances. Equation (2) is replaced by the equation for the conservation of energy

$$P^\circ = p_1^\circ + m_2 + p_3^\circ \tag{3}$$

from which it follows that if P°, the total energy, is given, only one energy, p_1° or p_3°, is independent. Resonances are now characterized by particular values of p_1° or p_3°. An immediate consequence of (3) is that the overlap region in P° reduces to a single point (neglecting the resonance width). On the basis of this it would appear that any structure due to

overlapping resonances would be more likely to occur in static models than in nonstatic models. Another simplification in the static case is that the Dalitz plot which is an area in the nonstatic case reduces to a line in the static case. Equivalently, a nontrivial two-dimensional integral in the nonstatic case means a nontrivial one-dimensional integral for the corresponding static case.

B. The Lee Model

The basic Lee model contains three fields: V, N, θ, where V and N represent static particles and θ represents a particle with relativistic kinematics. By restricting the kinematics to $V \rightleftarrows N\theta$ (no antiparticles), the Lee model is an exactly soluble field theory.[16] Chen-Cheung and Sommerfield[1] (C&S) have introduced an extension of the basic Lee model in which there are two V-fields, V_1 and V_2, with unrenormalized $(V_j N\theta)$ couplings g_1 and g_2 respectively. V-fields which are internally produced always occur in a linear combination given by $\overline{V} = \lambda(g_1 V_1 + g_2 V_2)$ where λ is a constant. C&S chose λ = 1 but we will find it convenient to take λ = 1/g which has the effect that $\overline{V}$ does not go to zero in the limit of zero coupling.[17] We will also choose this linear combination for external V-lines. For certain choices of the Lee model parameters the $\overline{V}$ propogator in momentum space, $\hat{G}_{\overline{V}}(p)$, has a pole in p° at $p^\circ = m$ with m below the (Nθ) threshold and a pole on the second sheet in p° at $p^\circ = \beta \simeq m^* - i\Gamma/2$ where $\Gamma > 0$ and m^* above the (Nθ) threshold. We choose g so that the residue of $-i\hat{G}_{\overline{V}}(p)$ at the stable particle pole is 1. We will adopt the convention that

V will stand for the stable V-particle and V* will stand for the unstable V-particle. This will only apply to external lines where the reduction procedure has picked out one or the other; the only V-field that appears in the propogators is $\overline{V}$. Further background and definitions along with expressions for various amplitudes and cross sections can be found in Appendix A.

The production process

$$\theta + V \rightarrow \theta + N + \theta \qquad \text{(c)}$$

has a three-particle final state with overlapping resonances. We will also be interested in the quasi-process

$$\theta + V \rightarrow \theta + V^* \qquad \text{(d)}$$

where the unstable V* is treated as though it were stable. (The notation (c) and (d) comes from Appendix A.) Figure 1a below is essentially Fig. 3 from Ref. 1 except that an "experimental" version of $\sigma_{(d)}(\omega)$ has been added and the corrected value for Γ has been used.[18] It was concluded by C&S that the anomalous peak in the V* cross section was indicative of the fact that process (d) is not well defined when the final state resonances overlap. Figure 1b contains the results of our calculations of these cross sections. We find no anomalous peaks and a reasonably good agreement between the production cross section and the resonance cross sections. In the following paragraphs we will outline the calculations of the

resonance cross sections, and we will isolate the erroneous assumptions made by C&S.

For the theoretical version of the resonance cross section we and C&S both start from Eq. (A15). A discussion of the assumptions involved in arriving at (A15) can be found below in Appendix B. There remains the question of how one chooses to evaluate $|F(\bar{m}^*,\omega)|^2$. We have calculated it three ways which turn out to agree with each other to within 10-15% and which agree with $\sigma_{(c)}(\omega)$ to about the same extent, and we can not really say that one method is more correct than the others. The first way is to take literally the derivation of (A15) given in the first part of Appendix B and to use the lowest order term in g^2 for $F(\bar{m}^*,\omega)$ which is just the Born approximation given by[3]

$$F(\bar{m}^*,\omega)_{\text{Born}} = g^2/(\omega - m^*) \ . \tag{4}$$

The resonance cross section using (4) is labelled (1) in Fig. 1b. The second method is to accept (A15) as given and to evaluate $F(\bar{m}^*,\omega)$ exactly. We have done this using Eq. (A10) and the Yale DCS computer, and the result is labelled (2) in Fig. 1b. The third method is that used by C&S and would probably be classified as lying somewhere between the other two in justification. In this method we note that implicit in the derivation of (A15) is the assumption that $F(\bar{\nu},\omega)$ is slowly varying for ν near m^* so that

$$F(\bar{m}^*,\omega) \simeq {}^{II}F(\bar{\beta},\omega) \tag{5}$$

where β is defined above and II denotes the second sheet in ν for F. As a check of (5) we can look at F_{Born}. We are actually interested in $|F(\overline{m}^*,\omega)|^2$, and we find

$$|{}^{II}F(\overline{\beta},\omega)_{Born}|^2 \simeq g^2/((\omega - m^*)^2 + \Gamma^2/4) \quad . \tag{6}$$

Combining (4) and (6) we find

$$|F(\overline{m}^*,\omega)_{Born}|^2 \simeq |{}^{II}F(\overline{\beta},\omega)_{Born}|^2[1 - \Gamma^2/(4(\omega - m^*)^2)]$$

so that (5) appears valid to the lowest order in Γ. An exact expression[19] for ${}^{II}F(\overline{\beta},\omega)$ can be obtained as follows: One rewrites the integral in (A10) as a contour integral in the complex ν plane using the fact that $K(\nu)^{-1}$ has a cut running from μ to ∞. $F(\omega',\omega)$ can be continued down into the lower half of the second sheet in $\overline{\omega}'$ and evaluated at $\overline{\omega}' = \beta$ by distorting the integration contour. The multiplicative factor ${}^{II}K(\beta)$, which is zero, eliminates everything except the singularity in the integral resulting from a pinch in the contour. The zero and the singularity cancel each other giving

$${}^{II}F(\overline{\beta},\omega) = g^2H(\overline{\beta},\beta,\omega)/D(\overline{\beta}) \quad . \tag{7}$$

Combining (5) and (7) we have

$$F(\overline{m}^*,\omega) \simeq g^2H(\overline{\beta},\beta,\omega)/D(\overline{\beta}) \quad . \tag{8}$$

The further assumption made by C&S is that if β in the function on the right hand side of (8) is replaced by ν, the resulting expression is slowly varying in ν, so that

$$H(\bar{\beta},\beta,\omega)/D(\bar{\beta}) \simeq H(\bar{m}^*,m^*,\omega)/D(\bar{m}^*) \quad . \tag{9}$$

Combining (8) and (9) leads to Eq. (B2) in Ref. 1 for $F(\bar{m}^*,\omega)$. It should be clear from the above that (9) does not follow from any previously made assumptions or approximations. We will now show that (9) is invalid for ω such that $\bar{m}^*$ is near m^*, which is the region of the resonance overlap. From Eq. (A11b) it is clear that $H(\bar{\nu},\nu,\omega)$ is indeed slowly varying in ν for ν near m^* and ω in this region, and using Eq. (A5) we find

$$H(\bar{\beta},\beta,\omega) \simeq H(\bar{m}^*,m^*,\omega)$$

and

$$D(\bar{\beta}) \simeq Z^{*-1}[\bar{\beta} - m^* + i\Gamma/2]$$

$$\simeq Z^{*-1}[\bar{m}^* - m^* + i\Gamma\ ],$$

while

$$D(\bar{m}^*) \simeq Z^{*-1}[\bar{m}^* - m^* + i\Gamma/2].$$

Thus at the overlap evergy, $\bar{m}^* = m^*$, the right hand side of (9) is a factor of 2 larger than the left hand side leading to a factor of 4 in the ratio of the corresponding cross sections which is close to the ratio of ${}^{Th}\sigma_{(d)}(\omega)$ to $\sigma_{(c)}(\omega)$ for ω such that $\bar{m}^* = m^*$ in Fig. 1a.

There is some ambiguity as to how one should proceed in obtaining an alternative to the assumption made by C&S. It seems to be in the spirit of the approximations made so far to treat functions of $\bar{\nu} = \bar{\beta}$ as a power series in $\bar{\beta} - \bar{m}^* = i\Gamma/2$ expanded about $\bar{\nu} = \bar{m}^*$. The method giving the best results of those tried is to keep the lowest order term in Γ in both the numerator and the denominator while interpreting the combination $Z^{*-1}\Gamma$ as independent of Γ. Again this leads to $H(\bar{\beta},\beta,\omega) \simeq H(\bar{m}^*,m^*,\omega)$ for the numerator. For the denominator we have

$$D(\bar{\beta}) = D(\bar{m}^*) + i\ \Gamma/2 \frac{d}{d\nu} D(\nu)\Big|_{\nu = \bar{m}^*} + \ldots \quad .$$

Now

$$Z^{*-1} = \frac{d}{d\nu} \operatorname{Re} D(\nu)\Big|_{\nu = m^*}$$

and

$$\frac{d}{d\nu} \operatorname{Re} D(\nu)\Big|_{\nu = \bar{m}^*} = Z^{*-1} + [\bar{m}^* - m^*] \frac{d}{d\nu} [\frac{d}{d\nu} \operatorname{Re} D(\nu)]\Big|_{\nu = \bar{m}^*} + \ldots$$

Hence we have

$$F(\bar{m}^*,\omega) = g^2 H(\bar{m}^*,m^*,\omega)/\tilde{D}(\bar{m}^*)$$

where

$$\tilde{D}(\bar{m}^*) = \operatorname{Re} D(\bar{m}^*) + i[\operatorname{Im} D(\bar{m}^*) + \operatorname{Im} D(m^*)]. \qquad (10)$$

The resonance cross section using (10) is labelled (3) in Fig. 1b.

We will now show by another method how the invalid result of C&S can arise. This method has a particular relevance because it has been used in a nonstatic model[8] to produce a peak in the 3π spectrum which we show in Section IIIB to be anomalous.

The fact that the method is invalid is easier to see here because of the static kinematics. The method is to apply Eq. (B11) as an approximation for the case $g^2 \neq 0$

$$|D(\nu)|^{-2} \mathrm{Im}\, D(\nu) \simeq \pi Z^* \delta(\nu - m^*) \tag{11}$$

to evaluate the production cross section given by Eq. (A13).

$$\sigma_{(c)}(\omega) = \frac{g^2 |u(\omega)|^2}{2\pi k(\omega)} \int_{\mu}^{\bar{\mu}} d\nu\, \mathrm{Im}\,[D(\nu)^{-1}]\, \mathrm{Im}\,[D(\bar{\nu})^{-1}]\, |H(\nu,\bar{\nu},\omega)|^2 \tag{A13}$$

$$\rightarrow \frac{Z^* g^2 |u(\omega)|^2}{2k(\omega)} \int_{\mu}^{\bar{\mu}} d\nu\, |H(\bar{\nu},\nu,\omega)|^2 \{\delta(\nu - m^*)\, \mathrm{Im}\,[D(\bar{\nu})^{-1}]$$

$$+ \delta(\bar{\nu} - m^*)\, \mathrm{Im}\,[D(\nu)^{-1}]\} \tag{11'}$$

$$= \frac{Z^* g^2 |u(\omega)|^2 |u(\bar{m}^*)|^2 k(\bar{m}^*)}{4\pi k(\omega)} \frac{|H(\bar{m}^*, m^*, \omega)|^2}{|D(\bar{m}^*)|^2} \theta(\bar{m}^* - \mu)$$

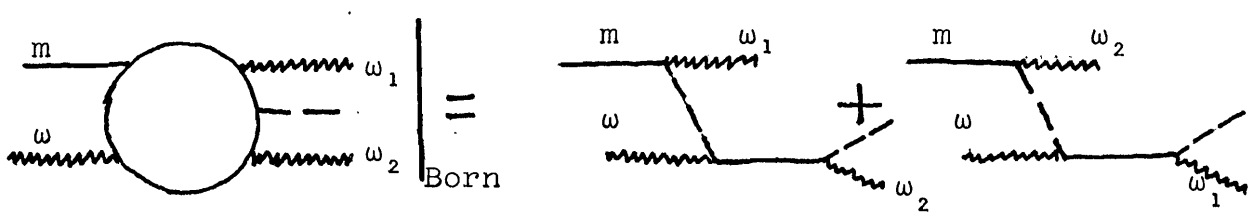
which is the same as the result of C&S. The error here is that (11) is a valid approximation for $\mathrm{Im}\,[D(\nu)^{-1}]$ only if the remainder of the integrand is slowly varying over the interval $m^* - \kappa\Gamma \leq \nu \leq m^* + \kappa\Gamma$ with $\kappa \geq 1$. For ω in the vicinity of the overlap, this condition is clearly not satisfied.

The "experimental" resonance cross section referred to above is just the resonance-cut cross section discussed in Section I. This cross section can be obtained by calculating $d\sigma_{(c)}(\omega)/d\nu \equiv A(\nu,\omega)$ and integrating $A(\nu,\omega)$ over the resonance region. Since the dynamics are static and since the θ's are identical, a resonance can occur when the energy of one of the

θ's is near m^* or near $\bar{m}^*$. For $|\bar{m}^* - m^*| \equiv x \geq 2\Gamma$ ($\kappa=\kappa'=1$), the differential cross section has two separated resonance regions, and the area of both of these should be included when calculating the resonance-cut cross section. For $x < 2\Gamma$ the regions overlap, and the overlap should only be counted once. Counting the overlap region once for each peak corresponds to an experimentalist counting some events twice relative to other events, and this is what C&S did in their calculation. Of course, one is at liberty to define the selection region any way he likes, the only question is how one interprets the results. **As explained in Section I neither of** these methods will necessarily lead to a good approximation of the total production cross section, but the symmetric method in which all events are weighted equally has a better chance than the other. The resonance-cut cross section using our method for defining the selection region is included in Fig. 1b, and we see that while it no longer has the peak of the resonance-cut cross section in Fig. 1a, it still has some structure that does not exist in the total cross section.

C. The Born Approximation of the Lee Model

There is no evidence of any dynamical structure in Fig.1b for the production process (c) which is not included in the Born approximation by which we mean that F, the $V\theta$ scattering amplitude, is approximated by one-N exchange giving the following graphical form for (c)



As discussed in Appendix B we can anticipate that the iterative terms will produce little effect since m* and Z* are shifted only slightly from their values for $g^2 = 0$.

Since the stable V-particle is far below threshold, $\hat{G}_{\overline{V}}(p)$ will be dominated by the V*, so we use Eq. (A5) as an approximation for all ω

$$[-i\hat{G}_{\overline{V}}(p)]^{-1} = D(\omega) \to Z^{*-1}[\omega - m^* + i\Gamma/2] \quad . \tag{A5}$$

except that in order to keep the kinematics correct, we use

$$\mathrm{Im}\, D(\omega)^{-1} = -|D(\omega)|^{-2}\, \mathrm{Im}\, D(\omega) \to -|D(\omega)|^{-2}\Gamma\, k(\omega)/[2k(m^*)] \ ,$$

and we have set $u(\omega) = 1$.

With these approximations we have

$$\sigma_{(c)}(\omega)_{\mathrm{Born}} = \int_{\mu}^{\overline{\mu}} d\nu\, A(\nu,\omega)_B$$

$$\mathrm{Exp}\,\sigma_{(d)}(\omega)_{\mathrm{Born}} = \int_{\text{resonance region}} d\nu\, A(\nu,\omega)_B$$

$$\mathrm{th}\,\sigma_{(d)}(\omega)_{\mathrm{Born}} = \frac{g^2\Gamma k(\overline{m}^*)}{2k(m^*)k(\omega)[\omega - m^*]^2}\theta(\overline{m}^* - \mu)$$

where

$$A(\nu,\omega)_B = [8\pi k(\omega)]^{-1}[k(m^*)]^{-2}g^2\Gamma^2 k(\nu)k(\overline{\nu})$$
$$\times\, Z^{*-2}|[(\nu - \omega)D(\nu)]^{-1} + [(\overline{\nu} - \omega)D(\overline{\nu})]^{-1}|^2 \ .$$

These cross sections have been plotted in Fig. 1c. Comparing Fig. 1c to Fig. 1b we see that little has been lost by these approximations.

A nice feature of the Born approximation is that the three methods of the previous section for calculating $F(\overline{m}^*,\omega)$ all reduce to the same result - the first and second methods are trivially the same, it is the equivalence between them and the third which is interesting.

D. The Product Form of the Scattering Amplitude in the Lee Model

A production amplitude to a final state which includes overlapping resonances can always be written both in a linear form such as Eq. (A12a) and as a product such as Eq. (A12b). We have treated the linear form as the more "fundamental" in that the Born approximation of the previous section and our theoretical resonance approximation have been based on it. Amado[2] has proposed that the dynamics of a state involving overlapping resonances are more clearly presented in the product form, which seems to imply that this is the form upon which approximations should be based. Kacser[20] has given some examples which show how the basis for Amado's assumption is model dependent, but the Lee model discussed above is one of Amado's examples and it is not discredited by Kacser. A sketch of the logic which leads to Amado's conclusion is as follows: In either the isobar or in the usual interpretation of the Faddeev equations for three-particle systems[21], terms in the scattering amplitude ending with the interaction of a given pair of particles are believed to carry the major information

concerning interactions of that pair. Amado finds that when an appropriate coherent combination of all terms contributing to the process is taken, a factor carrying the features of the given pair's final-state interactions appears multiplying the entire amplitude. Thus, when there are two overlapping resonances, the total amplitude contains the product of two such factors. Equation (A12b) appears to be consistent with this interpretation since each of the D's in the denominator carries the elastic scattering phase of a two-particle system, and from Eq. (A11b) it is clear that the numerator $H(\omega_1,\omega_2,\omega)$ is slowly varying in ω_1 and ω_2.

It might appear possible that such a product form could lead to an enhancement in ω at the resonance overlap because if we had $H = 1$ such an enhancement would occur since $[D(\nu)D(\bar{\nu})]^{-1}$ contains a nearby double pole when $\nu = m^*$ and $\bar{m}^* = m^*$. Amado argues that no such enhancement occurs because of unitarity. While we agree with the conclusion, we disagree slightly with his argument, so we will restate it here. Unitarity is used in the form of Eq. (A15)

$$[k(\omega)]^{-1}|u(\omega)|^2 \operatorname{Im} F(\omega,\omega) = \sigma_{(b)}(\omega) + \sigma_{(c)}(\omega) \ . \qquad \text{(A15)}$$

From Ref. 1 we borrow the fact that H can be written in the following form

$$H(\nu,\bar{\nu},\omega) = C(\omega)H'(\nu,\bar{\nu},\omega) \quad ,$$

where

$$C(\omega) = 2[2 - K(\omega)Q(\omega)]^{-1}$$

with $Q(\omega)$ given by Eq. (12a) of Ref. 1. Also, we use the fact that $F(\omega,\omega)$ can be written as

$$F(\omega,\omega) = C(\omega)[\omega - m]Q(\omega)/2 \equiv C(\omega)F'(\omega,\omega) \quad .$$

Using these we can rewrite (A15)

$$\mathrm{Im}\, F(\omega,\omega) = |C(\omega)|^2\{[4\Pi]^{-1}|u(\omega)|^2 k(\omega)|F'(\omega,\omega)|^2 + \int_{\mu}^{\bar{\mu}} d\nu N(\nu,\bar{\nu},\omega)|D(\bar{\nu})|^{-2}|D(\nu)|^{-2}\} \tag{12}$$

where

$$N(\nu,\bar{\nu},\omega) = [2\Pi]^{-1}g^2\ \mathrm{Im}\ [D(\nu)]\ \mathrm{Im}\ [D(\bar{\nu})]\ |H'(\nu,\bar{\nu},\omega)|^2 \quad .$$

The function $|C(\omega)|^2$ corresponds to Amado's $|\Delta(E)|^{-2}$ in Eq. (19) of Ref. 2, and any enhancement in ω for the elastic scattering cross section would occur in $|C(\omega)|^2$. The form of the Lee model which we have been using contains no such peak, but if we had included an unstable W-particle as described in Appendices A and B, all processes in the $V\theta$ channel would have a peak in ω and the structure would be contained in $|C(\omega)|^2$. Also contained in $C(\omega)$ is the so-called Peierls singularity[22] which was shown by Goebel[23] to lie on the wrong sheet in ω to fit the criterion of a resonance. Pagnamenta[24,25] found that the pole in $C(\omega)$ was actually accompanied by a nearby zero, and we have looked at $|C(\omega)|^2$ using the computer and found it to be essentially constant in ω. Equation (12) could

be rewritten with $|C(\omega)|^2$ cancelled from both sides leaving two slowly varying terms and the integral, which means that the integral must be slowly varying too. Thus, although N is slowly varying in ν, it must contain an ω dependence that cancels the peak in the denominator. The only difference between our argument and Amado's is that he claims the de-enhancing comes from $|C(\omega)|^2$ and relates the fact of no enhancement to the absence of a Peierls singularity.

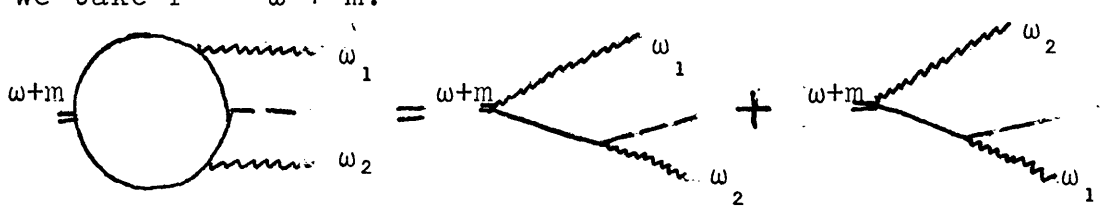
We have thus found no cause for disagreement with Amado's product form for the production amplitude. However, we have not been able to find any method of approximation based on this form - in fact, an attempt above at a resonance approximation gave quite bad results. We also found for the given parameterization of the Lee model, that good results were obtained in resonance and Born approximation which are related to the linear form of the amplitude. We have looked at parameterizations of the Lee model for which the Born approximation does not work, but we have found that in such cases Γ/m^* is too large to be taken as describing a resonance; for example, if g^2 is doubled while α,B, and m are kept fixed, Γ/m^* increases by a factor of 1.6. Our tentative conclusion is that when Γ/m^* is small enough to warrant the two-particle final-state interaction being described as a resonance, then the isobar model will be adequate for describing overlapping resonances. We should point out that such a conclusion can at best be but tentative in providing a prescription for dealing with processes in the real world due to the limited

applicability of the Lee model.

E. A Static Model Independent of the Mode of Production

The model of Section IIC was motivated by its being an approximation to the production process (c) of the Lee model. This model could also be interpreted simply as a production process proceeding by a one-particle exchange and utilizing static kinematics with a three-particle final state which contains two identical particles and two overlapping (static) resonances. We will now show that the relations among the various cross sections of Section IIC are not strongly dependent on the mode of production. We will do this by removing all information concerning the origin of the three-particle final state. Thus, for example, the three-particle system might be a subsystem of an n-particle final state where, for some dynamical reason, the interactions between this subsystem and the other n-3 particles can be ignored.

The production process is then replaced by a point vertex which only serves to give the system a four-momentum P^{μ} where we take $P^{o} = \omega + m$.



In the spirit of Appendix A, an effective amplitude is

$$\langle k_1, q, k_2 | P \rangle = [(2\Pi)^9 4\omega_1\omega_2]^{-\frac{1}{2}}\, i(2\Pi)^4 \delta(P-q-k_1-k_2) T(\omega_1,\omega_2), \quad (13)$$

where

$$T(\omega_1,\omega_2) = 1/D(\omega_1) + 1/D(\omega_2) \quad .$$

Equation (13) is essentially the same as Eq. (A12) except that we have left off the initial particle wave functions here and we have chosen $u(\omega) = g = 1$ (g can be considered as simply a scale factor). The cross sections are replaced by effective cross sections which are essentially the total transition probability per unit time

$$\tau(\omega) = 1/2 \int_{\mu}^{\bar{\mu}} d\nu k(\nu) k(\bar{\nu}) |T(\nu,\bar{\nu})|^2.$$

Note that we are retaining much of the Lee model notation, and we will also retain the particular choices for m, m^*, and Γ used above. Note that the cross sections in Section IIC are written so that they are independent of Z^*, so nothing is lost if we set $Z^* = 1$ if we use those results as a guide when making the resonance approximation. Then we have

$$D(\omega) = \omega - m^* + i\Gamma/2$$

and the theoretical resonance approximation is given by

$$Th_{\tau}(\omega) = 2\Pi\Gamma^{-1} k(m^*) k(\bar{m}^*) \theta(\bar{m}^* - \mu)$$

which is just two-particle phase space with the resonance as one of the particles.

We define a quantity $^{o}\tau(\omega)$ which is $\tau(\omega)$ except that the interference term in $|T(\nu,\bar{\nu})|^2$ has been omitted

$$^{\circ}\tau(\omega) = 1/2 \int_{\mu}^{\bar{\mu}} d\nu\ k(\nu)k(\bar{\nu})[|D(\nu)|^{-2} + |D(\bar{\nu})|^{-2}]$$

$$= \int_{\mu}^{\bar{\mu}} d\nu\ k(\nu)k(\bar{\nu})|D(\nu)|^{-2} \quad .$$

As described in Section I, $^{\circ}\tau$ is a possible candidate for the resonance cross section, and $^{\circ}\tau$ represents an intermediate step in a derivation of the theoretical resonance approximation as discussed in Appendix B.

These effective cross sections are plotted in Fig. 2. We see that $\tau(\omega)$ contains no structure in the overlap region and is indistinguishable from $^{Th}\tau(\omega)$. Comparing $^{\circ}\tau$ with τ we see that the interference term is both positive and small. It is interesting to note that $^{Th}\tau$ is a better approximation of τ than it is of $^{\circ}\tau$. The resonance-cut cross section, $^{Exp}\tau$, has essentially the same relation to τ as we had in the Lee model - a reasonably good approximation of τ up to the upper limit of the overlap region and then it drops off more rapidly.

This model illustrates clearly why the resonance approximation applied directly to the product form leads to an anomalous peak.

$$|D(m^*)T(m^*,\bar{m}^*)|^2 = 1 + [3\Gamma^2/4]/[(\bar{m}^*-m^*)^2 + \Gamma^2/4].$$

This leads to

$$^{Th}\tau(\omega)^{BAD} = {}^{Th}\tau(\omega)\{1 + [3\Gamma^2/4]/[(\bar{m}^*-m^*)^2 + \Gamma^2/4]\}.$$

SECTION III: Overlapping Resonances in Nonstatic Models

A. The Isobar Model: Derivation

The model we will now discuss is essentially a nonstatic fully relativistic version of the model in Section IIE. The only assumption we will make about the production process is that it leads to a three-particle system with a single value of the total angular momentum and parity J^{η}. Although an experimentalist can not usually isolate eigenstates of J^{η}, different eigenstates add incoherently. Since we are only interested in coherent effects and experimental situations in which a single eigenstate is believed to be dominant, we feel that it is legitimate to limit the model to a single value of J^{η}. Also we will introduce no structure in M, the three-particle mass, which does not follow directly from the final-state interactions since one of the questions we want to answer is whether or not final-state interactions resulting from overlapping resonances can cause enhancements in M.

We will first develop the formalism for a three-particle system with arbitrary masses m_1, m_2, m_3 and a single two-particle resonance in the (12)3 channel with spin = j. Next we will extend this model to include overlapping resonances in the (12)3 and (32)1 channels. We will discuss the case both for #1 and #3 identical and different. In both cases we reduce the calculation of the effective cross sections to not more than a two-dimensional integral. We derive the resonance approximation for this model, and we end this subsection with some remarks concerning an extension of the model to three overlapping two-particle resonances, which introduces

nothing new.

For spinless particles with nonzero masses a three particle system can be fully described in terms of the three three- momenta $[\vec{p}_1, \vec{p}_2, \vec{p}_3]$ or any equivalent set of nine independent variables. For the above set we can define a three-particle state $|\vec{p}_1, \vec{p}_2, \vec{p}_3\rangle$ by

$$|\vec{p}_1, \vec{p}_2, \vec{p}_3\rangle = \tilde{A}_1^\dagger(\vec{p}_1)\tilde{A}_2^\dagger(\vec{p}_2)\tilde{A}_3^\dagger(\vec{p}_3)|0\rangle \quad . \tag{14}$$

where $\tilde{A}_n^\dagger$ is the creation operator for the nth particle. A general form for a transition amplitude T to such a three-particle state can be written as

any initial state

$$T = T(\vec{p}_1, \vec{p}_2, \vec{p}_3) = \langle\vec{p}_1, \vec{p}_2, \vec{p}_3|\tilde{T}|in\rangle \tag{15}$$

what does or look like in terms of T

with the operator $\tilde{T}$ given by

$$\tilde{T} = \prod_{n=1}^{3}\{\int d^3p_n'/[2p_n'^0]\}T'(\vec{p}_1', \vec{p}_2', \vec{p}_3')\tilde{A}_1^\dagger(\vec{p}_1')\tilde{A}_2^\dagger(\vec{p}_2')\tilde{A}_3^\dagger(\vec{p}_3')$$

$$\times \sum_{initial}\tilde{\chi}_{initial} \tag{16}$$

with

$$\sum_{initial}\tilde{\chi}_{initial}|in\rangle = |0\rangle \tag{17}$$

where the function T′ contains any correlations among the three particles given by the dynamics. Substituting (14) into (15) and using (16) and (17) we find

$$T = \prod_{n=1}^{3} \{\int d^3p_n'/[2p_n^{\circ\prime}]\} T'(\vec{p}_1',\vec{p}_2',\vec{p}_3') \langle \vec{p}_1,\vec{p}_2,\vec{p}_3 | \vec{p}_1',\vec{p}_2',\vec{p}_3' \rangle \quad . \qquad (18)$$

out in

We choose our state normalization so that a completely uncorrelated state with no identical particles will be given by

$$T' = T = 1 \quad \text{(no identical particles, no correlations).} \qquad (19)$$

This leads to

$$\langle \vec{p}_1,\vec{p}_2,\vec{p}_3 | \vec{p}_1',\vec{p}_2',\vec{p}_3' \rangle = \prod_{n=1}^{3} [2p_n^{\circ}\delta^{(3)}(\vec{p}_n-\vec{p}_n')] \quad . \qquad (20)$$

An equivalent set of variables to the above which we will find to be useful is the set proposed by Wick[4] $[P,\theta_3,\phi_3,s_{12},\theta_{12},\phi_{12}]$. P is the total four-momentum which in the overall center of momentum frame (the (12)3 cmf) reduces to $\vec{P} = 0$ and $P^{\circ} = [-P^{\mu}P_{\mu}]^{1/2} \equiv M$. In this frame the "standard orientation" has the three momenta lying in the xz plane with $\hat{z} = -\hat{p}_3$ and $\hat{p}_1 \cdot \hat{x} \geq 0$. The angles (θ_3,ϕ_3) give the actual orientation of $\vec{P}_{12} \equiv \vec{p}_1+\vec{p}_2 = -\vec{p}_3$ with respect to this coordinate system. $\theta_3,\phi_3 = 0$ here The invariant two-particle mass $[s_{12}]^{1/2}$ is defined by $s_{12} = [-P^{\mu}_{12}P_{12\mu}]$. In the (12) cmf the angles (θ_{12},ϕ_{12}) give the orientation of $\hat{p}_1$ relative to the standard orientation defined as above except that here it is defined with reference to the (12) cmf. Note that the two standard orientations have the same $\hat{z}$ and are related by a Lorentz transformation along $\hat{z}$. This choice makes it possible to treat the $\hat{z}$-projection of the two-particle angular momentum $j_{(12)}$ (measured <u>in the</u> (12) cmf) as a helicity for the (12) system. The following identity is easily proved

$$\prod_{n=1}^{3}\int d^3p_n/[2p_n^\circ] = 1/4\int d^4P\int_{(m_1+m_2)^2}^{(M-m_3)^2} p_{12}q_3 ds_{12}/[m_{12}M]$$

$$\times \int d(\cos\theta_{12})\int d\phi_{12}\int d(\cos\theta_3)\int d\phi_3 \qquad (21)$$

where[26]

$$m_{12}^2 = s_{12}\ , \quad p_{12} = \gamma(s_{12},m_1^2,m_2^2)/[2m_{12}]\ ,$$

and

$$q_3 = \gamma(M^2,\ s_{12},m_3^2)/[2M]$$

with

$$\gamma(a,b,c) = \{a^2+b^2+c^2-2[ab+ac+bc]\}^{1/2}\ .$$

We will find it convenient to introduce the following notation

$$\int_{-1}^{1} d(\cos\theta_{12})\int_0^{2\Pi} d\phi_{12} \equiv \int d\Omega_{12}$$

and

$$\int_0^{2\Pi} d\phi_3\int_{-1}^{1} d(\cos\theta_3)\int_0^{2\Pi} d\phi_{12} = \int d\Omega_3\int_0^{2\Pi} d\phi_{12} \equiv \int dU_{R_3}\ .$$

The last grouping is useful because the rotation (operator)

$$\tilde{R}_3 \equiv \tilde{R}(\phi_3,\theta_3,\phi_{12}) = e^{-i\tilde{J}_z\Phi_3}e^{-i\tilde{J}_y\theta_3}e^{-i\tilde{J}_z\phi_{12}} \qquad (22)$$

for arbitrary $(\phi_3,\theta_3,\phi_{12})$ maps a state in the standard orientation in the (12)3 cmf given by $|P,0,0,s_{12},\theta_{12},0\rangle$ into the state $|P,\theta_3,\phi_3,s_{12},\theta_{12},\phi_{12}\rangle$.[27] Using Eq. (21) and requiring that Eq. (19) holds fixes the normalization of these states to be

normalization of

$$\langle P,\theta_3,\phi_3,s_{12},\theta_{12},\phi_{12}|P',\theta_3',\phi_3',s_{12}',\theta_{12}',\phi_{12}'\rangle \tag{23}$$

$$= 4Mm_{12}/[p_{12}q_3]\delta^{(4)}(P-P')\delta(s_{12}-s_{12}')\delta(\Omega_3-\Omega_3')\delta(\Omega_{12}-\Omega_{12}')$$

where

$$\delta(\Omega-\Omega') = \delta(\cos\theta-\cos\theta')\delta(\phi-\phi') \quad .$$

We can replace the angular set $[\theta_3,\phi_3,\theta_{12},\phi_{12}]$ by an angular momentum set $[J,\Lambda,j,\lambda]$ where J and $j = j_{12}$ have already been defined, while Λ and λ are respectively the $\hat{z}$-projections of J and j measured with respect to the standard orientation. We will define the state using the angular momentum set as a projection on the state using the angular set

inserting complete set of states

helicity state or

$$|P,J,\Lambda,s_{12},j,\lambda\rangle = N_J N_j \int dU_{R_3}\int d(\cos\theta_{12}) \tag{24}$$

$$\underbrace{\bar{D}^J_{\Lambda\lambda}(R_3)}_{\langle J_u|R_z|J\lambda\rangle}\, d^j_{\lambda 0}(\theta_{12})\,\underbrace{|P,\theta_3,\phi_3,s_{12},\theta_{12},\phi_{12}\rangle}_{\tilde{R}(\phi_3,\theta_3,\phi_{12})|p\,0\,0\,\cos\theta_{12}\,0\rangle}$$

where

$$N_L = [(2L+1)/(4\Pi)]^{1/2}$$

and $\bar{D}^J(R_3) = \bar{D}^J(\phi_3,\theta_3,\phi_{12})$ is the complex conjugate of the rotation matrix.[28] Equation (24) also preserves condition (19), and using the orthogonality relation Eq. (C4) for the D^J and d^j we find[29]

normalization of angular set

$$\langle P,J,\Lambda,s_{12},j,\lambda|P',J',\Lambda',s_{12}',j',\lambda'\rangle \tag{25}$$

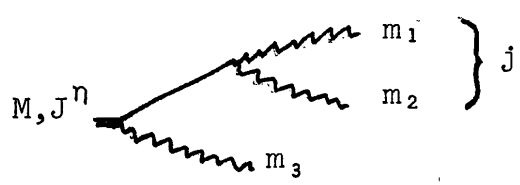
$$= 4Mm_{12}/[p_{12}q_3]\delta^{(4)}(P-P')\delta(s_{12}-s_{12}')\delta_{JJ'}\delta_{\Lambda\Lambda'}\delta_{jj'}\delta_{\lambda\lambda'} \quad .$$

The parity operator $\tilde{\eta}$ has the following effect on the angular momentum state[30]

$$\tilde{\eta}|P,J,\Lambda,s_{12},j,\lambda\rangle = (-1)^{J+1}|P,J,\Lambda,s_{12},j,-\lambda\rangle$$

where we evaluate P in the (12)3 cmf so that it is invariant under parity, and we have used the fact that the three particles are pseudoscalars. Thus the angular momentum state is an eigenstate of parity only if $\lambda = 0$ in which case $\eta = (-1)^{J+1}$ which is usually called the natural parity. One can, of course, form eigenstates of parity by taking linear combinations of these states of the form $|\lambda\rangle \pm |-\lambda\rangle$ where the + leads to a natural parity eigenstate and the - leads to an unnatural parity eigenstate. Unless $j = 0$ or $J = 0$ the choice of a particular linear combination depends upon ones choice of a dynamical model. Note that we had no such choice in the static models of the previous section.

The expression $\tilde{T}|in\rangle$ corresponding to the following diagram



is given by the following weighted linear combination of the angular momentum states

what is F^{η}_{J} ?

$$\tilde{T}|in\rangle = \sum_{\substack{J'j' \\ \Lambda'\lambda'}} \int d^4P' \int ds'_{12}\, p'_{12}\, q'_3 / [4m'_{12}M']\, F^{\eta}_{J'\Lambda'j'\lambda'}(M', s'_{12})$$

one-parameter of initial state

$$\times\, \delta_{JJ'}\delta_{jj'}\, |P', J', \Lambda', s'_{12}, j', \lambda'\rangle \qquad (26)$$

a way of putting in dynamics?

$$\tilde{T} = \prod_{n=1}^{3} \left\{ \int \frac{d^3 p'_n}{2 p'^0_n} T'(p'_1, p'_2, p'^3) \tilde{A}^{+}_{1} \cdots \times |0\rangle = \right.$$

Substituting Eq. (26) into Eq. (16) and using Eq. (24) we find

General matrix element:

left with one Λ since no sum

$$T = T^{\eta}_{Jj}(M,s_{12},\theta_{12},R_3) = N_J N_j \sum_{\Lambda\lambda} F^{\eta}_{J\Lambda j\lambda}(M,s_{12})\bar{D}^{J}_{\Lambda\lambda}(R_3)d^{j}_{\lambda 0}(\theta_{12}). \tag{27}$$

We assume the weight function can be written in the form

contains 2 particle resonance propagator
Dandy prescription

model enters here

$$F^{\eta}_{J\Lambda j\lambda}(M,s_{12}) = f_j(s_{12})B^{\eta}_{Jj\lambda}(q_3) . \tag{28}$$

Using either two-particle unitarity or a comparison of two-particle and one-particle phase space[61] we find f to be given by

$$f_j(s_{12}) = [(m_{12}/p_{12})\ \mathrm{Im}\ \Delta_j(s_{12})]^{1/2}/\Delta_j(s_{12}) \tag{29}$$

where Δ_j is the inverse propagator for the resonance. For p_{12} near threshold $f_j(s_{12})\ \alpha\ p_{12}^{j}$. We further assume that B can be written

$$B^{\eta}_{Jj\lambda}(q_3) = \sum_{|J-j|}^{J+j} {}_L\ b^{\eta}_{Jj\lambda L}q_3^{L}. \tag{30}$$

Just L

Nonrelativistically L is just the relative orbital angular momentum between #3 and the (12) system, and Fabri[31] showed that for low energies the lowest possible values of L compatible with a given J,j, and η would be favored (where j is here interpreted as the relative orbital angular momentum of #1 and #2 measured in the same reference system as L). Dalitz[32] obtained essentially the same results by arguing that only the lowest powers in the $|\vec{p}_n|$ should occur which were compatible

with a given J^{η}. Zemach[33] has rephrased Dalitz's model in relativistic terms and Werle[30] has done the same for Fabri. In the relativistic generalization of Fabri's model one must express L in a covariant way, which has been done by MacFarlane[6], McKerrell[5] and Werle[30]. One then finds that the L-state (or canonical state) can be written as the following linear combination of helicity states [5,30]

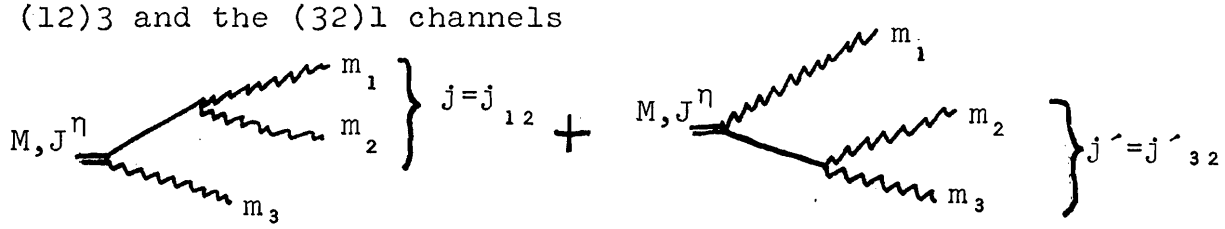
$$|P,J,\Lambda,s_{12},j,L\rangle = N_L/N_J \sum_{\lambda} \langle L0j\lambda|J\lambda\rangle |P,J,\Lambda,s_{12},j,\lambda\rangle \qquad (31)$$

where $\langle L0j\lambda|J\lambda\rangle$ is a Clebsch-Gordon coefficient. The L-state turns out to be an eigenstate of parity with eigenvalue $\eta = (-1)^{j+L+1}$ which is one reason why these states are considered useful. If only a single value of L contributes to T we have

$$B_{(\text{single } L)} = N_L/N_J \langle L0j\lambda|J\lambda\rangle q_3^L/\hat{q}_3^L \qquad (32)$$

where $\hat{q}_3$ is q_3 evaluated at some "average" value of M and for $m_{12} = m^*$, and it serves to keep the dimensionality of B independent of L.

We will now extend our results to resonances in both the (12)3 and the (32)1 channels



If #1 and #3 are not identical, $\tilde{T}|in\rangle$ of Eq. (26) has a second

term reflecting the second diagram above, and the resulting T will just be the coherent sum of two terms with arbitrary relative magnitude and phase

$$T = T^{\eta}_{Jj_{12}}(M,s_{12},\theta_{12},R_3) + Xe^{iY}\, T^{\eta}_{Jj'_{12}}(M,s_{32},\theta_{32},R_1), \qquad (33)$$

where we have included the subscripts in j and j′ to emphasize that they refer to different systems so that even if j = j′, the system will not be an eigenstate of j. If #1 and #3 are identical, $\tilde{T}|in\rangle$ of Eq. (26) should be multiplied by a factor of 1/2 to avoid double counting in the sum over states, and the state normalizations as given in (20), (23), and (25) must be symmetrized in #1 and #3; for example, (25) becomes

$$\langle P,J,\Lambda,s_{12},j_{12},\lambda_{12}|P',J',\Lambda',s'_{12},j'_{12},\lambda'_{12}\rangle$$

$$= 4Mm'_{12}/[p'_{12}q'_3]\,\delta^{(4)}(P-P')\,\delta_{JJ'}\,\delta_{\Lambda\Lambda'}$$

$$\times[\delta(s_{12}-s'_{12})\,\delta_{j_{12}j'_{12}}\,\delta_{\lambda_{12}\lambda'_{12}} + \delta(s_{32}-s'_{12})\,\delta_{j_{32}j'_{12}}\,\delta_{\lambda_{32}\lambda'_{12}}].$$

With these symmetrized normalizations and having $\tilde{T}|in\rangle$ multiplied by 1/2 we still have condition (19). Carrying out the calculations as above leads to Eq. (33) with X = 1, Y = 0, $j_{12} = j'_{32}$ and an overall factor of 1/2. This overall factor would prove undesirable later on, so we take advantage of the fact that it is the <u>functional</u> dependence of T relative to T′ in Eq. (19) that matters. Hence we leave $\tilde{T}|in\rangle$ as it is in

Eq. (26) which leads to

$$T = 2T' = 2 \text{ (two identical particles, no correlations)} \tag{34}$$

but no overall factor of 1/2 in Eq. (33) when there are identical particles.

By analogy with the static model in Section IIE we define an effective cross section which is essentially the transition probability per unit time integrated over all possible final states

$$\tau_j^{\eta}(M) = 1/4 \int p_{12} q_3 ds_{12}/[m_{12} M] \int d(\cos\theta_{12}) \int dU_{R_3} |T|^2 . \tag{35}$$

As written in Eq. (33) T is a function of both the (12)3 set of variables and the (32)1 set. When calculating τ this causes no problem for the incoherent part of $|T|^2$ since we need only change the variables of integration to fit whichever set appears in the term. However both sets appear in the interference term, so that if we want to keep the resonances coherent when calculating τ we need to express the (32)1 set in terms of the (12)3 set. In Appendix C we write expressions for $s_{32} = s_{32}(M, s_{12}, \cos\theta_{12})$ and $\cos\theta_{32} = \cos\theta_{32}(M, s_{12}, s_{32})$ so that the only function we have left is $\overline{D}^J_{\Lambda\lambda}(R_1)$ which must be rewritten in terms of the (12)3 basis.

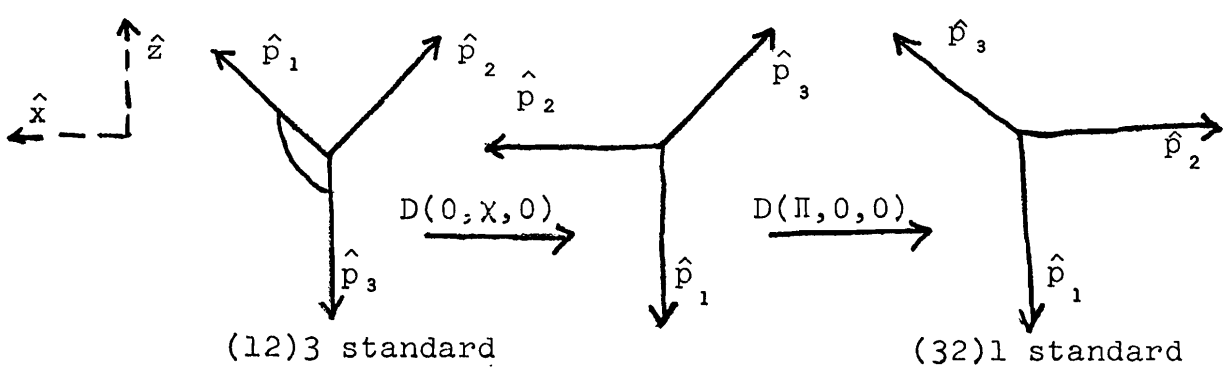
An arbitrary set of the three momenta $\tilde{\vec{p}}_1, \tilde{\vec{p}}_2, \tilde{\vec{p}}_3$ in the (12)3 cmf can be arrived at by performing some rotation $\tilde{R}_3$ on the momentum operators in the (12)3 standard orientation.

This same set could also be obtained by first applying the rotation $\tilde{R}'$ which maps the momenta from their configuration in the (12)3 standard orientation to the configuration corresponding to the (32)1 standard orientation and then applying the rotation $\tilde{R}_1$. Thus we have $\tilde{R}_3 = \tilde{R}_1\tilde{R}'$, or $\tilde{R}_1^{-1} = \tilde{R}'\tilde{R}_3^{-1}$. The same relations clearly hold for the eigenvalues of the rotation operators, and since the D-matrices are a representation of the rotation group we can write

$$\bar{D}^J_{\Lambda\lambda}(R_1) = D^J_{\lambda\Lambda}(\tilde{R}_1^{-1}) = \sum_{\lambda'} D^J_{\lambda\lambda'}(\tilde{R}')D^J_{\lambda'\Lambda}(\tilde{R}_3^{-1})$$

$$= \sum_{\lambda'} \bar{D}^J_{\Lambda\lambda'}(R_3)D^J_{\lambda\lambda'}(R'). \qquad (36)$$

The rotation $\tilde{R}'$ interchanges the directions of $\tilde{\hat{p}}_1$ and $\tilde{\hat{p}}_3$ in a fixed coordinate system. The same holds true for the eigenvalues which we prefer to work with. The rotation can be done in two steps: First $\hat{p}_1$ is mapped into the original direction of $\hat{p}_3$ which involves a rotation through an angle χ about the $\hat{y}$-axis. Now $-\hat{p}_1$ points along $\hat{z}$ but $\hat{p}_3.\hat{x} \leq 0$, so we rotate the system through an angle Π about the $\hat{z}$-axis which gives us the (32)1 standard orientation.



Then

$$D^J_{\lambda\lambda'}(R') = D^J_{\lambda\lambda'}(\Pi,\chi,0) = (-1)^\lambda d^J_{\lambda\lambda'}(\chi) \ . \tag{37}$$

Substituting (37) into (36) and then substituting the result into (27) gives

$$T^\eta_{Jj'}(M,s_{32},\theta_{32},R_1) = N_J N_{j'} \sum_{\substack{\Lambda\lambda \\ \lambda'}} f_{j'}(s_{32}) B^\eta_{\substack{Jj \\ \lambda'}}(q_1) \times \overline{D}^J_{\Lambda\lambda'}(R_1) d^J_{\lambda\lambda'}(\chi) d^{j'}_{0\lambda}(\theta_{32}), \tag{38}$$

where we used one of the symmetry relations for the d-matrices given in Eq. (C2) which in this case reads

$$(-1)^\lambda d^{j'}_{\lambda 0}(\theta_{32}) = d^{j'}_{0\lambda}(\theta_{32}) \ .$$

Substituting (38), (28), and (27) into (23), we have

$$T^\eta_J(M,(12)3 \text{ set})$$

$$= N_J \sum_{\substack{\Lambda\lambda \\ \lambda'}} \overline{D}^J_{\Lambda\lambda'}(R_3)[N_j f_j(s_{12})\delta_{\lambda\lambda'} B^\eta_{Jj\lambda}(q_3) d^j_{\lambda 0}(\theta_{12})$$

$$+ Xe^{iY} N_{j'} f_{j'}(s_{32}) B'^\eta_{Jj'\lambda}(q_1) d^J_{\lambda\lambda'}(\chi) d^j_{0\lambda}(\theta_{32})] \ . \tag{39}$$

The prime on B in the second term is there because it might be a different function from the B in the first term. (One could also put a prime on the f in the second term if the two resonances are different since although the functional form may be the same, the resonance parameters will differ.)

Using (39) we see that the integration over U_{R_3} in Eq. (35) can be done trivially, and that because of the orthogonality relation (C4) the double sums on Λ and λ' reduce to single sums with the resulting sum on Λ just giving a factor of 2J+1. Equation (35) then becomes

effective cross section

$$\tau_J^\eta(M) = [2J+1]/[8M]\int p_{12}q_3 ds_{12}/m_{12}\int d(\cos\theta_{12})$$

$$\times\sum_{\lambda'}\Big|[2j+1]^{1/2} f_j(s_{12})\xi_J^{\lambda'} B^\eta_{Jj\lambda'}(q_3) d^j_{\lambda' 0}(\theta_{12}) \qquad (40)$$

$$+ Xe^{iY}\sum_{\lambda}[2j'+1]^{1/2} f_{j'}(s_{32}) B^{\eta}_{Jj'\lambda}(q_1) d^J_{\lambda\lambda'}(\chi) d^{j'}_{0\lambda}(\theta_{32})\Big|^2$$

where

$$\xi_J^\lambda = \begin{cases} 1 & J \geq |\lambda| \\ 0 & \text{otherwise} \end{cases}$$

Equation (40) is only a two-dimensional integral and can be transformed quite simply into an integral over the usual Dalitz plot parameters s_{12} and s_{32} using the following relation which follows from Eq. (C5)

$$d(\cos\theta_{12}) = -1/2\ m_{12}/[Mp_{12}q_3]ds_{12}$$

which gives

$$1/M\int ds_{12}p_{12}q_3/m_{12}\int d(\cos\theta_{12}) = 1/[2M^2]\int ds_{12}\int ds_{32}. \qquad (41)$$

Thus the quantity $|\ |^2$ in (40) is just the Dalitz plot density.

If one wants an approximation to τ in which the resonances are treated as stable particles, one treats the two terms in (39) as though they were incoherent and uses the following:

$$|\Delta(s)|^{-2}\mathrm{Im}\Delta(s) \rightarrow \Pi\delta(s-m^{*2}) \quad .$$

The incoherent terms in $|T|^2$ are eigenstates in the two-particle angular momentum so that no angular projection is needed and the angular integrations are trivially done using the relations from Appendix C. The result is

$$\mathrm{Th}_{\tau_J^\eta}(M) = \Pi[2J+1]/[4M]\{\theta(M-m_3-m_{12}^*)q_3^* \sum_{-j}^{j}{}_\lambda |\xi_J^\lambda B_{Jj\lambda}^\eta(q_3^*)|^2$$

$$+ \theta(M-m_1-m_{32}^*)X^2 q_1^* \sum_{-j'}^{j'}{}_\lambda |\xi_J^\lambda B_{Jj'\lambda}^{\prime\eta}(q_1^*)|^2 \qquad (42a)$$

$$\equiv \mathrm{Th}_{\tau_J^\eta}(M)_{12} + \mathrm{Th}_{\tau_J^\eta}(M)_{32} \qquad (42b)$$

where $q_1^* = q_1(m_{32}=m_{32}^*)$ and $q_3^* = q_3(m_{12}=m_{12}^*)$.

The above results are easily extended to the case of three overlapping two-particle resonances in a three-particle system. Following the same procedure as above one can reexpress the third resonance term in the amplitude in the same basis as the other terms. In Appendix C some of the angles are defined for the third term given in the (31)2 basis. The only other relation which is needed is

$$\bar{D}^J_{\Lambda\lambda}(R_2) = \sum_\mu \bar{D}^J_{\Lambda\mu}(R_3)d^J_{\mu\lambda}(\chi')$$

where χ' is the angle between the $\hat{z}$-axes of the (12)3 and the (31)2 standard orientations. Note that the resonance terms can only interfere two at a time even when there are three terms.

B. An Example with Two Identical Particles

Our first example is the 3Π system in which the first and third are identical, and the overlapping resonances are identical ρ-mesons.[59] Our main reason for choosing this example is to see whether or not a model like ours can produce any structure in M in the $A_1(1080)$ region since there are papers by two sets of authors that claim it can.[7-9]

We first consider the problem in the way it was treated by Chang[7]. He simplified things slightly by taking $J = j = 0$ (and $\eta = -1$). As remarked above, this simplification removes any possibility of an ambiguity in the function B of Eq. (30) giving $B = 1$. Using Eq. (41) we can write Eq. (40) as follows:

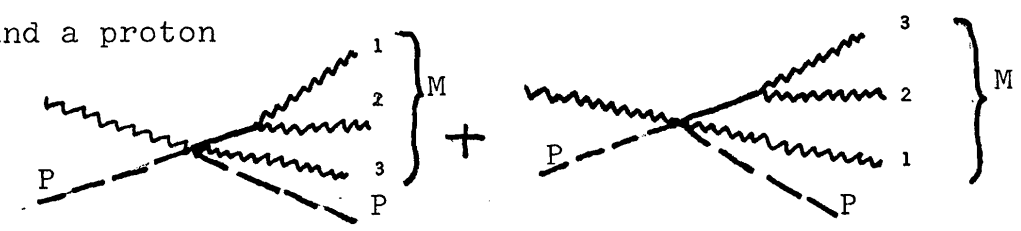
$$\tau_0^-(M) = [16M^2]^{-1}\int ds_{12}\int ds_{32}|f_0(s_{12}) + f_0(s_{32})|^2$$

$$= 1/[16M^2p^*]m^*\Gamma\int ds_{12}\int ds_{32}|1/\Delta_0(s_{12}) + 1/\Delta_0(s_{32})|^2.$$

If we write

$$\Delta_0(s) = s - m^{*2} + im^*\Gamma \qquad ,$$

we have the essentials of Chang's model. Chang actually imbedded the 3Π state in a four-particle state where the fourth particle is a (scalar) proton, and he has an initial state of a Π and a proton



The presence of the proton has the effect of multiplying $\tau_0^-(M)$ by a function of M giving

$$R(M) = \gamma(Q^2, M^2, m_P^{\ 2})\tau_0^-(M)$$

where Q is the four-particle mass which he fixes at Q = 2.7 BeV, and m_P is the proton mass.

Chang divides R into $R_{incoherent} + R_{interference}$. He approximates $R_{incoh.}$ by our theoretical resonance approximation[34]

$$R(M)_{incoh.} \simeq {}^{Th}R(M) = \gamma(Q^2, M^2, m_P^{\ 2})\ {}^{Th}\tau_0^-(M) \qquad (43a)$$

$$= [2M]^{-1}\Pi\gamma(Q^2, M^2, m_P^{\ 2})q^*\theta(M-m-m^*) \quad . \qquad (43b)$$

He then argues that $R_{interf.}$ should be appreciable only for values of M such that s_{12} and s_{32} can simultaneously be equal to m^{*2} which has the effect of limiting M for the interference term to $M^2 \geq 2m^{*2} + m^2$. For this region he calculates the interference term exactly. Figure 3a is Fig. 2 from Ref. 7. We see that his R has a peak near the A_1 which comes from a peak in $R_{interf.}$. Figure 3b contains our results for the same quantities except that we have not restricted the region for M in $R_{interf.}$. We find no peak in either $R_{interf.}$ or in R, and we feel confident that we have made no error in the integration, and we can only conclude that Chang made an error in his calculations. We see that Chang is also wrong in his claim that

one can ignore the interference term below the overlap threshold. Chang goes on to say that his peak is further enhanced if one subtracts out N*events. He gives an approximation scheme for making such a cut. We chose to use a computer program[35] which performs the integrals by a Monte Carlo technique so that one can make the N* cut exactly, and still we found no peak. After having made our calculations we found out that it is "common knowledge" that Chang is wrong.[36] However, when we made a literature search we found numerous references to Chang that assumed he was right[9,37,38] and one paper[39] which in turn referred to an unpublished preprint by M. Sweig[40] in claiming that Chang is in error. However, the wording in this paper is such that one might conclude only that making an N* cut does not enhance a peak that indeed exists. Thus we feel the point is worth making that there is no structure at all in the A_1 region using Chang's model.

In Fig. 3c we plot the exact result for R(M) along with the theoretical resonance approximation and the exact result for the incoherent part of R (°R). We see that the resonance approximation is in reasonably good agreement with R, and that as we found in the static model, ^{Th}R turns out to be a better approximation of R than it is of °R.

We will now treat the same case except that we give the ρ its correct spin and we set $J^{\eta} = 1^{+}$, which is the popular favorite for the A_1. The choice for B in Eq. (30) is now model dependent:[41] hard-pion techniques lead to predictions for $A_1 \to \rho\Pi$ to be dominated by s-waves[42] so that only L = 0 would appear in Eq. (30); quark model calculations of pion

emission by quarks, neglecting recoil corrections, predict[43] that $A_1 \to \rho\Pi$ is purely transverse -- only $\lambda = \pm 1$; and chiral SU(2)XSU(2) predicts[44] that the decay should be longitudinal, which means only $\lambda = 0$ survives for each $\rho\Pi$ part of the amplitude. A difficulty in reaching a decision concerning the correctness of one (or none) of these alternatives is that there is enough other structure in the 3Π spectrum in the A_1 region that it is not now certain that there is an A_1.[45] All experimental analyses previous to summer 1968 had approached the A_1 from the point of view that if there is an A_1 and if it had $J^\eta = 1^+$, then it is characterized by an s-wave for each $\rho\Pi$ system even though there could be some d-wave with the justification being essentially a carry-over of the nonrelativistic angular momentum barrier that inhibits higher orbital angular momenta relative to lower ones. There is a recent paper from SLAC which favors the longitudinal mode of decay.[46] The formalism we have developed particularly with respect to our treatment of the two-particle resonance and the form of B in Eq. (30) is based on the simplest possible momentum dependence consistent with two-particle unitarity and the threshold behavior, so that the formalism is probably on its firmest footing when one uses only the lowest possible value of L. Keeping more than one L means that results will necessarily be dependent on the choice of the normalizing factor $\hat{q}$ of Eq. (32) which will have an effect of setting the scale between contributions for different values of L. We will therefore concentrate mostly on s-wave dominance.

For $L = 0$, $J = 1$, and $j = 1$ Eq. (32) gives

$$B_\lambda = [3]^{-1/2}\langle 001\lambda|1\lambda\rangle = [3]^{-1/2}, \quad \text{(s-wave)}$$

and

$$\tau_1^+(M) = 3/[8M]\int ds_{12}p_{12}q_3/m_{12}\int d(\cos\theta_{12})$$

$$\times\sum_\lambda |f_1(s_{12})d^1_{\lambda 0}(\theta_{12}) + f_1(s_{32})d^1_{0\lambda}(\theta_{32}+\chi)|^2$$

$$= 3/[8M]\int ds_{12}p_{12}q_3/m_{12}\int d(\cos\theta_{12})$$

$\Rightarrow \alpha = +1$

$$\times\{|f_1(s_{12})|^2+|f_1(s_{32})|^2+2\mathrm{Re}[f_1(s_{12})\bar{f}_1(s_{32})]$$

$$\times\cos(\theta_{12}+\theta_{32}+\chi)\} \quad . \qquad (44a)$$

If we use the simplest form of $\Delta_1(s)$ consistent with two-particle unitarity[61]

$$\Delta_1(s) = s - m^{*2} + im^*\Gamma m^* p^3/[sp^{*2}]^{3/2},$$

Eq. (44a) can then be written

$$\tau_1^+(M) = 3m^{*2}\Gamma/[8Mp^{*3}]\int ds_{12}p_{12}q_3/m_{12}\int d(\cos\theta_{12})$$

$$\times\{p_{12}^2|\Delta_1(s_{12})|^{-2} + p_{32}^2|\Delta_1(s_{32})|^{-2}$$

$$+ 2p_{12}p_{32}\mathrm{Re}[\Delta_1(s_{12})\overline{\Delta_1(s_{32})}]^{-1}\cos(\theta_{12}+\theta_{32}+\chi)\} \quad . \qquad (44b)$$

The resonance approximation is

$$^{Th}\tau_1^+(M) = 3\times{}^{Th}\tau_0^-(M) = [2M]^{-1}3\Pi q^*\theta(M-m^*-m) \quad . \tag{45}$$

The various effective cross sections are plotted in Fig.4 along with $^{o}\tau$ which is τ neglecting the interference term. Again we have no peaks in the A_1 region and $^{Th}\tau$ is again a reasonably good approximation to τ. It appears that spin has little effect on either the validity of the resonance approximation or on the general shape of τ_J.

It is interesting to calculate some of the two-particle projections for this model and to compare them with the projections for the hypothetical case in which there is a 3Π system with only a single ρ. Curves for $d\tau/ds_{12}$ vs. s_{12} and for $d\tau/d\Omega_{12}$ vs. $\cos\theta_{12}$ are given in Fig. 5 (for τ_1^+) for three different values of the three-particle mass, M = 1.1 BeV, 1.555 BeV, and 1.875 BeV. These differential cross sections can be obtained from experimental data by including both of the two-particle masses or angles which occur for each event. The two-particle mass spectra are given in Fig. 5a,b,c for a range of $\pm 2m^*\Gamma$ surrounding m^{*2}. The dashed curves are the distributions for the single-ρ case. We see that while there are some differences in the shape for the two models, the peak position and the width are not changed much. The angular distribution for the single-ρ case would be isotropic, or a straight horizontal line in Fig. 5d,e,f, and we see that $d\tau/d\Omega_{12}$ (labelled "total") differs considerably from this for all three values of M. Also included in these subfigures are some angular distributions which do not really treat the identical particles symmetrically; we include them because such curves

were given by Bouchiat and Flamand.[47] In the curves labelled "resonance projection" m_{12} is set equal to m^*, and in the curves labelled "one resonance band" we have only integrated s_{12} over a region $m^{*2} \pm 2m^*\Gamma$. We see that these projections have somewhat less structure than the "total" curves but they are still not isotropic. Our conclusion, then, is that the two-particle mass distributions in the resonance region are only slightly affected by approximating our coherent model by an incoherent model, but the angular distributions are affected to a fairly significant extent. Reference 46 has not used a coherent model in their analysis, so we question their interpretation of the $\rho\Pi$ angular distribution.

The longitudinal or zero helicity model would contain contributions from both L = 0 and L = 2 (L = 1 has the wrong parity). To get B_λ for this case one first decomposes a $\lambda = 0$ state into L-states, then multiplies each L-state by $[q/\hat{q}]^L$, and then rewrites the L-states in terms of helicity states. The result is

$$B_\lambda(q)\Big|_{\text{longit.}} = 1/3\sum_L [2L+1]\langle L010|10\rangle\langle L01\lambda|1\lambda\rangle[q/\hat{q}]^L. \quad (46)$$

Substituting (46) into (42a) and using some Clebsch-Gordon algebra we find

$$^{Th}\tau_1^+(M)\Big|_{\text{longit.}} = {}^{Th}\tau_0^-(M)[1 + 2[q^*/\hat{q}]^4] \ . \quad (47)$$

It is only for $q = \hat{q}$ that the right hand side of (46) reduces

to $\delta_{\lambda 0}$ so that our results look like a zero-helicity model. If there is an A_1 with central mass $M = M^*$, and if $\hat{q}$ is defined to be $\hat{q} = \gamma(M^{*2}, m^{*2}, m^2)/[2M^*]$, then the transition amplitude for each $\rho\Pi$ system becomes

$$T^{+}_{11}(M^*, s^*_{12}, \theta_{12}, R_3)\Big|_{\text{longit.}}$$

$$= 3/[4\Pi]\sum_{\Lambda} f_1(s^*_{12})\bar{D}^1_{\Lambda 0}(R_3)\cos\theta_{12} \quad .$$

Thus it is only when the A_1 and ρ are "on shell" that $\lambda = 0$ and our dynamical approximation for B lead to a $\cos^2\theta$ prediction for each $\rho\Pi$ system. From (47) it is clear that no matter how $\hat{q}$ is defined, this model will not produce any enhancements in the A_1 region.

Gleeson and Meggs (G&M) claim to have obtained an enhancement in the A_1 region from a model similar to ours. In their first paper[8] they make the nonstatic analogue to our BAD_τ for the static model of Section IIE, and in their second paper[9] they obtain enhancements when they calculate uncorrected resonance-cut cross sections with a very small selection region in s. Figure 6a below is Fig. 3 from Ref. 8, and our Fig. 6b is Fig. 6 from Ref. 9.

The resonance approximation in their first paper includes both a mass projection and a spin projection. The mass projection can be obtained by the method used in Section IIE or by setting $m_{12} = m^*$ in the amplitude of Eq. (39) and not integrating over s_{12} when calculating τ. The spin projection

is made by taking the amplitude - either before or after the mass projection - and projecting out the part that transforms as $j_{12} = 1$. Performing the calculations on a corrected version of their model we get essentially the same results as they. The spin projection is clearly not treating the identical systems symmetrically, but the results are not altered much when this projection is left out. Using the analogue to Eq. (11′) it can be argued that the mass projection is symmetric in its treatment of the two resonances.[48] However, such a projection is invalid for the same reasons as in the static models treated above: when the amplitude is written in a product form, the coefficient of $|\Delta(s_{12})|^{-2}$ is not slowly varying near the overlap threshold to allow a δ-function approximation like Eq.(11). Since the coefficient here involves the result of an integration one can not see this fact as readily as in the static model by just looking at the functional dependence.

In their second paper G&M consider the same basic model both with and without a spin projection, and they replace the mass projection with an integral over a region Δm_{12} which is centered about m*. This procedure corresponds to the resonance-cut procedure discussed in Section I except that they have not treated the identical particles symmetrically. As is shown in Fig. 6b they find that the resulting effective cross sections plotted against M get more peaked as Δm_{12} is decreased. Figure 6c contains our calculations for some resonance-cut effective cross sections in which we have used the corrected form of their model. We have included curves for both symme-

trized and unsymmetrized selection regions. Particularly for the symmetrized resonance-cut cross sections we find relatively more peaking near the overlap threshold as Δm is decreased, but the main point here is how one chooses to interpret these results, and we feel that G&M have attributed to them a significance which is unwarranted: as described in Section I cross sections based on resonance cuts are only a means of suppressing nonresonance events - which we do not have in our model - and should not be interpreted as cross sections for resonance production.

In addition to not treating the identical particles symmetrically, other errors and inconsistencies in the formulation of the model by G&M are as follows: They claim to want a relative s-wave for each $\rho\Pi$ system but they in effect use $B_\lambda = \delta_{\lambda 0}$ which is not consistent. They use the "recoupling coefficient"[4] which corresponds to a cyclic permutation of the particles rather than an interchange of #1 and #3. They use a normalization for the angular momentum states which leads to some extraneous momentum dependence in the amplitude.

C. An Example with No Identical Particles

In the next section we show explicitly that when our model is restricted to one value of L it is equivalent to the model used by two groups in studying the charged $K\Pi\Pi$ system near 1.3BeV.[11,12] When this state contains a charged K there are two overlapping resonances, a ρ and a K*, while if there is a neutral K there can be three overlapping resonances, a ρ and two K*'s. Our purpose here is simply to illustrate a possible application of the model we have developed, so we will choose

to work with the simpler two-resonance case. We will also simplify on reality by taking all spins to be zero; we justify this partly by the fact that our results with the 3Π system seemed to indicate that spin has only a secondary effect on τ and on the validity of resonance approximations, and partly by the fact that having all spins zero and using Eq. (D10) for $\tau_0(s)$ reduces the integrals to a single nontrivial integration which leads to a considerable saving in computer time and money.

Choosing $(m^*,\Gamma)_{12} = (m^*,\Gamma)_{K^*}$ and $(m^*,\Gamma)_{32} = (m^*,\Gamma)_\rho$,[60] we find

$$\tau_0^-(M) = [16M^2]^{-1}\int ds_{12}\int ds_{32}|f_{K^*}(s_{12})+Xe^{iY}f_\rho(s_{32})|^2 \quad ,$$

and

$$^{Th}\tau_0^-(M) = [4M]^{-1}\theta[q_3^*\theta(M-m_{K^*}-m_\Pi) + |X|^2 q_1^*\theta(M-m_\rho-m_K)] \quad .$$

If the KΠΠ-system transforms like a K under SU(3), the functions f and B have been normalized so that pure f-(d-) coupling (corresponding to an octet with C = +1(-1)) would lead to X = 1 and Y = 0(Π). Accordingly, in Fig. 7a,b,c we give plots of τ for X = 0.2,1,and 5, and for each value of X we include values of 0,Π/2,and Π for Y. Figures 7d,e,f contain plots of τ for the same three values of X and for Y = 0 along with the various resonance approximations: $^{Th}\tau$, $^{Th}\tau_{K^*}$, $^{Th}\tau_\rho$, $^0\tau$, $^0\tau_{K^*}$, and $^0\tau_\rho$ where $^0\tau$ is τ neglecting the interference

term, and ${}^0\tau = {}^0\tau_{K^*} + {}^0\tau_\rho$.

Figures 8 and 9 contain plots of $d\tau/ds$ vs. s for M = 1.3 BeV and 1.6 BeV respectively. Each figure contains six subfigures: the first three are the two-particle mass distributions for the K* mass, and the last three are the distributions for the ρ mass. Again each subfigure is for a different value of X and contains curves for the three values of Y. For reference we have included a curve (dashed line) for which the second resonance is left out of the amplitude. Note that for M = 1.3 BeV the ρ-mass distributions are quite affected by the choice of Y, a fact that might prove useful when trying to determine Y from the data.

D. Another Form for the Model

Experimental angular distributions in $\cos\theta_{12}$ can be compared directly with the predictions based on Eq. (39) since θ_{12} is an experimental observable. However, if for some reason one wanted to look at angular distributions in the overall cmf, the model in the form we have developed could not be used for an experimental analysis because the angle ϕ_3, θ_3, and ϕ_{12} are defined with respect to a coordinate system which is not observable. We chose this particular coordinate system so that $\lambda = j_{12_z}$ could be interpreted as a helicity for the (12) system. Transforming from this helicity-based coordinate system to an arbitrary coordinate system which can be set experimentally is equivalent to transforming from the helicity framework of Wick[4] to the canonical framework of MacFarlane[6]. We have already discussed the canonical frame-

work in that our dynamical approximation for B in Eq. (30) is based on it. In that section we used Ref. 5 to write the canonical states in terms of helicity states which simply involved using Clebsch-Gordon coefficients. We will now rewrite the helicity states and the amplitude T in terms of the observable angular sets of the canonical basis.

We define the angular state $|\theta_3^c,\phi_3^c,\theta_{12}^c,\phi_{12}^c>$ where the angles are defined as in Section IIIA except that here both sets are measured with respect to some arbitrarily-set fixed coordinate system instead of the standard orientations defined above. We can obtain a state in the canonical basis with $j_{12} = j$ and $j_z = n$ as follows:

$$|\theta_3^c,\phi_3^c,j,n>_{(c)} = N_j\int d\Omega_{12}^c \bar{D}_{n0}^j(\phi_{12}^c,\theta_{12}^c,0)|\theta_3^c,\phi_3^c,\theta_{12}^c,\phi_{12}^c> \quad .(48)$$

Since all the particles are spinless, a helicity state $|\theta_3^c,\phi_3^c,j,\lambda>$ can be defined by treating the $(\underline{1}2)$ system as a single particle with spin j (c.f. Ref. 30, Eq. (28.50))

$$|\theta_3^c,\phi_3^c,j,\lambda> = \sum_n D_{n\lambda}^j(\phi_3^c,\theta_3^c,0)|\theta_3^c,\phi_3^c,j,n>_{(c)} \quad . \qquad (49)$$

The helicity state $|J,\Lambda,j,\lambda>$ is then given by

$$|J,\Lambda,j,\lambda> = N_J\int d\Omega_3^c \bar{D}_{\Lambda\lambda}^J(\phi_3^c,\theta_3^c,0)|\theta_3^c,\phi_3^c,j,\lambda>$$

$$= N_J\sum_n\int d\Omega_3^c \bar{D}_{\Lambda\lambda}^J(\phi_3^c,\theta_3^c,0)D_{n\lambda}^j(\phi_3^c,\theta_3^c,0)|\theta_3^c,\phi_3^c,j,n>_{(c)} \quad . \qquad (50)$$

From Ref. 30 Eq. (28.52) we have

$$\bar{D}^{J}_{\Lambda\lambda} D^{j}_{n\lambda} = \sum_L [N_L/N_J]^2 \langle L,\Lambda-n,j,n|L\Lambda\rangle\langle L0j\lambda|J\lambda\rangle \bar{D}^{L}_{\Lambda-n,0} \qquad (51)$$

Combining Eq. (48) - Eq. (51) and using Eq. (31) we find that the helicity state can be written in the form

$$|J,\Lambda,j,\lambda\rangle = \sum_L [N_L/N_J]^2 \langle L0j\lambda|J\lambda\rangle$$

$$\times \int d\Omega^c_3 \int d\Omega^c_{12} \sum_n \langle L,\Lambda-n,j,n|J\Lambda\rangle Y^{\Lambda-n}_L(\theta^c_3,\phi^c_3) Y^n_j(\theta^c_{12},\phi^c_{12})$$

$$\times |\theta^c_3,\phi^c_3,\theta^c_{12},\phi^c_{12}\rangle \quad . \qquad (52)$$

From (53) and the inverse of Eq. (31)

$$|P,J,\Lambda,s_{12},j,\lambda\rangle = \sum_L N_L/N_J \langle L0j\lambda|J\lambda\rangle |P,J,\Lambda,s_{12},j,L\rangle$$

we find

$$|P,J,\Lambda,s_{12},j,L\rangle = N_L/N_J \int d\Omega^c_3 \int d\Omega^c_{12} \sum_n \langle L,\Lambda-n,j,n|J\Lambda\rangle$$

$$\times Y^{\Lambda-n}_L(\theta^c_3,\phi^c_3) Y^n_j(\theta^c_{12},\phi^c_{12}) |P,\theta^c_3,\phi^c_3,s_{12},\theta^c_{12},\phi^c_{12}\rangle \quad . \qquad (53)$$

If Eq. (52) is used in Eq. (26), we find that T for a single resonance in Eq. (27) is given by

$$T^{\eta}_{Jj}(M,s_{12},\theta^c_3,\phi^c_3,\theta^c_{12},\phi^c_{12}) = \sum_{\substack{\Lambda\lambda \\ Ln}} [N_L/N_J]^2 \langle L0j\lambda|J\lambda\rangle \qquad (54)$$

$$\times \langle L,\Lambda-n,j,n|J\Lambda\rangle Y^{\Lambda-n}_L(\theta^c_3,\phi^c_3) Y^n_j(\theta^c_{12},\phi^c_{12}) B^{\eta}_{Jj\lambda}(q_3) f_j(s_{12}) \quad .$$

For the single L case, Eq. (32) and some Clebsch-Gordon algebra simplifies Eq. (54) to

$$T^{\eta}_{JjL}(M,s_{12},\theta^c_3,\phi^c_3,\theta^c_{12},\phi^c_{12})$$

$$= N_L/N_J f_j(s_{12})[q_3/\hat{q}_3]^L \sum_{\Lambda n} \langle L,\Lambda-n,j,n|J\Lambda\rangle$$

$$\times\ Y_L^{\Lambda-n}(\theta^c_3,\phi^c_3)Y^n_j(\theta^c_{12},\phi^c_{12}) \quad . \qquad (55)$$

Up to a constant, Eq. (55) is the form used by the Yale[11] and the UCLA[12] groups in their studies of the charged KΠΠ system.

The total amplitude for the overlapping resonances is of the form of Eq. (33) with the amplitude for each channel given by (55). There is no easy way to derive an analogue to Eq. (39) in the canonical framework, which is why we believe the helicity form to be more useful for performing calculations.

Appendix A: Background on the Lee Model

We list below some definitions and results from the paper by Chen-Cheung and Sommerfield[1] leaving out most of the derivations. We also include some definitions and formulae for an extension of the Lee model based in part on some work done by Chen-Cheung.[50]

The Lee model Hamiltonian with two V-fields is

$$H(t) = \int d^3r\{ \sum_{j=1}^{2} m_j V_j^\dagger(\vec{r},t)V_j(\vec{r},t) + m_N N^\dagger(\vec{r},t)N(\vec{r},t)$$

$$+ 2\int d^3r' \theta^\dagger(\vec{r},t)\tilde{\omega}^2(\vec{r}-\vec{r}')\theta(\vec{r}',t)$$

$$+ \sum_{j=1}^{2} g_j \int d^3r' \rho(\vec{r}-\vec{r}')[V_j^\dagger(\vec{r},t)N(\vec{r},t)\theta(\vec{r}',t)$$

$$+N^\dagger(\vec{r},t)V_j(\vec{r},t)\theta^\dagger(\vec{r}',t)]\} \quad , \qquad \text{(A1)}$$

where

$$\omega^n(\vec{r}-\vec{r}') = [2\Pi]^{-3}\int d^3p \ \omega(p)^n \ e^{i\vec{p}.(\vec{r}-\vec{r}')} \quad ,$$

$$\rho(\vec{r}-\vec{r}') = [2\Pi]^{-3} \int d^3p \ u[\omega(p)] \ e^{ip.(\vec{r}-\vec{r}')} \quad ,$$

and

$$\omega(p) = [\vec{p}.\vec{p} + \mu^2]^{1/2} \quad .$$

The function $u(\omega)$ is a cut-off function which is necessary in the nonrelativistic form of the theory[51] if the renormaliztion integrals are to converge. It can also be interpreted as a vertex structure function or as a wave function[52]. In keeping with C&S we choose $u(\omega)$ real and essentially unity for ω in

the region of interest. Specifically we choose $u(\omega) = [1 + \omega^2/M^2\ ]^{-1/2}$ with $M = 10^3\mu$. The only nonvanishing equal-time commutators[53] are

$$[V_i(\vec{r},t),V_j^\dagger(\vec{r}',t)] = \delta_{ij}\delta(\vec{r}-\vec{r}')$$

$$[N(\vec{r},t),N^\dagger(\vec{r}',t)] = \delta(\vec{r}-\vec{r}')$$

$$[\theta(\vec{r},t),\theta^\dagger(\vec{r}',t)] = \tilde{\omega}^{-1}(\vec{r}-\vec{r}')/2 \quad .$$

As described in Section IIB we work with the linear combination $\overline{V} = [g_1V_1 + g_2V_2]/g$, and we choose the parameters so that there is a stable V-particle with mass m and an unstable V*-particle with complex mass $\beta \simeq m^* - i\Gamma/2$. Because of the selection rule $\overline{V} \not\leftrightarrow N\theta$, only differences between the $\overline{V}$-masses and m_N occur in any processes so that we lose no generality setting $m_N = 0$.

The approach used by C&S is to use the Heisenberg equations of motion for the fields to find equations of motion for the two-point Green's functions which are the propagators and for the four- and five-point functions which represent the scattering processes of interest. For the two-point functions we define

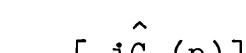
$$G_A(x,x') = \langle 0|(A(x)A^\dagger(x'))_+|0\rangle = [2\Pi]^{-4}\int d^4p\hat{G}_A(p)e^{ip\cdot(x-x')}.$$

The N-propagator and the θ-propagator are unchanged by renor-

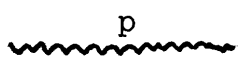
malization

$$[-i\hat{G}_N(p)]^{-1} = p^\circ + i\varepsilon \qquad [-i\hat{G}_\theta(p)]^{-1} = 4\omega(p)[p^\circ - \omega(p) + i\varepsilon] \ .$$

Graphical representations of $\hat{G}_N(p)$ and $\hat{G}_\theta(p)$ are given by

$\hat{G}_N(p)$ <=> 

and

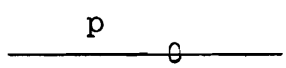
$\hat{G}_\theta(p)$ <=> 

.

The $\overline{V}$-propagator is changed by renormalization. Defining the dressed and undressed $\overline{V}$-propagators respectively by

$\hat{G}_{\overline{V}}(p)$ <=>

and

$\hat{G}^\circ_{\overline{V}}(p)$ <=> 

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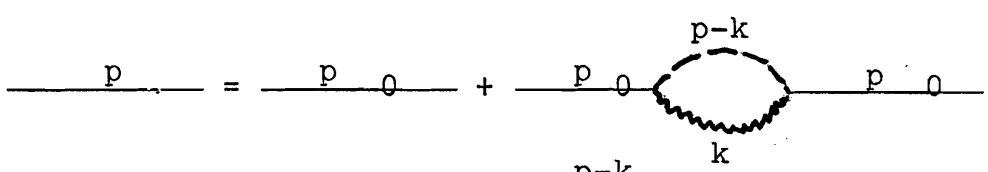
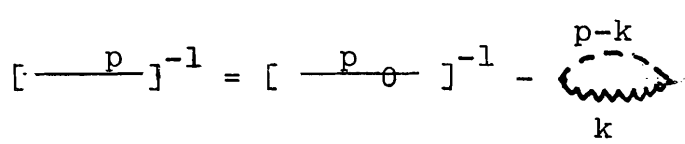
C&S find that $\hat{G}_{\overline{V}}(p)$ satisfies the following graphical equation



or



(A2)

where the vertex stands for $gu[\omega(k)]$ and the closed loop becomes $[2\Pi]^{-4}\int d^4k$. C&S show that (A2) can be written in the form

$$[-i\hat{G}_{\overline{V}}(p)]^{-1} = D(\omega) = (\omega-m)K(\omega) \qquad \text{(A3)}$$

where $\omega = p^o - m_N = p^o$ and

$$K(\omega) = 1 + (\omega - m)$$

$$\times \left[g^2/(4\Pi^2) \int_\mu^\infty d\omega' \frac{k(\omega')|u(\omega')|^2}{(\omega'-m)^2(\omega'-\omega-i\varepsilon)} - \frac{B}{\omega-\alpha}\right] \tag{A4}$$

where $k(\omega) = [\omega^2-\mu^2]^{1/2}$ and B,α are constants.

For $m = -20\mu$, $g^2 = 1.3$, $\alpha = 1.1$, and $B = 0.5$, the parameters for the V^* are given by $m^* = 21\mu$ and $\Gamma = 4.1\mu$. The $N\theta$ threshold is given by $\omega = \mu$, so that this parameterization has the stable V far below threshold with the result that the $N\theta$ two-particle mass spectrum will be dominated by the V^*. Also note that m^* is far above threshold, and Γ/m^* is about 0.2. Near $\omega = m^*$

$$[-i\hat{G}_{\overline{V}}(p)]^{-1} \simeq Z^{*-1}[\omega - m^* + i\Gamma/2] \tag{A5}$$

where

$$\Gamma = g^2/(2\Pi)\times Z^*|u(m^*)|^2 k(m^*) \text{ and } Z^{*-1} = \frac{d}{d\omega}[\mathrm{Re}\ D(\omega)]\Big|_{\omega=m^*} .$$

The processes of interest are

$$\theta + N \to \theta + N \tag{A6a}$$

$$\theta + V \to \theta + V \tag{A6b}$$

$$\theta + V \to \theta + N + \theta \tag{A6c}$$

$$\theta + V \to \theta + V^* \tag{A6d}$$

Process (A6d) is discussed in some detail in Appendix B. The

amplitudes for the first three processes are obtained by applying the usual reduction procedure which entails projecting the initial and final particles onto their mass shells; eg,, for (A5c) we have

$$\begin{aligned}
&{}^{N\ \theta\ \ \theta}_{\langle q,k_1,k_2}\ (\text{out})\,|{}^{V\ \theta}_{p,k}\ (\text{in})\rangle \\
&= \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 d^4x_5 \langle q|N^\dagger(x_1)|0\rangle\langle k_1|\theta^\dagger(x_2)|0\rangle \\
&\quad \times \langle k_2|\theta^\dagger(x_3)|0\rangle\langle 0|\overline{V}(x_4)|p\ (m)\rangle\langle 0|\theta(x_5)|k\rangle \\
&\quad \times \gamma_N(x_1)\gamma_\theta(x_2)\gamma_\theta(x_3)\tilde{\gamma}_V(x_4)\tilde{\gamma}_\theta(x_5) \\
&\quad \times \langle 0|(N(x_1)\theta(x_2)\theta(x_3)\overline{V}^\dagger(x_4)\theta^\dagger(x_5))_+|0\rangle \quad . \qquad \text{(A7)}
\end{aligned}$$

The wave functions for the particles are given by

$$\begin{aligned}
\langle 0|\overline{V}(x)|p\ (m)\rangle &= \langle p\ (m)|\overline{V}^\dagger(x)|0\rangle^* = [2\Pi]^{-\frac{3}{2}}\, e^{i(\vec{p}\cdot\vec{x}-mt)} \\
\langle 0|N(x)|q\rangle &= [2\Pi]^{-3/2}\, e^{i\vec{p}\cdot\vec{x}} \qquad \text{(A8)} \\
\langle 0|\theta(x)|k\rangle &= \{[2\Pi]^3\ 2\omega(k)\}^{-1/2}\, e^{i(\vec{k}\cdot\vec{x}-\omega(k)t)} \quad .
\end{aligned}$$

The γ's are inverse Green's functions satisfying equations of the form

$$\gamma_A(x)G_A(x,x') = \tilde{\gamma}_A(x')G_A(x,x') = \delta^{(4)}(x-x') \ .$$

Except for (A5a) the solutions for the Green's functions are found by going into momentum space; eg., for (A5c) we have

$$G(\overset{N}{x}_1,\overset{\theta}{x}_2,\overset{\theta}{x}_3,\overset{V}{x}_4,\overset{\theta}{x}_5) = [2\Pi]^{-20}\int d^4p_1 d^4p_2 d^4p_3 d^4p_4 d^4p_5$$

$$\times\ \hat{G}(\overset{N}{p}_1,\overset{\theta}{p}_2,\overset{\theta}{p}_3,\overset{V}{p}_4,\overset{\theta}{p}_5)\ e^{i(p_1\cdot x_1+p_2\cdot x_2+p_3\cdot x_3+p_4\cdot x_4+p_5\cdot x_5)}.$$

The cross section for a process is obtained from the scattering amplitude by subtracting off the no-interaction part (which is nonzero only for processes (a) and (b)), squaring the resulting amplitude, integrating over all possible final states, and dividing this result by the incident flux. We list below the graphical equations for the $\hat{G}$'s, the expressions for the transition amplitudes, and the expressions for the cross sections for the above processes.

(a) $\theta + N \rightarrow \theta + N$

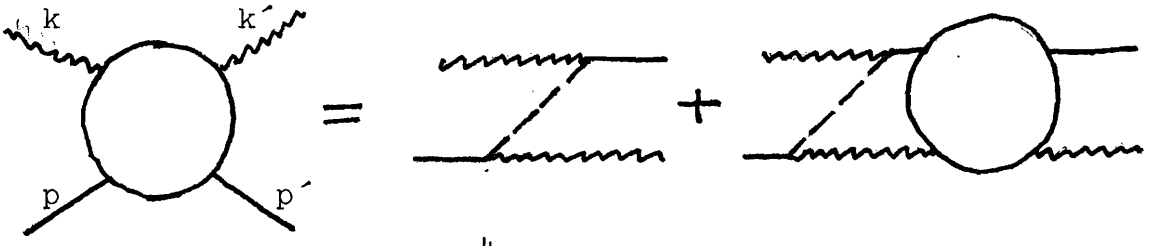
$$\langle q',k'|q,k\rangle' = \langle q',k'|q,k\rangle - \delta(\vec{q}-\vec{q}\,')\delta(\vec{k}-\vec{k}\,')$$

$$= \frac{i(2\Pi)^4\delta(q=k-q'-k')u(\omega)u(\omega')g^2}{[2\Pi]^6\times[4\omega\omega']^{1/2}\ D(\omega)}$$

$$\sigma_{(a)}(\omega) = g^4|u(\omega)|^2/[4\Pi]|D(\omega)|^{-2}$$

where $\omega = \omega(k) = k^o$ and $\omega' = \omega(k') = k'^o$.

(b) $\theta + V \to \theta + V$



$$\langle p',k'|p,k\rangle' = \frac{i[2\Pi]^4\,\delta(p+k-p'-k')u(\omega)u(\omega')F(\omega',\omega)}{[2\Pi]^6\times[4\omega\omega']^{1/2}} \qquad \text{(A9)}$$

$$\sigma_{(b)}(\omega) = |u(\omega)|^4/[4\Pi]|F(\omega,\omega)|^2$$

C&S find an exact solution for $F(\omega',\omega)$

$$F(\omega',\omega) = g^2K(\bar{\omega}')\left\{\frac{1}{(\omega'-m)K(\omega)} + \frac{1}{\Pi}\int_{\mu}^{\infty}\frac{d\nu}{\nu-\bar{\omega}'}\,\frac{H(\nu,\bar{\nu},\omega)}{D(\bar{\nu})}\,\mathrm{Im}\left[\frac{1}{K(\nu)}\right]\right\} \qquad \text{(A10)}$$

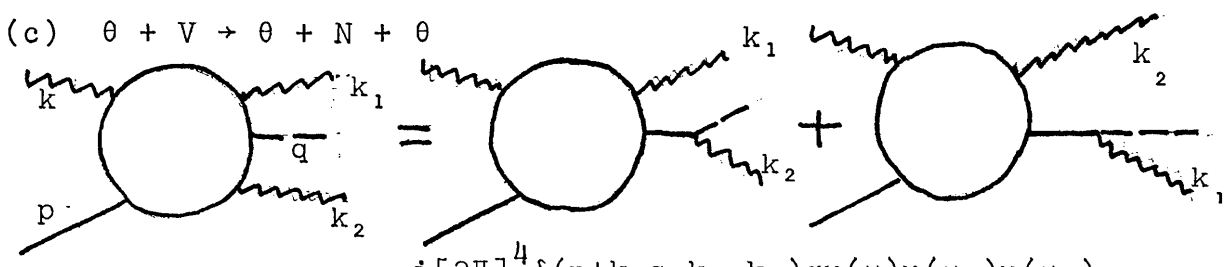
where

$$g^2H(\nu,\bar{\nu},\omega) = D(\nu)F(\nu,\omega) + D(\bar{\nu})F(\bar{\nu},\omega) \qquad \text{(A11a)}$$

$$= R(\omega) + S(\omega)[1/(\nu-\alpha) + 1/(\bar{\nu}-\alpha)]\ . \qquad \text{(A11b)}$$

R and S are known functions of ω, and $\bar{\nu} \equiv \omega+m-\nu$.

(c) $\theta + V \to \theta + N + \theta$



$$\langle k_1,q,k_2|p,k\rangle = \frac{i[2\Pi]^4\delta(p+k-q-k_1-k_2)gu(\omega)u(\omega_1)u(\omega_2)}{[(2\Pi)^{15}\ 8\omega_1\omega_2\omega]^{1/2}}$$

$$\times\ [F(\omega_1,\omega)/D(\omega_2) + F(\omega_2,\omega)/D(\omega_1)] \qquad \text{(A12a)}$$

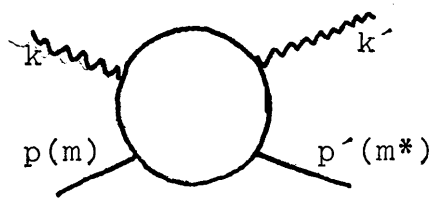
$$= \frac{i[2\Pi]^4 \delta(p+k-q-k_1-k_2) g^3 u(\omega) u(\omega_1) u(\omega_2)}{[(2\Pi)^{15}\ 8\omega_1\omega_2\omega]^{1/2}}$$

$$\times\ H(\omega_1, \omega_2, \omega)/[D(\omega_1)D(\omega_2)] \qquad \text{(A12b)}$$

$$\sigma_{(c)}(\omega) = g^2 |u(\omega)|^2/[2\Pi k(\omega)] \int_{\mu}^{\bar{\mu}} d\nu\ \mathrm{Im}[1/D(\nu)]\mathrm{Im}[1/D(\bar{\nu})]$$

$$\times\ |H(\nu, \bar{\nu}, \omega)|^2. \qquad \text{(A13)}$$

(d) $\theta + V \rightarrow \theta + V^*$



$$\langle p'\ (m^*), k' | p\ (m), k\rangle \qquad \text{(A14)}$$

$$= \frac{i[2\Pi]^4 \delta(p+k-p'-k') u(\omega) u(\omega') Z^{*1/2}\ F(\omega', \omega)}{[(2\Pi)^{12}\ 4\omega'\omega]^{1/2}}$$

$$\sigma_{(d)}(\omega) = Z^* k(\bar{m}^*)/[4\Pi k(\omega)] |u(\bar{m}^*)|^2 |F(\bar{m}^*, \omega)|^2\ \theta(\bar{m}^* - \mu)$$

$$\text{(A15)}$$

Unitarity leads to a form of the optical theorem for processes (b) and (c) given by

$$|u(\omega)|^2/k(\omega)\ \mathrm{Im}\ F(\omega, \omega) = \sigma_{(b)}(\omega) + \sigma_{(c)}(\omega). \qquad \text{(A16)}$$

We will now sketch briefly some of the effects which result from adding to the above Lee model a W-particle with the selection rule $W \rightleftarrows V_j \theta\ (j=1,2)$.[50] The extended model is used in Appendix B.

The total Hamiltonian is H(t) of Eq. (A1) plus $H^W(t)$, with $H^W(t)$ given by

$$H^W(t) = \int d^3r \{m_W^o W^\dagger(\vec{r},t)W(\vec{r},t) + \sum_{j=1}^{2} h_j \int d^3r' \; \rho(\vec{r}-\vec{r}')$$

$$\times [W^\dagger(\vec{r},t)V_j(\vec{r},t)\theta(\vec{r}',t) + V_j^\dagger(\vec{r},t)W(\vec{r},t)\theta^\dagger(\vec{r}',t)]\} \; .$$

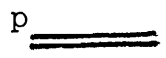
The only additional nonvanishing equal-time commutator is

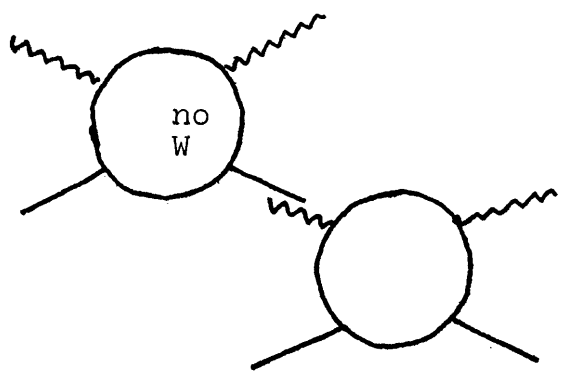
$$[W(\vec{r},t), W^\dagger(\vec{r}',t)] = \delta(\vec{r}-\vec{r}') \; .$$

For the Lee model with no W the linear combination of V_1 and V_2 which occurred in all internal V-lines was $\overline{V} = \lambda(g_1V_1+g_2V_2)$. This is no longer true if we include the W unless we choose $h_1/h_2 = g_1/g_2$, which we shall do.[54]

We will now write down a graphical solution to the $\overline{V}\theta$ Green's functions from which the scattering amplitudes for (b)-(d) can be obtained exactly as in the no-W case.[55] We start with some definitions.

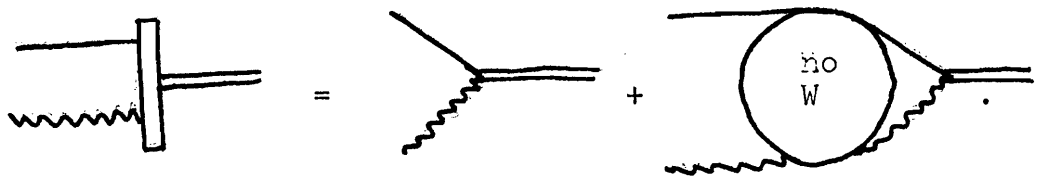
$\hat{G}^o_W(p)$ <=>

$\hat{G}_W(p)$ <=> 

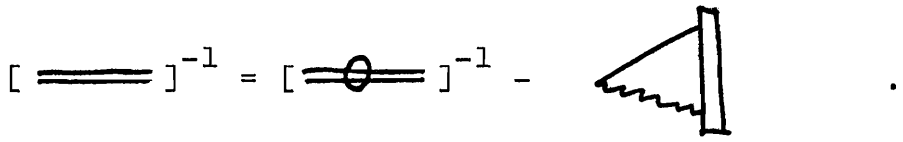
$\hat{G}_{\overline{V}\theta}$ no W <=> 

$\hat{G}_{\overline{V}\theta}$ <=>

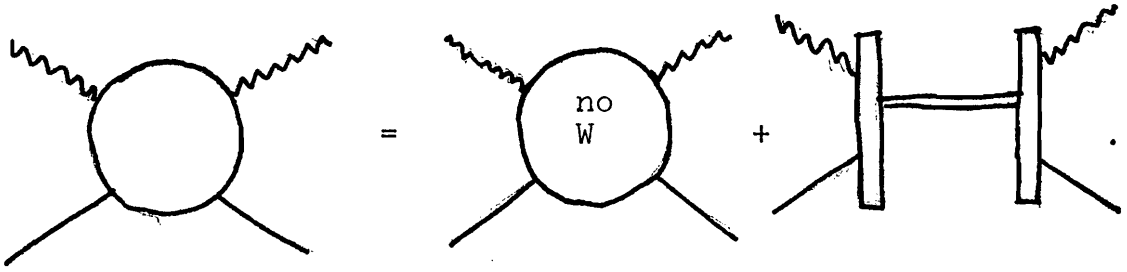
The $(\bar{V}\theta W)$ vertex function is given by



Using this one can derive the following equation for the renormalized W-propagator



The solution for the Green's function is then



Appendix B: Unstable Particles in the Lee Model

In this appendix we will derive the expression given in Eq. (A14) for the resonance-production process (A6d). We find it useful to work in the framework of the extended Lee model described towards the end of Appendix A in which we have a W-particle which couples to $V\theta$. We choose to do this because we will be considering the limit $g^2 \to 0$, and with no W, processes (c) and (d) vanish in this limit and (b) reduces to the no-interaction process. We could work with no W and replace the $g^2 = 0$ limit with a lowest-order-in-g^2 argument, but we prefer not to.

In the limit $g^2 \to 0$ with m, α, and B fixed[56] it can be shown that $g_1/g = g_2/g = g_1/g_2 = 1$ so that $\overline{V} = V_1 + V_2$. Also, from (A4)

$$\lim_{g^2 \to 0} D(\omega) = (\omega-m)(1-B[\omega-m]/[\omega-\alpha]),$$

and we find $m_o^* = [\alpha-Bm]/[1-B]$ and $Z_o^* = B/[1-B]$. Thus our choice of B = 0.5 has the effect that in this limit the V and V* are both stable particles with residues of unity for $1/D(\omega)$. For the parameterization used by C&S we find $m_o^* = 22.2\mu$ and $Z_o^* = 1$ which are both about 6% higher than their values for $g^2 = 1.3$. These results are probably a better indicator of the effective strength of the coupling than Γ/m^* or g/g_{max} and the fact that they are so small may help explain why the Born approximation is so successful.

Still in the $g^2 = 0$ limit, we can derive the V and V*

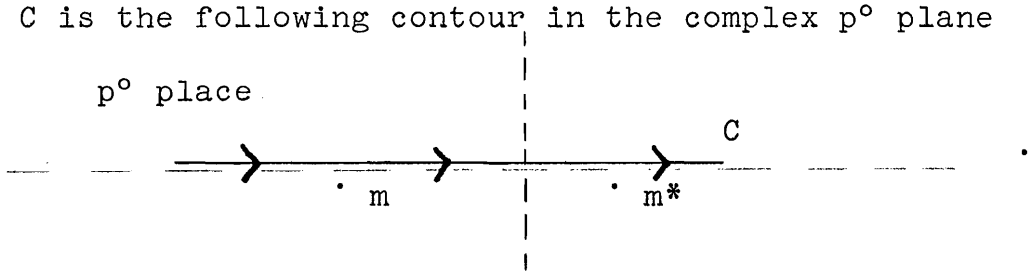
wave functions by the usual procedure of equating two expressions for $G_{\overline{V}}(x,0)$. We will go through this because we want to see how things are affected when $g^2 \neq 0$. The first way to write G is the direct way

$$G_{\overline{V}}(x,0) = [2\Pi]^{-4} \times \int d^4p \; \hat{G}_{\overline{V}}(p) \; \theta(x^\circ) \; e^{ip\cdot x}$$

$$= \delta(\vec{x}) \times i/[2\Pi] \times \int_C dp^\circ \; \theta(x^\circ) \; e^{-ip^\circ x^\circ}$$

$$\times \; \{(p^\circ - m)(1 - B[p^\circ - m]/[p^\circ - \alpha])\}^{-1}$$

where C is the following contour in the complex p° plane



The contour can be closed in the lower half plane and the contribution of the infinite semi-circle can be shown to vanish. The final result is then just the contribution from the two poles

$$G_{\overline{V}}(x,0) = \delta(\vec{x})\theta(x^\circ) \times [e^{-imx^\circ} + Z_o^* e^{-im_o^* x^\circ}] \quad . \qquad \text{(B1)}$$

The second method is to write $G_{\overline{V}}(x,0) = \langle 0|(\overline{V}(x)\overline{V}^\dagger(0))_+|0\rangle$ and to insert a complete set of states between the $\overline{V}$'s, which in this case is just the V and the V*

$$G_{\overline{V}}(x,0) = \int d^3p \; [\langle 0|\overline{V}(x)|m\rangle\langle m|\overline{V}^\dagger(0)|0\rangle +$$

$$+ \langle 0|\bar{V}(x)|m_o^*\rangle\langle m_o^*|\bar{V}^\dagger(0)|0\rangle]\theta(x^o)$$

$$= (2\Pi)^3\delta(\vec{x})\theta(x^o)$$

$$\times [\,|\langle 0|\bar{V}(0)|m\rangle|^2 e^{-imx^o} + |\langle 0|\bar{V}(0)|m_o^*\rangle|^2 e^{-im_o^* x^o}] \; . \tag{B2}$$

Equating terms with the same x^o dependence in (B1) and (B2) we find that to within a phase

$$\langle 0|\bar{V}(0)|m\rangle = [2\Pi]^{-3/2} \tag{B3}$$

$$\langle 0|\bar{V}(0)|m_o^*\rangle = [Z_o^*/(2\Pi)^3]^{1/2} \quad . \tag{B4}$$

Starting from the four-point function $\langle 0|(\bar{V}\theta\bar{V}^\dagger\theta^\dagger)_+|0\rangle$, projecting the initial $\bar{V}$ onto the V-mass shell, and projecting the two θ's onto their mass shells leads to

$$\langle k'|\bar{V}(x)|p\ (m),k\rangle$$

$$= iu(\omega')u(\omega)F(\omega',\omega)\hat{G}_{\bar{V}}(\omega')e^{ix\cdot(p+k-k')}/[(2\Pi)^9 4\omega\omega'] \quad . \tag{B5}$$

Because the V* is stable, process (d) is well defined, and the amplitude can be written

$$\langle p'\ (m_o^*),k'|p\ (m),k\rangle$$

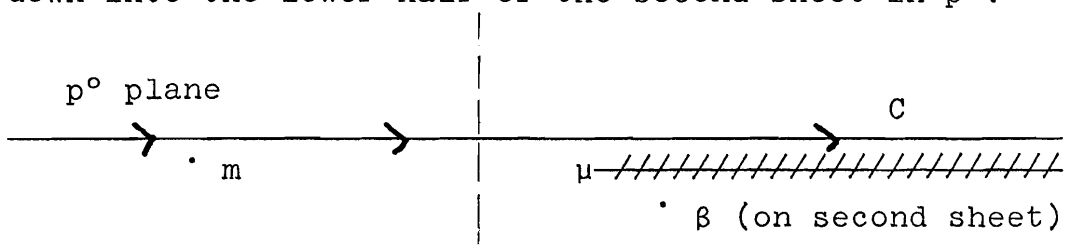
$$= \int d^4x\langle p'(m_o^*)|\bar{V}^\dagger(x)|0\rangle\gamma_{V^*}(x)\langle k'|\bar{V}(x)|p\ (m),k\rangle \tag{B6}$$

Using (B4) and (B5) we can perform the integral in (B6) and the result is Eq. (A14) except that m* and Z* are given here by their values for $g^2 = 0$.

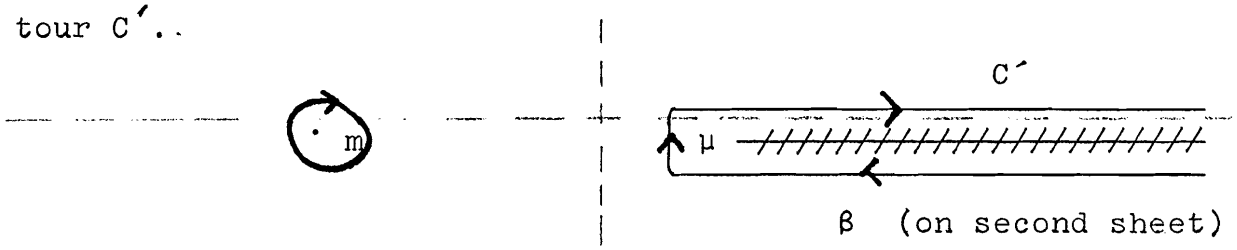
Now we consider the above procedure for $g^2 \neq 0$. We start with the two expressions for the $\overline{V}$ propagator. First we write G directly

$$G_{\overline{V}}(x,0) = \delta(\vec{x})i/[2\Pi]\int_C dp^\circ\, \theta(x^\circ)e^{-ip^\circ x^\circ}/D(p^\circ) \quad .$$

There is now a cut from μ to ∞, and the pole at m_0^* has now moved down into the lower half of the second sheet in p°.



Again we can close the contour in the lower half plane, and we can still ignore the contribution from the infinite semi-circle. We find C can be replaced by a pole at m and a contour C′.



With a little algebra we can put G into the form

$$G_{\overline{V}}(x,0) = \delta(\vec{x})\theta(x^\circ)[e^{-imx^\circ} + 1/\Pi\int_\mu^\infty dp^\circ e^{-ip^\circ x^\circ}\, \frac{\mathrm{Im}\, D(p^\circ)}{|D(p^\circ)|^2}]. \tag{B7}$$

The complete set of states we insert for the second method of evaluating G are now the stable V and the ($N\theta$) continuum which

we denote by $|k(\omega),N>$. We find

$$G_{\overline{V}}(x,0) = [2\Pi]^3\delta(\vec{x})\theta(x^\circ)\{|<0|\overline{V}(0)|m>|^2 e^{-imx^\circ}$$

$$+ 4\Pi\int_\mu^\infty d\omega\ \omega k(\omega)|<0|\overline{V}(0)|k(\omega),N>|^2 e^{-i\omega x^\circ}\} \quad . \qquad (B8)$$

If we equate terms in (B7) and (B8) with the same x° dependence, we find (B3) again and

$$|<0|\overline{V}(0)|k(\omega),N>|^2 = \text{Im } D(\omega)|D(\omega)|^{-2}/[(2\Pi)^5 k\omega] \ . \qquad (B9)$$

The quantity $<0|\overline{V}(0)|k(\omega),N>$ can be obtained directly by applying the reduction procedure to the three-point function $<0|(\overline{V}N^\dagger\theta^\dagger)_+|0>$ with the result

$$<0|\overline{V}(0)|k(\omega),N\ (in)> = gu(\omega)/[(2\Pi)^3(2\omega)^{1/2}D(\omega)] \qquad (B10)$$

which is consistent with (B9) since from (A4)

$$\text{Im } D(\omega) = g^2|u(\omega)|^2 k(\omega)/[4\Pi] \quad .$$

Using (A5) we find

$$\lim_{g^2\to 0} \text{Im } D(\omega)|D(\omega)|^{-2} = \Pi Z_o^* \delta(\omega - m_o^*) \ . \qquad (B11)$$

Combining (B11), (B9), (B8), (B4), and (B2)

$$\lim_{g^2\to 0} 4\Pi\int_{\mu}^{\infty} d\omega\, k(\omega)\omega\,|\langle 0|\overline{V}(0)|k(\omega),N\rangle|^2 e^{-ix^o\omega}$$

$$= |\langle 0|\overline{V}(0)|m_o^*\rangle|^2 e^{-ix^o m_o^*} \qquad \text{(B12)}$$

which is a further consistency check.

We run into trouble when we try to derive an expression for the resonance transition amplitude (A14) for $g^2 \neq 0$. Equation (B5) still holds, but neither the wave function $\langle p\ (m^*)|\overline{V}^\dagger(x)|0\rangle$ nor the inverse Green's function $\gamma_{V^*}(x)$ in Eq. (B6) are well defined. (We can not continue the right hand side of (B10) to $\omega = \beta$ on the second sheet because it is singular there.)

Thus we find that the resonance-production transition amplitude given in (A14) is only well defined in the limit $g^2 \to 0$. However, the cross section (A15) can be viewed as an approximation to the total production cross section, $\sigma_{(c)}$, for $g^2 \neq 0$. We will now consider this point in some detail.

To be rigorous, $\sigma_{(d)}$ should be viewed as an approximation to the Born approximation of $\sigma_{(c)}$, and we will use this fact in motivating our derivation of $\sigma_{(d)}$ from $\sigma_{(c)}$. We start out by rewriting $\sigma_{(c)}$ as the sum of two terms, an incoherent part and an interference part. This decomposition is obtained by considering the linear form for the transition amplitude given by (A12a) rather than the product form of (A12b), and the incoherent part of the cross section gotten by treating the two terms in the amplitude as incoherent. The point of such a decomposition is that it can be shown that the interference

term is of higher order in g^2 than the incoherent part and can therefore be dropped.[62] We have then

$$\sigma_{(c)}(\omega)\Big|_{\text{incoh.}} = |u(\omega)|^2/[4\Pi k(\omega)]$$
$$\times\ 1/\Pi\int_{\mu}^{\bar{\mu}} d\nu\ \text{Im}\ D(\nu)\,|D(\nu)|^{-2}\,|u(\bar{\nu})|^2 k(\bar{\nu})\,|F(\bar{\nu},\omega)|^2 .$$

If $k(\bar{\nu})\,|F(\bar{\nu},\omega)|^2$ is slowly varying near $\nu = m^*$ and if g^2 is small, we can use (B11) as an approximation to get

$$\sigma_{(c)}(\omega)\Big|_{\text{incoh.}} = \sigma_{(d)}(\omega)[1 + O(\Gamma/m^*)].$$

The fact that process (c) has overlapping final-state resonances has not entered into our discussion; when we have $g^2 \to 0$, process (c) is replaced by process (d) which has only stable particles in its final state, and when (d) is viewed as an approximation of (c) for $g^2 \neq 0$ we lose any reference to the overlap when we drop the interference part of $\sigma_{(c)}$.

We end this appendix by pointing out that we can replace m_0^* by m^* and Z_0^* by Z^* in the above discussion by the usual technique of adding contact terms of order g^2 to the Hamiltonian. Doing this is in keeping with our observation that Born approximations are more successful if one uses renormalized coupling constants and masses rather than the unrenormalized quantities.

Appendix C: Appendix to Section III

In this appendix we list some of the properties of the rotation D-matrices and we give some definitions of expressions used in performing calculations in Sections IIIB and IIIC.

The convention we have used for the rotation matrices is that of Rose[57] and is given by

$$D^J_{\lambda\mu}(\alpha,\beta,\gamma) = \langle J\lambda|\tilde{R}(\alpha,\beta,\gamma)|J\mu\rangle$$

$$= e^{-i\alpha\lambda} d^J_{\lambda\mu}(\beta) e^{-i\gamma\mu}$$

where $\tilde{R}(\alpha,\beta,\gamma)$ is a rotation operator (see Eq. 22). All of our references use this convention except McKerrell[5] who uses Edmonds' convention[58] which differs from that of Rose in the following way

$$D^J(\alpha,\beta,\gamma)_{(\text{Edmonds})} = D^J(-\alpha,-\beta,-\gamma)_{(\text{Rose})} \quad .$$

Using Rose's convention we have

$$Y^J_\lambda(\theta,\phi) = N_J \bar{D}^J_{\lambda 0}(\theta,\phi,0) = [(2J+1)/(4\Pi)]^{1/2} \bar{D}^J_{\lambda 0}(\theta,\phi,0) \tag{C1}$$

where the bar stands for complex conjugation. Some useful symmetries of the d-matrices are as follows

$$d^J_{\lambda\mu}(\theta) = \bar{d}^J_{\lambda\mu}(\theta) = d^J_{\mu\lambda}(-\theta) = (-1)^{\lambda-\mu} d^J_{\mu\lambda}(\theta)$$

$$= (-1)^{\lambda-\mu} d^J_{-\lambda,-\mu}(\theta) = (-1)^{J+\lambda} d^J_{\lambda,-\mu}(\theta). \tag{C2}$$

Tables of explicit values for $d^J_{\lambda\mu}(\theta)$ can be found in Werle[30]

and Rose[57] among the references we have used.

If $\tilde{R}_1$ and $\tilde{R}_2$ are two rotations and $\tilde{R} = \tilde{R}_1\tilde{R}_2$, the fact that the D-matrices are a representation of the rotation group gives

$$D^J_{\lambda\mu}(\tilde{R}) = \sum_\nu D^J_{\lambda\nu}(\tilde{R}_1)D^J_{\nu\mu}(\tilde{R}_2) \ . \tag{C3}$$

Using $D^J_{\lambda\mu}(\tilde{R}) = D^J_{\mu\lambda}(\tilde{R}^{-1})$ and $D^J_{00}(\tilde{1}) = 1$, Eq. (C3) gives

$$\sum_\mu \bar{D}^J_{\mu 0}(\tilde{R})D^J_{\mu 0}(\tilde{R}) = \sum_\mu D^J_{0\mu}(\tilde{R}^{-1})D^J_{\mu 0}(\tilde{R}) = D^J_{00}(\tilde{1}) = 1 \ .$$

In Section III we often use the eigenvalue R instead of the operator $\tilde{R}$ in the argument of D. An orthogonality relation we use is

$$N_J N_L \int_0^{2\Pi} d\alpha \int_{-1}^{1} d(\cos\beta) \int_0^{2\Pi} d\gamma \ \bar{D}^J_{\Lambda\Omega}(\alpha,\beta,\gamma)D^L_{\lambda\omega}(\alpha,\beta,\gamma) = 2\Pi\delta_{JL}\delta_{\Lambda\lambda}\delta_{\Omega\omega} \ . \tag{C4}$$

A little algebra leads to the following results for writing variables in the (32)1 basis or the (31)2 basis in terms of the (12)3 basis.

$$s_{32} = [2s_{12}]^{-1}\{[m_2^2-m_1^2][M^2-m_3^2]-4Mm_{12}p_{12}q_3\cos\theta_{12} + s_{12}[M^2+m_1^2+m_2^2+m_3^2-s_{12}]\} \tag{C5}$$

$$s_{31} = [2s_{12}]^{-1}\{[m_1^2-m_2^2][M^2-m_3^2]+4Mm_{12}p_{12}q_3\cos\theta_{12} + s_{12}[M^2+m_1^2+m_2^2+m_3^2-s_{12}]\}$$

$$\cos\theta_{32} = -\hat{p}_3\cdot\hat{p}_1\Big|_{\substack{(32)\\ \text{cmf}}} = [4Mm_{32}p_{32}q_1]^{-1}\{[M^2-m_1^2][m_2^2-m_3^2] + s_{32}[M^2+m_1^2+m_2^2+m_3^2-2s_{12}-s_{32}]\}$$

$$\cos\chi = \hat{p}_3\cdot\hat{p}_1\Big|_{\substack{(12)3\\ \text{cmf}}} = [4M^2q_1q_3]^{-1}\{s_{12}s_{32}+[M^2-m_1^2]s_{12}$$
$$+[M^2-m_3^2]s_{32}-[M^2+2m_2^2-m_1^2-m_3^2]M^2+m_1^2m_3^2\}$$

$$\cos\theta_{31} = -\hat{p}_3\cdot\hat{p}_2\Big|_{\substack{(31)\\ \text{cmf}}} = \left(\text{ interchange 1 and 2 in } \cos\theta_{32}\right)$$

$$\cos\chi' = \hat{p}_3\cdot\hat{p}_2\Big|_{\substack{(12)3\\ \text{cmf}}} = \left(\text{ interchange 1 and 2 in } \cos\chi\right) .$$

Footnotes and References

1. F.S.Chen-Cheung and C.M.Sommerfield, Phys. Rev. 152, 1401 (1966). See Ref. 18 for some comments and corrections.

2. R.D.Amado, Phys. Rev. 158, 1414 (1967).

3. We use the physical masses and coupling constants in our Born approximations.

4. G.C.Wick, Ann. Phys. 18, 65 (1962).

5. A.McKerrell, Nuovo Cimento 34, 1289 (1964).

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7. N.P.Chang, Phys. Rev. Letters 14, 806 (1965).

8. A.M.Gleeson and W.J.Meggs, Nuovo Cimento 55, 584 (1968).

9. A.M.Gleeson and W.J.Meggs, International Atomic Energy Agency Report No. IC/68/56 (submitted to Nuovo Cimento).

10. There is an enhancement in the $K\Pi\Pi$ spectrum around 1.3 BeV, but there is some controversy as to whether this enhancement is caused by zero, one, two, or three resonances.

11. C.Y.Chien et al., Phys. Letters 28B, 143 (1968).

12. T.W.Ludlam, K^--Proton Interactions at 12.6 Gev/c, Ph.D. Dissertation, Yale University (1969).

13. C.Lovelace, Phys. Letters 28B, 264 (1968).

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15. S.Y.Fung et al., Phys. Rev. Letters 21, 47 (1968).

16. The Lee model with all particles treated relativistically is still soluble - in principle.

17. Setting $\lambda = 1/g$ changes the form of some of the expressions given in Ref. 1. See Appendix A below.

18. J.A.Snoke, Phys. Rev. 179 (to be published).

19. The same results can also be obtained by working directly with $H(\nu,\bar{\nu},\omega)$ of Eq. (A17).

20. I.J.R.Aitchison and C.Kacser, Phys. Rev. 173, 1700 (1968).

21. L.D.Faddeev, Soviet Phys.-JETP. 12, 1014 (1961).

22. R.L.Peierls, Phys. Rev. Letters 6, 641 (1961). Also Ref. 37.

23. C.Goebel, Phys. Rev. Letters 13, 143 (1964).

24. A.Pagnamenta, Nucl. Phys. 87, 801 (1967).

25. Pagnamenta's major example actually has the wrong sign for the imaginary part of the inverse amplitude to be considered a resonance. Some minus sign errors hide this fact.

26. $p_{12}=|\vec{p}_1-\vec{p}_2|/2$ evaluated in the (12) cmf, and $q_3=|\vec{p}_3|$ evaluated in the (12)3 cmf.

27. Another way of saying this is that three angles are necessary to give the orientation of a plane.

28. Appendix C below contains some properties of the D-matrices.

29. Because of Eq. (19) our normalization differs from Wick's.[4]

30. J.Werle, Relativistic Theory of Interactions (John Wiley and Sons, Inc., New York, 1966).

31. E.Fabri, Nuovo Cimento 11, 479 (1954).

32. R.H.Dalitz, Phil. Mag. 44, 1068 (1953).

33. C.Zemach, Phys. Rev. 133, B1201 (1964).

34. Note that $^{Th}\tau_0^-(M)$ is just the two-particle phase space with the ρ treated as though it were a stable particle.

35. P.F.Slattery, Resonance Production in π^+p Collisions at 7 Bev/c, Ph.D. Dissertation, Yale University (1967).

36. C.M.Sommerfield, private communication.

37. C.Schmid, Phys. Rev. 154, 1363 (1967).

38. S.U.Chung, Phys. Rev. 165, 1491 (1968).

39. Aachen-Berlin-CERN Collaboration, Phys. Letters 22, 112 (1966).

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41. H.Harari, Rapporteur Talk at Vienna, 14th International Conference on High Energy Physics (CERN,Geneva, 1968).

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43. H.J.Lipkin, Phys. Rev. 159, 1303 (1967).

44. F.J.Gilman and H.Harari, Phys. Rev. 165, 1803 (1968).

45. N.Barash-Schmidt et al., Rev. Mod. Phys. 41, 109 (1969).

46. J.Ballam et al., Phys. Rev. Letters 21, 934 (1968).

47. C.Bouchiat and G.Flamand, Nuovo Cimento 23, 13 (1962).

48. The mass projection can be viewed as a projection either on T or on $|T|^2$ and the latter is easily symmetrized in #1 and #3.

49. J.Boguta, preprint Physikalisches Institut Universität Bonn, April 1969.

50. F.S.Chen-Cheung, Phys. Rev. 159, 1278 (1967), and J.R. Bronzan, Phys. Rev. 172, 1429 (1968).

51. F.J.Ynduráin (J. Math. Phys. 7, 1133 (1966)) finds for a relativistic form of the Lee model that one can do away with the cutoff without getting divergences or a trivial S-matrix.

52. R.D.Amado, Brandeis Summer Institute in Theoretical Physics 1967 (to be published).

53. Often V and N are taken to be spin-zero fermions so that the commutation relations given above for these fields are replaced by anticommutation relations. Since we never have more than a single V or N at a time, the two approaches are equivalent.

54. Chen-Cheung (Ref. 50) has overlooked this point.

55. Process (a) is unaffected by the addition of a W.

56. As g^2 changes with m, B, and α kept constant, the ratio g_1/g_2 changes so that if one wants to keep $g_1/g_2 = h_1/h_2$ he must change the W parameters at the same time he changes g^2.

57. M.E.Rose, Elementary Theory of Angular Momentum (John Wiley and Sons, Inc., New York, 1957).

58 A.R.Edmonds, Angular Momentum in Quantum Mechanics (Princeton, 1957).

59. We use $m^* = 770$ MeV and $\Gamma = 128$ MeV for the ρ.

60. We use $m^* = 892$ MeV and $\Gamma = 50$ MeV for the K^*.

61. J. D. Jackson, Nuovo Cimento 34, 1644 (1964).

62. The density in the overlap region of the Dalitz plot is therefore about a factor of two times the density in the nonoverlap parts of the resonance bands - which runs counter to the intuition of many physicists which would have it be about a factor of four. An intuitive argument which leads to the correct result is to consider nonidentical overlapping resonances and to let Γ/m^* go to zero for one of the resonances. In this limit the interference term must vanish relative to the incoherent part.

Figure Captions

Figure 1. (See pp. 15-23) The Lee model. 1a is production process cross sections using the methods of Ref. 1. 1b is the same curves using our methods. 1c is the Born approximation. The units for σ are $10^{-6}\mu^{-2}$. $\omega_0=2m^*-m_V=62\mu$.

Figure 2. (pp. 27-29) Effective cross sections for static model of Section IIE.

Figure 3. (pp. 44-46) 3π model of Ref. 7. 3a is Chang's results. 3b is our results. 3c contains three versions of the resonance cross section.

Figure 4. (pp. 46-49) 3π model with $J^\eta=1^+$, $j_\rho=1$, and each $\rho\pi$ in a relative s-wave.

Figure 5. (pp. 51-53) 3π model of Ref. 8,9. 6a,b are their results. 6c is our results.

Figure 6. (pp. 53-55) $K\pi\pi$ system with $J=j=0$, one K^* and one ρ. 7a,b,c are effective cross sections for various choices of the relative magnitude and phase between the two channels. 7d,e,f contain resonance approximations for different choices of the relative magnitude.

Figure 8. (p. 55) Two-particle mass spectra for model used in Fig. 7 for M = 1.3 BeV.

Figure 9. (p. 55) Same as Fig. 8 but with M = 1.6 BeV.

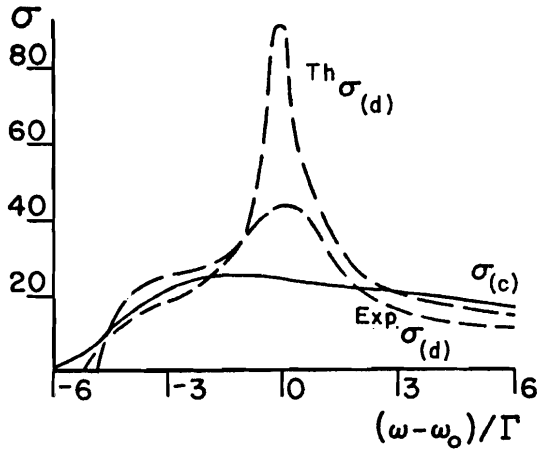

FIGURE 1a

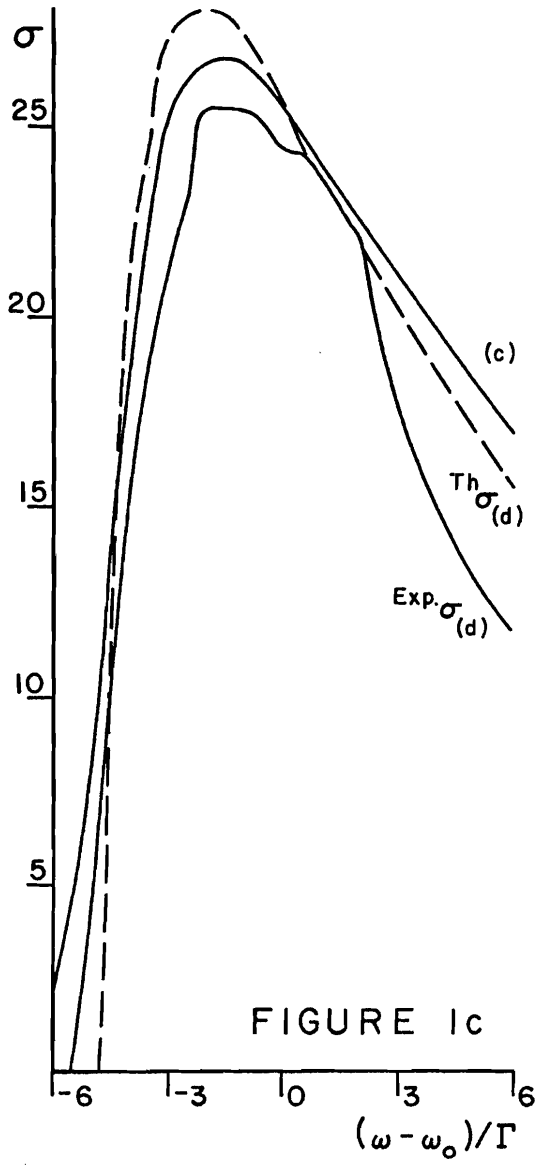

FIGURE 1c

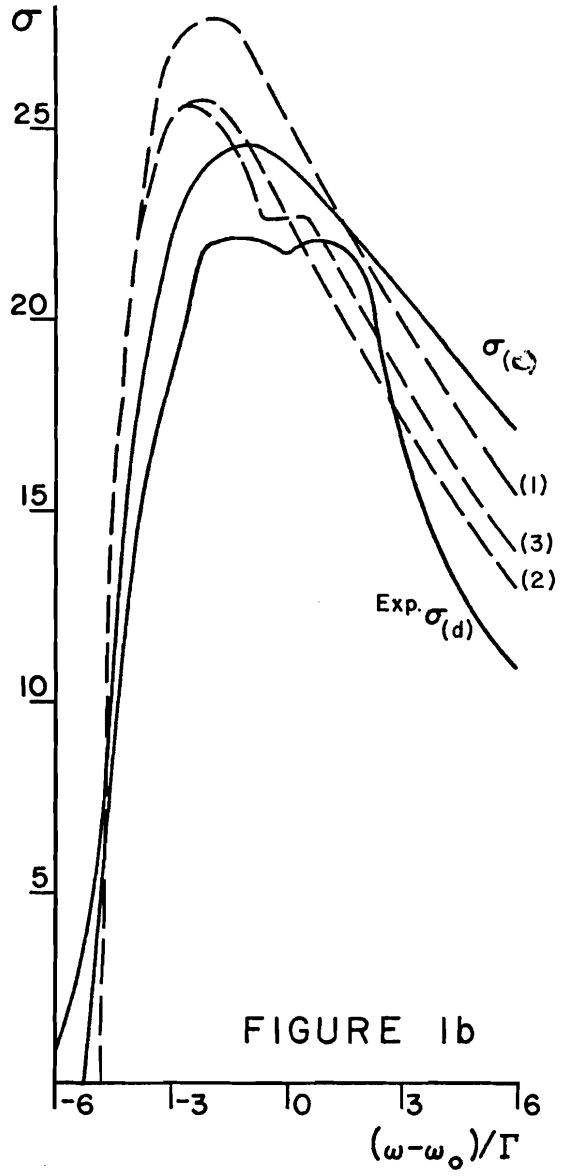

FIGURE 1b

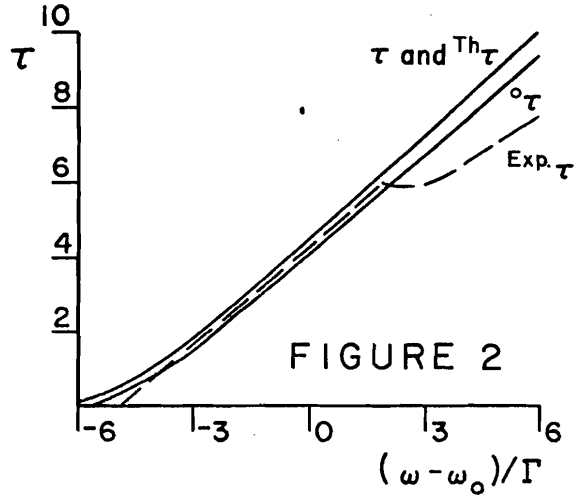

FIGURE 2

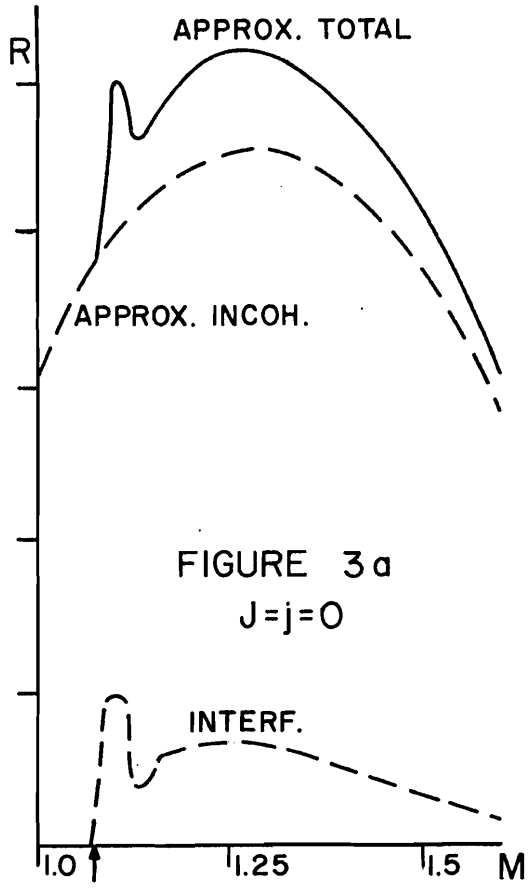

FIGURE 3a
J=j=0

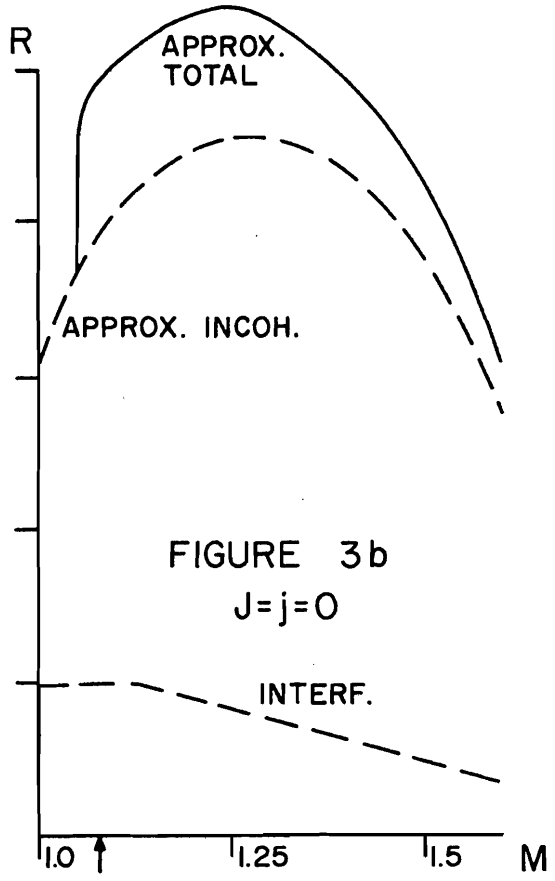

FIGURE 3b
J=j=0

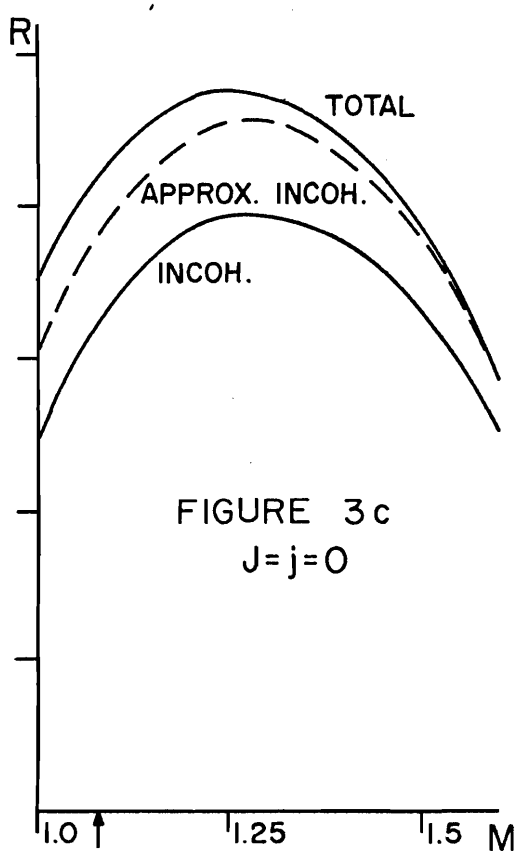

FIGURE 3c
J=j=0

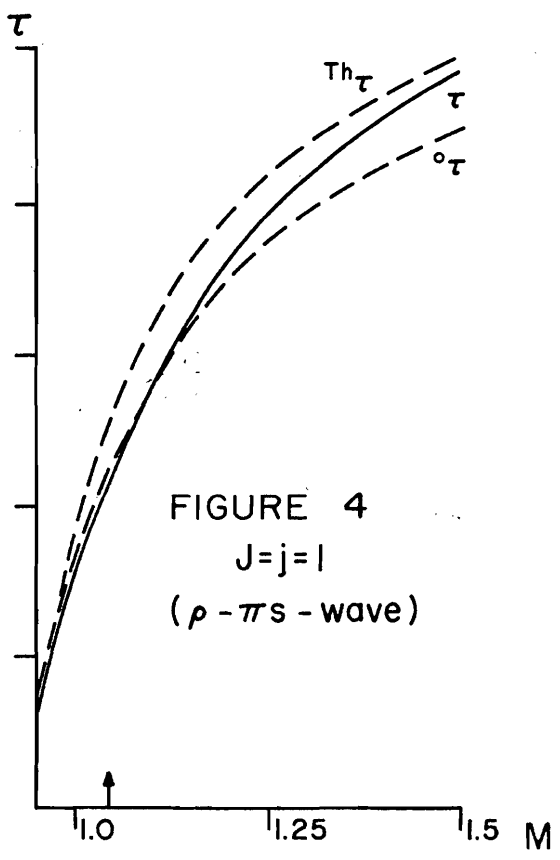

FIGURE 4
J=j=1
(ρ-πs-wave)

FIGURE 5

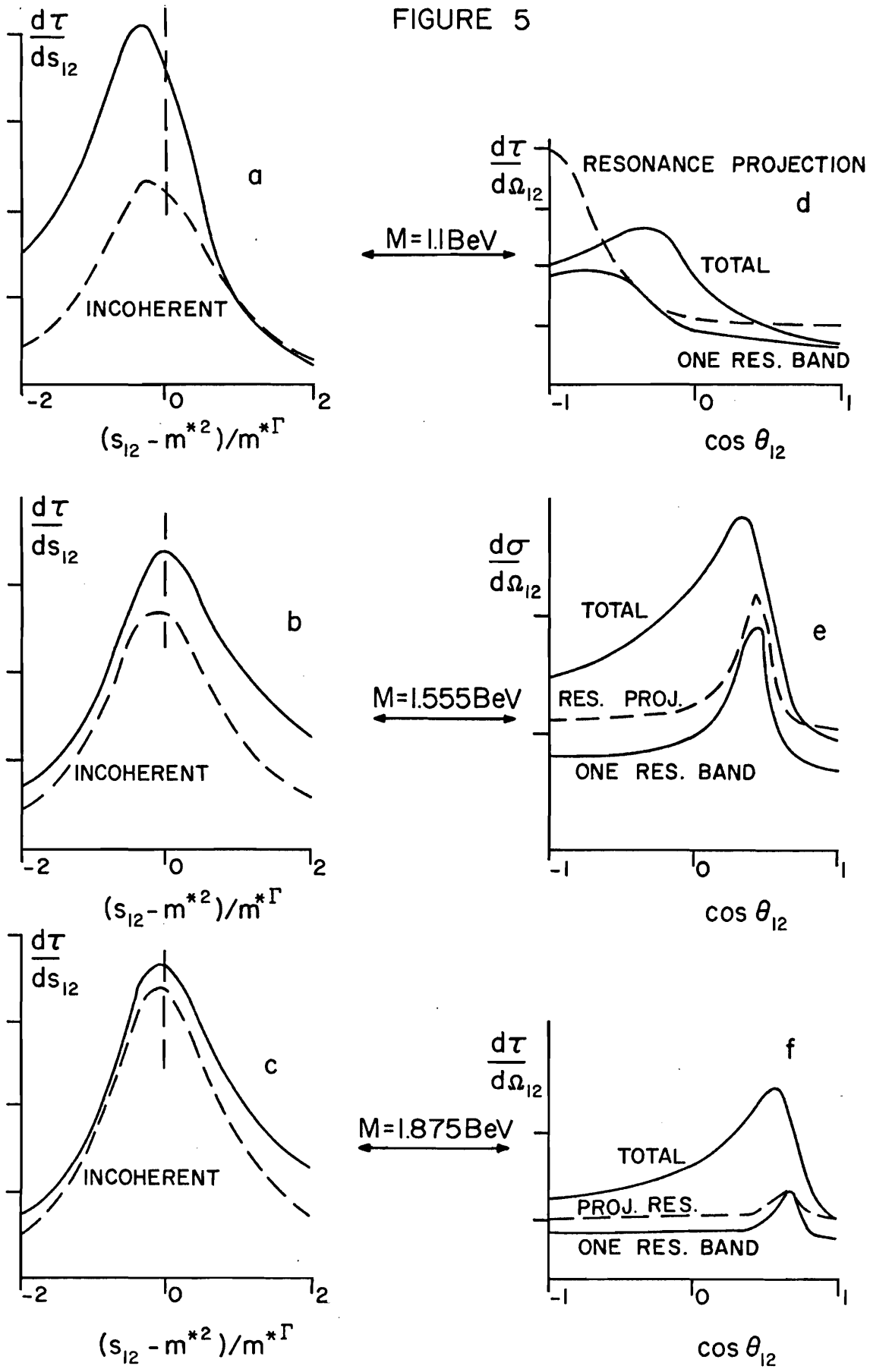
dτ/ds12
a
INCOHERENT
-2
0
2
(s12 - m*2)/m*Γ
dτ/dΩ12
RESONANCE PROJECTION
d
M=1.1BeV
TOTAL
ONE RES. BAND
-1
0
1
cos θ12
dτ/ds12
b
INCOHERENT
-2
0
2
(s12 - m*2)/m*Γ
dσ/dΩ12
TOTAL
e
M=1.555BeV
RES. PROJ.
ONE RES. BAND
-1
0
1
cos θ12
dτ/ds12
c
INCOHERENT
-2
0
2
(s12 - m*2)/m*Γ
dτ/dΩ12
f
M=1.875BeV
TOTAL
PROJ. RES.
ONE RES. BAND
-1
0
1
cos θ12

FIGURE 6

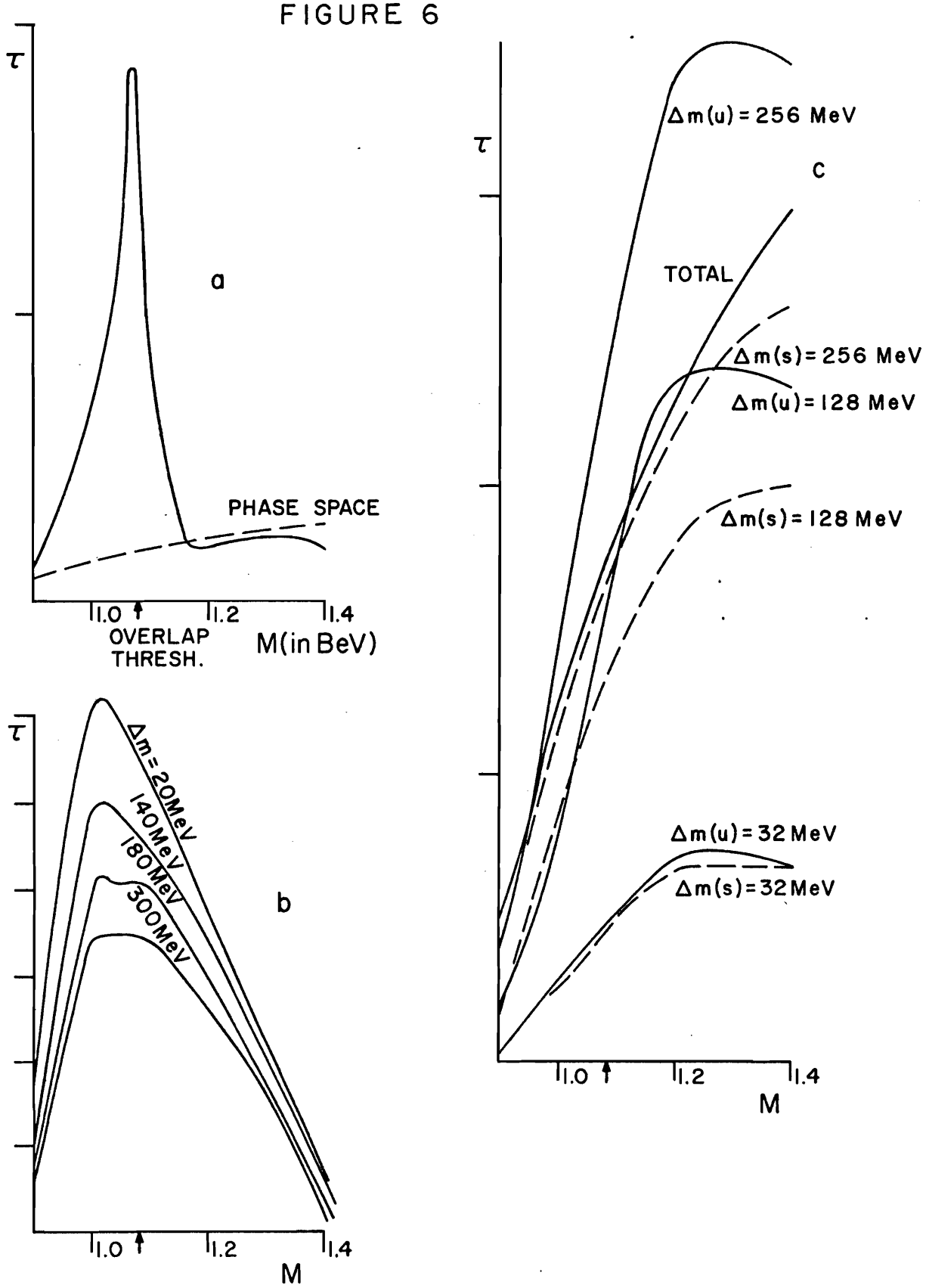
τ
a
PHASE SPACE
1.0
1.2
1.4
OVERLAP
THRESH.
M(in BeV)
τ
Δm=20MeV
140MeV
180MeV
300MeV
b
1.0
1.2
1.4
M
τ
Δm(u) = 256 MeV
c
TOTAL
Δm(s) = 256 MeV
Δm(u) = 128 MeV
Δm(s) = 128 MeV
Δm(u) = 32 MeV
Δm(s) = 32 MeV
1.0
1.2
1.4
M

FIGURE 7

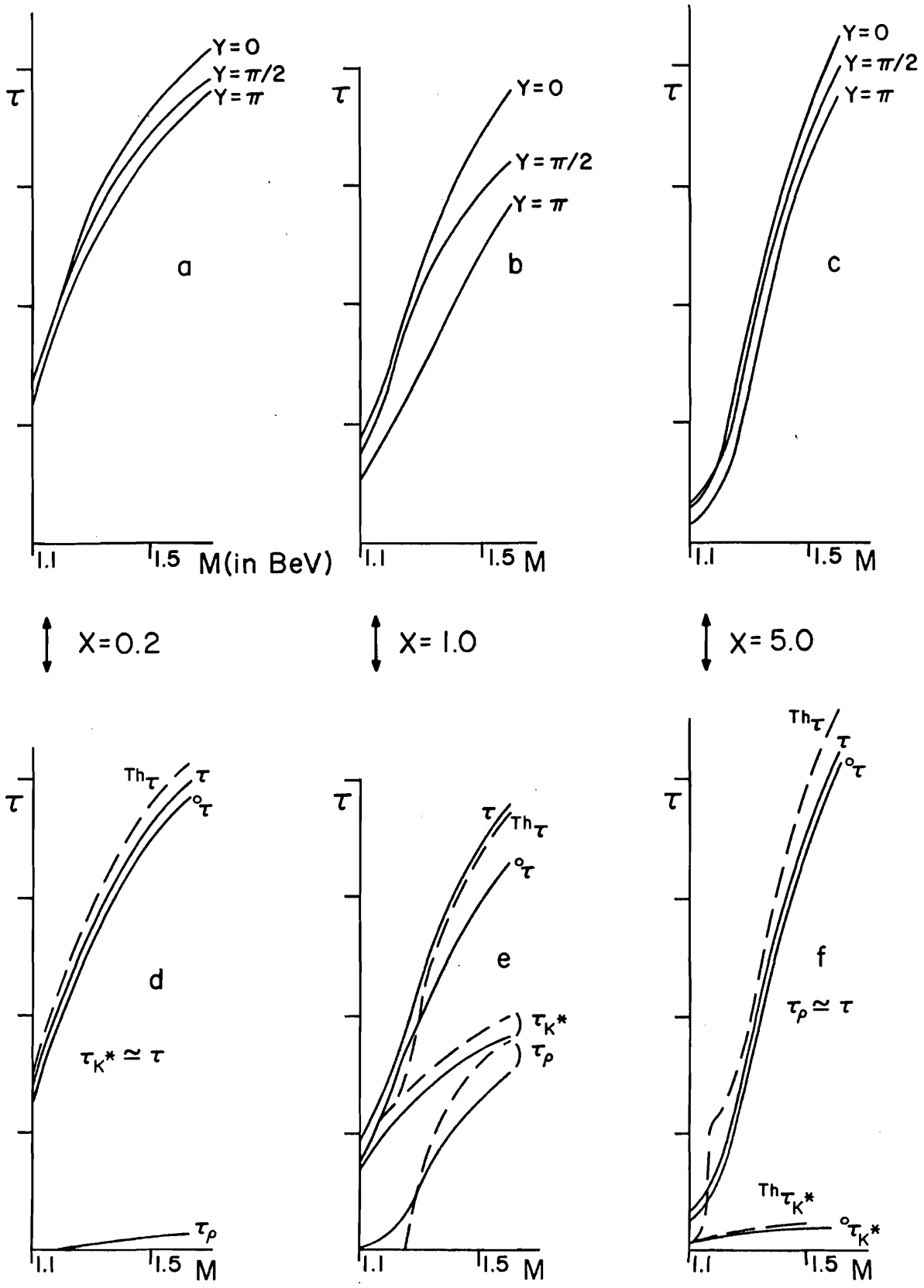
τ
Y=0
Y=π/2
Y=π
a
1.1
1.5
M(in BeV)
b
c
M
X=0.2
X=1.0
X=5.0
Thτ
τ
°τ
d
τK* ≃ τ
τρ
e
τK*
f
τρ ≃ τ
ThτK*
°τK*

FIGURE 8
M = 1.3 BeV

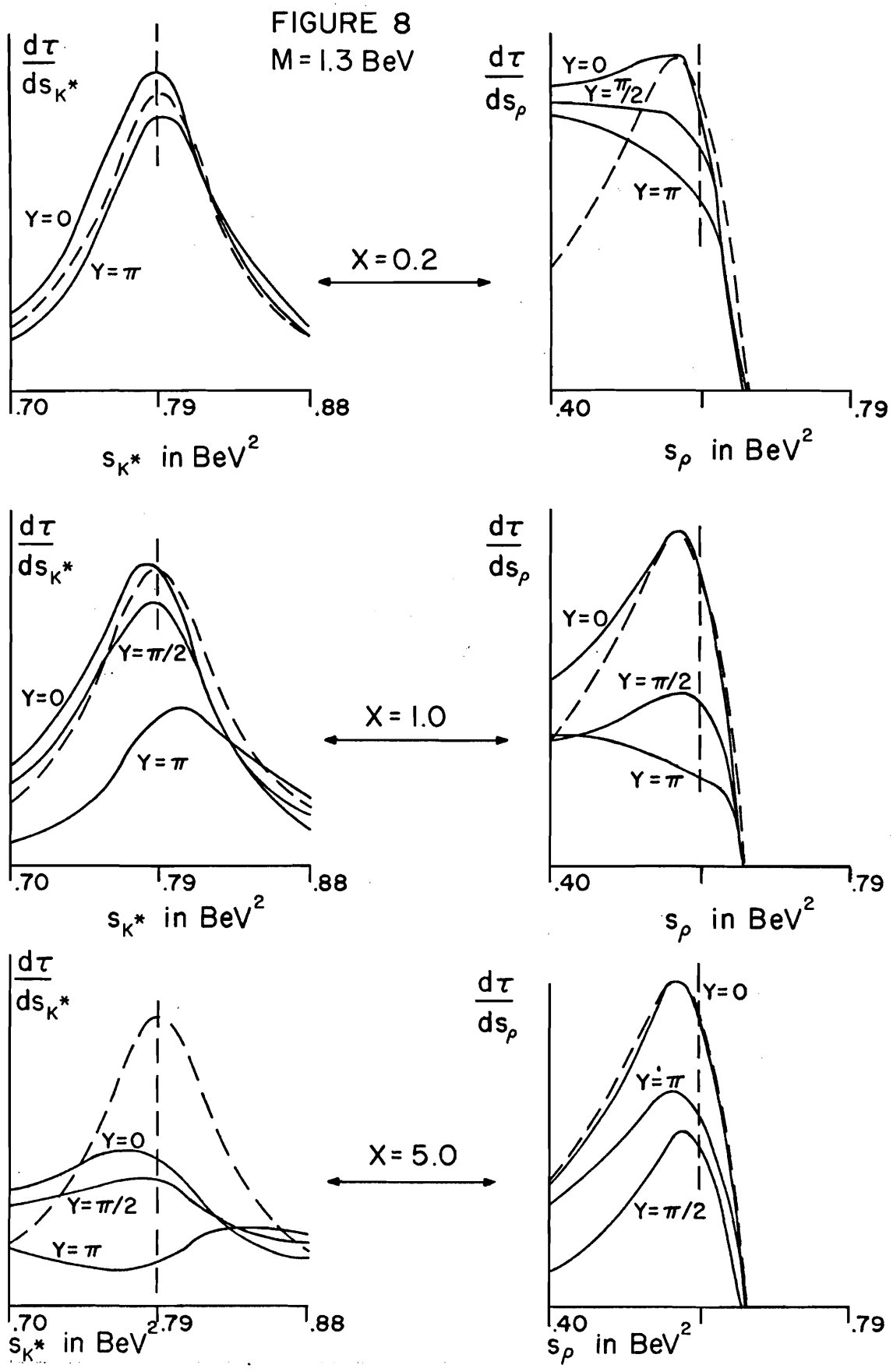
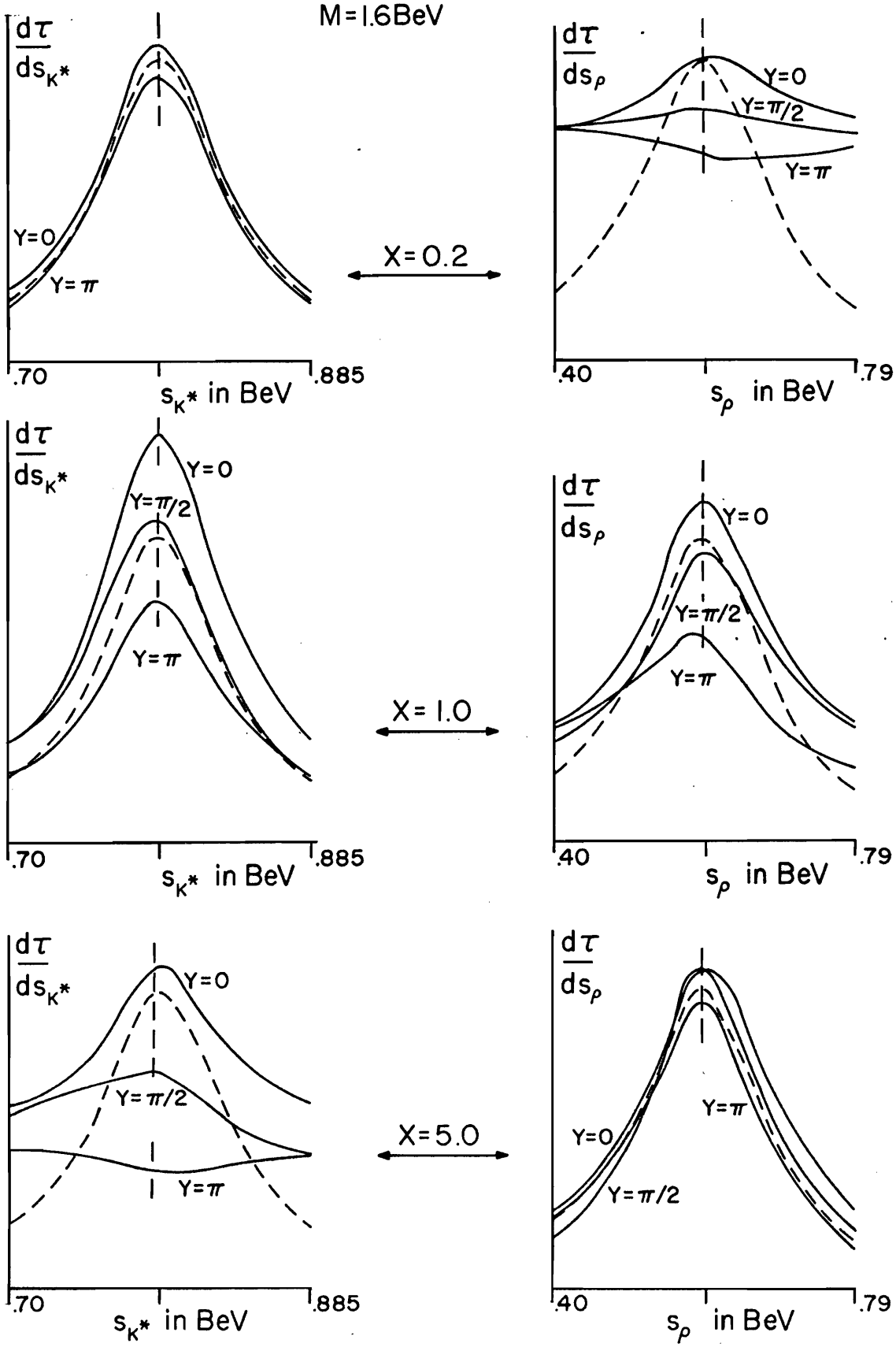
FIGURE 9
M=1.6BeV