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Quickly Available Graphics of Static, Electromagnetic Field Distributions Given by Conformal Maps Using *Mathematica*

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Dedicated to all those people who don't neither posess one of these computer things nor even have to work in front of one (or more) day in day out, however, do not miss anything or, moreover, are still happy <u>without</u> having none.

Acknowledgement and Technical Report

This diploma thesis was written at the *Technical University of Graz | Institute of Theoretical Physics | Section for Mathematical Physics and Electrodynamics* on the NeXTstation Color Computer of Prof.Schnizer (NeXT System Release 2.1; Workspace Version 218; Postscript Version 2.1.55.3) with the help of the text editing language CTEX: NeXT Version 3.1a & LATEX Version 2.09 .
The *Mathematica* computer progams of chapter 2 were designed using the NeXTstation (*Mathematica* Version 2.0) as well as the HP workstation (Model apollo 715/50 [n. A2094A]; *Mathematica* Version 2.2) of the institute.

At this place I want to thank Professor Schnizer for placing his personal NeXTstation at my disposal. Although nearly always distressed by university administration, by preparing and reading lectures, he had an open ear to all my questions and even helped me by trying to solve the arising problems personally.

I also want to thank my fellow-students who worked with me at the institute and which I had a lot of fun with.

A disk containing the *Mathematica* programs can be obtained from :

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Quickly Available Graphics of Static, Electromagnetic Field Distributions Given by Conformal Maps Using *Mathematica*

Conformal maps based on the Schwarz-Christoffel transformation map the infinite plane condenser on other simple plane configurations rigorously. The field distributions between the electrodes as well as the field and charge distributions on the electrodes can be computed from analytic expressions. This thesis contains ready-to-use *Mathematica* programs plotting these physical quantities for 11 condenser or pole piece configurations: The sharp or round corner of an infinite rectangular pole piece, a pole piece of finite thickness, the end of a condenser consisting of infinitely thin plates or of planes with arbitrary thickness, two parallel infinitely thin strip lines, a channel with a rectangular sharp bend, a channel with a finite step, an infinitely thin stripline in line with a stripline of inifnite width at each side. The design philosophy was to provide electrical engineers neither being acquainted with conformal maps nor with Mathematica with a tool for getting computer generated graphics fast and easily. The parameters defining the geometry of the configurations and the voltage may be chosen at will within limits indicated in each program. These programs can also be applied in magnetostatics, hydro- and aerodynamics and can be modified to deal with new configurations.

Griffbereite *Mathematica* Programme für Graphik konform abgebildeter statischer elektromagnetischer Feldverteilungen

Konforme Abbildungen, welche mittels Schwarz-Christoffel Transformation gewonnen werden, bilden den unendlichen ebenen Plattenkondensator auf Kondensatoren einfacher, planer Geometrie exakt ab. Die Feldverteilung zwischen den Platten sowie Feld und Ladungsdichte auf diesen können damit analytisch berechnet werden. Diese Arbeit enthält einfache *Mathematica* Programme, welche Bilder dieser physikalischen Groessen fuer 11 Kondensator- oder Polschuhformen zeichnen: z.B. einen unendl. Polschuh mit rechtwinkeliger spitzer oder runder Ecke, einen Polschuh endlicher Breite, das Ende eines Plattenkondensators mit unendlich dünnen und endlich dicken Platten, zwei unendlich dünne Streifenleiter, ein Kanal mit einem rechtwinkeligen Knick, einen Kanal mit endlicher Stufe, eine Streifenleitung zwischen zwei halbunendlich langen Streifenleitern. Ingenieure und Elektrotechniker können damit ohne Vorkenntnisse von *Mathematica* und konformen Abbildungen auf dem Computer Bilder für statische Felder schnell erhalten. Die Kondensatorabmessungen, wie Plattenabstand, Plattendicke, und die Spannung zwischen den Platten können in grossen Bereichen selbst vorgegeben werden. Die graphischen Ergebnisse gelten nicht nur für die Elektro- und Magnetostatik, sondern auch für Hydro- und Aerodynamik. Weiters können die Programme für eigene konforme Abbildungen adaptiert werden.

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Introduction

This " Diplomarbeit " (diploma thesis) contains *Mathematica* programs for problems of electrostatics that include conformal maps. Conformal mapping is an important means to solve problems in Electro- and Magnetostatics as well as in Hydro- and Aerodynamics, because it helps to reduce complicated mathematical problems in complex space to simpler ones. Especially graphics (of conformal maps) show their behaviour in a very distinct way, which was the motivation for this thesis.
When you leaf through the thesis you see a lot of graphics plotted by the programming language *Mathematica*. Although some of these graphics look quite complicated, the pertinent *Mathematica* programs are comparatively short.

Mathematica , a language for symbolic computation, has a lot of implemented graphics commands as well as routines for elementary and special functions as, for example, for elliptic integrals and functions. Together with the package Graphics'ConformalMap' *Mathematica* is immensely helpful to plot graphics of conformal maps.

The first chapter of this thesis gives a short survey of the mathematic and electrostatic concept of conformal mapping. The most important *Mathematica* commands , necessary to produce the graphics, are explained.
The following chapter contains the mathematical treatment of the conformal function as well as the *Mathematica* programs and ensuing graphics for a dozen problems. Those programs deal with the graphics output of some frequently used condenser configurations and their corresponding electric field strengths and surface charge densities. Finally the difficulties which arise when you want to round the corners of the condensers treated in chapter 2 are shown.

Although the thesis always talks in electrostatic terms the field distribution for magnetic pole pieces looks the same as that for condensers of equal form. Then the electrostatic quantities (electric field strength, surface charge density) have to be translated to magnetic terms. Even in Hydro- and Aerodynamics the laminal stream around edges and the sudden change of the radius of a streamed through tube leads to the same pictures.

You can run the *Mathematica* programs instantly to produce the printed graphics on your own computer display. Furthermore as soon as you understand how my *Mathematica* programs are constructed you may write your own programs for similar problems.
For readers with no or little knowledge of *Mathematica* this paper also explains how the *Mathematica* objects used in the thesis work. Nevertheless you do not learn *Mathematica* by reading the paper!

It was the intention of the autor to render each section describing a *Mathematica* program as selfcontained as possible. This entails a certain amount of repetitions.

1 The Conformal Map

This chapter is devided into three sections that treat the mathematic and elecrodynamic basics as well as the basics of those *Mathematica* commands that occur in the programs of the next chapter.

1.1 The Mathematic Concept

At the beginning I give a short mathematic abstract of conformal mapping including the main definitions and rules, followed by a glimpse to elliptic functions and integrals. This brief mathematic section should suffice not only to understand the examples given in this paper, but also to design your own *Mathematica* programs for similar cases that are not treated in chapter 2.

Conformal Map: A function $w = f(z)$ which is single-valued and holomorphic in a domain $\mathbf{D}_z$ of the complex Z-plane (Z $= \{z|\ z \in \mathbb{C}\}$[1]) and whose total derivative $f'(z) \neq 0$ is called a conformal map.
Two curves C_1 and C_2 in $\mathbf{D}_z$ whose tangents, provided they exist, intersect in the point $z_0 \in \mathbf{D}_z$ at an angle α, will intersect at the same angle α when they are transformed to the range $\mathbf{R}_w = f(\mathbf{D}_z)$, the image of $\mathbf{D}_z$ on the complex W-plane, by the conformal map $w = f(z)$.
The inverse function $z = f^{-1}(w)$ of a conformal map $f(z)$ always exists[2] and is again conformal in $\mathbf{R}_w$.

So, when I want to know whether a function $f(z)$ is conformal, I have to verify that it fulfills two conditions according to the definition given above:

- Firstly it has to be holomorphic in a set $\tilde{\mathbf{D}}_z \subset Z$-plane. Therefore $f(z)$ has to satisfy the Cauchy-Riemann differential equations for all z of $\tilde{\mathbf{D}}_z$.
- Secondly I have to find all zeros z_0 of the function's first derivative that are within $\tilde{\mathbf{D}}_z$. In these points no inverse function $z_0 = f^{-1}(w_0)$ of $w_0 = f(z_0)$ exists. The set $\tilde{\mathbf{D}}_z$ without all z_0 is called the domain $\mathbf{D}_z$ for whose elements $f(z)$ is conformal.

---

[1] Representation of the compex numbers $\mathbb{C}$ by a Gaussian plane named Z with its elements z.

[2] However, to write $f^{-1}(w)$ down explicitely usually encounters a lot of trouble because either you have to calculate the first n coefficients of the Bärgmann-Lagrange series for an nth order approximation of the inverse function or you have to find an exact solution by trial and error.

Conformal functions are "very well behaved" since they do not have any singularities and have a unique inverse function for all elements of the range $\mathbf{R}_w$. Moreover a pair of distinct lines in the Z-plane is mapped to a pair of distinct lines on the W-plane. So the electric field lines which are parallel to the iy-axis on the Z-plane do not cross on the W-plane either. So for the **Infinite Plane Condenser** the equipotential lines which are parallel to the x-axis on the Z-plane cross the electric field lines on the W-plane at right angle too. This is very convenient when calculating the electric field strength or the surface charge density of the condenser plates.

Some terms mentioned in the definition and others emerge throughout the thesis. To avoid confusion I will always use them in the same way :

General Notations :

- The real numbers $\mathbb{R} = \{t \mid -\infty < t < \infty\}$ do not contain infinite points.
- The complex numbers $\mathbb{C} = \{z = x + iy \mid x, y \in \mathbb{R}\}$ do not contain the infinite point either.
- The term Z-plane with the elements $z = x + iy$ is always used for the complex plane of the **Infinite Plane Condenser**.
- So W-plane – with the elements $w = u + iv$ – is used for condensers that are obtained from the **Infinite Plane Condenser** by a conformal map $f(z)$. An exception is the following one:
- T-plane – elements $t = r + is$ – denotes the complex plane with $\Im m(t) \geq 0$. This is the basis for the Schwarz-Christoffel transformation explained later on.
- The set $\tilde{\mathbf{D}}_\lambda$ $(\lambda = z, t)$ for whose elements λ the function $f(\lambda)$ is holomorphic is a subset of the $\Lambda-$ plane $(\Lambda = Z, T)$.
- The Domain $\mathbf{D}_\lambda$ is a subset[3] of $\tilde{\mathbf{D}}_z$ restricted to the elements λ for which $f(\lambda)$ is conformal. $\mathbf{D}_\lambda = \{\lambda | f(\lambda) \text{ conformal}\} = \tilde{\mathbf{D}}_\lambda \setminus \{\lambda_0 | f'(\lambda_0) = 0\}$.
- All conformal maps of this thesis are called $f(z)$, their inverse functions $f^{-1}(w)$ or $f^{-1}(t)$ respectively.

---

[3] A domain $\mathbf{D}$ is an open subset of Riemann's sphere. Nevertheless, for simplificity I will always regard its boundary $\partial\mathbf{D}$ a part of the domain.

- All curves on the Z-plane are called $\mathcal{C}_i$, their parametric representations of the parameter ζ are denoted by $C_i(\zeta)$. The images on the W-plane are called $\hat{\mathcal{C}}_i$ and $\hat{C}_i(\omega)$ respectively.

- (a, b) denotes an interval which is open to both sides.

- $[a, b]$ is a closed interval: i.e. it contains the points a and b of the boundary: $[a, b] = (a, b) \cup \{ a, b \}$.

> **Riemann's Mapping Theorem:** Any simply connected domain $\mathbf{D}$ of the complex plane with connected boundary that consists of at least two points can be transformed schlicht[4] to the interior of the unit circle by a holomorphic function in $\mathbf{D}$.

Riemann's theorem is the fundamental theorem of conformal mapping in simply connected domains. It states that conformal functions exist for any realistic (physical) problem with connected boundary[5] but it does not show how to find them.

For certain classes of problems, however, special prescriptions lead to the conformal map wanted. One of these is the Schwarz-Christoffel formula which maps a condenser whose electric field is in the upper half of the complex plain to our special configurations.

> **Schwarz-Christoffel Formula:** The integral
>
> $$w = f(t) = A \int_{t_0}^{t} \prod_{i=1}^{n} (t - t_i)^{-\alpha_i} \, dt \; + B \qquad (1)$$
>
> with the supplementary conditions:
> $t_1 < t_2 < \ldots < t_n$; $0 < |\alpha_i| \leq 1$ and $\sum_{i=1}^{n} \alpha_i = 2$ with $\alpha_i, t_i \in \mathbb{R}$ for the coefficients t_i and α_i is a single-valued and holomorphic map in the complex plane with nonnegative imaginary part $\tilde{\mathbf{D}}_t = \{t \mid \Im m(t) \geq 0\}$ which transforms the set $\hat{\mathbf{D}}_t = \{t \mid \Im m(t) > 0\}$ to the interior of a polygon whose edges are at $w_i = f(t_i)$. The coefficients $\alpha_i \pi$ are the angles of these edges (cf. Figure 1). A and B are complex constants.

Having the definition of conformal mapping in mind I deduce that the map in eq.(1) defines a conformal function for all $t \in \tilde{\mathbf{D}}_t \setminus \{t_0 | f'(t_0) = 0\} = \mathbf{D}_t$. The points t_0 are the points t_i of

---

[4] A function f(z) is called schlicht in the neighbourhood $\mathbf{U}(z_0)$ of the point z_0 when it can be inverted univalent in the neighbourhood $\mathbf{U}(w_0)$ of the point $w_0 = f(z_0)$.

[5] A theory for conformal maps in multiply connected domains exists but we do not need it here.

the real r-axes (= boundary of $\tilde{\mathbf{D}}_t$) whose corresponding angles $\alpha_i\pi$ on the W-plane are negative. Compared to this all points of the interior of $\tilde{\mathbf{D}}_t$ are conformal. You just have to solve eq. (1). This, though always possible in principle, may meet some difficulties which must be overcome by hard labour.

The numbering of the w_i is usually done counterclockwise and the angles α_i are the (outer angles $-\pi)/\pi$ of the lines meeting at w_i (see Figure 1). So when the edge of w_i points outwards (seen from the interior of the polygon) the corresponding angle $\alpha_i > 0$ is positive, when it points inwards $\alpha_i < 0$ is negative (cf. α_1 and α_4 of Figure 1).

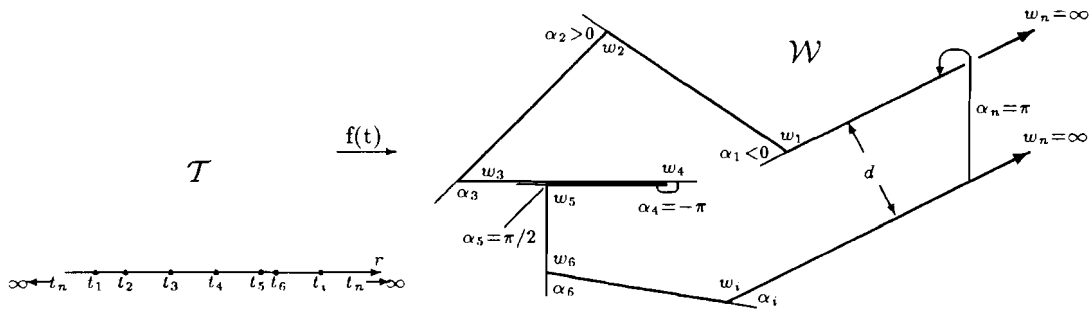

Figure 1: Mapping by the Schwarz-Christoffel Formula

When I know the location of the edges w_i and the values of all α_i on the W-plane I can calculate the values of the points t_i on the T-plane and the constants A and B in eq.(1) in the following way :

As the theory of conformal mapping shows you may choose three points t_i (i.e assign real numbers including infinity to them) arbitrarily to fix the mapping function $w = f(t)$ completely.
I choose t_n to be at infinity which reduces the product in our integral by one (the last) term and obtain

$$w = f(t) = A \int_{t_0}^{t} \prod_{i=1}^{n-1} (t - t_i)^{-\alpha_i} \, dt \; + B \tag{2}$$

Moreover let t_m $(1 \leq m \leq n-1)$ be the arbitrary lower limit of integration. We get :

$$B = w_m \tag{3}$$

$$w_{m+1} = A \int_{t_m}^{t_{m+1}} f'(r)\, dr + w_m = f(t_{m+1})^7 \quad \text{with} \quad r \in \mathbb{R}^8 \tag{4}$$

Inserting (3) into (4) I obtain a symbolic solution for A as a function of the remaining t_i. Those t_i can then be found recursively

$$w_{m+2} = A \int_{t_{m+1}}^{t_{m+2}} f'(x)\, dx + w_{m+1} = f(t_{m+2}) \ , \quad \ldots \tag{5}$$

and be inserted into equation(2) to complete the calculation of the conformal map $w = f(t)$.

When I know the distance d of two parallel polygon lines that extend both to infinity (cf. fig.1) I get a solution for the constant A (whose calculation is usually harder than that of B) by[9]

$$A = -\frac{\Delta w}{i\pi} \prod_{j \neq p} (t_p - t_j)^{+\alpha_j} \tag{6}$$

where Δw is the distance on the W-plane when changing from one parallel line to the other; so $d = |\Delta w|$. t_p is the pre-image of the corner at infinity (cf. corner n of fig.1). When I choose $t_p = \infty$ eq. (6) equals

$$A = \frac{\Delta w}{i\pi} \ . \tag{7}$$

The procedure to find the remaining t_j is similar to the one from above. Here, however, B still has to be calculated; i.e. eq. (3) does not hold any more.

---

[6] Proof (using equation(1)): $w_m = A \int_{t_0}^{t_0} \prod_{i=1}^{n-1} (t - t_i)^{-\alpha_i}\, dt + B = 0 + B = B$.

[7] When I decide upon m to be $n-1$, $m+1$ will then be 1 since t_n can no longer be found in our integral.

[8] The integration in the complex plain, can be reduced to one on the real r-axis (cf.[du]).

[9] cf. [wb] or [fl].

For polygons with at least two connected edges the function $w = f(t)$ involves elliptic integrals and functions of which I give a short summary as far as they turn up in the problems 2.12 - 2.14.

Doubly Periodic Functions: A function f(z) which is holomorphic in the whole complex plane $\mathbb{C}$ except for some isolated singularities and fulfils

$$f(z) = f(z + n\omega_1 + m\omega_2) \quad n, m \in \mathbb{N}, \quad \omega_1, \omega_2 \neq 0 \quad \text{and} \quad n\omega_1 \neq m\omega_2 \quad (8)$$

for any combination of integer n and m is called a doubly periodic function with the periods ω_1 and ω_2.
The smallest periods ω_1, ω_2 that fulfill the periodicity condition given above are called primitive. The set of all periodic elements $n\omega_1 + m\omega_2$ has no finite accumulation point unless the function is constant.

We call the domain connecting the points $z = 0, z = \omega_1, z = \omega_1 + \omega_2$ and $z = \omega_2$ without the lines between $z = \omega_1$ to $z = \omega_1 + \omega_2$ and $z = \omega_2$ to $z = \omega_1 + \omega_2$ and excluding $z = \omega_1$ and ω_2 the fundamental parallelogram. This subdomain of the complex plane contains all information about our doubly periodic function due to periodicity and to the absence of a finite accumulation point for our periodic elements.

Elliptic Functions: An elliptic function is a meromorphic[10] doubly periodic function. The order of an elliptic function is the sum of the multiplicities of the poles within the fundamental parallelogram.

Theorems for Elliptic Functions:
1) The sum of the residues of the poles in the fundamental parallelogram is allways zero so there are no elliptic function of zero and first order.
2) Every value of an elliptic function of nth order is assumed exactly n times in the fundamental parallelogram.

The simplest elliptic functions are those of second order :
a) The elliptic function with one pole of second order with a zero residue is Weierstrass's $\wp$ function .
b) The elliptic functions with two poles and residues of opposite sign lead to the Jacobi elliptic functions and others.

---

[10] meromorphic = holomorphic except for poles

Altough any elliptic function can be represented by the Weierstrassian $\wp$ and its derivative $\wp'$ Jacobi's elliptic functions are a useful tool in mathematical physics and easier to apply than $\wp$. They are the inverse functions of Legendre's first elliptic integral for different arguments.

Elliptic Integrals: An integral of the form

$$\int R(z,\zeta)\, dz \quad \text{with } R(z,\zeta) \text{ a rational function of z and } \zeta$$
$$\text{and } \zeta = \sqrt{a_0 z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4}$$

the square root of a polynomial with simple zeros and a_0 and a_1 not both zero is called an elliptic integral.
This elliptic integral can allways be separated into an integral of rational functions and three basic types of elliptic integrals:

$$\int \frac{dz}{\zeta} \quad , \quad \int \frac{z\, dz}{\zeta} \quad , \quad \int \frac{dz}{(z-c)\zeta} \quad \text{with c a zero of } \zeta^2.$$

There are several forms of writing down the three basic elliptic integrals; but the elliptic integrals in Legendre's normal form only play an important role in physical applications.

Legendre's (Jacobi's) Incomplete Elliptic Integrals:

First kind: $\mathrm{F}(\arcsin y, k) = \int_0^y \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} = \mathrm{F}(\varphi, k) = \int_0^\varphi \frac{d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}}$

Second kind: $\mathrm{E}(\arcsin y, k) = \int_0^y \frac{\sqrt{1-k^2z^2}}{\sqrt{1-z^2}}\, dz = \mathrm{E}(\varphi, k) = \int_0^\varphi \sqrt{1-k^2\sin^2\vartheta}\, d\vartheta$

Third kind: $\Pi(\arcsin y, \alpha^2, k) = \int_0^y \frac{dz}{(1-\alpha^2z^2)\sqrt{(1-z^2)(1-k^2z^2)}} = \Pi(\varphi, \alpha^2, k) = \int_0^\varphi \frac{d\vartheta}{(1-\alpha^2z^2)\sqrt{1-k^2\sin^2\vartheta}}$

The variable k is called the **modulus** which also appears as one argument of Jacobi's elliptic functions. Its value is not restricted, though in most problems of mathematical physics k will be real and even $0<k<1$. The complementary modulus is defined by $k' = \sqrt{1-k^2}$. So the complementary modulus of k' is again the original modulus k. [as] and *Mathematica* , however, use the **parameter** $m = k^2$ and the complementary parameter $m_1 = 1-k^2 = k'^2$ as an argument.
The variable α^2 of the third integral must be real.
The argument y or φ of the elliptic integrals may be complex but they usually are real and $0<y\leq 1$, $0<\varphi\leq\pi/2$ respectively.

Legendre's (Jacobi's) Complete Elliptic Integrals:

First kind: $\mathrm{K}(k) \quad = \mathrm{F}(\pi/2, k) \quad = \int_0^{\pi/2} \frac{d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}}$

Second kind: $\mathrm{E}(k) \quad = \mathrm{E}(\pi/2, k) \quad = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\vartheta}\; d\vartheta$

Third kind: $\Pi(\alpha^2, k) = \Pi(\pi/2, \alpha^2, k) = \int_0^{\pi/2} \frac{d\varphi}{(1-\alpha^2 z^2)\sqrt{1-k^2\sin^2\vartheta}} \quad [\alpha^2 \neq 1]$

Sometimes K(k) and E(k) are shortened to K and E, the complete elliptic integrals of the complementary modulus k' are then marked by a prime; i.e K' and E'.

Jacobi's elliptic functions can be obtained by the so called Jacobi amplitude $\mathrm{am}(u, k)$ which is singly-periodic with the period $\omega = 4i\mathrm{K}'$ and nearly periodic with 2 K since $\mathrm{am}(u + 2\mathrm{K}, k) = \mathrm{am}(u, k) + \pi$. It is the inverse to the elliptic integral of the first kind for the same modulus

$$\mathrm{am}^{-1}(\varphi, k) = \mathrm{F}(\varphi, k) \quad .$$

When I wrap a function periodic with 2π as sine and cosine around the Jacobi amplitude I obtain Jacobi's elliptic functions. Usually three functions $\mathrm{sn}(u, k)$,$\mathrm{cn}(u, k)$ and $\mathrm{dn}(u, k)$ are defined by :

$\mathrm{sn}(u, k) = \sin(\mathrm{am}(u, k))$, called **sine amplitude**,
$\mathrm{cn}(u, k) = \cos(\mathrm{am}(u, k))$, **cosine amplitude**, and finally
$\mathrm{dn}(u, k) = (1 - \mathrm{k}^2\ \mathrm{sn}^2(u, k))^{1/2}$ called the **delta amplitude**.

All the other Jacobi functions as, for example, $\mathrm{sd}(u, k) = \mathrm{sn}(u, k)/\mathrm{dn}(u, k)$ or $\mathrm{nd}(u, k) = 1/\mathrm{dn}(u, k)$ are just ratios of these three.

At the end of this chapter I place a rule which may be useful when you want to change my *Mathematica* programs a bit.

> **Schwarz's Reflection Principle:** Let $w = f(z)$ be holomorphic in the domain $\tilde{\mathbf{D}}$ which has a segment of a line or circle as boundary curve $\mathcal{C}$. Let $f(z)$ be continuous on this curve and map $\mathcal{C}$ to a curve $\mathcal{C}'$ which is again a segment of a line or circle.
> The reflection of the domain $\tilde{\mathbf{D}}$ at the curve $\mathcal{C}$ gives $\tilde{\mathbf{D}}'$ – the image of $\tilde{\mathbf{D}}$ – for which the function $w^* = f^*(z^*)$ is again holomorphic. When $\mathcal{C}$ is the real axis w^* is given by $\overline{w} = f(\overline{z})$ the bars denoting complex conjugation.

Condensers that are symmetric with the u-axis can easily be obtained when I apply the theorem to the condensers in the second chapter one of whose plates is always the u-axis except for the Sharp Bend Condenser that may also be reflected at the iv-axis (cf. figs 9, 12, 17, 22 and 33).

1.2 Conformal Mapping in Electrostatics

This short section points out how the real quantities of electrodynamics can be transformed to a complex space so that they still describe the physics correctly.

I start my considerations in the three dimensional real space $\mathbb{R}^3$ with the co-ordinates x, y, z. Imagine a condenser, whose infinitely thin plates (made of an ideal conductor) are infinite in at least one dimension – say z. In this case the electric field strength $\vec{E}$ between the charged plates is independent of this variable z. With that $\vec{E}$ becomes $\vec{E} = (E_x, E_y, 0)$. Without loss of information I reduce the dimension of the real space $\mathbb{R}^3$ by one. The remaining two components of $\vec{E} = (E_x, E_y)$ can be combined in the following way

$$E = E_x + iE_y \tag{9}$$

to form the new complex electric field strength E whose real (E_x) and imaginary parts (E_y) are the two components of the real electric field $\vec{E}$. When I choose a holomorphic complex potential Φ so that

$$\Phi = \Phi_1 + i\Phi_2 \tag{10}$$

with the real potential Φ_1 and the equipotentials Φ_2 of the real electric field $\vec{E} = \nabla\Phi_1$. The complex electric field can then be written in a different form

$$E(z) = -\overline{\left(\frac{d\Phi}{dz}\right)} \tag{11}$$

whereby the bar denotes complex conjugation and $z = x + iy$ is an element of the complex Z-plane.

Let $z \in \mathbf{D}_z$ be transformed to $w \in \mathbf{R}_w$ by the conformal map $f(z) = w = u + iv$. For the new complex variable w eqs (10) and (11) are transformed as:

$$\Phi(z) = \Phi_1(x,y) + i\Phi_2(x,y) \xrightarrow{f(z)} \Psi(w) = \Psi_1(u,v) + i\Psi_2(u,v) \quad ; \tag{12 a}$$

$$E(z) = -\overline{\left(\frac{d\Phi(z)}{dz}\right)} \xrightarrow{f(z)} E(w) = -\overline{\left(\frac{d\Psi(w)}{dw}\right)} \quad . \tag{12 b}$$

Then the old and new real potentials (Φ_1 and Ψ_1) are related to each other by

$$\Delta_w \Psi_1 = \frac{1}{|f'(z)|^2}\, \Delta_z \Phi_1 \tag{13}$$

whereby Δ_w and Δ_z are the Laplacians of the *W*- and *Z*-plane respectively and $f'(z)$ is the total derivative of the conformal function f with respect to z. The same equation holds for the equipotentials; i.e. the subscript 1 may be replaced by 2. To get this formula I assumed that the old and new complex potential are pointwise equal ($\Psi(w) = \Phi(z)$), but, in general, Φ and Ψ are not the same functions of their arguments z and w.
For the charged condenser plates I may compute a surface charge density η by adaption of its definition in real vector space to our complex notation using equation (11).

$$\eta \;=\; -\vec{\mathrm{n}} \cdot \vec{\mathrm{E}} \;\hat{=}\; -\,\varepsilon_0\, \Re e\left\{ n_z\, \frac{d\Phi(z)}{dz} \right\} \;, \tag{14}$$

with $\vec{n}$ the unit vector normal to the ideally conducting surface pointing to the other plate and ε_0 the dielectric constant of the vacuum ($\varepsilon_0 = 8.85 \cdot 10^{-12} \frac{As}{Vm}$). $n_z = n_x + i n_y$ is the complex representation of $\vec{\mathrm{n}} = (\mathrm{n}_x, \mathrm{n}_y)$. The surface charge density transformed to the *W*-plane reads:

$$\begin{aligned} \eta = -\varepsilon_0\, \Re e\left\{ n_w\, \frac{d\Psi(w)}{dw} \right\} \qquad &= -\varepsilon_0\, \Re e\left\{ n_w\, \frac{d\Psi(w(z))}{dz}\frac{dz}{dw} \right\} \\ = -\varepsilon_0\, \Re e\left\{ n_w\, \frac{d\Phi(z)}{dz}\frac{1}{dw/dz} \right\} &= -\varepsilon_0\, \Re e\left\{ n_w\, \frac{d\Phi(z)}{dz}\frac{1}{f'(z)} \right\} \end{aligned} \tag{15}$$

Here n_w is the normal unit vector on the *W*-plane. The condition $f'(z) \neq 0$ – so that η stays finite – is fulfilled as $f(z)$ is conformal.
Although the first line of the previous equation describes the surface charge density on the *W*-plane as a function of w the transformation on the following line makes η a function of z. To get η in terms of w, z must be expressed by w: i.e. I have to find $z = f^{-1}(w)$.
As mentioned in section 1.1 that will normally be very difficult and laborious. The command **ParametricPlot[ ]** which occurs in every *Mathematica* program and that is described in the next section will do that for you.

1.3 Application of *Mathematica* to Conformal Mapping

This diploma thesis was especially written for people who have little experience with *Mathematica* although not even that is needed to produce the graphics presented in chapter 2.
Nevertheless it is usefull to understand how the graphics commands in these *Mathematica* programs work. At least two of the commands: **af [x_]** , **Clear[ ]** , **FindRoot[ ]**, **Needs["Graphics'ComplexMap'"]** , **CartesianMap[ ]** , **ParametricPlot[ ]**, **Show[ ]** and **Show[GraphicsArray[ ]]** occur in every program of chapter 2. So I will explain what they do.

The built-in *Mathematica* functions as, e.g., sin x (**Sin[x]**), arccos x (**ArcCos[x]**) or exp x (**Exp[x]**) are named as usual.(Note the capital letters.) Some functions as, e.g., ln x (**Log[x]**) or artanh x (**ArcTanh[x]**) are slightly different.
The argument **m** used in the elliptic integrals (**EllipticK[m], EllipticE[φ,m], ...**), in the elliptic functions (**JacobiSN[u,m], ... , InverseJacobiDN[y,m], ...**) and the Jacobi amplitude (**JacobiAmplitude[u,m]**) is the square of the modulus k: $\mathbf{m} = k^2$ (cf. p.8).

af [x_] = ***expression*** defines a pure function named af (short for auxiliary function) of the argument **x**, which must occur in ***expression*** at least once. ***Expression*** can be any expression that contains other *Mathematica* commands mathematic or not. In our case this will always be an expression of some built-in mathematic functions.
The argument **x_** is a dummy variable; it need not necessarily be a number but can be another expression as **s = ArcSinh[t]**. So **af [s]** returns ***expression*** from above replacing all **x** by **ArcSinh[t]**. If **t** is given a value (e.g. a complex number) **af [s]** evaluates symbolically for this value or number. Adding **// N** calculates the numeric value for the above expression.
There is another way to define a pure function: ***expression*** & **[x]** is the same as **af [x_]** = ***expression*** when you replace all variables **x** in ***expression*** by the sign **#** .

The command **Clear[*symb1, symb2, ...*]** clears the values and/or definitions attached to the symbols ***symb1, symb2, ...*** . These symbols may be e.g. variables or pure functions.

FindRoot[*lhs* == *rhs*, {x, x1, x2}, *options*] numerically searches for a root of the equation ***lhs* − *rhs* = *0*** near **x1** and **x2** where **lhs** is any mathematic function of the argument **x** and **rhs** is a complex number. Specifying **x1** and **x2 FindRoot[ ]** uses a variation of the secant method; specifying **x1** only (and omitting **x2**) induces *Mathematica* to use Newton's method.
If **FindRoot[ ]** found a root this will be returned as a replacement rule { **x** −> ***complex number***}. This root, however, must not necessarily be the only one of ***lhs* − *rhs* = *0*** ; in

general it is just the closest one to the starting point(s) **x1** (and **x2**).
To replace the value of the variable **x** by the solution { **x** –> ***complex number***} of **FindRoot[ ]** I add **x** = **x** /. { **x** –> ***complex number***} .
In some case the default options for **FindRoot[ ]** may not be sufficient for *Mathematica* to find a zero so you can change the maximum number of iterations by **MaxIterations** –> ***number***, the working precision and the accuracy of the iteration by **WorkingPrecision** –> ***number*** and **AccuracyGoal** –> ***number*** (For further options cf. [ma] p.794)

<< **~path/hilfsfile** reads in and evaluates the auxiliary file Hilfsfile. The part **~path/** represents the path to Hilfsfile in terms of the UNIX operating system but is not restricted to UNIX. This path has to be specified by the user. For further details see section 2.2 .

Needs["Graphics`ComplexMap`"] loads the package Graphics`ComplexMap` when it is not in the kernel yet. The package defines two graphics commands **CartesianMap[ ]** and **PolarMap[ ]** of which we will need the first one only.

CartesianMap[*f*, { x_0, x_1, dx }, { y_0, y_1, dy }, ***options*** **]** plots the image of the Cartesian coordinate lines in the given range ($x_0, x_1; y_0, y_1$) under function f. f has to be a pure function of one complex argument $z = x + iy$. The stepwidth dx and dy of the variables x (y) between x_0 and x_1 (y_0 and y_1) is optional. In case you omit dx and/or dy **CartesianMap[ ]** chooses dx and dy so that the number of lines in each direction x and y is equal to the value of the option **PlotPoints** of **Plot3D[ ]**. (On most computers **PlotPoints** has the default value 15.)
However, specifying dx and/or dy overrules the option **PlotPoints** .
The function f must not necessarily be conformal as in our case since **CartesianMap[ ]** can deal with singularities and extremely high and low values of a complex function $f(x+iy)$[11].
The warning massages, however, are still printed.
The default options, except for the one fixing the number of lines (**Options[Plot3D, PlotPoints]**) are those of **Graphics[ ]**.
You can change the options of **Graphics[ ]**. (cf. [ma p. 802]) by resetting the local options in **CartesianMap[ ]** directly or by resetting the global options using **SetOptions[Graphics,** ***options*** –> ***rules*** **]**, which is not so good an idea .
The main part of **CartesianMap[ ]** draws the electric field and equipotential lines whereas the option
PlotRange –> **All** makes *Mathematica* plot all calculated points of **CartesianMap[ ]** (The default option **Automatic** includes just those points of the calculation that are not to far from the "centre" of the picture.);
AxesLabel –> { "***textu***", "***textv***" } adds the texts ***textu***, ***textv*** to the u- and iv-axes respectively;
Axes –> **None** surpresses the drawing of axes and the option

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[11] For further details see [mae] section 1.7 .

Epilog -> { **Thickness[.011]**, { **Line[**{{u_1, v_1},..., {u_j, v_j}}**]**,
Line[{{u_k, v_k},..., u_n, v_n}}**]** }}
draws the condenser planes about thrice the thickness of the fieldlines by connecting the points (u_i, v_i) in the **Line[ ]** suboption by lines.

ParametricPlot[{f_u, f_v}, {*t, tmin, tmax* }**]** produces a parametric plot with u and v co-ordinates f_u and f_v generated as a function of parameter t ranging between ***tmin*** and ***tmax***. The list {f_u, f_v } can be exchanged by {{f_u, f_v}, {g_u, g_v}, ...} to get several curves $f(t), g(t), \ldots$ together in one picture.
ParametricPlot[ ] is used to plot the surface charge density and the electric field along the condenser planes, where f_u is the surface charge density/el. field of the argument u and f_v is the real part $\Re e\{\hat{C}\}$ or $\Im m\{\hat{C}\}$ of the curve $\hat{C}$ that represents the condenser plane in the complex W-plane as a function of v. Thus the parametric curve gives the graphical image of the surface charge density/field of the argument u and v in the complex W-plane in the range $x = x_{min}$ (=***tmin***) and $x = x_{max}$ (=***tmax***).
The option
PlotRange -> {{***umin,umax***},{***vmin,vmax***}} specifies the range of the graphics to be displayed where ***umin, umax ; vmin, vmax*** are the minimum/maximum values of the functions f_u and f_v that are plotted;
AxesOrigin -> {u_0, v_0} moves the origin to the point (u_0, v_0) of the $\mathbb{R}^2$;
the option
AxesLabel -> { "***textx***", "***texty***" } is the same as in **CartesianMap[ ]**;
For reference the option
Epilog -> { **Text[...], Text[...]** } adds some text to the graphics and
PlotLabel -> "***text*** " adds some text heading the picture.
The option
DisplayFunction -> **Identity** lets *Mathematica* calculate the curves. However, it does not display the graphics. In the *Mathematica* programs of chapter 2 the names given to these "invisible" pictures have the extension "**inv**". If you want to change some more options of **ParametricPlot[ ]** consult the lists on page 802, 839 and 842 of [ma].

Show[*graphics, options***]** displays the graphics ***graphics*** using the new options ***options***. The option **DisplayFunction -> $DisplayFunction** makes an "invisible" graphics visible.

Show[GraphicsArray[{ *list* }, ***options*** **]**, ***options*** **]** draws an array of all pictures contained in the list ***list*** using the options ***options*** for the whole array.
One important option of **GraphicsArray [ ]** is
GraphicsSpacing -> ***number*** . This option adds the space *number* between the graphics opjects given in { ***list*** } . The second ***options*** belong to the **Show[ ]** command which displays the array of graphics designed by **GraphicsArray [ ]**. These two ***options*** are a bit different in what they do but to explain their difference needs some experience with *Mathematica* .

For reference the option
Epilog –> {**Text[...]** } adds some text to the graphics and
PlotLabel –> "***text*** " adds some text to the picture.
Frame –> **True** frames the graphics array. If you do not want a frame for all sides but just for, say, the second and third one you can replace **True** by { **False, True, True, False** }. Then **FrameLabel** –> { "", "***text1***", "***text2***", "" } puts the text ***text1*** and ***text2*** to the second and third frame line. The two quotation marks "" represent an "empty" text printed to the first and fourth frame line. If you chose some "real" text instead *Mathematica* would nevertheless ignore this text since it cannot put text to a frame line which does not exist.
The **PlotRange** –> {***vmin, vmax*** } command cuts the v-range of the picture produced by **GraphicsArray [ ]** (default range is {{**0, 1**},{**0, ratio between the length of the** u**- and** v**-axes in internal *Mathematica* units** }}) leaving the u-range untouched.
To repace old values of options in some or all graphics of the list by new ones you add: (***option*** –> ***old value***) –> (***option*** –> ***new value***) to the options of the **Show[ ]** command.

2 The *Mathematica* Programs

This is the main chapter which gives you some hints how to execute my programs best. It is followed by the sections for various conformal maps and graphics. The last section gives an outlook to the difficulties that arise when one wants to round the corners in a condenser configuration treated by the Schwarz-Christoffel transformation.

2.1 How to Use the Programs

The *Mathematica* programs are designed in a way that even users with no experience in the computer language *Mathematica* may have the programs' graphics plotted provided they type the programs correctly. Before you do that, however, you should read section 2.1 to 2.3 in which many things are explained that occur in the programs later on.

All sections of chapter 2 except for number 2.1 and 2.2 are arranged in the same way. They contain text and pictures in the following order:

- about the conformal map $w = f(z)$: poles of $f(z)$, $f'(z)$ and its zeros , domain $\mathbf{D}_z$
- the image of the condenser-plates in the W-plane (picture).
- calculation of the parametric curve $\hat{C}$ representing the condenser planes and the normal unit vectors in W-space.
- calculation of the complex electric field E and the surface charge density η in terms of x and y.
- what the electric field and the equipotential lines look like
- how will the electric field strength and the surface charge density on the plates behave
- *Mathematica* program
- coments on and tricks used in this section's *Mathematica* program
- tips for variants of the *Mathematica* program
- the graphics that can be generated by the *Mathematica* program

The *Mathematica* programs are quite similar one to another:

▲ At the beginning the package Graphics'ComplexMap is loaded and the dimensions of the condenser as, e.g., thickness of the plates, their distance and their voltage must be specified by hand.
The user makes his/her choice whether he/she wants the pictures of the surface charge density or the electric field plotted.
▲ When the conformal function contains any elliptic functions or integrals **FindRoot[ ]** searches for the value of the parameter m belonging to the input data.
▲ Then all functions needed in the graphics commands are defined.
Values that specify some graphics parameters are calculated.
The auxiliary file Hilfsfile which contains some additional variables used in the plots is loaded.
▲ The main part consists of the commands **ParametricPlot[ ]** and **CartesianMap[ ]** in the order:
ParametricPlot(s) for the upper plate , **CartesianMap** , **ParametricPlot**(s) for the lower plate.

When you run the programs you should divide them into these four sequences so that you do not have to run through the whole program when an error in one of these parts arises. This will save you a lot of trouble, time and nerves.

However, in my *Mathematica* programs there is one operation where difficulties may arise. This is the **FindRoot[ ]** command followed by the **ReplaceAll[ ]** $\hat{=}$ **/.** command. Even with *Mathematica* it is not possible to devise programs which work for any values of the condenser parameter.
After the dimensions of the condenser in the first part of the program have been specified **FindRoot[ ]** searches for a solution of its argument that is a function of those new values. Then the program replaces the value of the variable – say **a** – by the solution found by **FindRoot[ ]** .
For a tested range of condenser diameters a list of parameters for each **FindRoot[ ]** command is given at the end of each program.
If you want the graphics for values of parameters outside the ranges listed there you have to find out the parameters and options in **FindRoot[ ]** yourself. Unfortunately one error in finding the solution by **FindRoot[ ]** entails a number (256) of errors when *Mathematica* tries to replace the old value of **a** by the new incorrect one.
To prevent this you have to divide the program right behind the first command :
fra = FindRoot[*lhs == rhs*, {*a*, *amin*, *amax*}, *options*] .
So the other part will begin with:
; a = a /. fra ;
Now run the program up to this **FindRoot[ ]** command and change the parameters and options within **FindRoot[ ]** until *Mathematica* gives you no more warning messages. Then merge the two parts again and divide after the second **FindRoot[ ]** command. Do

this until you reach the last **FindRoot[ ]** and merge the parts. The program should run without trouble.
This procedure looks very tedious but it is the quickest way to get the wanted results.

As mentioned before the options in **ParametricPlot[ ]** and **CartesianMap[ ]** can be changed too. The following changes will be the most needed:

for **ParametricPlot[ ]** :

- The original labeling of the axes done by the option **Epilog** –> {...} should sometimes be modified by hand when using other "starting values" for your condenser. Therefore the numbers within the first { } in the **Text[ ]** suboption must be changed. These numbers are the co-ordinates of the text "***text***" in the graphics.
  If you want to render any text inoperative as an option you just surround the option <u>and</u> the preceding comma by (* , ***option*** *) in this case (*, **Epilog** –> {...}*). *Mathematica* disregards everything wrapped by (* *); later on you may use the option again after erasing the (* *).

- Some graphics showing the electric field and surface charge density are labeled with ticks in SI units using the specifications (thickness, distance, voltage) from above. If you wanted to use different units instead, you have to change the variable **mult** in the Hilfsfile to your purpose.

- If you want to change the range of the plot to look at a certain detail you may do this by changing the co-ordinates in the **PlotRange** option. For further details cf. section 1.3 or [ma, p. 413ff]

for **CartesianMap[ ]** :

- Pictures for special parts of the condenser are plotted by changing the **PlotRange** option as indicated in section 1.3 and [ma, p. 413ff]

- When you want to change the number of gridlines you can do this by adding:
  " , **PlotPoints** –> ***number*** " (don't forget the comma !)
  to the options or by changing dx and dy in **CartesianMap[ ]** (cf. section 1.3).

- Due to roundoff errors during the calculation the points near the condenser plates may have wrong values which lead to zigzag lines across the picture. If that is true you have to modify the y co-ordinates in **CartesianMap[ ]** so that you add some tiny number to the lower limit y_0 and subtract it from the upper limit y_1. The tiny numbers for the upper and lower limit do not have to be of the same value and these values must be found out by trial and error.

Each program can be used exclusively without parts of any other program except for the auxiliary program Hilfsfile presented in section 2.2 . So you just have to type those programs you really want to use. When you want to use more than one program at the same time you should not try to scratch out lines that occur in any of the programs. You might get funny graphics you did not want to get. Normally you should not get into more troubles than that the graphics are of different size and not exactly one below the other.

There are some rules however which have to be strictly followed when typing and executing my programs:

1. Note that all *Mathematica* commands begin with capital letters. Even the parts of compound words start with capital letters. Nevertheless compound words must in any case be written without space in between.
2. Do not forget to type all brackets. There must be the same number of opening as well as closing brackets (whether they are round, curly or square brackets.) *Mathematica* would give an error message otherwise.
3. Do not try to modify the program before you have not tried to run exactly the one discussed in the thesis. If an error arises you should check whether it results from a slip committed in typing the program or from your computer which may not like certain commands or options due to an older *Mathematica* version than 2.0 or a bad installation of *Mathematica* .
4. It is advisable to start a brandnew *Mathematica* session when you run these programs. Otherwise some specifications of options in the *Mathematica* commands **Graphics[ ]** and **ParametricPlot[ ]** that are not the default ones – they are assumed when working with my original programs – could still be in the kernel.

These four items are valid for all programs presented in this paper. Sometimes some warning messages appear which can be ignored when the program ran correctly for at least one time. These messages are:

General::spell: Possible spelling error: new symbol name "*sym*" is similar to existing symbols *list of symbols*.

General::spell1: Possible spelling error: new symbol name "sym_1" is similar to existing symbol "sym_2".

Since I often used quite similar names for variables and functions in the *Mathematica* programs these warning messages may occur. They will not do any harm: so ignore them ! There is another message you cannot do much about. It occurs when the working precision

in **FindRoot[ ]** is above the working precision of a function ***f*** (usually 16 -19 digits) used as the argument of **FindRoot[ ]** .

FindRoot::precw: Warning: The precision of the argument function (*f*) is less than WorkingPrecision (*number*).

Unless you are an expert in *Mathematica* ignore it too. For further information about these and other error messages see: *Mathematica* Warning Messages which came with your copy of *Mathematica* .

2.2 Information About **Hilfsfile** and its *Mathematica* Program

This short section explaines the way the auxiliary file Hilfsfile works.

When you run one of the *Mathematica* programs (cf. section 2.5 to 2.14 except for 2.8) you have to decide whether you want the graphics to be plotted for the electric field strength or for the surface charge density along the condenser planes. Specifying the variable **decide** in the program makes this switch between the electric field (**ef**) and the surface charge density (**scd**). The line << **~path/hilfsfile** in the progams reads in and evaluates the file Hilfsfile where **~path/** represents the UNIX commands for the path: home directory → Hilfsfile. This path has to be specified by hand. Easiest would be to use the home directory. Then << **~path/hilfsfile** reduces to << **hilfsfile**. When you execute the *Mathematica* programs by a computer that is not run by the UNIX operating system you may have to modify **~path/** to the proper commands of your operating system.

I will explain Hilfsfile with the aid of its computer program.

```
Clear[branch,mult,sgn,tex1,tex2,tex3,fr,frl,frtex]
Which[decide === ef,
      mult = V 1000/a; sgn = -1; branch = 1;
      tex1 = "E[V/m]"; tex3 = "E";
      fr = {False,False,True,False};
      frl = {"","",frtex,""},
      decide === scd,
      mult = V 8.85 10^-9/a; sgn = 1; branch = 0;
      tex1 = "eta[C/m]";
      tex3 = FontForm["h",{"Symbol",10}];
      fr = {True,False,False,False};
      frl = {frtex,"","",""},
      decide =!= scd && decide =!= ef,
      Print["Your input is wrong !"]] ;

tex2 = FontForm["h | E",{"Symbol",10}] ;

(* This must be an ASCII text file *)
```

In the first line all previous specifications for the variables used in Hilfsfile are cleared. In the following branching command **Which[ ]** the current value of the variable **decide** is compared to the three possibilities: **ef**, **scd**, neither **ef** nor **scd** . If the value equals **ef** Hilfsfile exports the graphics parameters needed for plotting the electric field. If the value

equals **scd** Hilfsfile exports the graphics parameters needed in the main program for plotting the surface charge density. If the value is non of those Hilfsfile sends a warning.

The variables contained in the **Which[ ]** command are:

- **mult** defines a multiplication operator which specifies the scale of the y-axes in SI units. **mult** of **scd** equals $1000\frac{\epsilon_0 V_0}{a}$. The factor 1000 derives from the scaling of the u- and iv-axes in mm (1 meter = 1000 millimetres) for a given voltage of V_0 Volt and a plate distance a mm.
  mult of **ef** equals $1000V_0\ /a$ (cf. eqs (27)).
- **sgn** defines a multiplication operator which specifies the scale's sign on the y-axis (cf. eqs (27)).
- **branch** is used to specify one limit (lower limit for the upper — upper limit for the lower plate) of the **PlotRange** in the **GraphicsArray[ ]** .
- **tex1** defines the text containing the SI units to label the y-axes with.
- **tex3** is the same as **tex1** without units.
- **fr** specifies the frame lines to be drawn in the **GraphicsArray[ ]**.
- **frl** tells *Mathematica* to add the text **frtex** (specified in the main program) to the frame line in **GraphicsArray[ ]**.

tex2 defines the text for graphics without ticks that are the same for the electric field as well as for the charge density.

2.3 *Mathematica* Program for the Infinite Plane Condenser

$f(z) = \text{Identity}(z) = z$

The *Infinite Plane Condenser* is the basis for all other condenser configurations since its electric field and all other related quantities are known. These may then be mapped conformally to the other condensers presented in the following sections.

The Infinite Plane Condenser is a condenser whose two parallel plates run from minus to plus infinity for two of the three Cartesian co-ordinates x,y,z. The electric field $\vec{E}$ between these plates is homogenous and can be calculated from the voltage applied to these plates and for the distance d.

$$\vec{E} = -\vec{\nabla}\Phi \quad \text{with} \quad \Phi = \frac{\Delta V \ \mathrm{y}}{\mathrm{d}} \tag{16}$$

Φ denotes the potential between the plates and ΔV is the potential difference of the plates. This expression corresponds to one condensor plane located in the xz-plane and the other one in a plane parallel to the first one at the distance d.
Since the electric field is homogenous the surface charge density defined in equation (14) is constant.

$$\eta = \mp\varepsilon_0 \ \Delta V/\mathrm{d} \tag{17}$$

The minus, plus sign respectively applies to the positively, to the negatively charged plate respectively.

Up to now in this section I talked about electric quantities in real space $\mathbb{R}^3$. Now I switch to the complex plane. As in section 1.2 I drop the z component of the $\mathbb{R}^3$ space. The remaining components x and y become the real (x) and imaginary part (y) of the elements z (don't confuse it with the third real component z !) in the complex plane.
For many condenser configurations conformal maps are known which transform the problem to that of a two dimensional Infinite Plane Condenser. When you know such a function[12] the electric field, the potential, the surface charge density and the electric field strength on the surface of this condenser can be computed a lot easier. The Infinite Plane Condenser which I choose to work with on the complex plane is one whose plates are located on the x-axis and at $z = x + i\pi$, a line parallel to the x-axis at the distance $\mathrm{d} = i\pi$. After this choice I map the Infinite Plane Condenser to the Flat Condenser, which matter is the content of the next section. To this configuration in the upper half of the complex plane I apply the Schwarz-Christoffel formula to get all other condenser configurations treated in this thesis.

The conformal map for the Infinite Plane Condenser is the identity function. It is conformal in the whole complex plane. The domain for the Infinite Plane Condenser would therefore

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[12] A standard book for conformal mapping is [ko].

be $\mathbf{D}_z = \mathbb{C}$.
This domain, however, is oversized, since it is larger than the size of the **Infinite Plane Condenser** which is the starting-point for all other configurations.
That is why I restrict the domain $\mathbf{D}_z$ to $\mathbf{D}_{id} := \{z|\ z = x + iy \text{ with } x \in \mathbb{R} \wedge y \in [0, \pi]\}$.
So the domains for further conformal functions can only be subdomains of $\mathbf{D}_{id}$. Moreover zeros of the first derivative of the following conformal maps that are not within this domain $\mathbf{D}_{id}$ will not be mentioned.
Nevertheless you have to be careful when choosing the domain for the computer programs. You might get false graphics results especially for conformal maps including square roots, elliptic functions and integrals due to errors arising from numerical calculation.

The potential of equation (16) must be slightly modified when changing to complex coordinates: It must be holomorphic in the domain $\mathbf{D}_{id}$ and must fulfill the boundary conditions. Thus the correct expression for Φ is

$$\Phi(z) = -i\frac{V_0\, z}{\pi} \quad . \tag{18}$$

Since the complex potential for the lower plate ($z = x$) equals $\Phi = -i\frac{V_0 x}{\pi}$ and $\Phi = V_0 \ -i\frac{V_0 x}{\pi}$ for the upper plane ($z = x + i\pi$) we see that the voltage at the upper plate is V_0 and is zero at the lower plate.

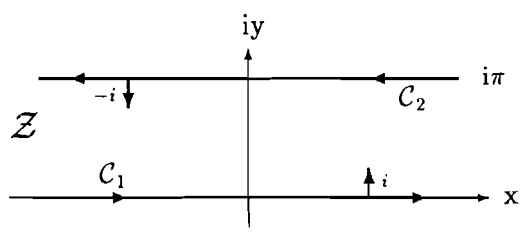


Figure 2: Curve C and normal vectors of the **Infinite Plane Condenser**

Figure 2 represents the plane condenser in domain $\mathbf{D}_z = \mathbf{D}_{id}$ indicated by the caligraphic $\mathcal{Z}$. Apart from the condenser planes entitled C_1 and C_2 you see two tiny arrows pointing upwards and downwards and some triangles giving a direction to the lines that represent the two dimensional planes (The second dimension was skipped when changing to the complex plane.).

The arrows are the unit vectors n_z normal to the condenser planes. Of course there is a unit vector for every point on each plane, but since all n_z of one plate are the same I plotted one representative for each plane. They are constructed as follows:

Let $C(s)$ be the parametric representation of a curve C with s the parameter of $C(s)$. By definition the normal vector $N_C(s_0)$ in the point s_0 of the curve $C(s)$ is normal to

the tangent vector $T_C(s_0)$ in the same point. The tangent vector $T_C(s_0)$, however, can be calculated when I know the curve $C(s)$. It is the total derivative of $C(s)$ in the point s_0

$$T_C(s_0) = \frac{dC(s)}{ds}\Big/_{s=s_0} \quad . \tag{19}$$

On the Gaussian plane the multiplication of T_C with one of the complex numbers $e^{i(2k+1)\pi/2}$ ($k \in \mathbf{Z}$) leads to a set of normal vectors N_C^k. This is evident when you consider that the multiplication of a complex number by $e^{i\varphi}$ can be interpreted as the rotation of the corresponding vector by the angle α. I choose $k = 0$[13]. Thus the normal unit vector in the point s_0 becomes

$$n_C = i\left(\frac{dC(s)}{ds}\Big/\left|\frac{dC(s)}{ds}\right|\right)_{s=s_0} \quad . \tag{20}$$

The denominator in equation (20) is the modulus of the numerator added to normalize the normal vector N_C^0.
With the knowledge of one parametric representation for the condenser plates I may calculate the normal unit vectors immediately. As you will see all our curves which are closed loops on Riemann's sphere result clearly from the curve for the infinite plain condenser[14].
For the problem of the **Infinite Plane Condenser** the parametric curve $C(s)$ consists of two parts C_1 and C_2

$$C_1(x) = \{x|\ x\epsilon(-\infty,\infty)\} \qquad , \qquad C_2(x) = \{-x + i\pi|\ x\epsilon(-\infty,\infty)\} \tag{21}$$

where the parameter s of the curve $C(s)$ was replaced by the variable x.
To understand the minus sign in $C_2(x)$ you have to look at the image of the infinite condenser planes on the Riemann sphere (they have the shape of a deformed figure eight, crossing at the north pole) or at figure 2:
Imagine two lines connecting the two parts $\mathcal{C}_1$ and $\mathcal{C}_2$ forming a closed loop with these. I now run along $\mathcal{C}_1$ to the point where the fictious line at the right leaves to $\mathcal{C}_2$; I follow that line up to $\mathcal{C}_2$; then I follow $\mathcal{C}_2$ in the negative sense to the left auxiliary line; I get down to $\mathcal{C}_1$ again to complete the loop. When you let the two auxiliary lines tend to plus and minus infinity you get expressions for $\mathcal{C}_1$ and $\mathcal{C}_2$ that are equal to equation (21).
If I chose the plus sign instead, I would again get the shape of an eight form on Riemann's sphere, but I would have to follow the upper loop of the eight form in the unusual sense so that the normal vector pointed to the outside of the loop. This would mean that the

---

[13] Since $\mathbf{D}_{id}$ is not the whole complex plane k is restricted to $k\epsilon\{-1,0\}$. I choose $k = 0$ whose multiplication factor $e^{i\pi/2}$ turns T_C counterclockwise into N_C which is the mathematical direction for closed loops and of the complex polar angle φ. Choosing $k = -1$ the surface charge density η changes sign which does not lead to new solutions since V_0 in $\eta_{k=0}$ can be replaced by $-V_0$ to get the same results.

[14] Which is quite evident since the conformal functions which map the **Infinite Plane Condenser** to all others are single-valued. Even for points on the planes that do not belong to the domain the limit from the condenser's interior is single-valued.

charge on the plates had the same sign and the two plates would therefore not form a condenser. Mathematically speaking the loop integral of the surface charge density would not equal zero. This cannot be correct since the field between the plates does not have any singularities which the Cauchy integration formula would demand. Hence the tangent and normal (unit) vectors read

$$T_z = \frac{dC(s)}{ds} = \begin{cases} 1 \text{ for part } C_1(s) \\ -1 \text{ for part } C_2(s) \end{cases} \tag{22}$$

$$n_z = e^{i\pi/2} \cdot T_z = i\frac{dC(s)}{ds} = \begin{cases} i \text{ for part } C_1(s) \\ -i \text{ for part } C_2(s) \end{cases} \tag{23}$$

The subscript $_z$ of T and n again indicates the domain $\mathbf{D}_z$ wherein the above vectors are defined.

With the help of eqs (23), (18) and (15) the surface charge density of the condenser plates can be calculated as

$$\eta = -\varepsilon_0 \, \Re e\left\{\mp i \, \frac{d}{dz}\left(-i\frac{V_0 \, z}{\pi}\right)\right\} = \pm\frac{\varepsilon_0 V_0}{\pi} \quad . \tag{24}$$

As I mentioned before η has a constant value, whereby the plus sign in the equation belongs to the upper and the minus sign to the lower plate. In physical terms this means that the charge density is the same all over both planes; yet the densities are oppositely charged. Since I always use this Infinite Plate Condenser as a basis to calculate all the others except for those in section 2.13 and 2.14 I insert its potential (eq. (18)) into formula (1.2) of the surface charge density in the W-plane.

$$\eta_w(\hat{\mathcal{C}}) = -\varepsilon_0 \, \Re e\left(n_w \, \frac{d}{dz}\left(-\, iV_0 \, z/\pi\right)\frac{1}{f'(z)\big/_{z\in\mathcal{C}}}\right) = \varepsilon_0 \, \Re e\left(n_w \, \frac{iV_0}{\pi}\frac{1}{f'(z)\big/_{z\in\mathcal{C}}}\right)$$

$$= -\frac{\varepsilon_0 V_0}{\pi} \, \Im m\left(n_w \, \frac{1}{f'(z)\big/_{z\in\mathcal{C}}}\right) = \eta(\hat{\mathcal{C}}) \tag{25}$$

You see that the surface charge density $\eta(\hat{\mathcal{C}}) = \eta(\hat{\mathcal{C}} \subset \mathbf{R}_W)$ for plates in the W-plane tends to infinity for the limit $z \longrightarrow z_0$ when $f'(z_0) = 0$. Nevertheless it stays finite in $\mathbf{R}_w$ since $z_0 \notin \mathbf{D}_z$. Fortunately all inner points $w_i \in \mathbf{R}_w$ on the W-plane have a finite electric field as $f(z)$ is conformal in $\mathbf{D}_z$.

The electric field strength E_w on the W-plane can be obtained by the same procedure. I have to insert equation (18) into (12 b) and obtain

$$
\begin{aligned}
E = E_u + iE_v &= -\overline{\left(\frac{d\Psi}{dw}\right)} = -\frac{iV_0}{\pi}\,\frac{1}{\overline{f'(z)}\big/_{z\in C}} \\
&= \frac{V_0}{\pi}\,\Im m\left\{\frac{1}{\overline{f'(z)}\big/_{z\in C}}\right\} - i\frac{V_0}{\pi}\,\Re e\left\{\frac{1}{\overline{f'(z)}\big/_{z\in C}}\right\} \\
&= -\frac{V_0}{\pi}\,\Im m\left\{\frac{1}{f'(z)\big/_{z\in C}}\right\} - i\frac{V_0}{\pi}\,\Re e\left\{\frac{1}{f'(z)\big/_{z\in C}}\right\} \quad . \qquad (26)
\end{aligned}
$$

If the nomal unit vectors n_w are simply ± 1 or $\pm\, i$ as in all cases except for the **One Rounded Corner of 90° Condenser** the relation between the electric field strength and the surface charge density is quite simple.

$$
\begin{aligned}
\eta_{n_w=1}(v) &= \varepsilon_0\, E_u(v) \\
\eta_{n_w=-1}(v) &= -\varepsilon_0\, E_u(v) \\
\eta_{n_w=i}(u) &= \varepsilon_0\, E_v(u) \\
\eta_{n_w=-i}(u) &= -\varepsilon_0\, E_v(u) \qquad (27)
\end{aligned}
$$

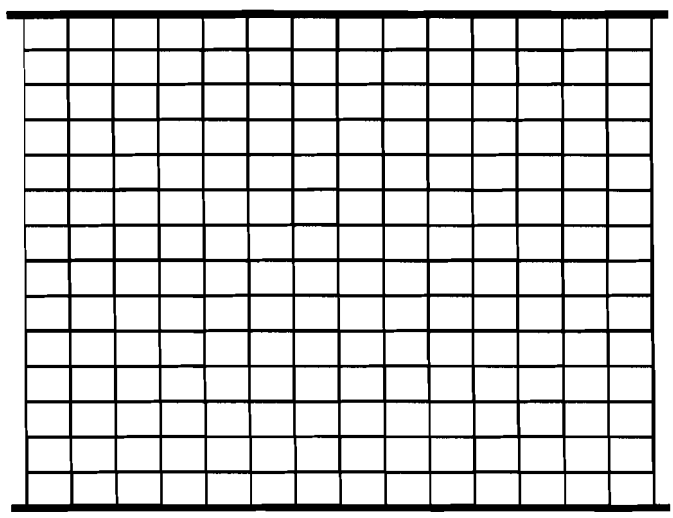
This equations show that the surface charge density and the electric field on the plates are the same apart from the factor $\pm\varepsilon_0$. This permits one to create one *Mathematica* program for both the field and charge density (cf. sections 2.1 & 2.2).

As everything to make up the *Mathematica* program has been collected time has come to present it.

Mathematica program 1: for the Infinite Plane Condenser

```
Needs["Graphics`ComplexMap`"]

a1 = CartesianMap[# &,{-2,2},{0,Pi}, Axes -> None,
      Epilog -> { Thickness[.011],
                  { Line[{{-2.1,Pi},{2.1,Pi}}], Line[{{-2.1,0},{2.1,0}}] }} ] ;
```

This first *Mathematica* program consists of just two commands (apart from the options) and I do not suppose you would like to vary it.
Therefore I show the graphics plotted by the program straight away:



(a1)

Figure 3: Graphics produced by the *Mathematica* program 1 for the Infinite Plane Condenser.

The Infinite Plane Condenser is not very instructive: so we go to the next one.

2.4 *Mathematica* Program for the Flat Condenser $t = e^z$

This section as the previous one is more of an explanatory character because it prepares the foundation for the other problems as indicated in section 2.3. The condenser plates are located on the r-axis which is the real axis with the one plane extending from minus infinity to zero, the other one (oppositely charged) extending from zero to infinity with a gap at zero[15]. The conformal map which does the transformation is: $t = e^z$. The equation $f(t) = A\int_{t_0}^{t}(t-t_0)^{-1}dt + B = A\ln(t-t_0) + B$ maps the upper half of the T-plane to the Infinite Plane Condenser . I choose t_0 to be $t_0 = 0$. When I let $t = \pm 1$ be transformed to $z = 0$ and $z = i\pi$ respectively I get $0 = A\ln 1 + B = B$ and $i\pi = A\ln(-1) = i\pi A \Rightarrow A = 1$. So the transformation reads $z = \ln t$ [16] . Its inverse function $f^{-1}(z) = e^z = t$ gives the conformal map as written above.

The conformal map $t = f(z) = e^z$ is holomorphic in the whole complex plain $\mathbb{C}$. Its first derivative $f'(z) = f(z)$ gets zero for values $z = -\infty$ $(m \in \mathbb{Z})$ which is not in $\mathbf{D}_{id}$.
With that the domain $\mathbf{D}_z$ is $\mathbf{D}_z = \mathbf{D}_{id}$

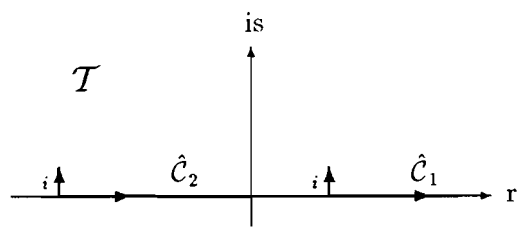


Figure 4: Curve $\hat{C}$ and normal vectors of the Flat Condenser

The image $\hat{C}$ of the curve C in the T-plane is shown in the picture above. As C it consists of two parts $\hat{C}_2$ and $\hat{C}_1$. $\hat{C}_1$ is the image of C_1 and $\hat{C}_2$ the image of C_2 in the T-plane since

$$f(C_1) = e^x \geq 0 \qquad \Longrightarrow \hat{C}_1(r) = f(C_1) = r \quad \text{for} \quad r > 0 \ ;$$

$$f(C_2) = e^{-x+i\pi} = -e^{-x} \leq 0 \Longrightarrow \hat{C}_2(r) = f(C_2) = r \quad \text{for} \quad r < 0 \ .$$

---

[15] The derivative of the conformal map $f(z) = f'(z) = e^z$ is zero for $z = -\infty$. Therefore $t = 0$ is not an element of the domain $\mathbf{D}_t$ which fits our physical purpose well because we can place an ideal insulator at the zero-point to prevent the charged particles to move from one plate to the other without spoiling the mathematical description.

[16] The function $f(t) = \ln t = z$ is multi-valued since its inverse is periodic. To get a unique solution for $\ln t$ I have to cut the range. I choose a new range with $\mathbf{R}_t = \{z = x + iy \mid x \in \mathbb{R}, y \in (-\pi, \pi]\}$. This is a superset of the plane condenser's domane $\mathbf{D}_{id}$. The new function f is therewith $\mathrm{Ln}t = z$ the principal branch of the logarithm.

Here $f(C_i)$ denotes the image of the curve C_i located in the Z-plane under the function f. Since $f(C_1)$ is real and monotonically increasing from 0 to infinity for increasing x it has to be aquivalent to the positive branch of the r-axes. The real negative and with x increasing function $f(C_2)$ must therefore be the negative branch of the r-axes. With these parametric curves for the condenser plates I compute the unit normal vectors in the W-plane.

They are calculated to $n_t = i\frac{dr}{dr} = i$ which is the same result as for the lower plate of the Infinite Plane Condenser . Inserting the derivative for the points of the curve $\mathcal{C}$ and the normal vectors n_w into equation (25) I get the surface charge density for the two parts $\hat{\mathcal{C}}_1$ and $\hat{\mathcal{C}}_2$ in terms of x:

$$\eta_{\hat{\mathcal{C}}_1}(x) = -\frac{\varepsilon_0 V_0}{\pi}\,\Im m\left\{i\frac{1}{e^x}\right\} \quad = -\frac{\varepsilon_0 V_0}{\pi}\,\frac{1}{e^x} \qquad \text{for} \quad x \in \mathbb{R}: \quad \hat{C}_1(x) = e^x \quad ;$$

$$\eta_{\hat{\mathcal{C}}_2}(x) = -\frac{\varepsilon_0 V_0}{\pi}\,\Im m\left\{i\frac{1}{-e^{-x}}\right\} = +\frac{\varepsilon_0 V_0}{\pi}\,\frac{1}{e^{-x}} \quad \text{for} \quad x \in \mathbb{R}: \quad \hat{C}_2(r) = -e^{-x} \quad .$$

In this special case I can even write down the surface charge density as a function of r when I insert the representations for $\hat{\mathcal{C}}_1$ and $\hat{\mathcal{C}}_2$ in terms of x. In all the other problems this cannot be done by analytic expressions. This is performed by the *Mathematica* command **ParametricPlot[ ]**.

$$\eta_{\hat{\mathcal{C}}_1}(r) = -\frac{\varepsilon_0 V_0}{\pi}\,\frac{1}{r} \qquad \text{for} \quad r > 0 \quad ;$$

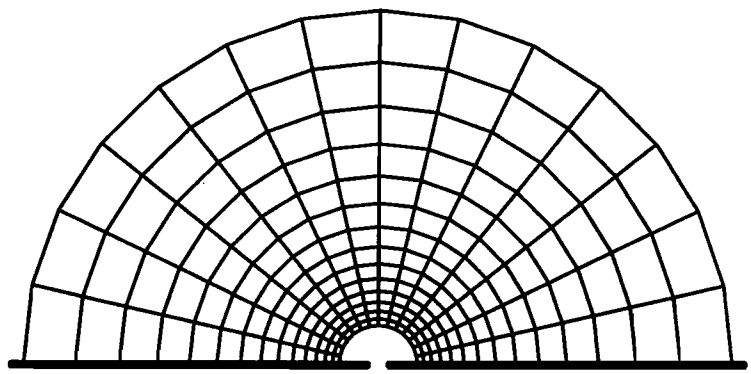
$$\eta_{\hat{\mathcal{C}}_2}(r) = -\frac{\varepsilon_0 V_0}{\pi}\,\frac{1}{r} \qquad \text{for} \quad r < 0$$

Mathematica program 2: for the Flat Condenser

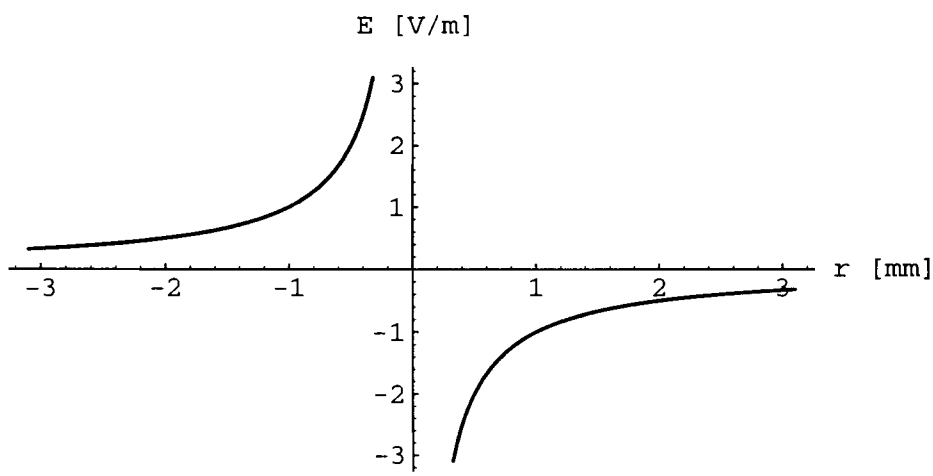
```
Needs["Graphics`ComplexMap`"]

Clear[V]
V = Pi/1000 (* Voltage between the plates in Volt *) ;

af20[z_] = V 1000/Pi Exp[z] ;

a2 = CartesianMap[af20, {-1.1,1.1},{0,Pi}, Axes -> None,
      Epilog -> { Thickness[.011],
                { Line[{{-3.1,0},{-.1,0}}], Line[{{.1,0},{3.1,0}}] }} ] ;

b2 = ParametricPlot[{{ af20[t], -af20[-t] }, { -af20[-t], af20[t] }},
      {t, -1.1314, 1.1314}, AspectRatio -> .5,
      AxesLabel -> {"r[mm]","E[V/m]"}] ;
```

This program like all the following ones consists of a combination of **ParametricPlot[ ]** and **CartesianMap[ ]**. The first one draws the surface charge density, the latter one the condenser planes with their electric field. No special tricks for the design of this program were needed except for the use of the option **AspectRatio** which renders the output of **ParametricPlot[ ]** as large as the graphics of **CartesianMap[ ]**. Since I do not suppose that you will want to change this program no variants are mentioned.

Here are the Graphics:



(a2)



(b2)

Figure 5: Graphics produced by the *Mathematica* program for the Flat Condenser. The labels indicate that parts of the program which produced the pictures.

2.5 *Mathematica* Program for the Semi-Infinite Plane Condenser $w = \frac{a}{\pi}(1 + z + e^z)$

This is the first realistic problem which deals with the Semi-Infinite Plane Condenser. The upper plate extends from minus infinity to zero and ends there. I call this a semi-infinite plate to distinguish it from the infinite plate. The lower plate is at the u-axis. Although all pictures of this section show only one semi-infinite plate at $u + ia$ and an infinite plate the problem is compatible with that of two semi-infinite plates at $u \pm ia$ as shown in the variations of the *Mathematica* program.

The conformal map for this problem is holomorphic for all $z \in \mathbb{C}$ and the first derivative $f'(z) = \frac{a}{\pi}(1 + e^z)$ is zero for $z = i\pi$. With that the domain $\mathbf{D}_z$becomes $\mathbf{D}_z = \mathbf{D}_{id}\backslash\{i\pi\}$.

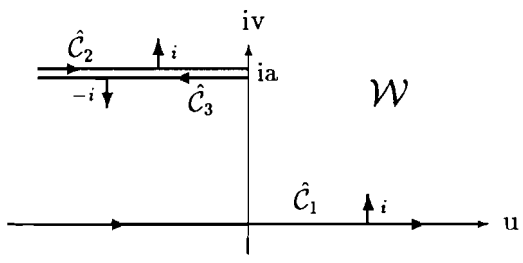


Figure 6: Curve $\hat{C}$ and normal vectors of the Semi-Infinite Plane Condenser

The image of the curve C in the complex W-plane is given in the picture above. It consists of three parts. $\hat{C}_1$ is the image of C_1, $\hat{C}_2$ the image of C_2 for $x \leq 0$ and $\hat{C}_3$ the image of C_2 for $x \geq 0$ on the W-plane.
Allthough both curves $\hat{C}_2$ and $\hat{C}_3$ are at the same distance ia from the u-axis they are drawn as seperate lines for clarity.

$$f(C_1) = \frac{a}{\pi}(1 + x + e^x) \quad \Longrightarrow \quad \hat{C}_1(u) = f(C_1) \quad = \quad u \quad \text{for } u \in \mathbb{R} \ ;$$

$$f(C_2) = \frac{a}{\pi}(1 - x + i\pi - e^{-x}) \Longrightarrow \begin{cases} \hat{C}_2(u) = f(C_2(x < 0)) & = \ u + ia \quad \text{for } u < 0 \ , \\ \hat{C}_3(u) = f(C_2(x > 0)) & = -u + ia \quad \text{for } u > 0 \ . \end{cases}$$

The curves $f(C_i(x))$ can be transformed to the parametric curves $\hat{C}_j(u)$ of the condenser plate in the W-plane as indicated in the previous section by simply studying their monotonicity for the real and imaginary part. The minus sign in $\hat{C}_3(u)$ has to be interpreted similar to the one for the upper plate of the Infinite Plane Condenser .

So the unit normal vectors are calculated to $n_w(\hat{C}_1) = n_w(\hat{C}_2) = i$ and $n_w(\hat{C}_3) = -i$. With

these the surface charge density in terms of x is

$$\eta_{\hat{C}_1}(x) = -\frac{\varepsilon_0 V_0}{\pi}\,\Im m\left\{\frac{i\ \ \pi}{a(1+e^x)}\right\} \qquad = -\frac{\varepsilon_0 V_0}{a}\,\frac{1}{1+e^x}$$

$$\text{for}\ \ x \in \mathbb{R}\ :\quad \hat{C}_1(x) = \tfrac{a}{\pi}(1+x+e^x)\ \ ;$$

$$\eta_{\hat{C}_2}(x) = -\frac{\varepsilon_0 V_0}{\pi}\,\Im m\left\{\frac{i\ \ \pi}{a(1+e^{-x+i\pi})}\right\} = -\frac{\varepsilon_0 V_0}{a}\,\frac{1}{1-e^{-x}}$$

$$\text{for}\ \ x < 0\ :\quad \hat{C}_2(x) - ia = \tfrac{a}{\pi}(1-x-e^{-x})\ \ ;$$

$$\eta_{\hat{C}_3}(x) = -\frac{\varepsilon_0 V_0}{\pi}\,\Im m\left\{\frac{-i\ \ \pi}{a(1+e^{-x+i\pi})}\right\} = +\frac{\varepsilon_0 V_0}{a}\,\frac{1}{1-e^{-x}}$$

$$\text{for}\ \ x > 0\ :\quad \hat{C}_3(x) - ia = \tfrac{a}{\pi}(1-x-e^{-x})\ \ .$$

The surface charge density and the electric field of the upper plate becomes infinite in $w = ia$ for both the top and bottom of this plate.
This pole results from the sudden change of direction in the point $w = ia$. In this case I cannot get rid of the infinite values since the application of the modified Schwarz-Christoffel formula will not lead to a smooth curve as the plates are infinitely thin. Their ensuing singularity is a consequence of the sharp edge. For a plate of finite thickness (cf. the next section), however, a mapping function which produces an electric field and a surface charge density that are finite all over the plates may be constructed.
The asymptotic behaviour of the electric field and charge density on the plates for $u \to \pm\infty$ can be considered quite easily:
For $u \to -\infty$ field and charge density at the bottom of the upper as well as at the lower plate have to approach their values of the Infinite Plane Condenser: these values equal $-\frac{V_0}{a}$ For the field and $\pm\frac{\varepsilon_0 V_0}{a}$ for the charge density. The plus sign in the last term holds for the upper, the minus sign for the lower plate.
for $u \to -\infty$ on the top of the upper plate and for $u \to +\infty$ on the lower plate the field and charge density tend to zero since the distances between the charges of these plates tend to infinity.

Mathematica program 3: for the Semi-Infinite Plane Condenser

```
Needs["Graphics`ComplexMap`"]

Clear[V,a,b,c,d,e,t,frc,decide]
V = 1 (* Voltage between the plates in Volt *);
a = 3 (* Distance of the plates in mm :
         for all  1.1 10^-5 < a < 2 10^2 (scd) and 1.1 10^-5 < a < 3 10^5 (ef) *) ;
decide = scd (* Decide whether you want the electric field (ef) or
                surface charge density (scd) of the plates *) ;

af30[z_,t_] = a/Pi (1 + z + t) ;
af31[t_] = 1/(1 + t) ;
t[z_] = Exp[z] ;

b = -5.7155 ; d = af30[-c,t[-c]]; e = af30[b,t[b]] ;
frc = FindRoot[e - af30[-c,-t[-c]],{c,-2,-3}]; c = c/.frc ;

<< ~path/hilfsfile

a3 = ParametricPlot[{ Re[af30[I Pi - x,-t[-x]]], -mult af31[-t[-x]] },
      {x,.001,c}, PlotRange -> {All,{0,mult 5}},
      AxesLabel -> {"u[mm]",tex1}, PlotLabel -> "Upper Plate: Top" ] ;

a3inv = ParametricPlot[{ -Re[af30[I Pi - x, -t[-x]]], -mult af31[-t[-x]] },
         {x,.001,c}, PlotRange -> {{-e,0},mult 5{branch sgn,1}},
         DisplayFunction -> Identity ] ;

b3 = ParametricPlot[{ Re[af30[I Pi - x, -t[-x]]], sgn mult af31[-t[-x]] },
      {x,-b,-.1}, PlotRange -> {{e,0},{0,sgn mult 5}},
      AxesLabel -> {"u[mm]",tex1}, PlotLabel -> "Upper Plate: Bottom" ] ;

c3 = CartesianMap[af30[#,t[#]] &,{-c,b},{0,Pi},
      PlotRange -> All, AxesLabel -> {"u[mm]","iv[mm]"},
      Epilog -> { Thickness[.011],
                  { Line[{{d,0},{e,0}}], Line[{{e,a},{0,a}}] }} ] ;

d3 = ParametricPlot[{ af30[x,t[x]], -mult af31[t[x]] }, {x,-c,b},
       AxesLabel -> {"u[mm]","Lower Plate"},
       Epilog -> { Text[tex1,{-e/8,-mult/3},{0,0},{0,1}] } ] ;
```

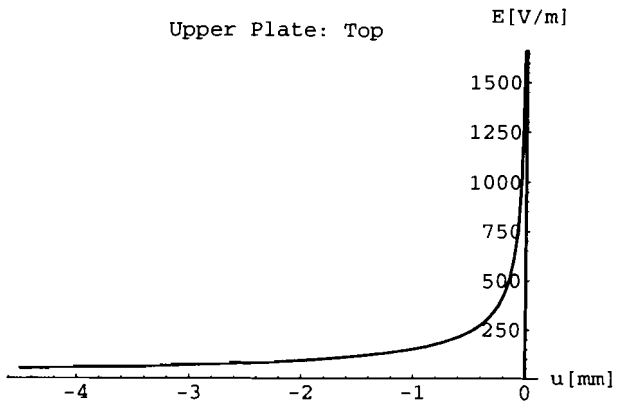
```
Show[GraphicsArray[{b3,a3inv}, GraphicsSpacing -> 0 ]/.
      (Ticks -> Automatic) -> (Ticks -> None) /.
      (AxesLabel -> {"u[mm]",tex1}) -> (AxesLabel -> {None,None}) /.
      (PlotLabel -> "Upper Plate: Bottom") -> (PlotLabel -> None) /.
      (PlotRange -> {{e,0},{0,sgn mult 5}}) ->
        (PlotRange -> {{e,0},mult 5{branch sgn,1}}),
      Epilog -> { Text["u ->                          <- u",
                       {.5,If[decide == ef,.19,,.04]}],
                  Text[tex3,{.47,.27}] }, PlotLabel -> "Upper Plate"] ;

Show[c3, Axes -> None] ;
Show[d3, Ticks -> None, AxesLabel -> {"u","Lower Plate"},
     Epilog -> { Text[tex2,{-e/8,-mult/3},{0,0},{0,1}] } ] ;
```

Comments:

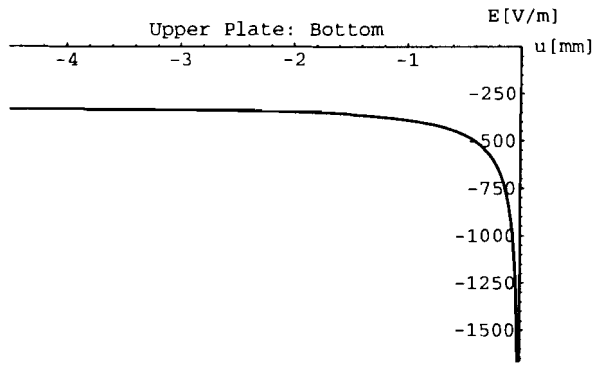
The first pictures (**a3**, **b3**, **c3**, **d3** except for **b3inv**) are drawn with ticks (**b3inv** is calculated but stays invisible). The following three **Show[ ]** commands produce pictures without ticks to see the qualitative behaviour of the curves whereby the first i.e. the **Show[GraphicsArray[ ]]** command puts the pictures of the upper plate **a3**, **b3inv** into one picture without ticks. The image of **b3inv** looks like the picture of **b3** with the u-axis pointing into the other direction. This change of direction is indicated by the reverse arrow in the **FrameLabel** option.

Variants:

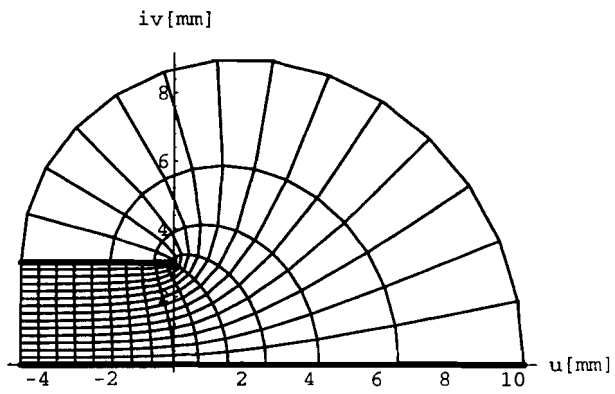
- it is possible to change **ParametricPlot[ ]** and **CartesianMap[ ]** as described in section 2.1.
- since you will normally deal with condensers that consist of two semi-infinite plates you can obtain that condenser by replacing the limits for the y co-ordinate: {**0,Pi**} in **CartesianMap[ ]** by {**-Pi,Pi**}. The surface charge density of the lower plane will than be of opposite sign. In order to avoid confusion by the surface charge density of the middle plate that is not any more existent, wrap (* *) around **d3 = ParametricPlot[...]** ; and **Show[d3, ...]** ;. To get the line for the lower condenser plate instead of that of the middle plate replace **Line[{{d,0},{e,0}}]** in the **Epilog** option of **c3** by **Line[{{e,-a},{0,-a}}]**.



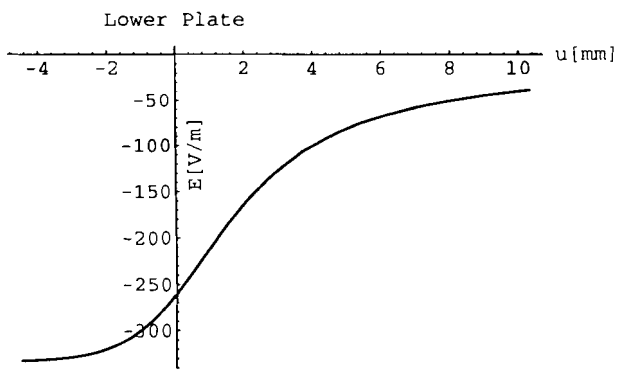
(a3)



(b3)

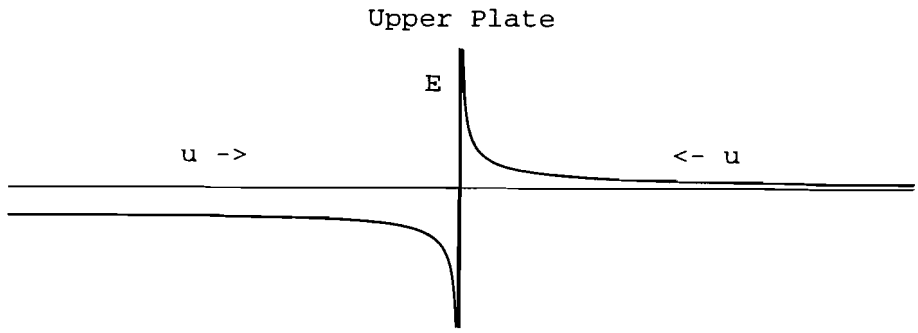


(c3)

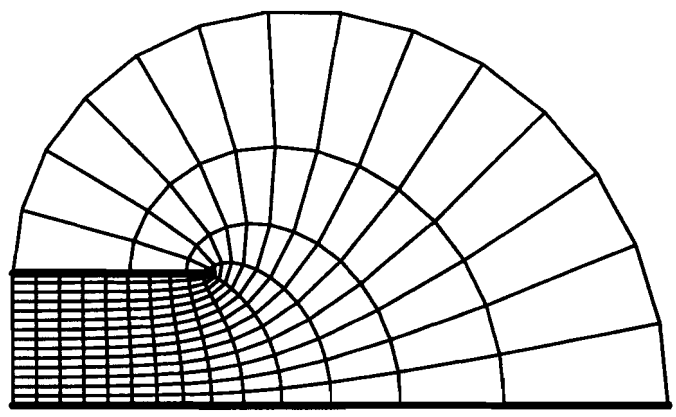


(d3)

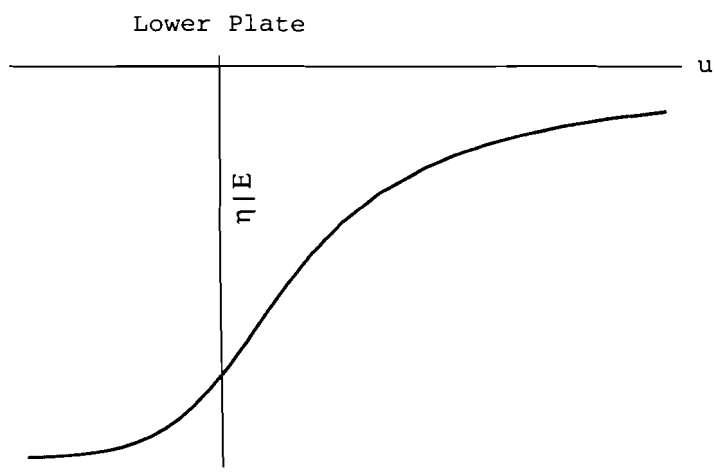
Figure 7: Graphics with ticks plotted by the *Mathematica* program for the Semi-Infinite Plane Condenser. The labels indicate the parts of the program which produced the pictures.



first **Show[ ]**



second **Show[ ]**



third **Show[ ]**

Figure 8: Graphics without ticks plotted by the three **Show[ ]** commands of the *Mathematica* program for the Semi-Infinite Plane Condenser.

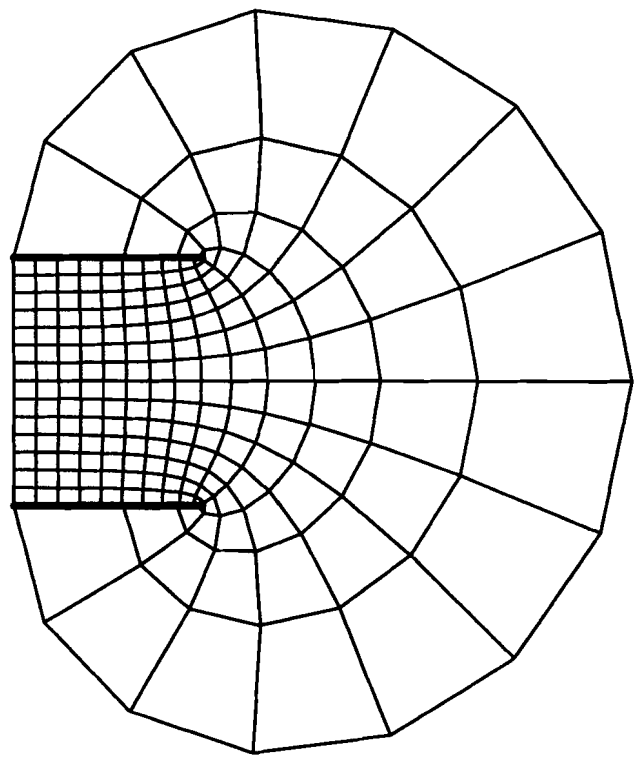


Figure 9: Symmetric Semi-Infinite Plane Condenser without ticks which is produced by the variant of the *Mathematica* program for the same condenser parameters as used in the previous pictures.

2.6 *Mathematica* Program for the Thick Semi-Infinite Plane Condenser

$$w = \frac{a}{\pi\sqrt{p}}\left\{\sqrt{t+1}\,\sqrt{t+p} + (p+1)\,\ln\left[\sqrt{t+1} + \sqrt{t+p}\right] - 2\sqrt{p}\,\ln\left[\sqrt{p}\,\sqrt{t+1} + \sqrt{t+p}\right] + \sqrt{p}\,\ln t\right\} - \frac{b}{\pi}\ln(p+1)$$

$$\text{with} \quad t = e^z \quad \text{and} \quad \frac{b}{a} = \frac{(\sqrt{p}-1)^2}{2\sqrt{p}}$$

This section contains a variant of the previous problem with a condenser plate of finite thickness b which is at a distance a from the u-axis. For thin plates the electric field looks nearly the same as in the previous problem. For thick plates and near the end of the upper plate, however, the field shows some new aspects. The map which is given in [fl, p. 76]

$$f(z) = \frac{a}{\pi\sqrt{p}}\left\{\sqrt{t+1}\,\sqrt{t+p} + (p+1)\,\ln\left[\sqrt{t+1} + \sqrt{t+p}\right] - 2\sqrt{p}\,\ln\left[\sqrt{p}\,\sqrt{t+1} + \sqrt{t+p}\right] + \sqrt{p}\,\ln t\right\} - \frac{b}{\pi}\ln(p+1)$$

with $t = e^z$ and $\frac{b}{a} = \frac{(\sqrt{p}-1)^2}{2\sqrt{p}}$ is holomorphic in $\mathbb{C}$ when $\sqrt{p} \neq 0$ (which is always satisfied) and real (a > b). The first derivative $f'(z) = \frac{a}{\pi\sqrt{p}}\sqrt{1+e^z}\,\sqrt{p+e^z}$ is zero at $z = i\pi$ and $z = i\pi + \ln p$. Therefore the domain $\mathbf{D}_z$ becomes $\mathbf{D}_z = \mathbf{D}_{id}\backslash\{i\pi, i\pi + \ln p\}$.

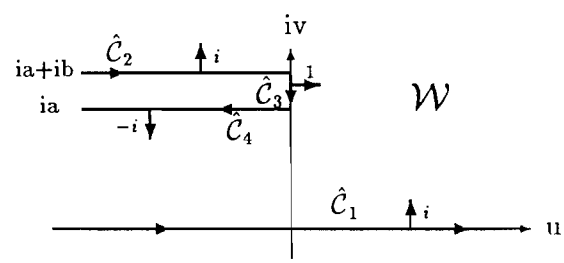


Figure 10: Curve $\hat{C}$ and normal vectors of the Thick Semi-Infinite Plane Condenser

As seen in the picture above the condenser's curve $\hat{C}$ on the W-plane consists of four parts: $\hat{C}_1$ is the image of C_1 , $\hat{C}_2$ of $C_2(x < -\ln p)$, $\hat{C}_3$ of $C_2(-\ln p < x < 0)$ and $\hat{C}_4$ of $C_2(x > 0)$.

$$f(C_1) = \frac{a}{\pi\sqrt{p}}\Big\{\sqrt{e^x+1}\,\sqrt{e^x+p} + (p+1)\,\ln\Big(\sqrt{e^x+1}+\sqrt{e^x+p}\Big) - \\ -2\sqrt{p}\,\ln\Big(\sqrt{p}\,\sqrt{e^x+1}+\sqrt{e^x+p}\Big) + \sqrt{p}\,x\Big\} - \frac{b}{\pi}\ln(p+1)$$

$$\Longrightarrow \quad \hat{C}_1(u) = f(C_1) = u \quad \text{for} \quad u \in \mathbb{R} \;\; ;$$

$$f(C_2(x<-\ln p)) = \frac{a}{\pi\sqrt{p}}\Big\{-\sqrt{e^{-x}-1}\sqrt{e^{-x}-p} + (p+1)\,\ln\Big(i(\sqrt{e^{-x}-1}+\sqrt{e^{-x}-p})\Big) - \\ -2\sqrt{p}\,\ln\Big(i(\sqrt{p}\,\sqrt{e^{-x}-1}+\sqrt{e^{-x}-p})\Big) + \sqrt{p}\,(i\pi - x)\Big\} - \frac{b}{\pi}\ln(p+1)$$

$$= \frac{a}{\pi\sqrt{p}}\Big\{-\sqrt{e^{-x}-1}\,\sqrt{e^{-x}-p} + (p+1)\,\ln\Big(\sqrt{e^{-x}-1}+\sqrt{e^{-x}-p}\Big) - \\ -2\sqrt{p}\,\ln\Big(\sqrt{p}\,\sqrt{e^{-x}-1}+\sqrt{e^{-x}-p}\Big) - \sqrt{p}\,x\Big\} - \frac{b}{\pi}\ln(p+1) + i(a+b)$$

$$\Longrightarrow \quad \hat{C}_2(u) = f(C_2(x<-\ln p)) = u + i(a+b) \quad \text{for} \quad -\infty < u < 0 \;\; ;$$

$$f(C_2(-\ln p < x < 0)) = \frac{a}{\pi\sqrt{p}}\Big\{i\sqrt{e^{-x}-1}\,\sqrt{p-e^{-x}} + (p+1)\,\ln\Big(i\sqrt{e^{-x}-1}+\sqrt{p-e^{-x}}\Big) - \\ -2\sqrt{p}\,\ln\Big(i\sqrt{p}\,\sqrt{e^{-x}-1}+\sqrt{p-e^{-x}}\Big) + \sqrt{p}\,(i\pi - x)\Big\} - \frac{b}{\pi}\ln(p+1)$$

$$= \frac{ia}{\pi\sqrt{p}}\Big\{\sqrt{e^{-x}-1}\,\sqrt{p-e^{-x}} + (p+1)\,\arctan\Big[\frac{\sqrt{e^{-x}-1}}{\sqrt{p-e^{-x}}}\Big] - \\ -2\sqrt{p}\,\arctan\Big[\frac{\sqrt{p}\,\sqrt{e^{-x}-1}}{\sqrt{p-e^{-x}}}\Big] + \sqrt{p}\,\pi\Big\}$$

$$\Longrightarrow \quad \hat{C}_3(v) = f(C_2(x<-\ln p)) = -iv \quad \text{for} \quad -(a+b) < v < -a$$

$$f(C_2(x>0)) = \frac{a}{\pi\sqrt{p}}\Big\{-\sqrt{1-e^{-x}}\,\sqrt{p-e^{-x}} + (p+1)\,\ln\Big(\sqrt{1-e^{-x}}+\sqrt{p-e^{-x}}\Big) - \\ -2\sqrt{p}\,\ln\Big(\sqrt{p}\,\sqrt{1-e^{-x}}+\sqrt{e^{-x}-p}\Big) - \sqrt{p}\,x\Big\} - \frac{b}{\pi}\ln(p+1) + ia$$

$$\Longrightarrow \quad \hat{C}_4(u) = f(C_2(x>0)) = -u \quad \text{for} \quad 0 < u < \infty \;\; .$$

Here I used a formula given in [bs, §3.4.4.2.2] to write down the logarithm for complex arguments as a sum of its real and imaginary part.
Since the images of the cuve $C(x)$ are a lot longer now I plotted the functions $f(C)$ with *Mathematica*'s **Plot[ ]** command to find out their monotonicity. When the function increases with x $\hat{C}$ follows the positive u or v direction. When it decreases it runs against those directions.

The unit normal vectors are calculated to $n_w(\hat{\mathcal{C}}_1) = n_w(\hat{\mathcal{C}}_2) = i, n_w(\hat{\mathcal{C}}_3) = 1$ and $n_w(\hat{\mathcal{C}}_4) = -i$. The surface charge density in terms of x is

$$\eta_{\mathcal{K}_1}(x) = -\frac{\varepsilon_0 V_0}{\pi}\ \Im m\left\{ i\frac{\pi\sqrt{p}}{a\sqrt{1+e^x}\,\sqrt{p+e^x}}\right\} \quad = -\frac{\varepsilon_0 V_0}{a}\ \frac{\sqrt{p}}{\sqrt{1+e^x}\,\sqrt{p+e^x}}$$

$$\text{for} \quad x \in \mathbb{R}: \quad \hat{C}_1(x) = f(C_1(x)) \ ;$$

$$\eta_{\mathcal{K}_2}(x) = -\frac{\varepsilon_0 V_0}{\pi}\ \Im m\left\{ i\frac{-\pi\sqrt{p}}{a\sqrt{e^{-x}-1}\,\sqrt{e^{-x}-p}}\right\} \ = \ \frac{\varepsilon_0 V_0}{a}\ \frac{\sqrt{p}}{\sqrt{e^{-x}-1}\,\sqrt{e^{-x}-p}}$$

$$\text{for} \quad x < -\ln p: \quad \hat{C}_2(x) = \Re e\{f(C_2(x < -\ln p))\} \ ;$$

$$\eta_{\mathcal{K}_3}(x) = -\frac{\varepsilon_0 V_0}{\pi}\ \Im m\left\{ 1\frac{\pi\sqrt{p}}{ia\sqrt{e^{-x}-1}\,\sqrt{p-e^{-x}}}\right\} = \ \frac{\varepsilon_0 V_0}{a}\ \frac{\sqrt{p}}{\sqrt{e^{-x}-1}\,\sqrt{p-e^{-x}}}$$

$$\text{for} \quad -\ln p < x < 0: \quad \hat{C}_3(x) = \Im m\{f(C_2(-\ln p < x < 0))\} \ ;$$

$$\eta_{\mathcal{K}_4}(x) = -\frac{\varepsilon_0 V_0}{\pi}\ \Im m\left\{ -i\frac{\pi\sqrt{p}}{a\sqrt{1-e^{-x}}\,\sqrt{p-e^{-x}}}\right\} = \ \frac{\varepsilon_0 V_0}{a}\ \frac{\sqrt{p}}{\sqrt{1-e^{-x}}\,\sqrt{p-e^{-x}}}$$

$$\text{for} \quad x > 0: \quad \hat{C}_4(x) = \Re e\{f(C_2(x > 0))\} \ .$$

The electric field and the surface charge density on the lower, on the top and the bottom of the upper plate behave as in the previous problem. At the front of the upper plate they tend to infinity both for $iv = ia$ and $iv = i(a+b)$ but they are not symmetric since the points with smaller values of iv are nearer to the oppositly charged lower plate than those points with larger values. In addition the charge density and the electric field tend to infinity as $\rho^{-1/3}$ (ρ is the distance from the corner); whereas theses quantities become infinite as $\rho^{-1/2}$ for the infinitely thin plate.

Mathematica program 4: for the Thick Semi-Infinite Plane Condenser

```
Needs["Graphics`ComplexMap`"]

Clear[V,a,b,c,d,e,f,g,p,t,sp,frc,frd,decide]
V = 1 (* Voltage between the plates in Volt *);
a = 3 (* Distance of the plates in mm *);
b = .5 (* Plate's thickness in mm *);
decide = scd (* Decide whether you want the electric field (ef) or
                surface charge density (scd) of the plates *) ;

Solve[(sp -1)^2/(2 sp) - N[b]/a == 0,sp] ;
sp = sp/.% //N ;
If[First[%] > Last[%],sp = First[%], sp = Last[%]]; p = sp^2 ;

af40[t_] = a/(Pi sp) (Sqrt[t + 1] Sqrt[t + p] +
                 (p + 1) Log[Sqrt[t + 1] + Sqrt[t + p]] -
                 2 sp Log[sp Sqrt[t + 1] + Sqrt[t + p]] +
                 sp Log[t]) - b/Pi Log[p - 1] ;
af41[t_] = sp/(Sqrt[t + 1] Sqrt[t + p]);
t[z_] = Exp[z] ;

e = af40[t[c]]; f = af40[t[d]]; g = Im[af41[-t[.5 Log[p]]]] ;
frc = FindRoot[e + 1.5(a + b), {c,-6,-5}]; c = Re[c]/.frc ;
frd = FindRoot[e + N[a + b] I - af40[-t[d]],{d,Log[p] + .5,5},
       MaxIterations -> 300]; d = Re[d]/.frd ;

<< ~path/hilfsfile

a4 = ParametricPlot[{ Re[af40[-t[-x]]], -mult Re[af41[-t[-x]]] },
     {x,-d,-Log[p] - .00001}, PlotRange -> {{e,0},{0,mult 3}},
     AxesLabel -> {"u[mm]",tex1}, PlotLabel -> "Upper Plate: Top" ] ;

a4inv = ParametricPlot[{ -Re[af40[-t[-x]]], -mult Re[af41[-t[-x]]] },
         {x,-d,-Log[p] -.00001}, PlotRange -> {{-e,0},mult 3{branch sgn,1}},
         DisplayFunction -> Identity ] ;

b4 = ParametricPlot[{ Im[af40[-t[-x]]], -mult Im[af41[-t[-x]]] },
      {x,-Log[p] + .00001,-.00001}, PlotRange -> {{a,a + b},{0,mult 7}},
      AxesLabel -> {"iv[mm]",tex1}, PlotLabel -> "Upper Plate: Front" ] ;
```

```
c4 = ParametricPlot[{ Re[af40[-t[-x]]], sgn mult Re[af41[-t[-x]]] },
      {x,.000001,-c}, PlotRange -> {{e,0},{0,sgn mult 3}},
      AxesLabel -> {"u[mm]",tex1}, PlotLabel -> "Upper Plate: Bottom" ] ;

d4 = CartesianMap[af40[t[#]] &,{c,d},{0,Pi},
      PlotRange -> All, AxesLabel -> {"u[mm]","iv[mm]"},
      Epilog -> { Thickness[.011],
                 { Line[{{1.02 f,0},{1.02 e,0}}],
                   Line[{{1.02 e,a},{0,a},{0,a + b},{1.02 e,a + b}}] }} ] ;

e4 = ParametricPlot[{ af40[t[x]], -mult af41[t[x]] }, {x,c,d},
      PlotRange -> {{e,f},{-mult,0}},
      AxesLabel -> {"u[mm]",""}, PlotLabel -> "Lower Plate",
      Epilog -> { Text[tex1,{f/20,-.4 mult},{0,0},{0,1}] } ] ;

Show[GraphicsArray[{c4,Show[b4, PlotRange -> {{a,a + b},mult 3{branch sgn,1}},
                        Ticks -> None, AxesLabel -> None, PlotLabel -> None,
                        DisplayFunction -> Identity],
                     a4inv}, GraphicsSpacing -> 0] /.
      (Ticks -> Automatic) -> (Ticks -> None) /.
      (AxesLabel -> {"u[mm]",tex1}) -> (AxesLabel -> {None,None}) /.
      (PlotLabel -> "Upper Plate: Bottom") -> (PlotLabel -> None) /.
      (PlotRange -> {{e,0},{0,sgn mult 3}}) ->
        (PlotRange -> {{e,0},mult 3{branch sgn,1}}),
      PlotRange -> {0,.2}, PlotLabel -> "Upper Plate",
      Epilog -> { Text[tex3,{.69,.18}],
                  Text["u ->            (iv) ->             <- u",
                        {.5,If[decide == ef,.13,,.04]}] } ] ;

Show[d4, Axes -> None] ;
Show[e4, Ticks -> None, AxesLabel -> {"u",""},
     Epilog -> {Text[tex2,{-f/20,-.5 mult},{0,0},{0,1}] } ] ;
```

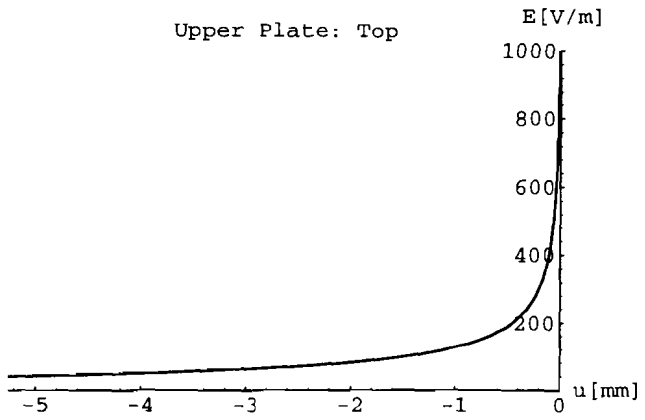
Comments:

In the first part of the program a new command **Solve[ ]** is used to solve an equation symbolically; here the equation **(sp − 1)^2/(2 sp) − N[b]/a == 0** given above is solved for the variable **sp**. A list of solutions for the equation (of two elements for this problem) is returned. Hence the **If[ ]** command chooses the solution with larger value to work with in the further program. As in the previous section the first pictures (**a4, b4, c4, d4, e4** except for **a4inv**) are plotted with ticks. The following three **Show[ ]** commands produce pictures without ticks to display the qualitative behaviour of the curves whereby the first, i.e. the **Show[GraphicsArray[ ]]**, command gathers the pictures of the upper plate **a4inv, b4, c4** in one picture without ticks. The pictures, however, can only be obtained for a certain range of values for the plate distance **a** and the plate thickness **b** :

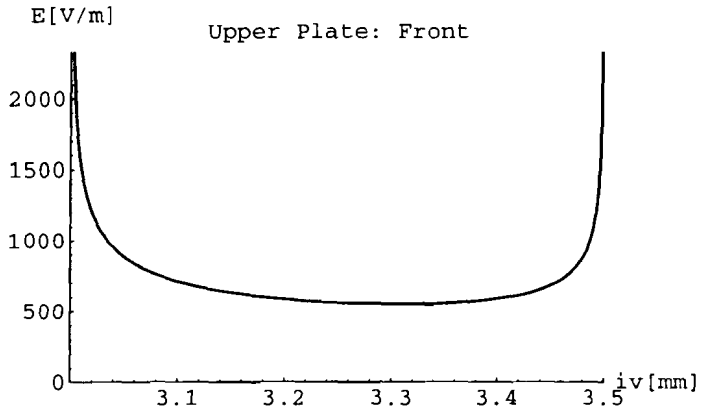
a	10^−4 to 10^−2	10^−2 to .5	.5 to 4	4 to 40	40 to 600	600 to 870	870 to 10^5
a/b	13 to 500	13 to 700	6 to 700	20 to 700	50 to 700	100 to 700	50 to 650

Variants:

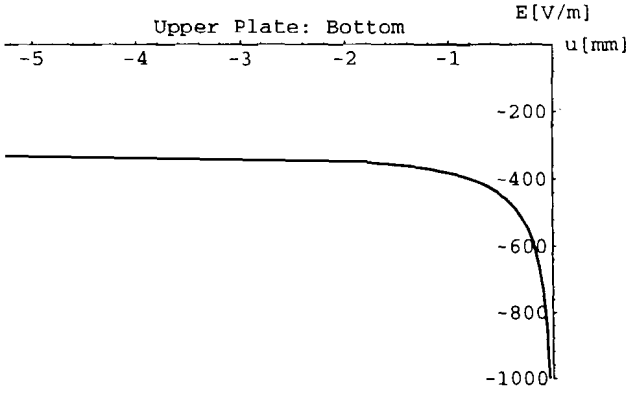
- it is possible to change **ParametricPlot[ ]** and **CartesianMap[ ]** as described in section 2.1.
- since you will normally deal with condensers that consist of two thick semi-infinite plates you can obtain that condenser by replacing the limits for the y co-ordinate : {**0,Pi**} in **CartesianMap[ ]** by {**−Pi,Pi**}. The surface charge density of the lower plane will than be of opposite sign. In order not to arise confusion by the surface charge density of the central plate, that is not any more existent, wrap (* *) around **e4 = ParametricPlot[...] ;** and **Show[e4, ...] ;**.
  To get the line for the lower condenser plate instead of that of the central plate replace **Line[{{1.02 f,0},{1.02 e,0}}]** in the **Epilog** option of **d4** by **Line[{{1.02 e,−a},{0,−a},{0,−a − b}, {1.02 e,−a − b}}]**.



(a4)

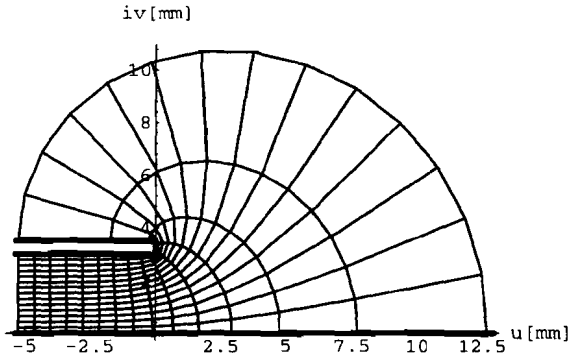


(b4)

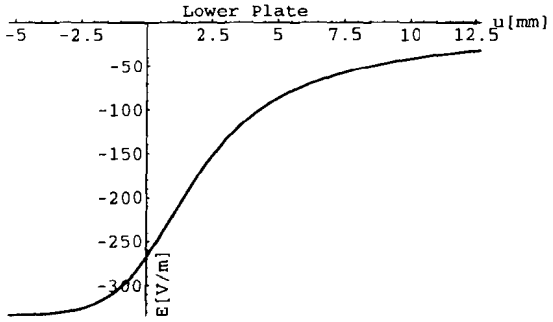


(c4)

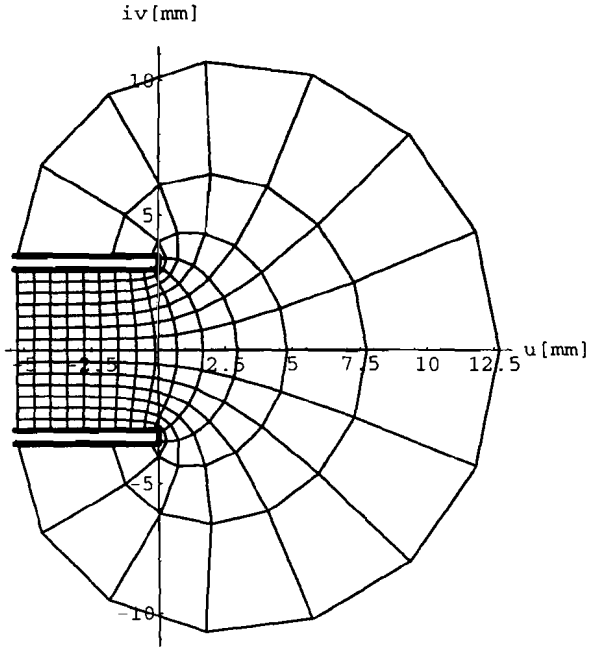
Figure 11: Graphics with ticks produced by the *Mathematica* program for the Thick Semi-Infinite Plane Condenser. The labels indicate the parts of the program that plotted the pictures.



(d4)

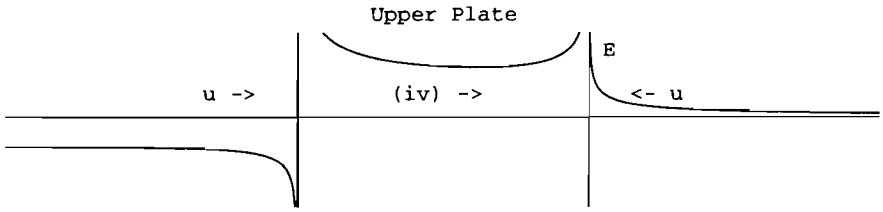


(e4)

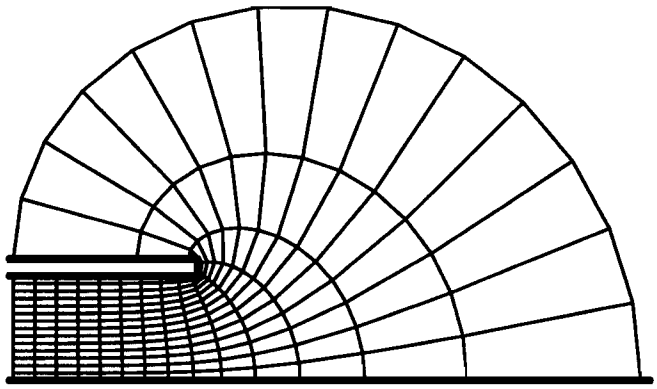


variant

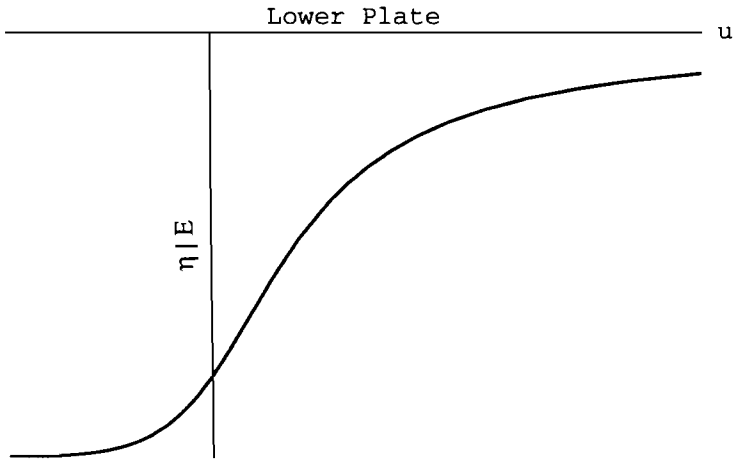
Figure 12: Graphics with ticks produced by the *Mathematica* program for the Thick Semi-Infinite Plane Condenser. The third picture was plotted using the variant for the same condenser parameters as before. The labels indicate the parts of the program that plotted the pictures.



first **Show[ ]**



second **Show[ ]**



third **Show[ ]**

Figure 13: Graphics without ticks produced by the *Mathematica* program for the Thick Semi-Infinite Plane Condenser.

2.7 *Mathematica* Program for the One Corner of 90° Condenser

$$w = \frac{a}{\pi}\left\{i\pi + 2i\sqrt{e^z - 1} - \ln\left[\frac{1 + i\sqrt{e^z - 1}}{1 - i\sqrt{e^z - 1}}\right]\right\}$$

This section is about a condenser whose lower plate is the u-axis. The upper plate is parallel to the positive part of the u-axis located at a distance ia with a 90° corner at $w = ia$; there it turns to $i\infty$ along the iv-axis. It may be considered as the limiting case of the Thick Aperture Condenser of section 2.11 for large values of the thickness $2b$. As in sections 2.5 and 2.6 one gets a condenser with two unconnected edges by application of Schwarz's reflection principle (cf. the variants of the *Mathematica* program).
The problem is described in [vb, p. 142f] with the difference of an additional term ia in the conformal function to get the two-corner version more easily.
The mapping function for this problem is holomorphic for any $z \in \mathbb{C}$; the first derivative $f'(z) = \frac{ia}{\pi}(e^z - 1)^{1/2}$ is zero for $z = 0$. With that the domain $\mathbf{D}_z$becomes $\mathbf{D}_z = \mathbf{D}_{id}\backslash\{0\}$.

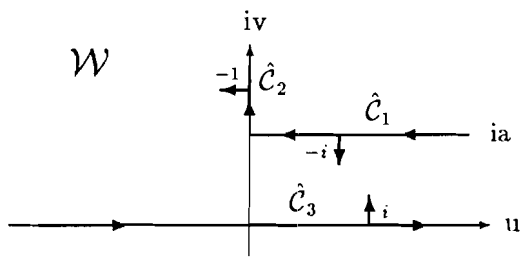


Figure 14: Curve $\hat{C}$ and normal vectors of the One Corner of 90° Condenser

The image of the curve C in the W-plane is shown in the picture above. Again it consists of three parts $\hat{C}_1$, $\hat{C}_2$ and $\hat{C}_3$. $\hat{C}_1$ is the image of C_1 for $x < 0$, $\hat{C}_2$ the image of C_1 for $x > 0$ and $\hat{C}_3$ the image of C_2 on the W-plane since :

$$f(C_1(x|x<0)) = \frac{a}{\pi}\left\{i\pi - 2\sqrt{1-e^x} - \ln\left[\frac{1-\sqrt{1-e^x}}{1+\sqrt{1-e^x}}\right]\right\}$$

$$\Longrightarrow \hat{C}_1(u) = f(C_1(x<0)) = -u + ia \ : u < 0 \ ;$$

$$f(C_1(x|x>0)) = \frac{ia}{\pi}\left\{\pi + 2\sqrt{e^x - 1} - \arctan\left(\frac{2\sqrt{e^x-1}}{2-e^x}\right)\right\}$$

$$\Longrightarrow \hat{C}_2(v) = f(C_1(x>0)) = iv : v > a \ ;$$

$$f(C_2) = \frac{a}{\pi}\left\{ i\pi - 2\sqrt{e^{-x}+1} + \ln\left[\frac{\sqrt{e^{-x}+1}+1}{\sqrt{e^{-x}+1}-1}\right]\right\}$$
$$\Longrightarrow \hat{C}_3(u) = f(C_2) = u : u \in \mathbb{R}$$

The normal unit vectors are calculated to $n_w(\hat{C}_1) = -i$, $n_w(\hat{C}_2) = -1$ and $n_w(\hat{C}_3) = i$. The surface charge density in terms of x is

$$\eta_{\hat{C}_1}(x) = -\frac{\varepsilon_0 V_0}{\pi}\,\Im m\left\{-i\frac{\pi/a}{\sqrt{1-e^x}}\right\} = +\frac{\varepsilon_0 V_0}{a}\,\frac{1}{\sqrt{1-e^x}}$$
$$\text{for } x<0: \quad \hat{C}_1(x) = f(C_1(x<0)) - ia \;;$$

$$\eta_{\hat{C}_2}(x) = -\frac{\varepsilon_0 V_0}{\pi}\,\Im m\left\{-1\frac{i\,\pi/a}{\sqrt{e^x-1}}\right\} = +\frac{\varepsilon_0 V_0}{a}\,\frac{1}{\sqrt{e^x-1}}$$
$$\text{for } x>0: \quad \hat{C}_2(x) = f(C_1(x>0)) \;;$$

$$\eta_{\hat{C}_3}(x) = -\frac{\varepsilon_0 V_0}{\pi}\,\Im m\left\{i\frac{\pi/a}{\sqrt{1+e^{-x}}}\right\} = -\frac{\varepsilon_0 V_0}{a}\,\frac{1}{\sqrt{1+e^{-x}}}$$
$$\text{for } x\in\mathbb{R}: \quad \hat{C}_3(x) = f(C_2) \;.$$

The surface charge density and the electric field on the upper plate get infinite at $w = ia$. $\eta_{\hat{C}_1}$ tends to $\frac{\varepsilon_0 V_0}{a}$ (the field to $-\frac{V_0}{a}$) for $u \to \infty$ and $\eta_{\hat{C}_2}$ and the field tend to zero for $v \to \infty$. The surface charge density and the field on the lower plate both tend to zero for $u \to -\infty$ and to $-\frac{\varepsilon_0 V_0}{a}$, $-\frac{V_0}{a}$ respectively for $u \to +\infty$.
The singularity on the upper plate is of the order $u^{-1/3}$ and $v^{-1/3}$ respectively. To prove this the function $f(C_1(x))$ has to be inverted for the limit $x \to 0$. Inserting this into $\eta_{\hat{C}_1}$ and $\eta_{\hat{C}_2}$ proves the edge condition to be valid in this case.
Proof for the horizontal part of the upper plate:

$$\hat{C}_1(u) = -u + ia = f(C_1(x<0)) = \frac{a}{\pi}\left\{ i\pi - 2\sqrt{1-e^x} - \ln\left[\frac{1-\sqrt{1-e^x}}{1+\sqrt{1-e^x}}\right]\right\} \bigwedge_{u<0} \;.$$

The second approximation of $\hat{C}_1(u)$ for $x \to 0^-$ leads to (the first one gives a constant for the variable u and is therefore useless) :

$$\hat{C}_1(u \to 0^+) \approx \frac{a}{\pi}\left\{ i\pi - 2\sqrt{-x} + 2\sqrt{-x} + \frac{2}{3}(-x)^{3/2}\right\} = ia + \frac{2}{3}(-x)^{3/2} \;;$$
$$\Longrightarrow \hat{C}_1 - ia = -u = \tfrac{2}{3}(-x)^{3/2} \quad \Longrightarrow \sqrt{-x} \sim (-u)^{1/3} \;.$$

Here I used the series expansion of the exponential function

$$e^x = 1 + x + \ldots \Longrightarrow \sqrt{1-e^x} \approx \sqrt{-x} > 0$$

and that of the natural logarithm ([bs, p.33])

$$\ln\left(\frac{1-t}{1+t}\right) = -\ln\left(\frac{1+t}{1-t}\right) = -2\left[t + \frac{t^3}{3} + \ldots\right] \quad .$$

Inserting the approximation of u into the surface charge density gives

$$\eta_{\hat{C}_1}(x) \sim \frac{\varepsilon_0 V_0}{a} \frac{1}{\sqrt{1-e^x}} \approx \frac{1}{\sqrt{-x}} \sim (-u)^{-1/3} \quad .$$

Proof for the vertical part of the upper plate:

$$\hat{C}_2(v) = iv = f(C_1(x>0)) = \frac{ia}{\pi}\left\{\pi + 2\sqrt{e^x-1} - \arctan\left(\frac{2\sqrt{e^x-1}}{2-e^x}\right)\right\} \bigwedge_{v>a} \quad .$$

The first approximation of $\hat{C}_2(v)$ for $x \to 0^+$ leads to:

$$\hat{C}_2(v \to 0^+) \approx \frac{ia}{\pi}\left\{\pi + 2\sqrt{x} - \frac{2\sqrt{x}}{1-x}\right\} = \frac{ia}{\pi}\left\{\pi + 2x^{3/2}\right\} \quad ;$$

$$\Longrightarrow \Im m\{C_1 - ia\} \sim x^{3/2} \sim v \quad \Longrightarrow \sqrt{x} \sim (v)^{1/3} \quad .$$

Here I used the series expansion of the exponential function and of the arcus tangent ([bs, p.34])

$$\arctan t = t - \frac{t^3}{3} + \ldots$$

Inserting the approximation of v into the surface charge density gives

$$\eta_{\hat{C}_2}(x) \sim \frac{\varepsilon_0 V_0}{a} \frac{1}{\sqrt{e^x-1}} \approx \frac{1}{\sqrt{x}} \sim v^{-1/3} \quad .$$

Nevertheless for realistic condensers field and charge density have to stay finite. For this purpose the corner has to be rounded which is shown in the next section.

Mathematica program 5: for the One Corner of 90° Condenser

```
Needs["Graphics`ComplexMap`"]

Clear[V,a,b,c,d,e,f,g,decide]
V = 1 (* Voltage beween the plates in Volt *);
a = 3 (* Distance of the plates in mm :
         for 1.5 10^-5 < a < 2 10^2 (scd) and 1.5 10^-5 < a < 2 10^5 *) ;
decide =  ef (* Decide whether you want the electric field (ef) or
                surface charge density (scd) of the plates *) ;

af50[z_] = a/Pi (I (Pi + 2 Sqrt[Exp[z] - 1]) -
                 Log[(1 + I Sqrt[Exp[z] - 1])/
                 (1 - I Sqrt[Exp[z] - 1])]) ;

b = 3.37465 ; c = Re[af50[I Pi + b]] ; d = Re[af50[I Pi + e]] ;
e = -8.46758 ; f = Re[af50[e]] ; g = Im[af50[b]] ;

<< ~path/hilfsfile

a5 = ParametricPlot[{ af50[x]/I , sgn mult/Sqrt[Exp[x] - 1] },
      {x,.1,b}, PlotRange -> {All,{0,3 sgn mult}},
      AxesOrigin -> {a,0}, AxesLabel -> {"iv[mm]",tex1},
      PlotLabel -> "Upper Plate: Vertical" ] ;

a5inv = ParametricPlot[{ -af50[x]/I, sgn mult/Sqrt[Exp[x] - 1] },
         {x,.1,b}, PlotRange -> {{-a,-g},{0,3 sgn mult}},
         AxesOrigin -> {-a,0}, DisplayFunction -> Identity ] ;

b5 = ParametricPlot[{ af50[x] - a I, sgn mult/Sqrt[1-Exp[x]] }, {x,e,-.001},
      PlotRange -> {{0,f},{0,3 sgn mult}}, AxesLabel -> {"u[mm]",tex1},
      PlotLabel -> "Upper Plate: Horizontal" ] ;

c5 = CartesianMap[af50, {e,b},{0,Pi-.001}, PlotRange -> All,
      AxesLabel -> {"u[mm]","iv[mm]"},
      Epilog -> { Thickness[.011],
                 { Line[{{0,g},{0,a},{f,a}}], Line[{{c,0},{d,0}}] }} ] ;
```

```
d5 = ParametricPlot[{ af50[-x + I Pi], -mult/Sqrt[Exp[-x] + 1] },
      {x,-b,-e}, PlotRange -> {{c,d},{-V 10^-13,-mult}},
      AxesLabel -> {"u[mm]",""}, PlotLabel -> "Lower Plate",
      Epilog -> { Text[tex1,{f/12,-.6 mult},{0,0},{0,1}] } ] ;


Show[GraphicsArray[{a5inv,b5}, GraphicsSpacing -> 0]/.
      (Ticks -> Automatic) -> (Ticks -> None)/.
      (AxesLabel -> {"u[mm]",tex1}) -> (AxesLabel -> {None,None})/.
      (PlotLabel -> "Upper Plate: Horizontal") -> (PlotLabel -> None),
      PlotRange -> {0,.3}, PlotLabel -> "Upper Plate",
      Epilog -> { Text[tex3,{.47,.05}] },
      Frame -> fr, FrameTicks -> None,
      frtex = "<- (iv)                        u ->"; FrameLabel -> frl] ;

Show[c5, Axes->None] ;
Show[d5, Ticks -> None, AxesOrigin -> {0,0}, AxesLabel -> {"u",""},
     Epilog -> { Text[tex2,{f/10,-.6 mult},{0,0},{0,1}] } ] ;
```

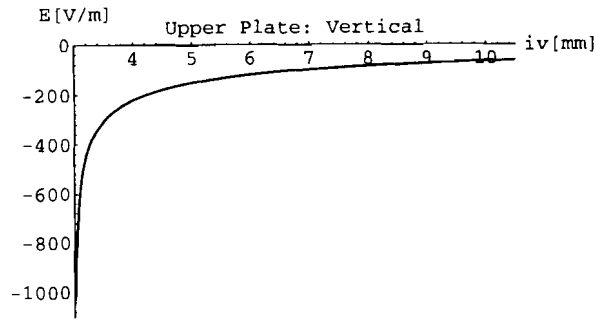
Comments:

As usual the first pictures (**a5, b5, c5, d5** except for **a5inv**) are plotted with ticks. The following three **Show[ ]** commands produce pictures without ticks to see the qualitative behaviour of the curves whereby the first, i.e. the **Show[GraphicsArray[ ]]**, command gathers the pictures of the upper plate **a5inv,b5** in one picture without ticks.

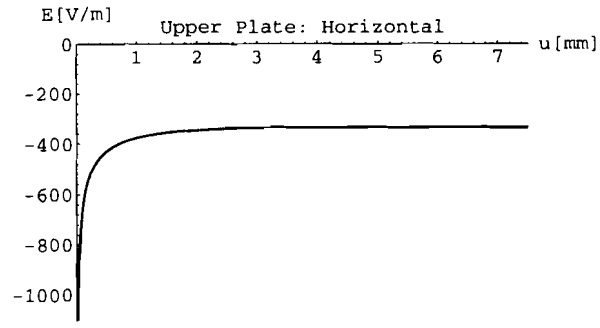
Variants:

- it is possible to change **ParametricPlot[ ]** and **CartesianMap[ ]** as described in section 2.1.

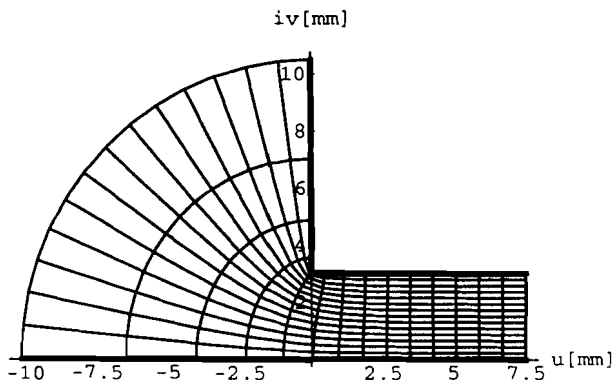
- since you will normally deal with condensers that consist of two corners you can obtain that one by application of the Schwarz reflection principle. You have to delete the ", **Line[{{c,0},{d,0}}]** " suboption (don't forget the comma !) in the **Epilog** option of **c5** and to add another picture:
  c5low = CartesianMap[Conjugate[af50[#]] &, {e,b},{0,Pi - .001},
  PlotRange -> All, Prolog -> {{ Thickness[.011],
  Line[{{0,-g},{0,-a},{f,-a}}] }}] ;
  The **Prolog** option does the same as **Epiolg**, however, not after but before the picture is plotted. That is why a second pair of curly brackets is wrapped around the **Thickness** and **Line** suboptions to prevent that all lines are drawn so thick.
  To display both the upper (**c5**) and lower (**c5low**) part of the Two Corners of 90° Condenser in one picture you add:
  Show[c5, c5low] ; .
  If I used the **Epilog** option in **c5low** instead of **Prolog** I would get a Two Corners of 90° Condenser with just one thick line since *Mathematica* executes the **Epilog** option of the first graphics in the list of the **Show[ ]** command only.
  This picture is drawn with ticks. To get the "pure" version add:
  ", **Axes -> None**" as an option to the **Show[ ]** command. The surface charge density of the lower plane will than be of opposite sign. In order to avoid confusion by the surface charge density of the middle plate that is not any more existent wrap (* *) around **d5 = ParametricPlot[...]** ; and **Show[d5, ...]** ; .



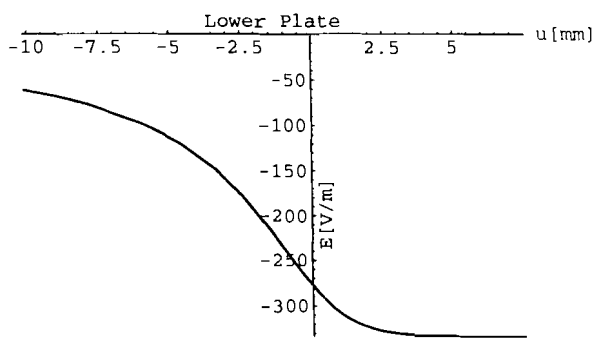
(a5)



(b5)

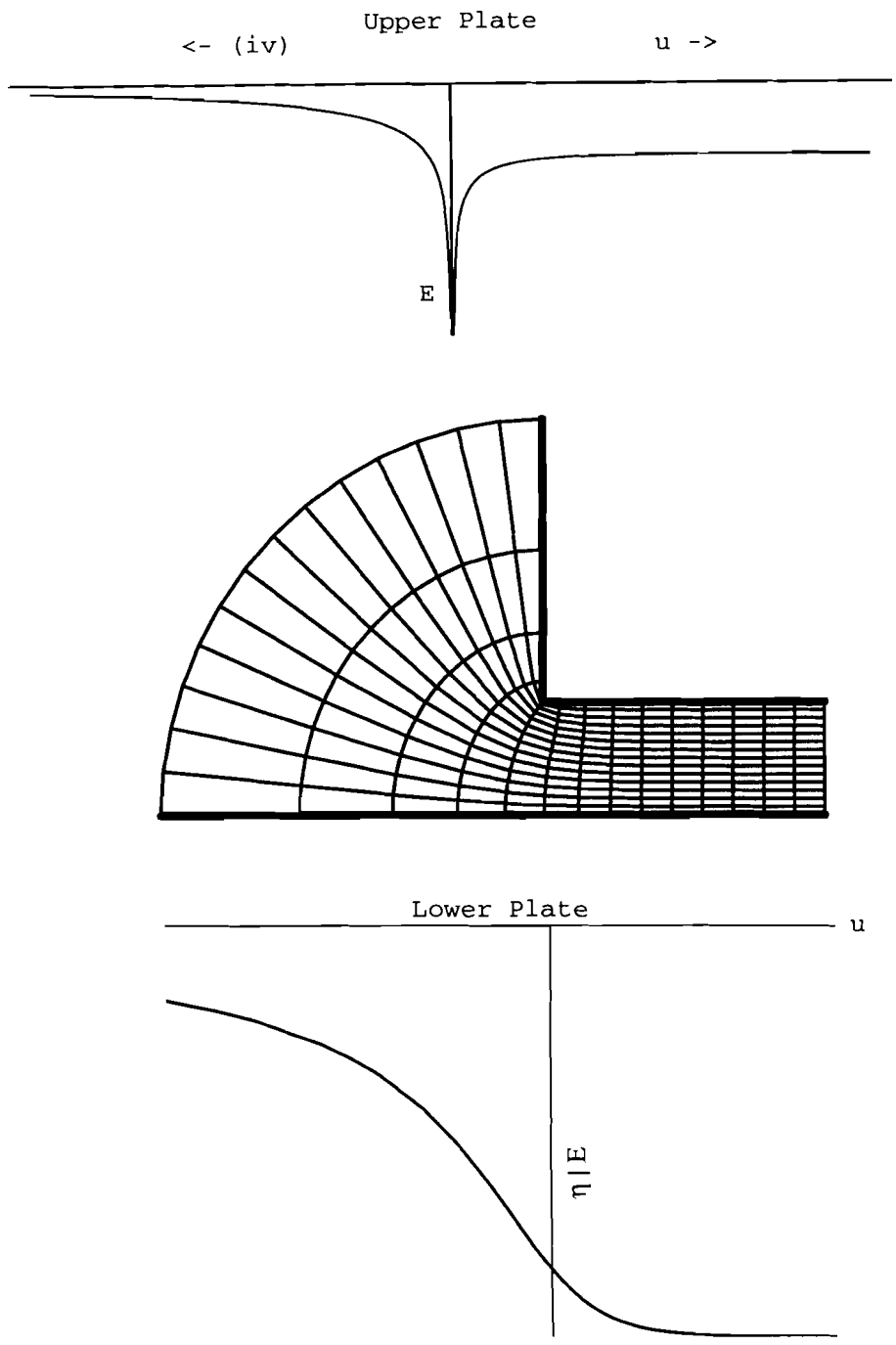


(c5)



(d5)

Figure 15: Graphics with ticks produced by the *Mathematica* program for the **One Corner of 90° Condenser**. The labels indicate that part of the program which plotted the pictures. The voltage betwen the plates is 1 V; the distance of the parallel branches is 3 mm.

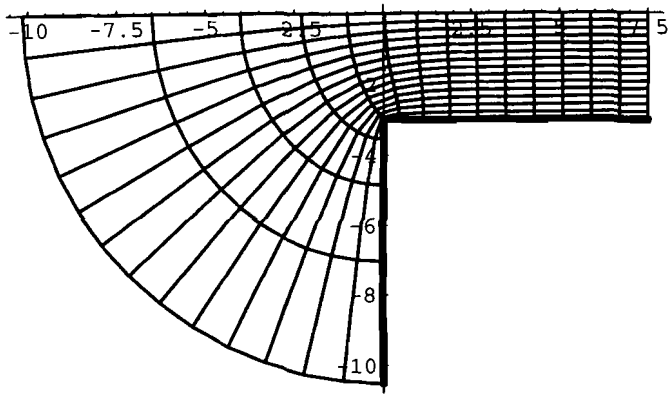


First **Show[ ]**

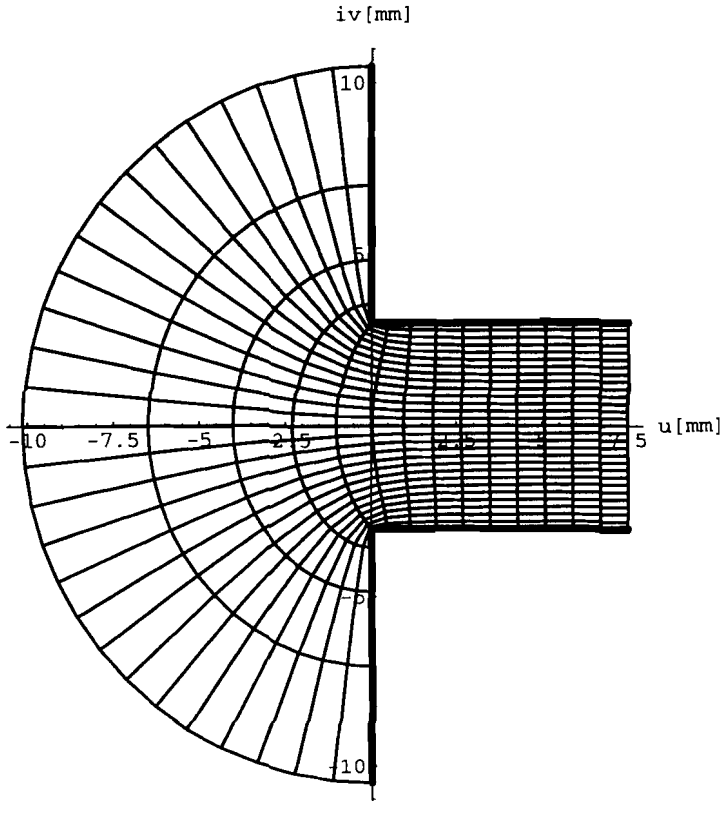
Second **Show[ ]**

Third **Show[ ]**

Figure 16: Graphics without ticks produced by the *Mathematica* program for the **One Corner of 90° Condenser** for the same condenser parameters as for the pictures on the previous page.



(c5low)



Show[c5,c5low]

Figure 17: Graphics produced by the variants of the *Mathematica* program for the One Corner of 90° Condenser for the same condenser parameters as for the pictures on the previous pages.

It is not only possible to have the electric field strength of the plates plotted, but I may also get a picture of the field for any equipotential line between the plates. Eq.(26) is still valid when I interpret the curve $\mathcal{C}$ as the equipotential curve which is a straight line on the Z-plane of the Infinite Plane Condenser.
All I have to do now is to insert the conformal function $f(z)\big/_{z\in\mathcal{C}}$ and its derivative $f'(z)\big/_{z\in\mathcal{C}}$ restricted to the curve $\mathcal{C}$ into the **ParametricPlot[ ]** command in the following way :

ParametricPlot[Re[$f(z)\big/_{z\in\mathcal{C}}$], −1000 V Im[$1/f'(z)\big/_{z\in\mathcal{C}}$], *options*] : or
ParametricPlot[Im[$f(z)\big/_{z\in\mathcal{C}}$], −1000 V Re[$1/f'(z)\big/_{z\in\mathcal{C}}$], *options*] ;

with $\mathcal{C}$ the equipotential line on the Z-plane of which I want to know the field on the W-plane.
Since I usually do not know the location of the equipotential line on the Z-plane I have to calculate one of its co-ordinates by the **FindRoot[ ]** command:
Let the equipotential curve on the W-plane include the point $w_? = u_? + iv_?$. To get the inverse image $z_? = x_? + iy_?$ with $w_? = f(z_?)$ I insert this equation into **FindRoot[ ]** in the form :

FindRoot[$w_? - f(z_?), \{z_?, z_0, z_1\}$]

With that the curve $\mathcal{C}$ is represented by $C(x) = x + i\,\Im m\{z_?\}$ which can be inserted into the **ParametricPlot[ ]** commands given above.

I can of course plot several graphs for different equipotential lines in one picture as is shown in figs 18 and 23 . In these four pictures I chose the 15 equipotentials of the electric field distributions of **c5** and **c6** respectively. Those equipotential curves are the two plates and 13 equipotentials in between.

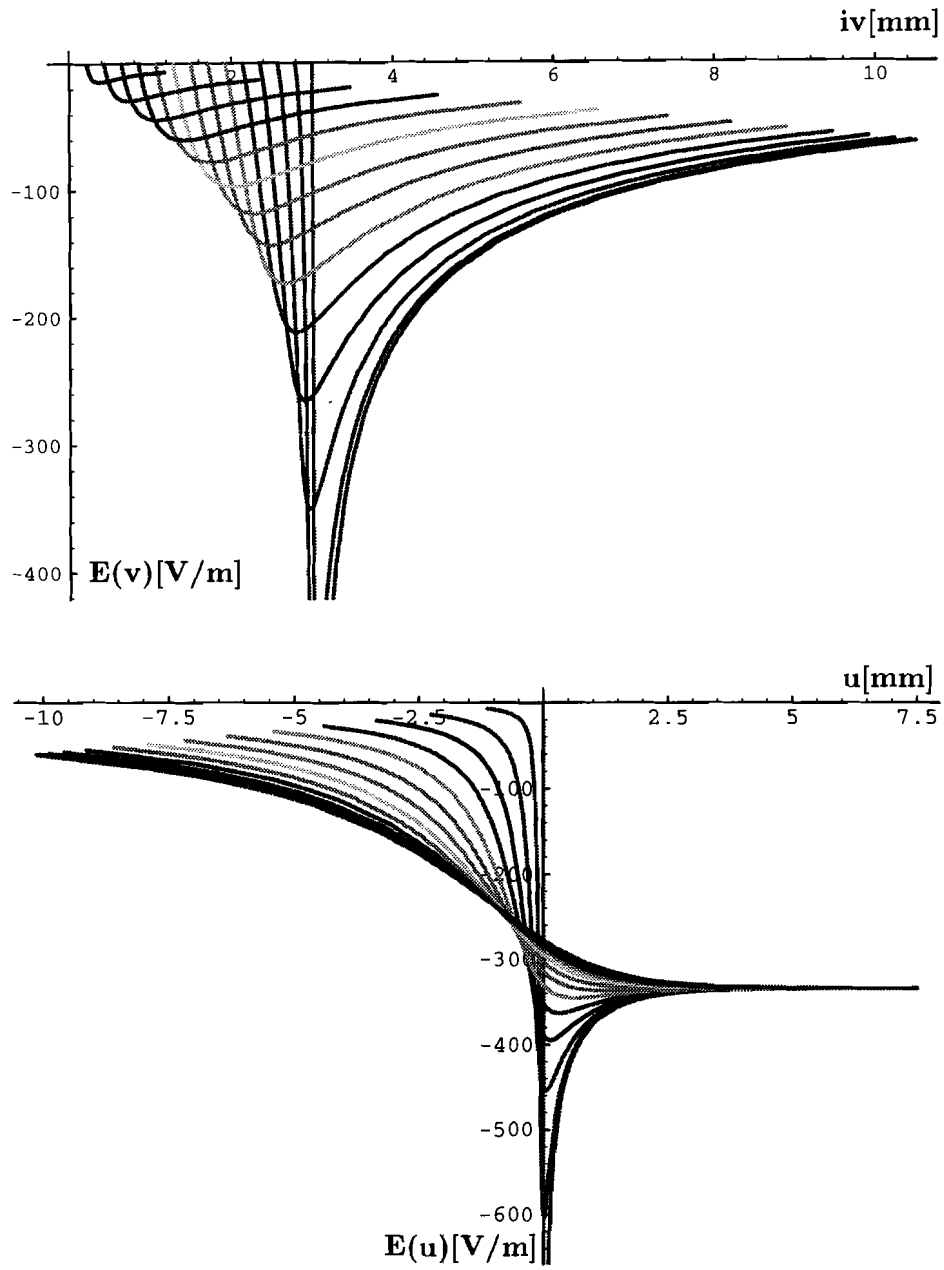


Figure 18: These graphics show the values of the electric field strength at 15 equipotential curves of the One Corner of 90° Condenser together in one picture.
These equipotentials are exactly those drawn in the pictures produced by **c5** or the second **Show[ ]** command for the same condenser parameters. The upper picture for the real part (E(v) = E_u) of the field shows how the field increases monotonically from zero at the lower plate (**d5**) to the value on the vertical part of the upper plate (**a5**) when I hop from the lower to the upper plate measuring the field on the plates and on 13 equipotentials in between. The lower picture for the imaginary part (E(u) = E_v) of the field shows how the field decreases on the left side whereas it increases in the center (it has to tend to the same constant at the right side) when moving from the lower plate to the horizontal part of the upper plate (**b5**). (cf. the pictures on page 71)

2.8 *Mathematica* Program for the Rounded One Corner of 90° Condenser

$$w = 2\,i\,A\left\{\sqrt{t-p} - \sqrt{p}\,\arctan\sqrt{\frac{t-p}{p}} + \lambda\left(\sqrt{t-q} - \sqrt{q}\,\arctan\sqrt{\frac{t-q}{q}}\right)\right\} + ia$$

$$\text{with} \quad -b = 2\,\lambda\,A\left\{1 - \sqrt{q}\,\mathrm{Artanh}\sqrt{\tfrac{1}{q}}\right\} \quad , \quad A = \frac{a}{\pi}\frac{1}{\sqrt{p}+\lambda\sqrt{q}}$$

$$\lambda = \left\{\frac{2a}{\pi b}\left[1 - \sqrt{p}\,\arctan\sqrt{\tfrac{1}{p}}\right] - \sqrt{p}\right\}\Big/\sqrt{q} \quad , \quad q = \varepsilon + p \quad \text{and} \quad t = e^z$$

This section deals with a variant of the previous problem. In contrast to the last condenser this one has a rounded instead of a sharp edge which causes the electric field and the surface charge density to stay finite all over the plates. The function is calculated in [we, p.373-376]. I multiplied it by the imaginary unit i to rotate the condenser in order to compare it with the previous one.

Although the conformal map is the same in [we] and this thesis (for $\varepsilon = 1$) the conclusions, however, are a bit different. For a radius $b = a/8$ of the smoothed edge, [we, p.375] gets a corresponding $\lambda = .787$. For the same b I compute λ to $\lambda = .800845$. To get a value of $\lambda = .787$ my b has to equal $b = a/7.1346$.
On page 376 [we] compares the contour of the rounded corner as the image of the conformal function with a quater circle. The contour looks a bit wavy whereas my contour looks very smooth (cf. the pictures on p. 72).
Since [we] was published in 1950 the numerical algorithm used then to solve the equation for λ may have been a bit inaccurate which may cause the differences I mentioned. Moreover I found that ε should be chosen as $\varepsilon = 0.96$.

The map

$$f(z) = 2\,i\,A\left\{\sqrt{t-p} - \sqrt{p}\,\arctan\sqrt{\frac{t-p}{p}} + \lambda\left(\sqrt{t-q} - \sqrt{q}\,\arctan\sqrt{\frac{t-q}{q}}\right)\right\} + ia$$

$$\text{with } A = \tfrac{a}{\pi}\tfrac{1}{\sqrt{p}+\lambda\sqrt{q}} \text{ and } t = e^z$$

is holomorphic on the whole Z-plane. The first derivative $f'(z) = i\,A[\sqrt{t-p} + \lambda\sqrt{t-q}]$ has no zeros. So the domain $\mathbf{D}_z$ becomes $\mathbf{D}_z = \mathbf{D}_{id}$

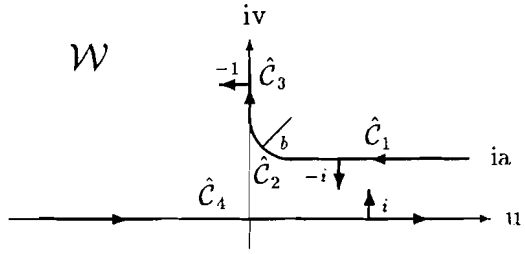


Figure 19: Curve $\hat{C}$ and some normal vectors of the **Rounded One Corner of 90° Condenser**

The picture looks quite similar to the one of section 2.7. Nevertheless there is one important difference: The normal unit vectors for the straight parts of the upper plane are the same as before. Yet those of the curved part are functions of the variables u and v which complicates the calculation of the surface charge density a lot.

The curve C in the W-plane consists of four parts. $\hat{C}_1$ is the image of C_1 for $x < \ln p$, $\hat{C}_2$ the image of C_1 for $\ln p < x < \ln q$, $\hat{C}_3$ the image of C_1 for $x > \ln q$ and $\hat{C}_3$ the image of C_2 on the W-plane:

$$f(C_1(x|x < \ln p))$$
$$= -2\,A\Big\{\sqrt{p - e^x} - \sqrt{p}\,\mathrm{Artanh}\sqrt{\tfrac{p-e^x}{p}} + \lambda\Big(\sqrt{q - e^x} - \sqrt{q}\,\mathrm{Artanh}\sqrt{\tfrac{q-e^x}{q}}\Big)\Big\} + ia$$
$$\Longrightarrow \hat{C}_1(u) = f(C_1(x < \ln p)) = u + ia : b < u < \infty \quad ;$$

$$f(C_1(x|\ln p < x < \ln q))$$
$$= 2\,i\,A\Big\{\sqrt{e^x - p} - \sqrt{p}\,\arctan\sqrt{\tfrac{e^x-p}{p}} + \lambda\Big(i\sqrt{q - e^x} - i\sqrt{q}\,\mathrm{Artanh}\sqrt{\tfrac{q-e^x}{q}}\Big)\Big\} + ia$$
$$\Longrightarrow \hat{C}_2(u,v) = f(C_1(\ln p < x < \ln q)) = -u + iv : -b < u < 0 \;\text{ and }\; a < v < (a+b)$$
$$\text{with approximatly } \; u^2 + v^2 = b^2 \quad ;$$

$$f(C_1(x|x > \ln q))$$
$$= 2\,i\,A\Big\{\sqrt{e^x - p} - \sqrt{p}\,\arctan\sqrt{\tfrac{e^x-p}{p}} + \lambda\Big(\sqrt{e^x - q} - \sqrt{q}\,\arctan\sqrt{\tfrac{e^x-q}{q}}\Big)\Big\} + ia$$
$$\Longrightarrow \hat{C}_3(u) = f(C_1(x > \ln q)) = iv : a < v < \infty \quad ;$$

$$f(C_2)$$
$$= -2\,A\Big\{\sqrt{e^{-x} + p} - \sqrt{p}\,\mathrm{Artanh}\sqrt{\tfrac{e^{-x}+p}{p}} + \lambda\Big(\sqrt{e^{-x} + q} - \sqrt{p+1}\,\mathrm{Artanh}\sqrt{\tfrac{e^{-x}+q}{q}}\Big)\Big\} + ia$$
$$\Longrightarrow \hat{C}_4(u) = f(C_2) = u : u \in \mathbb{R} \quad .$$

The normal unit vectors are calculated to $n_w(\hat{C}_1) = -i$, $n_w(\hat{C}_3) = -1$ and $n_w(\hat{C}_4) = i$. The fourth normal vector $n_w(\hat{C}_2)$ cannot be calculated exactly. To approximate this one I assume the rounded edge of the upper plate to be a quarter circle ($u^2 + v^2 = b^2$). Than the curve $\hat{C}_2(u, v)$ can be written in the form:

$$\hat{C}_2(u,v) = -u + iv = -u + i\sqrt{b^2 - u^2} = \sqrt{b^2 - v^2} + iv \quad .$$

Differentiation with respect to u or v respectively and multiplication by the imaginary unit gives

$$n_w(\hat{C}_2(u)) = \frac{u - i\sqrt{b^2 - u^2}}{\sqrt{b^2 - u^2}} = \frac{u}{v} - i \quad ;$$

$$n_w(\hat{C}_2(v)) = \frac{iv - \sqrt{b^2 - v^2}}{\sqrt{b^2 - v^2}} = \frac{iv}{u} - 1 \quad .$$

Here u is the real and v the imaginary part of the function $f(C_1(x|\ln p < x < \ln q))$. This expression is rather lengthy but the simplest one I could find. To get a result even more accurate one must approximate the rounded corner by an ellipse ($u^2/\alpha + v^2/\beta = 1$) with individual α and β for every ratio a/b.

The surface charge density in terms of x is

$$\eta_{\hat{C}_1}(x) = -\frac{\varepsilon_0 V_0}{\pi}\, \Im m\left\{-i\,\frac{-1}{A[\sqrt{p - e^x} + \lambda\sqrt{q - e^x}]}\right\} = -\frac{\varepsilon_0 V_0}{\pi A}\,\frac{1}{\sqrt{p - e^x} + \lambda\sqrt{q - e^x}}$$

for $x < \ln p$: $\hat{C}_1(x) = f(C_1(x < \ln p)) - ia$;

$$\eta_{\hat{C}_2(u)}(x) = -\frac{\varepsilon_0 V_0}{\pi}\, \Im m\left\{\frac{-1(\frac{u}{v} - i)}{A[i\sqrt{e^x - p} - \lambda\sqrt{q - e^x}]}\right\} = -\frac{\varepsilon_0 V_0}{\pi A}\,\frac{\frac{u}{v}\sqrt{e^x - p} - \lambda\sqrt{q - e^x}}{(e^x(1 - \lambda^2) - p + \lambda^2 q)}$$

for $\ln p < x < \ln q$ and $v = \sqrt{b^2 - u^2}$: $\hat{C}_2(x) = u = -\Re e\{f(C_1(\ln p < x < \ln q))\}$;

$$\eta_{\hat{C}_2(v)}(x) = -\frac{\varepsilon_0 V_0}{\pi}\, \Im m\left\{\frac{-1(\frac{iv}{u} - 1)}{A[i\sqrt{e^x - p} - \lambda\sqrt{q - e^x}]}\right\} = -\frac{\varepsilon_0 V_0}{\pi A}\,\frac{\frac{v}{u}\lambda\sqrt{q - e^x} - \sqrt{e^x - p}}{(e^x(1 - \lambda^2) - p + \lambda^2 q)}$$

for $\ln p < x < \ln q$ and $u = \sqrt{b^2 - v^2}$: $\hat{C}_2(x) = v = \Im m\{f(C_1(\ln p < x < \ln q))\}$;

$$\eta_{\hat{C}_3}(x) = -\frac{\varepsilon_0 V_0}{\pi}\, \Im m\left\{-1\,\frac{1}{Ai[\sqrt{e^x - p} + \lambda\sqrt{e^x - q}]}\right\} = -\frac{\varepsilon_0 V_0}{\pi A}\,\frac{1}{\sqrt{e^x - p} + \lambda\sqrt{e^x - q}}$$

for $x > \ln q$: $\hat{C}_3(x) = \Im m\{f(C_1(x > \ln q))\}$;

$$\eta_{\hat{C}_4}(x) = -\frac{\varepsilon_0 V_0}{\pi}\, \Im m\left\{i\,\frac{-1}{A[\sqrt{e^{-x} + p} + \lambda\sqrt{e^{-x} + q}]}\right\} = +\frac{\varepsilon_0 V_0}{\pi A}\,\frac{1}{\sqrt{e^{-x} + p} + \lambda\sqrt{e^{-x} + q}}$$

for $x \in \mathbb{R}$: $\hat{C}_4(x) = f(C_2(x))$.

The surface charge density and electric field on the condenser plates are finite now although the division of the representations of charge density and field for the round part of the upper plate is a great disadvantage that cannot be prevented. To get charge density and field at this part one has to add these quantities by hand or again try to approximate the relation between u and v using the equation for circles or ellipses.

To show that the approximation by a circle should suffice your purposes the curve of the upper condenser plate for three values of ε is plotted on page 72.
The curves for $\varepsilon = 0.95$ and $\varepsilon = 0.96$ are very close to a quarter circle, although the point $f(p)$ moves to the left for smaller values of ε. That is why I choose $\varepsilon = 0.96$ since $f(p)$ is closer to the circle than in the $\varepsilon = 0.95$ case. For $\varepsilon = 1$ the points $f(p)$ and $f(q)$ are located exactly on the circle. This causes the electric field strength and the surface charge density at these points to be as close as possible to the case when the curve were an exact quarter circle.
You can also see that the connections of the curved and the straight parts of the upper plate are continuous but not smooth. So the graphs for the field and the charge density at the connecting points $(f(p), f(q))$ cannot be smooth either. The field and charge density in the interior of the condenser, however, has to be smooth.

Mathematica program 6: for the Rounded One Corner of 90° Condenser

```
Needs["Graphics`ComplexMap`"]

Clear[V,a,b,c,d,e,p,A,eps,frc,frd,frp,lambda,decide]
V = 1 (* Voltage between the plates in Volt *) ;
a = 3 (* Distance of the plates in mm :
         for any 1.5 10^-5 < a < 2 10^2 (scd) and 1.5 10^-5 < a < 2 10^5 (ef) *);
b = a/3 (* Radius of circular arc in mm :
           for any a/50000 < b < a/2 *) ;
decide = ef (* Only the electric field (ef) is available *) ;

A = a/Pi 1/(Sqrt[p] + lambda Sqrt[q]) ;
lambda = ((2 a/(Pi b)(1 - Sqrt[p] ArcTan[Sqrt[1/p]]) -
                      Sqrt[p]))/Sqrt[q]; q = p + eps; eps = .96 ;
af60 = 2 lambda A(1 - Sqrt[q] ArcTanh[Sqrt[1/q]]) ;

frp = FindRoot[af60 == -b, {p,.1,.5},
       AccuracyGoal -> (6 - Floor[N[Log[10,b]]]),
       WorkingPrecision -> $MachinePrecision - If[b < 1,Floor[N[Log[10,b]]],0]] ;
p = p/.frp ;

t[z_] = Exp[z] ;
af61[t_] = 2 I A (Sqrt[t - p] - Sqrt[p] ArcTan[Sqrt[(t - p)/p]] +
                   lambda(Sqrt[t - q] - Sqrt[q]*
                          ArcTan[Sqrt[(t - q)/q]])) + I a ;
af62[t_] = 1/A 1/(Sqrt[t - p] + lambda Sqrt[t - q]) ;

e = Re[af61[t[c]]]; f = Im[af61[t[d]]]; g = Re[af61[-t[d]]] ;
frc = FindRoot[e - 2.5 a, {c,-5,-10}]; c = c/.frc ;
frd = FindRoot[f - 3.5 a, {d,6,10}]; d = d/.frd ;

<< ~path/hilfsfile

a6 = ParametricPlot[{ Im[af61[t[x]]], sgn mult Re[af62[t[x]]] },
       {x,Log[p],d}, PlotRange -> {All,{0,sgn mult Pi/a 1.3}},
       AxesLabel -> {"iv[mm]",tex1}, PlotLabel -> "Upper Plate: Vertical"] ;

b6 = ParametricPlot[{ Re[af61[t[x]]], -sgn mult Im[af62[t[x]]] },
       {x,c,Log[1 + p]}, PlotRange -> {All,{0,sgn mult Pi/a 2}},
       AxesLabel -> {"u[mm]",tex1}, PlotLabel -> "Upper Plate: Horizontal"] ;
```

```
c6 = CartesianMap[af61[t[#]] &, {c,d},{0,Pi}, PlotRange -> All,
       AxesLabel -> {"u[mm]","iv[mm]"},
       Epilog -> { Thickness[.011],
                  { Line[{{g,0},{e,0}}], Line[{{e,a},{b,a}}],
                    Line[{{0,a + b},{0,f}}],
                    Circle[{b,a + b},b,{-Pi,-Pi/2}] }} ];

d6 = ParametricPlot[{ af61[-t[-x]], mult Im[af62[-t[-x]]] },
       {x,-c,-d}, PlotRange -> {0,-1.1 mult Pi/a},
       AxesLabel -> {"u[mm]","Lower Plate"},
       Epilog -> { Text[tex1,{.07 e,-.6 mult Pi/a},{0,0},{0,1}] } ] ;

Show[a6, Ticks -> None, AxesLabel -> {"iv",tex3}] ;
Show[b6, Ticks -> None, AxesLabel -> {"u",tex3}] ;
Show[c6, Axes -> None] ;
Show[d6, Ticks -> None, AxesLabel -> {"u","Lower Plate"},
     Epilog -> { Text[tex2,{.07 e,-.4 mult Pi/a},{0,0},{0,1}]} ] ;
```

Comments:

The program is structured like the other ones. The pictures (**a6, b6, c6, d6**) are plotted with ticks. The following four **Show[ ]** commands produce these pictures without ticks. In this section, however, no common picture of the upper plate's components can be plotted since the electric fields or charge densities cannot be added by one or two *Mathematica* commands only.
Moreover only the electric field strength can be plotted. If you want the surface charge density you may include the very long n_w terms into the **ParametricPlot[ ]**commands by yourself (You have to replace u and v in $n_w(\hat{C}_2(u|v))$ by the real and imaginary part of the curve $f(C_1(\ln p < x < \ln q))$ so that the terms for the round part of the upper plate become very lengthy).

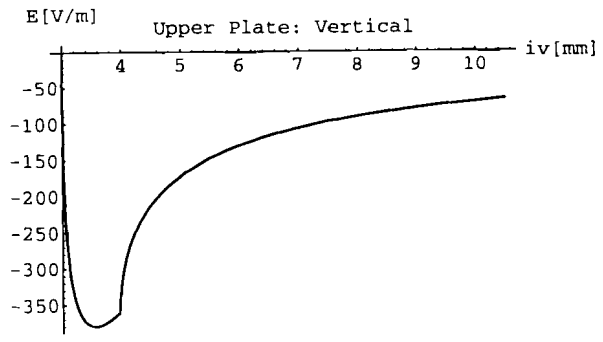
Variants:

- it is possible to change **ParametricPlot[ ]** and **CartesianMap[ ]** as described in section 2.1. The ranges of the iv co-ordinate in **a6,b6** as well as the starting values in **frp, frd** vary for different values of **b** or **b/a** respectively and have to be modified by hand:

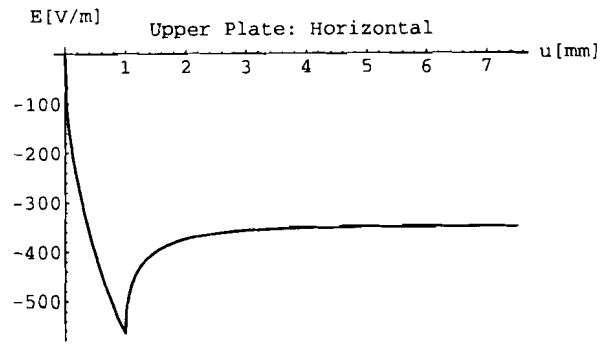
FindRoot[ ]	values of b	starting values	values of b	starting values
frp	**a/50000** to **a/300**	**{10, 50 }**	**a/300** to **a/2**	**{.1, .5 }**
frd	**a/50000** to **a/300**	**{6, 10 }**	**a/300** to **a/2**	**{2, 5 }**

Graphics	iv coordinates for **b = a/50000**	iv coordinates for **b = a/2**
a6	**{... , ... Pi/a 2.5 }**	**{... , ... Pi/a }**
b6	**{... , ... Pi/a 2.5 }**	**{... , ... Pi/a 2 }**

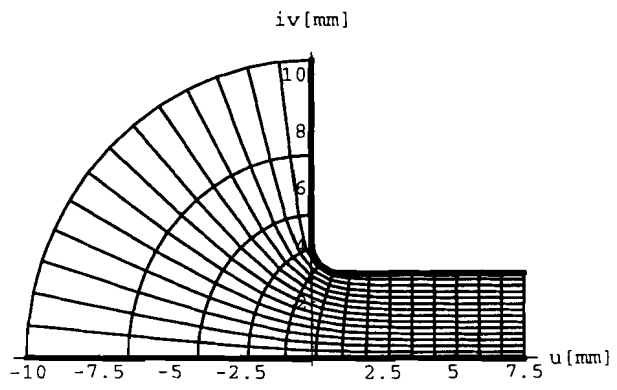
- since you will normally deal with condensers that consist of two rounded corners you can obtain that condenser by application of the Schwarz reflection principle. You have to delete the ", **Line[{{g,0},{e,0}}]** " suboption (don't forget the comma !) in the **Epilog** option of **c6** and to add another picture:
c6low = CartesianMap[Conjugate[af61[t[#]]] &, {c,d},{0,Pi},
PlotRange -> All, Prolog -> {{ Thickness[.011], { Line[{{e,-a},{b,-a}}],
Line[{{0,-a - b},{0,-f}}], Circle[{b,-a - b},b,{Pi/2,Pi}] } }}] ;
To display both the upper (**c6**) and lower (**c6low**) part of the Rounded Two Corners of 90° Condenser in one picture you add:
Show[c6, c6low] ; .
This picture is drawn with ticks. To get the "pure" version add the option ", **Axes -> None** " (don't forget the comma !) to the **Show[ ]** command. The surface charge density of the lower plane will than be of opposite sign. In order to avoid confusion by the surface charge density of the middle plate that is not any more existent wrap (* *) around **d6 = ParametricPlot[...] ;** and **Show[d6, ...] ;**
.



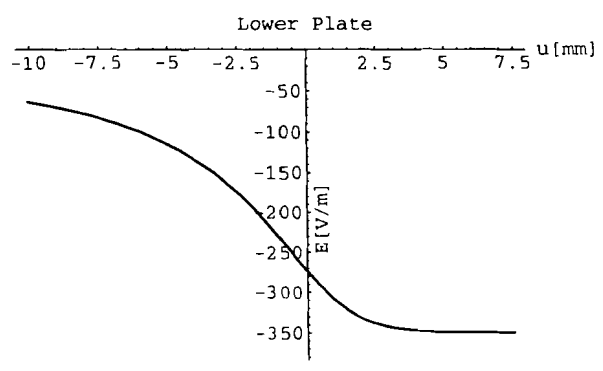
(a6)



(b6)

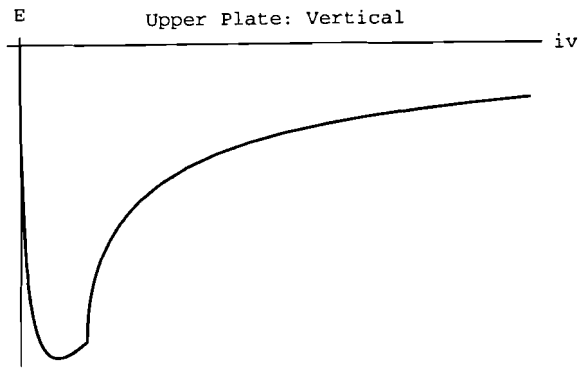


(c6)

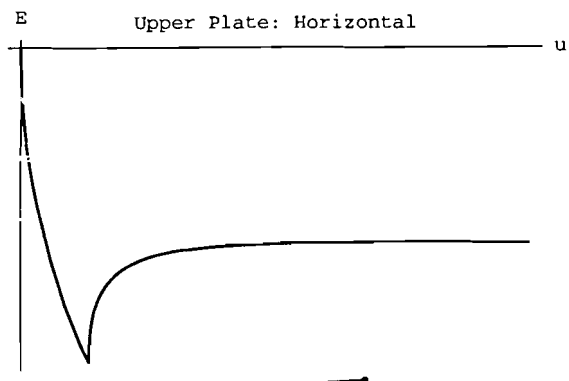


(d6)

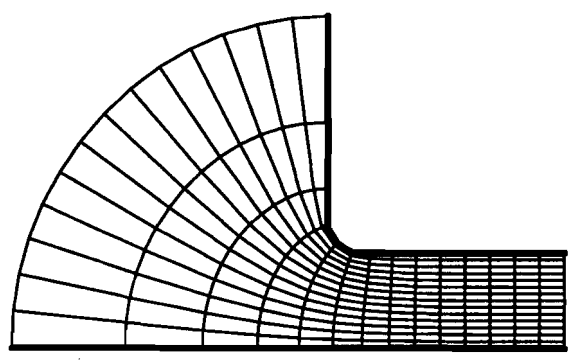
Figure 20: Graphics with ticks produced by the *Mathematica* program for the Rounded One Corner of 90° Condenser. The labels indicate that parts of the program which plotted the pictures. The voltage betwen the plates is 1 V; the distance of the parallel branches is 3 mm.



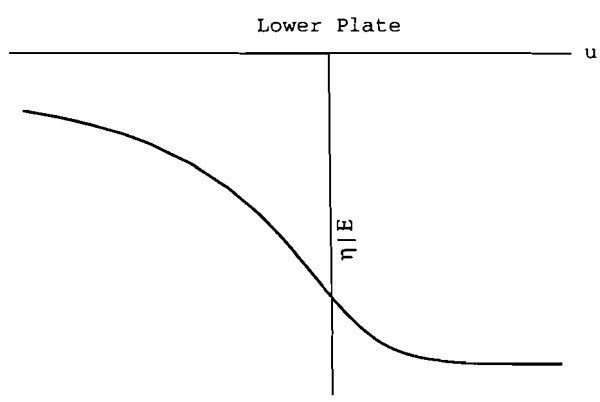
first **Show[ ]**



second **Show[ ]**

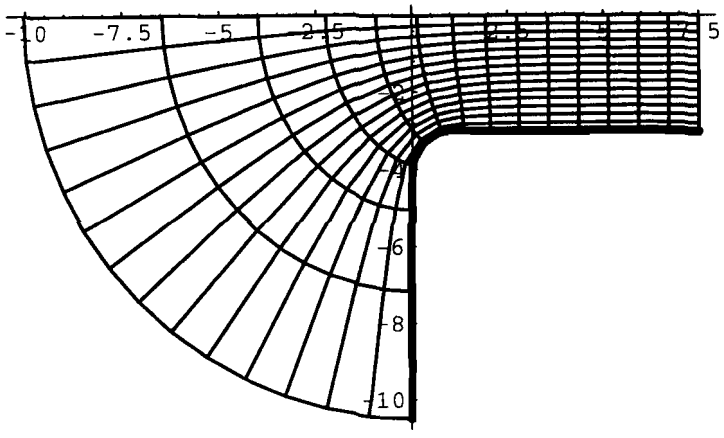


third **Show[ ]**

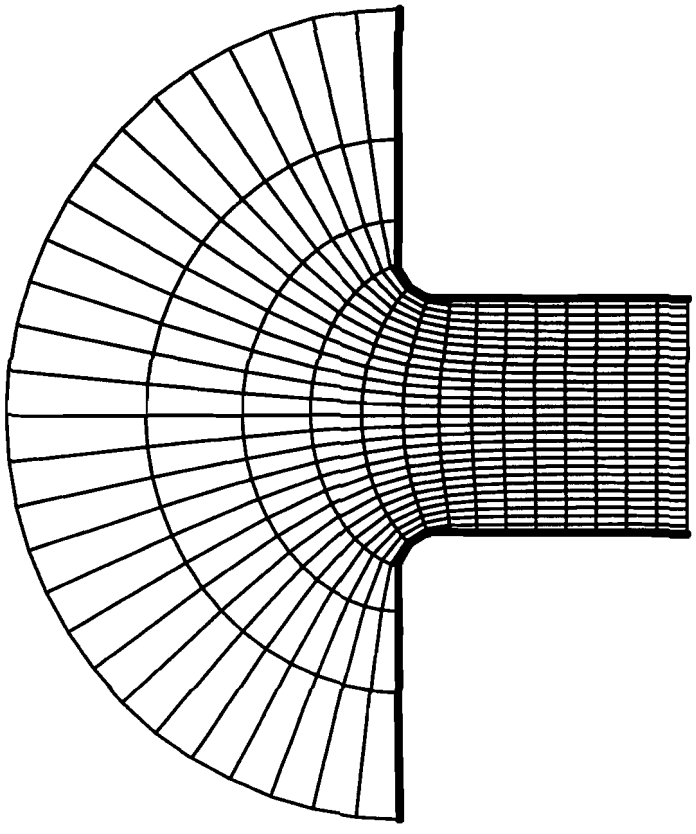


fourth **Show[ ]**

Figure 21: Graphics without ticks produced by the *Mathematica* program for the Rounded One Corner of 90° Condenser.

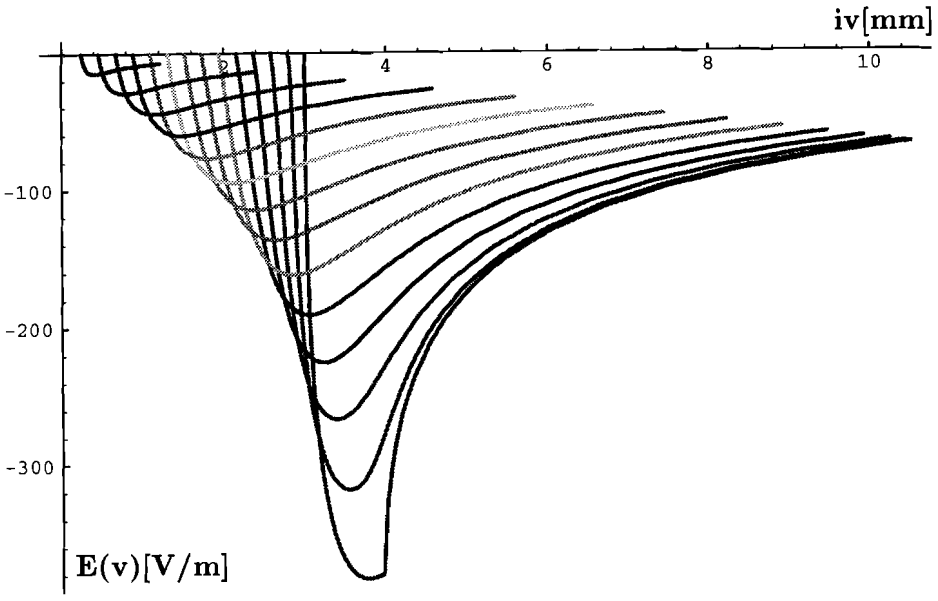


(c6low)



Show[c6,c6low,
Axes -> None]

Figure 22: Graphics produced by the variant of the *Mathematica* program for the Rounded One Corner of 90° Condenser for the same condenser parameters as on the previous pages.



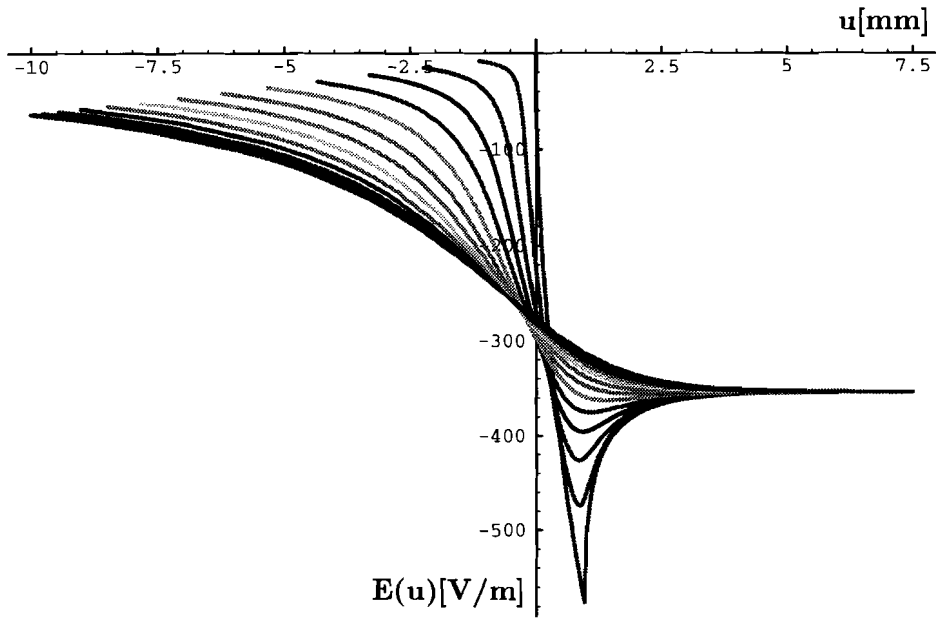


Figure 23: These graphics show the values of the electric field strength at 15 equipotential curves of the Rounded One Corner of 90° Condenser together in one picture.
These equipotentials are exactly those drawn in the pictures produced by **c6** or the second **Show[ ]** command for the condenser parameters used there. The upper picture for the real part (E(v) = E_u) of the field shows how the field increases monotonously from zero at the lower plate (**d6**) to the value on the vertical part of the upper plate (**a6**) when I hop from the lower to the upper plate measuring the field on the plates and on 13 equipotentials in between. The lower picture for the imaginary part (E(u) = E_v) of the field shows how the field decreases on the left side whereas it increases in the center (it has to tend to the same constant at the right side) when moving from the lower plate to the horizontal part of the upper plate (**b6**). For reference read the text on page 59. (cf. the pictures on page 60).

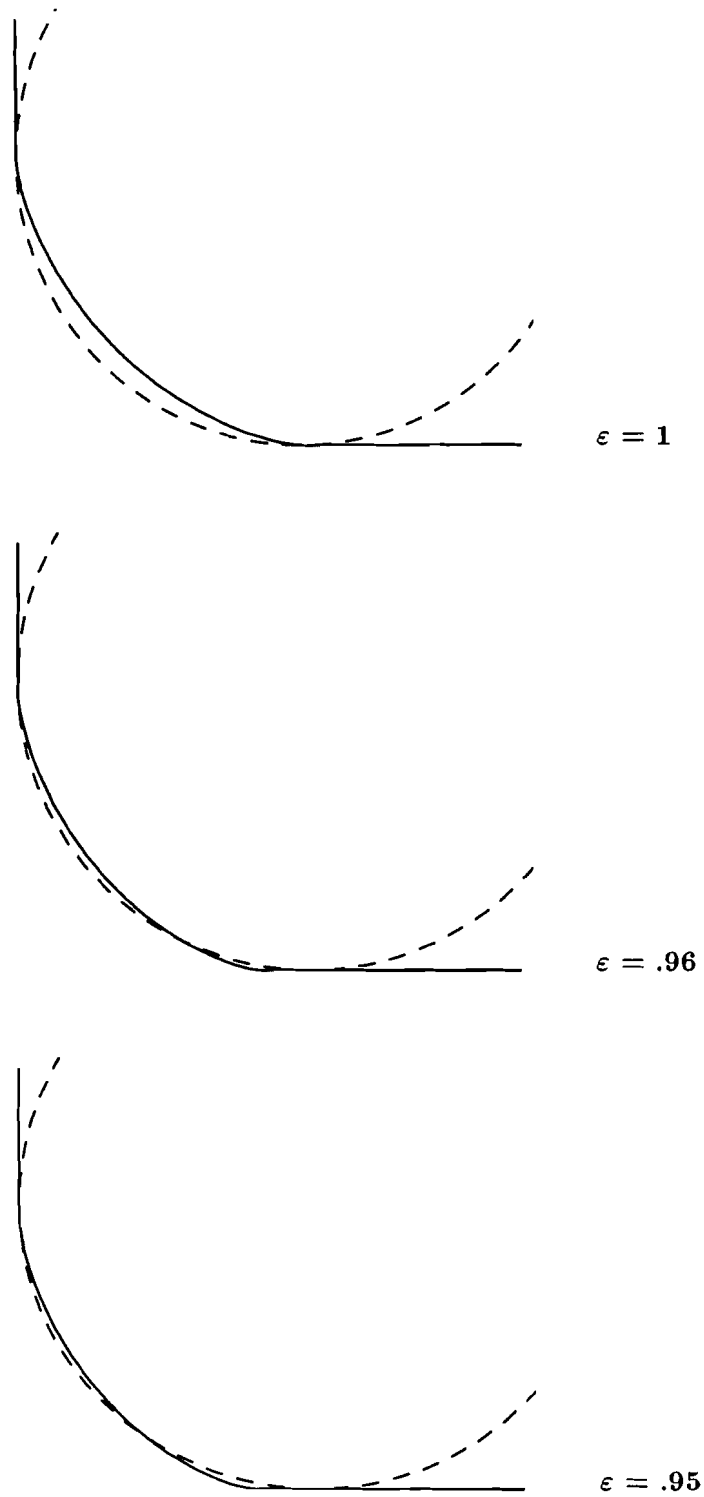


Figure 24: Approximation of the rounded corner (as given by the second equation on page 62) by a circle (dashed) for three different values of ε. [we] uses an $\varepsilon = 1$ but the default value for my program is $\varepsilon = 0.96$.

2.9 *Mathematica* Program for the One Step Condenser

$$w = \frac{2}{\pi}\left\{b\,\ln\left[\frac{\sqrt{p}\,\sqrt{t+1}+\sqrt{t+p}}{\sqrt{p-1}}\right] - a\,\ln\left[\frac{\sqrt{t+1}+\sqrt{t+p}}{\sqrt{p-1}}\right]\right\} - \frac{b}{\pi}\ln t$$

$$\text{with } t = e^z \quad \text{and} \quad p = \frac{a^2}{b^2}$$

This condenser is comparable to the Infinite Plane Condenser with a step in one plate. The first plate is located at the u-axis. The other plate follows the negativ u-axis at a distance $-ia$ with a corner of 90° at $w = -ia$, it aproaches the other plane via the negativ iv-axis and after bending for 90° at $-ib$ follows the positiv u-axes at a distance $-ib$. So the whole looks like a roofed step.
The problem is very shortly treated in [fl, p. 76]. I moved the plates down for $-ia$ units (i mm) so that the One Step Condenser can be reflected at the real u-axis which is easier than reflecting it at the line with $z = x + ia$ using the Schwarz reflection principle.

The conformal map

$$w = \tfrac{2}{\pi}\left\{b\,\ln\left[\sqrt{p}\,\sqrt{t+1}+\sqrt{t+p}\right] - a\,\ln\left[\sqrt{t+1}+\sqrt{t+p}\right] - (b-a)\,\ln\sqrt{p-1}\right\} - \tfrac{b}{\pi}\ln t$$

with $t = e^z$ and $p = \frac{a^2}{b^2}$ is holomorphic in $\mathbb{C}$ when $a > b > 0$ and a, b are positiv.
Its first derivative $f'(z) = -\frac{a\sqrt{1+e^z}}{\pi\sqrt{p+e^z}}$ has a zero at $z = i\pi$ and so the domain $\mathbf{D}_z$ is $\mathbf{D}_z = \mathbf{D}_{id}\backslash\{i\pi\}$.

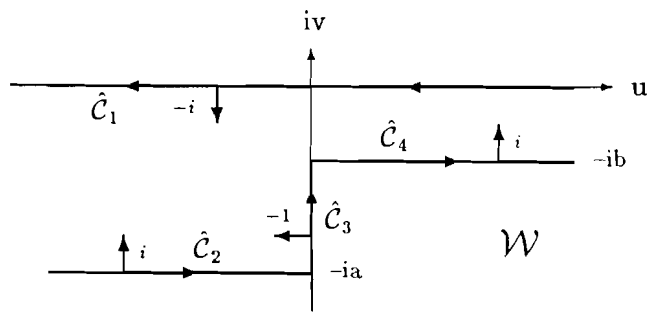


Figure 25: Curve $\hat{C}$ and normal vectors of the One Step Condenser

The condenser's curve $\hat{C}$ consists of four parts:
$\hat{C}_1$ is the image of C_1 , $\hat{C}_2$ that of $C_2(x < -\ln p)$, $\hat{C}_3$ that of $C_2(-\ln p < x < 0)$ and $\hat{C}_4$ is the image of $C_2(x > 0)$.

They are given by :

$$f(C_1) = \frac{2}{\pi}\left\{ b\ln\left[\frac{\sqrt{p}\sqrt{e^x+1}+\sqrt{e^x+p}}{\sqrt{p-1}}\right] - a\ln\left[\frac{\sqrt{e^x+1}+\sqrt{e^x+p}}{\sqrt{p-1}}\right] - \frac{bx}{2}\right\}$$

$$\Longrightarrow \quad \hat{C}_1(u) = f(C_1) = -u \quad \text{for} \quad u \in \mathbb{R} \ ;$$

$$f(C_2(x<-\ln p)) = \frac{2}{\pi}\left\{ b\ln\left[i\frac{\sqrt{p}\sqrt{e^{-x}-1}+\sqrt{e^{-x}-p}}{\sqrt{p-1}}\right] - a\ln\left[i\frac{\sqrt{e^{-x}-1}+\sqrt{e^{-x}-p}}{\sqrt{p-1}}\right]\right\} - \frac{b(i\pi-x)}{\pi}$$

$$= \frac{2}{\pi}\left\{ b\ln\left[\frac{\sqrt{p}\sqrt{e^{-x}-1}+\sqrt{e^{-x}-p}}{\sqrt{p-1}}\right] - a\ln\left[\frac{\sqrt{e^{-x}-1}+\sqrt{e^{-x}-p}}{\sqrt{p-1}}\right] + \frac{bx}{2}\right\} - ia$$

$$\Longrightarrow \quad \hat{C}_2(u) = f(C_2(x<-\ln p)) = u - ia \quad \text{for} \quad u<0 \ ;$$

$$f(C_2(-\ln p<x<0)) = \frac{2}{\pi}\left\{ b\ln\left[\frac{i\sqrt{p}\sqrt{e^{-x}-1}+\sqrt{p-e^{-x}}}{\sqrt{p-1}}\right] - a\ln\left[\frac{i\sqrt{e^{-x}-1}+\sqrt{p-e^{-x}}}{\sqrt{p-1}}\right]\right\} - \frac{b(i\pi-x)}{\pi}$$

$$= i\left\{-b + \frac{2}{\pi}\left[b\arctan\left(\frac{\sqrt{p}\sqrt{e^{-x}-1}}{\sqrt{p-e^{-x}}}\right) - a\arctan\left(\frac{\sqrt{e^{-x}-1}}{\sqrt{p-e^{-x}}}\right)\right]\right\}$$

$$\Longrightarrow \quad \hat{C}_3(v) = f(C_2(-\ln p<x<0)) = iv \quad \text{for} \quad -a<v<-b \ ;$$

$$f(C_2(x>0)) = \frac{2}{\pi}\left\{ b\ln\left[\frac{\sqrt{p}\sqrt{1-e^{-x}}+\sqrt{p-e^{-x}}}{\sqrt{p-1}}\right] - a\ln\left[\frac{\sqrt{1-e^{-x}}+\sqrt{p-e^{-x}}}{\sqrt{p-1}}\right] + \frac{bx}{2}\right\} - ib$$

$$\Longrightarrow \quad \hat{C}_4(u) = f(C_2(x>0)) = u - ib \quad \text{for} \quad u>0 \ .$$

Again I used [bs, §3.4.4.2.2] to write down the functions for the curve C as the sum of their real and imaginary parts.

The unit normal vectors are calculated to $n_w(\hat{C}_1) = -i, n_w(\hat{C}_2) = n_w(\hat{C}_4) = i$ and $n_w(\hat{C}_3) = -1$. The surface charge density in terms of x is

$$\eta_{\hat{C}_1}(x) = -\frac{\varepsilon_0 V_0}{\pi}\,\Im m\left\{-i\,\frac{-\pi\sqrt{p+e^x}}{a\sqrt{1+e^x}}\right\} = -\frac{\varepsilon_0 V_0}{a}\,\frac{\sqrt{p+e^x}}{\sqrt{1+e^x}}$$

$$\text{for} \quad x \in \mathbb{R}: \quad \hat{C}_1(x) = f(C_1(x)) \quad ;$$

$$\eta_{\hat{C}_2}(x) = -\frac{\varepsilon_0 V_0}{\pi}\,\Im m\left\{i\,\frac{-\pi\sqrt{e^{-x}-p}}{a\sqrt{e^{-x}-1}}\right\} = \frac{\varepsilon_0 V_0}{a}\,\frac{\sqrt{e^{-x}-p}}{\sqrt{e^{-x}-1}}$$

$$\text{for} \quad x < -\ln p: \quad \hat{C}_2(x) = f(C_2(x < -\ln p)) \quad ;$$

$$\eta_{\hat{C}_3}(x) = -\frac{\varepsilon_0 V_0}{\pi}\,\Im m\left\{-1\,\frac{-\pi\sqrt{p-e^{-x}}}{ia\sqrt{e^{-x}-1}}\right\} = \frac{\varepsilon_0 V_0}{a}\,\frac{\sqrt{e^{-x}-p}}{\sqrt{e^{-x}-1}}$$

$$\text{for} \quad -\ln p < x < 0: \quad \hat{C}_3(x) = f(C_2(-\ln p < x < 0)) \quad ;$$

$$\eta_{\hat{C}_4}(x) = -\frac{\varepsilon_0 V_0}{\pi}\,\Im m\left\{i\,\frac{-\pi\sqrt{p-e^{-x}}}{a\sqrt{1-e^{-x}}}\right\} = \frac{\varepsilon_0 V_0}{a}\,\frac{\sqrt{p-e^{-x}}}{\sqrt{1-e^{-x}}}$$

$$\text{for} \quad x > 0: \quad \hat{C}_4(x) = f(C_2(x > 0)) \quad .$$

The behaviour of the surface charge density and the electric field at the corner at the bottom of the step is opposite to that at the corner of the step's top. Surface charge density and electric field go to minus/plus infinity at the convex corner and to zero at the concave one.
Field and charge density of the upper and lower plate tend to $-\frac{\varepsilon_0 V_0}{a}$ and $\pm V_0\,/a$ respectively for $u \to -\infty$; to $-\frac{\varepsilon_0 V_0}{b}$ and $\pm V_0\,/b$ respectively for $u \to \infty$.
For the limit $b \to 0$ the step condenser approaches the Infinite Plane Condenser.

Mathematica program 7: for the One Step Condenser

```
Needs["Graphics`ComplexMap`"]

Clear[V,a,b,c,d,e,p,t,frc,frd,frtex,decide]
V = 1 (* Voltage between the plates in Volt *) ;
a = 3 (* Distance of the plates at the left in mm :
         for 2 10^-5 < a < 10^4 (ef) and 2 10^-5 < a < 2 10^2 (scds) *);
b = 2 (* Distance of the plates at the right in mm :
         for a/10 < b < a/1.0001 *);
decide = ef (* Decide whether you want the electric field (ef) or
                surface charge density (scd) of the plates *) ;

p = a^2/b^2;
af70[t_] = 2/Pi (b Log[a/b Sqrt[t + 1] + Sqrt[t + p]] -
                 a Log[Sqrt[t + 1] + Sqrt[t + p]] -
                 (b - a) Log[Sqrt[p - 1]]) -
           b/Pi Log[t] ;
af71[t_] = Sqrt[t + p]/Sqrt[t + 1] ;
t[z_] = Exp[z] ;

e = Re[af70[t[c]]] ;
frc = FindRoot[e + 1.5 a, {c,0,8},
       AccuracyGoal -> 5 - Floor[N[Log[10,a]]]]; c = c/.frc ;
frd = FindRoot[Re[af70[t[d]]] + e, {d,-10,-5}]; d = d/.frd ;

<< ~path/hilfsfile

a7 = ParametricPlot[{ Re[af70[t[x]]], sgn mult af71[t[x]] }, {x,d,c},
      PlotRange -> {{e,-e},{0,1.05 sgn mult a/b}},
      AxesLabel -> {"u[mm]","Upper Plate"},
      Epilog -> { Text[tex1,{-e/15,.5 sgn mult a/b},{0,0},{0,1}] } ] ;

b7 = CartesianMap[af70[t[#]] &,{d,c,(c - d)/50},{0,Pi,Pi/20},
      PlotRange -> All, AxesLabel -> {"u[mm]","iv[mm]"},
      Epilog -> { Thickness[.011], { Line[{{e,0},{-e,0}}],
                  Line[{{e,-a},{0,-a},{0,-b},{-e,-b}}] }} ] ;
```

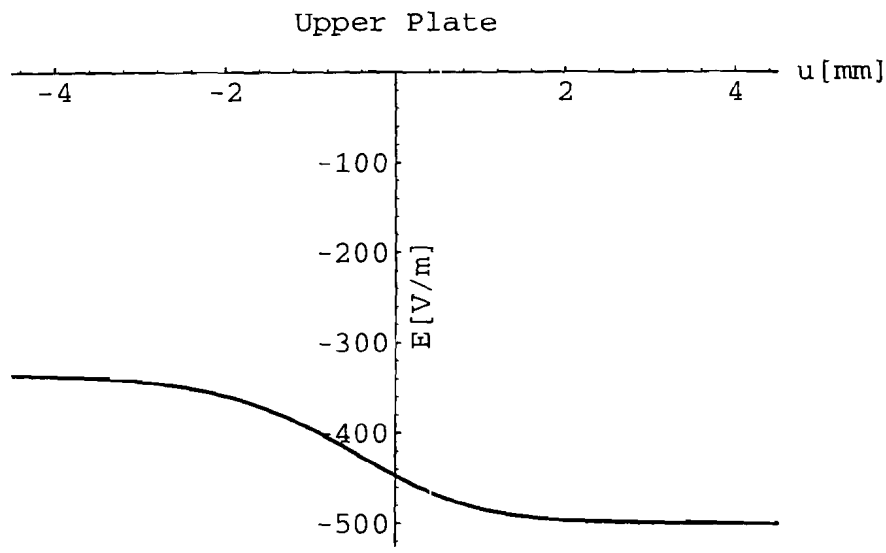
```
c7 = ParametricPlot[{ Re[af70[-t[-x]]], -mult Re[af71[-t[-x]]] },
      {x,-c,-Log[p] - 10^-7}, PlotRange -> {{e,0},{0,-mult}},
      AxesOrigin -> {0,0}, AxesLabel -> {"u[mm]",tex1},
      PlotLabel -> "Lower Plate: Left" ] ;

d7 = ParametricPlot[{ Im[af70[-t[-x]]], sgn mult Im[af71[-t[-x]]] },
      {x,-Log[p] + 10^-7,-10^-7}, AxesLabel -> {"iv[mm]",tex1},
      AxesOrigin -> {-b,0}, PlotRange -> {{-a,-b},{0,-6 sgn mult}},
      PlotLabel -> "Lower Plate: Middle" ] ;

e7 = ParametricPlot[{ Re[af70[-t[-x]]], -mult af71[-t[-x]] }, {x,10^-7,-d},
      PlotRange -> {{0,-e},{0,-1.5 a/b mult}}, AxesOrigin -> {0,0},
      AxesLabel -> {"u[mm]",tex1}, PlotLabel -> "Lower Plate: Right" ] ;

Show[a7, Ticks -> None, AxesLabel -> {"u","Upper Plate"},
     Epilog -> { Text[tex3,{-e/15,.5 sgn mult a/b}] } ] ;
Show[b7, Axes -> None] ;

Show[GraphicsArray[{c7,d7,e7}, GraphicsSpacing -> 0]/.
     (Ticks -> Automatic) -> (Ticks -> None) /.
     (AxesLabel -> {"u[mm]",tex1}) -> (AxesLabel -> {None,None}) /.
     (AxesLabel -> {"iv[mm]",tex1}) -> (AxesLabel -> {None,None}) /.
     (PlotLabel -> "Lower Plate: Left") -> (PlotLabel -> None) /.
     (PlotLabel -> "Lower Plate: Middle") -> (PlotLabel -> None) /.
     (PlotLabel -> "Lower Plate: Right") -> (PlotLabel -> None) /.
     (PlotRange -> {{e,0},{0,-mult}}) ->
       (PlotRange -> {{e,0},{-1.5 a/b branch mult,-1.5 a/b sgn mult}}) /.
     (PlotRange -> {{-a,-b},{0,-6 sgn mult}}) ->
       (PlotRange -> {{-a,-b},{-1.5 a/b branch mult,-1.5 a/b sgn mult}}) /.
     (PlotRange -> {{0,-e},{0,-1.5 a/b mult}}) ->
       (PlotRange -> {{0,-e},{-1.5 a/b branch mult,-1.5 a/b sgn mult}}),
     PlotRange -> {0,.21},
     Epilog -> { Text["u ->                iv ->              u ->",
                      {.5,If[decide == ef,.12,,.18]}] }] ;
```

Comments:

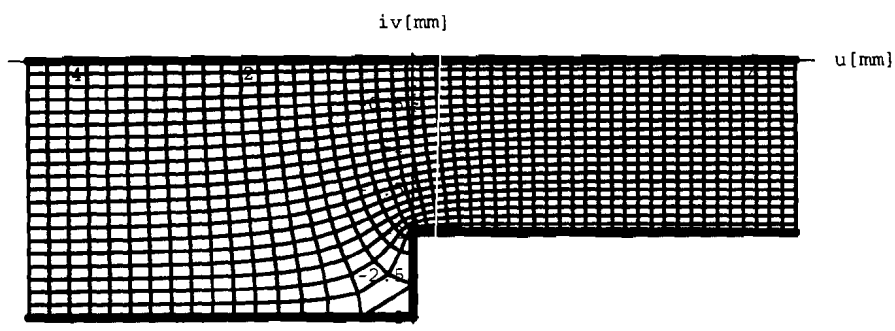
The graphics commands named **a7,b7,c7,d7,e7** plot the pictures of the electric field or the surface charge density and the field distribution with ticks. The three **Show[ ]** commands draw the pictures **a7** to **e7** without ticks whereby the third **Show[ ]** unites **c7,d7,e7** in one picture.

Variants:

- it is possible to change **ParametricPlot[ ]** and **CartesianMap[ ]** as described in section 2.1.
- The case of the Two Steps Condenser which is symmetric with the u-axis can be obtained by replaceing the limits for y: **{0,Pi,Pi/20}** in **CartesianMap[ ]** by **{-Pi,Pi,Pi/40}**. The surface charge density of the lower plane will than be of opposite sign. In order not to arise confusion by the surface charge density of the central plate, that is not any more existent, wrap (* *) around **a7 = ParametricPlot[...]** ; and **Show[a7, ...]** ; .
  To get the line for the lower condenser plate instead of that of the central plate replace **Line[{{e,0},{-e,0}}]** in the **Epilog** option of **b7** by **Line[{{e,a},{0,a},{0,b}, {-e,b}}]**.

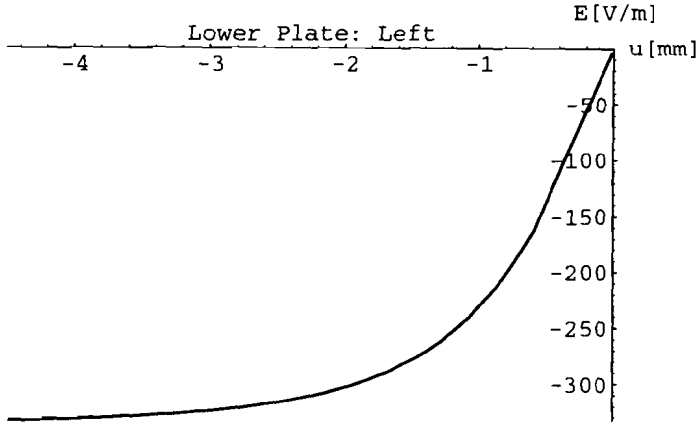


(a7)

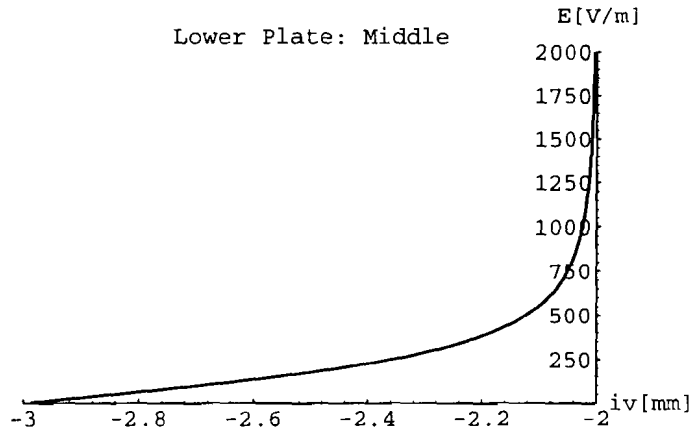


(b7)

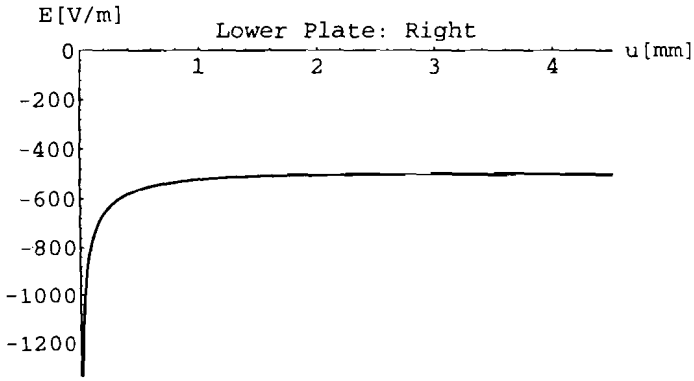
Figure 26: Graphics with ticks produced by the *Mathematica* program for the **One Step Condenser**. The labels indicate that parts of the program which plotted the pictures. The voltage between the plates is 1 V. The distance of the plates is 2, 3 mm respectively.



(c7)

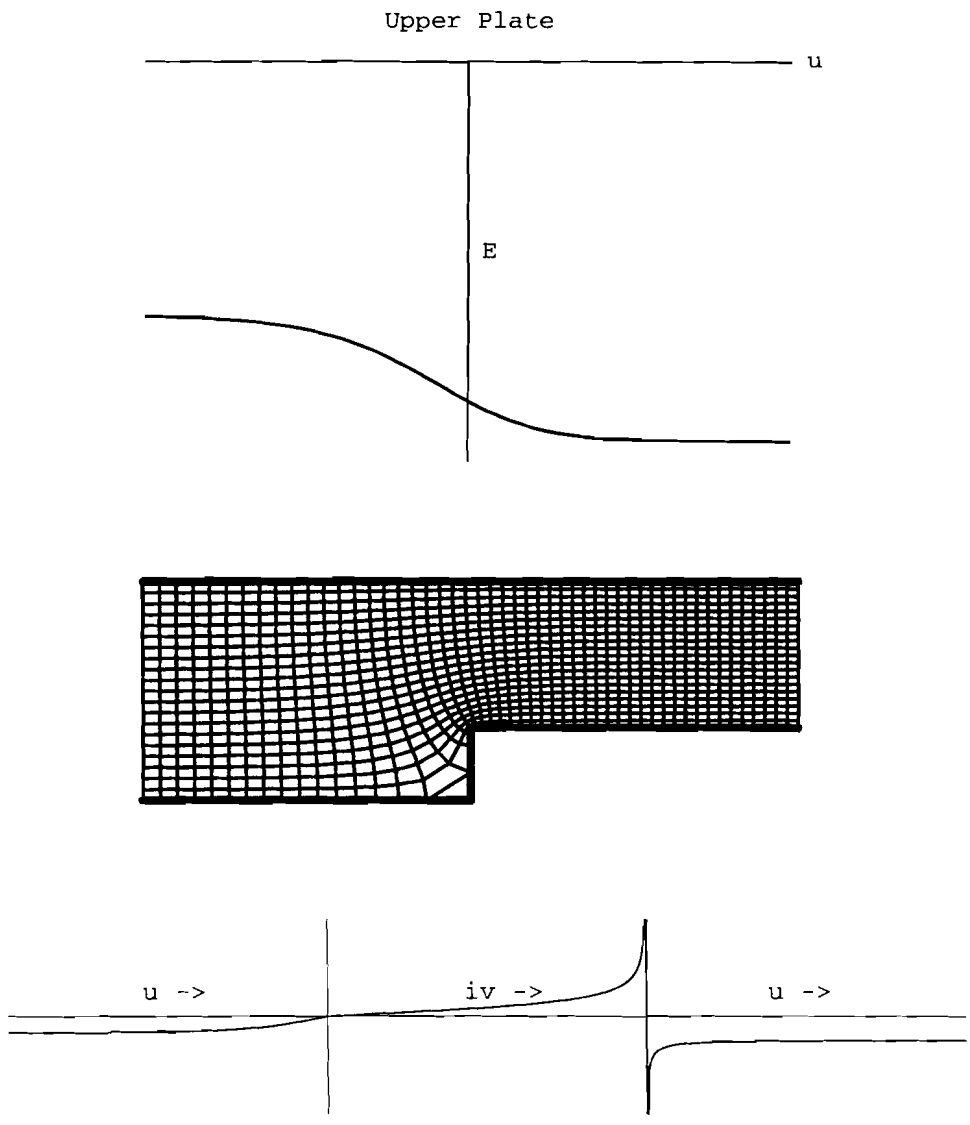


(d7)



(e7)

Figure 27: Graphics with ticks produced by the *Mathematica* program for the **One Step Condenser**. The labels indicate that parts of the program which plotted the pictures.



first **Show[ ]**

second **Show[ ]**

third **Show[ ]**

Figure 28: Graphics without ticks produced by the *Mathematica* program for the One Step Condenser. The labels indicate that parts of the program which plotted the pictures.

2.10 *Mathematica* Program for the Sharp Bend Condenser

$$w = \frac{2}{\pi}\left(b\,\ln\left[\frac{\sqrt{t+1}+\sqrt{t-q}}{\sqrt{q+1}}\right] - ai\,\ln\left[\frac{\sqrt{t-q}+i\sqrt{q}\sqrt{t+1}}{\sqrt{q+1}}\right]\right) + \frac{ia}{\pi}\ln t - ib$$

$$\text{with } t = e^z \quad \text{and} \quad q = \frac{a^2}{b^2}$$

Unlike most condensers in this thesis this one is not symmetric relative to the u- or iv-axis. It consists of two plates: one plate at the positive u-axis with an edge to the negative iv-axis at the origin; and the second plate below the positive u-axis at a distance b with an edge at $z = a - ib$ and a part that is parallel to the negative iv-axes at a distance a. So the whole thing looks a bit like a knee of a robot or a street with a very sharp bend (with edges instead of the curves).
The conformal map

$$w = \frac{2}{\pi}\left(b\,\ln\left[\frac{\sqrt{t+1}+\sqrt{t-q}}{\sqrt{q+1}}\right] - ai\,\ln\left[\frac{\sqrt{t-q}+i\frac{a}{b}\sqrt{t+1}}{\sqrt{q+1}}\right]\right) + \frac{ia}{\pi}\ln t - ib$$

with $t = e^z$and $q = \frac{a^2}{b^2}$ is holomorphic in $\mathbf{C}$ for $a \geq 0$ and $b > 0$.
The first derivative $f'(z) = \frac{b\sqrt{e^z-q}}{\pi\sqrt{e^z+1}}$ is zero at $z = \ln q$. So the domain $\mathbf{D}_z$ is $\mathbf{D}_z = \mathbf{D}_{id}\backslash\{\ln q\}$.

The picture of the Sharp Bend Condenser is shown below:

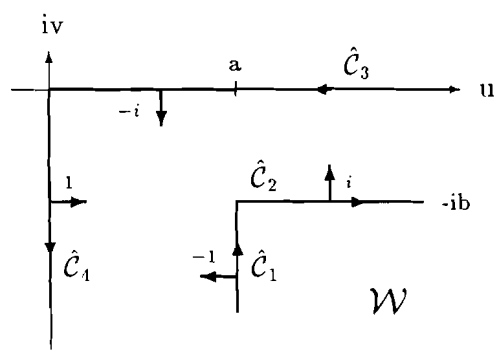


Figure 29: Curve $\hat{C}$ and normal vectors of the Sharp Bend Condenser

The curve $\hat{C}$ consists of four parts :
$\hat{C}_1$ is the image of $C_1(x < lnq)$, $\hat{C}_2$ the one of $C_1(x > lnq)$, $\hat{C}_3$ derives from $C_2(x < 0)$ and $\hat{C}_4$ from $C_2(x > 0)$.

$$f(C_1(x < \ln q)) = a + i\left\{\frac{2a}{\pi}\left(\frac{x}{2} - \ln\left[\sqrt{q - e^x} + \sqrt{q}\,\sqrt{1 + e^x}\right] + \ln\sqrt{1+q}\right) + \frac{2b}{\pi}\arctan\left(\frac{q - e^x}{1 + e^x}\right) - b\right\}$$

$$\Longrightarrow \quad \hat{C}_1(v) = f(C_1(x < \ln q)) = iv + a \quad \text{for} \quad -\infty < v < -b \quad ;$$

$$f(C_1(x > \ln q)) = \frac{2b}{\pi}\ln\left[\frac{\sqrt{e^x + 1} + \sqrt{e^x - q}}{1 + q}\right] + \frac{2a}{\pi}\arctan\left(\frac{\sqrt{q}\,\sqrt{e^x + 1}}{\sqrt{e^x - q}}\right) - ib$$

$$\Longrightarrow \quad \hat{C}_2(u) = f(C_1(x > \ln q)) = u - ib \quad \text{for} \quad a < u < \infty \quad ;$$

$$f(C_2(x < 0)) \quad = -a - \frac{2a}{\pi}\arctan\left(\frac{\sqrt{e^{-x} + q}}{\sqrt{q}\,\sqrt{e^{-x} - 1}}\right) + \frac{2b}{\pi}\ln\left[\frac{\sqrt{e^{-x} + q} + \sqrt{e^{-x} - 1}}{\sqrt{1 + q}}\right]$$

$$\Longrightarrow \quad \hat{C}_3(u) = f(C_2(x < 0)) = -u \quad \text{for} \quad -\infty < u < 0 \quad ;$$

$$f(C_2(x > 0)) \quad = i\left\{\frac{2}{\pi}\left[\frac{ax}{2} - a\,\ln\left[\frac{\sqrt{q + e^{-x}} + \sqrt{q}\,\sqrt{1 - e^{-x}}}{\sqrt{1 + q}}\right] + b\,\arctan\left(\frac{\sqrt{q + e^{-x}}}{\sqrt{1 - e^{-x}}}\right)\right] - b\right\}$$

$$\Longrightarrow \quad \hat{C}_4(v) = f(C_2(x > 0)) = -iv \quad \text{for} \quad 0 < v < \infty \quad .$$

Again I used the formula $\mathrm{Ln}\ z = \ln|z| + i\arg z$ to represent the images of the curve C as sums of real and imaginary terms.

The unit normal vectors are calculated to $n_w(\hat{C}_1) = -1, n_w(\hat{C}_2) = i, n_w(\hat{C}_3) = -i$ and $n_w(\hat{C}_4) = 1$. The surface charge density in terms of x is

$$\eta_{\hat{C}_1}(x) = -\frac{\varepsilon_0 V_0}{\pi}\ \Im m\left\{-1\,\frac{\pi\sqrt{1+e^x}}{ib\sqrt{q-e^x}}\right\} = -\frac{\varepsilon_0 V_0}{b}\ \frac{\sqrt{1+e^x}}{\sqrt{q-e^x}}$$

$$\text{for}\quad x < \ln q: \quad \hat{C}_1(x) = f(C_1(x < \ln q))\ \ ;$$

$$\eta_{\hat{C}_2}(x) = -\frac{\varepsilon_0 V_0}{\pi}\ \Im m\left\{i\,\frac{\pi\sqrt{e^x+1}}{b\sqrt{e^x-q}}\right\} \quad = -\frac{\varepsilon_0 V_0}{b}\ \frac{\sqrt{e^x+1}}{\sqrt{e^x-q}}$$

$$\text{for}\quad x > \ln q: \quad \hat{C}_2(x) = f(C_1(x > \ln q))\ \ ;$$

$$\eta_{\hat{C}_3}(x) = -\frac{\varepsilon_0 V_0}{\pi}\ \Im m\left\{-i\,\frac{\pi\sqrt{e^{-x}-1}}{b\sqrt{e^{-x}+q}}\right\} = \frac{\varepsilon_0 V_0}{b}\ \frac{\sqrt{e^{-x}-1}}{\sqrt{e^{-x}+q}}$$

$$\text{for}\quad x < 0: \quad \hat{C}_3(x) = f(C_2(x < 0))\ \ ;$$

$$\eta_{\hat{C}_4}(x) = -\frac{\varepsilon_0 V_0}{\pi}\ \Im m\left\{1\,\frac{\pi\sqrt{1-e^{-x}}}{ib\sqrt{q+e^{-x}}}\right\} = \frac{\varepsilon_0 V_0}{b}\ \frac{\sqrt{1-e^{-x}}}{\sqrt{q+e^{-x}}}$$

$$\text{for}\quad x > 0: \quad \hat{C}_4(x) = f(C_2(x > 0))\ \ .$$

In contrast to the last example the surface charge density and electric field are zero in the corner of the upper and infinite in the corner of the lower plate. Field and charge density on the horicontal upper and lower plate tend to the constant values $-\frac{V_0}{b}$ and $\pm\frac{\varepsilon_0 V_0}{b}$ for $u \to \infty$. They tend to $\frac{V_0}{a}$ and $\pm\frac{\varepsilon_0 V_0}{a}$ on the vertical part of the upper and lower plate for $v \to -\infty$.

Mathematica program 8: for the Sharp Bend Condenser

```
Needs["Graphics`ComplexMap`"]

Clear[V,a,b,c,d,e,q,t,frc,frd,sgn8,decide]
V = 1 (* Voltage between the plates in Volt *) ;
a = 3 (* Vertical plates' distance in mm :
         for 10^-5 < a <  2 10^2 (scd) and 10^-5 < a < 10^5 (ef) *);
b = a/2 (* Horizontal plates' distance in mm :
           for a/10 < b < a/1.0001 *);
decide = ef (* Decide whether you want the electric field (ef) or
                surface charge density (scd) of the plates *) ;

q = a^2/b^2 ;
af80[t_] = 2/Pi (b Log[(Sqrt[t + 1] + Sqrt[t - q])/Sqrt[q + 1]] -
                 a I Log[(Sqrt[t - q] + I a/b Sqrt[t + 1])/
                           Sqrt[q + 1]]) + I (a/Pi Log[t] - b) ;
af81[t_, sgn8_] = Sqrt[t + sgn8]/Sqrt[t - sgn8 q] ;
t[z_] = Exp[z] ;

e = Re[af80[-t[c]]] ;
frc = FindRoot[e - 1.5 a, {c,0,10}]; c = c/.frc ;
frd = FindRoot[Im[af80[-t[d]]] + e, {d,-5,-3}]; d = d/.frd ;

<< ~path/hilfsfile

a8 = ParametricPlot[{ Re[af80[-t[-x]]], sgn mult a/b Re[af81[t[-x],-1]] },
      {x,-c,-.000001}, PlotRange -> {{0,1.02 e},{0,1.01 sgn mult a/b}},
      AxesLabel -> {"u[mm]",tex1}, PlotLabel -> "Upper Plate: Horizontal" ] ;

b8 = ParametricPlot[{ Im[af80[-t[-x]]], mult a/b Im[af81[t[-x],-1]] },
      {x,.000001,-d}, PlotRange -> {{-1.02 e,0},{0,mult}},
      AxesLabel -> {"iv[mm]",tex1}, PlotLabel -> "Upper Plate: Vertical" ] ;

c8 = CartesianMap[af80[t[#]] &, {d,c,(c - d)/30},{0,Pi},
      PlotRange -> All, AxesLabel -> {"u[mm]","iv[mm]"},
      Epilog -> { Thickness[.011],
                  { Line[{{0,-1.02 e},{0,0},{1.02 e,0}}],
                    Line[{{a,-1.02 e},{a,-b},{1.02 e,-b}}] }} ] ;
```

```
d8 = ParametricPlot[{ Im[af80[t[x]]], sgn mult a/b Im[af81[t[x],1]] },
      {x,d,Log[q] - .000001},
      PlotRange -> {{-1.02 e,-b},{0,-3 sgn mult a/b}},
      AxesOrigin -> {-b,0}, AxesLabel -> {"iv[mm]",tex1},
      PlotLabel -> "Lower Plate: Vertical" ] ;

e8 = ParametricPlot[{ Re[af80[t[x]]], -mult a/b Re[af81[t[x],1]] },
      {x,Log[q] + .000001,c}, AxesLabel -> {"u[mm]",tex1},
      PlotRange -> {{a,1.02 e},{0,-3 mult a/b}},
      PlotLabel -> "Lower Plate: Horizontal" ] ;

Show[GraphicsArray[{b8,a8}, GraphicsSpacing -> 0]/.
      (Ticks -> Automatic) -> (Ticks -> None)/.
      (AxesLabel -> {"u[mm]",tex1}) -> (AxesLabel -> {None,None})/.
      (AxesLabel -> {"iv[mm]",tex1}) -> (AxesLabel -> {None,None})/.
      (PlotLabel -> "Upper Plate: Horizontal") -> (PlotLabel -> None)/.
      (PlotLabel -> "Upper Plate: Vertical") -> (PlotLabel -> None)/.
      (PlotRange -> {{0,1.02 e},{0,1.01 sgn mult a/b}}) ->
        (PlotRange -> {{0,1.02 e},{1.01 branch sgn mult a/b,mult}})/.
      (PlotRange -> {{-1.02 e,0},{0,mult}}) ->
        (PlotRange -> {{-1.02 e,0},{1.01 branch sgn mult a/b,mult}}),
      PlotLabel -> "Upper Plate",
      Epilog -> { Text[tex3,{.48,.3}],
                  Text["(iv) ->                          u ->",
                       {.5,If[decide == ef,.115 a/b,,.02]}] } ] ;

Show[c8, Axes -> None] ;
Show[GraphicsArray[{d8,e8}, GraphicsSpacing -> 0]/.
      (Ticks -> Automatic) -> (Ticks -> None)/.
      (AxesLabel -> {"u[mm]",tex1}) -> (AxesLabel -> {None,None})/.
      (AxesLabel -> {"iv[mm]",tex1}) -> (AxesLabel -> {None,None})/.
      (PlotLabel -> "Lower Plate: Horizontal") -> (PlotLabel -> None)/.
      (PlotLabel -> "Lower Plate: Vertical") -> (PlotLabel -> None)/.
      (PlotRange -> {{a,1.02 e},{0,-3 mult a/b}}) ->
        (PlotRange -> {{a,1.02 e},{-3 branch sgn mult a/b,-3 mult a/b}})/.
      (PlotRange -> {{-1.02 e,-b},{0,-3 sgn mult a/b}}) ->
        (PlotRange -> {{-1.02 e,-b},{-3 branch sgn mult a/b,-3 mult a/b}}),
      PlotLabel -> "Lower Plate",
      Epilog -> { Text[tex3,{.48,.28}],
                  Text["(iv) ->                          u ->",
                       {.5,If[decide == ef,.13,,.28]}] } ] ;
```

Comments:

The graphics commands named **a8,b8,c8,d8,e8** plot the pictures of the electric field or the surface charge density and the field distribution with ticks. The three **Show[ ]** commands draw the pictures **a7** to **e7** without ticks whereby the first and third **Show[ ]** unites **a8,b8** and **d8,e8** in one picture.

Variants:

- it is possible to change **ParametricPlot[ ]** and **CartesianMap[ ]** as described in section 2.1.
- This problem can be changed to form a condenser symmetric relative to the u- or iv- or both axes. The condenser symmetric with the u-axis is obtained by erasing the last term in the list of the first **Line[ ]** suboption in **c8**'s **Epilog** plus the preceding comma :

  {{0,–1.02 e},{0,0},{1.02 e,0}} $\Longrightarrow$ **{{0,–1.02 e},{0,0}}**

  To get the upper part of the condenser **c8up** I have to add :

  ```
  c8up = CartesianMap[Conjugate[af80[t[#]]] &, {d,c,1/4.49618},{0,Pi},
  Prolog -> {{ Thickness[.011], { Line[{{0,1.02 e},{0,0}}],
  Line[{{a,1.02 e},{a,b},{1.02 e,b}}] } }} ] ;
  ```

  and **Show[c8,c8up]** ; to get both parts in one picture with ticks. I chose the funny value of dx = **1/4.49618** so that *Mathematica* plots a field line intersecting with the origin.
  The condenser symmetric with the iv-axis is obtained by erasing the first term in the list of the first **Line[ ]** suboption in the original **c8**'s **Epilog** plus the following comma :

  {{0,–1.02 e},{0,0},{1.02 e,0}} $\Longrightarrow$ **{{0,0},{1.02 e,0}}**

  To get the left part of the condenser **c8left** I have to add :

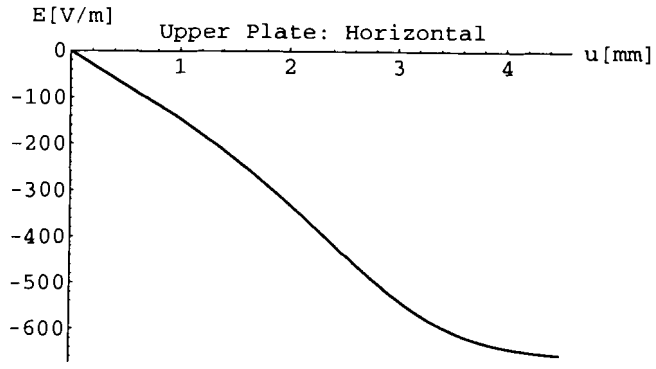
  ```
  c8left = CartesianMap[-Conjugate[af80[t[#]]] &, {d,c,1/4.49618},{0,Pi},
  Prolog -> {{ Thickness[.011], { Line[{{0,0},{-1.02 e,0}}],
  Line[{{-a,-1.02 e},{-a,-b},{-1.02 e,-b}}] } }} ] ;
  ```

  and **Show[c8,c8left]** ; to get both parts in one picture with ticks.

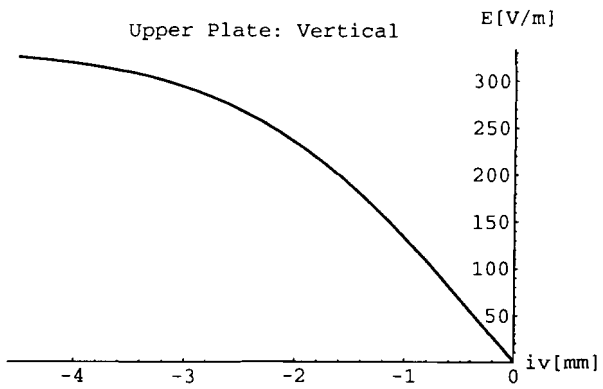
To get a condenser that is symmetric with both axes I have to add another part **c8leftup** to get a condenser that looks like a cross (cf. fig.33):

c8leftup = CartesianMap[–af80[t[#]] &, {d,c,1/4.4961},{0,Pi}] ;

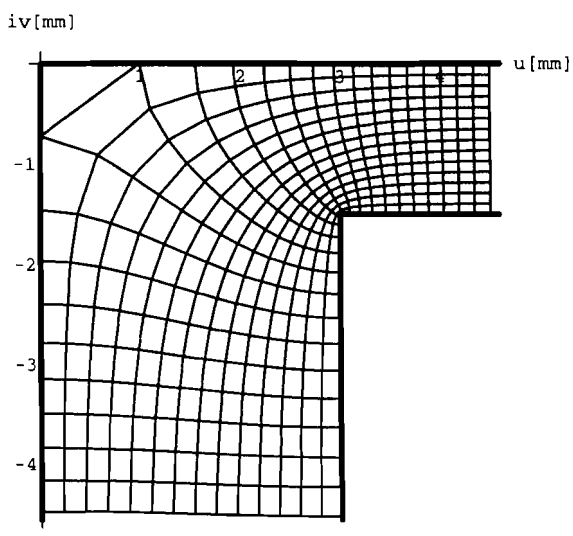
and one **Show[c8,c8up,c8left,c8leftup]** ; to get the whole condenser with ticks. The ticks can be omitted by adding **Axes –> None**.
Obviously I may construct a condenser consisting of three parts but I will not list all possibilities here (cf. fig.33).



(a8)

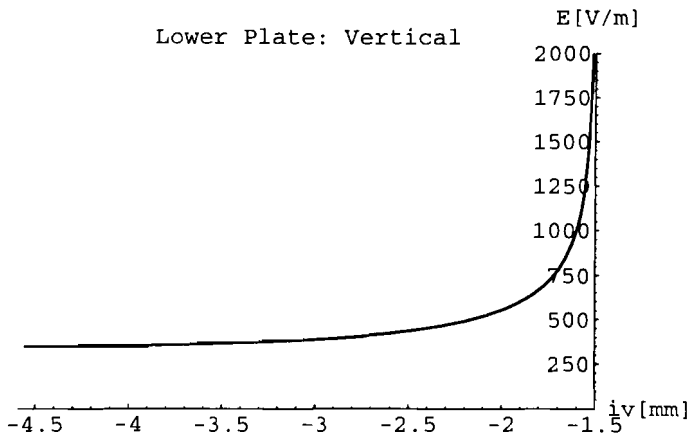


(b8)

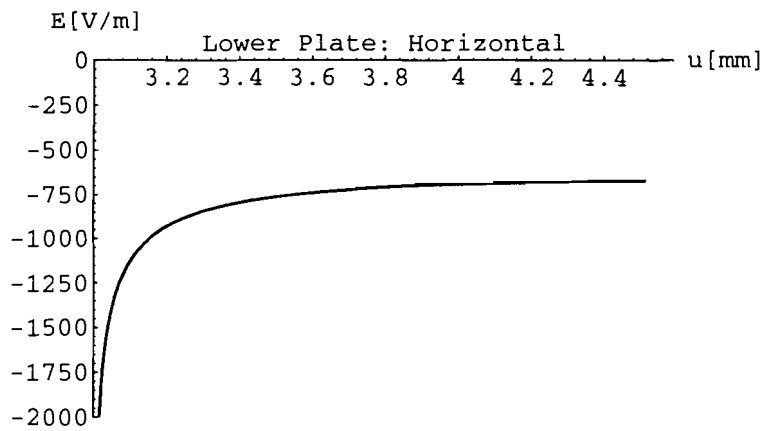


(c8)

Figure 30: Graphics with ticks produced by the *Mathematica* program for the **Sharp Bend Condenser**. The labels indicate that part of the program which plotted the pictures. The voltage between the plates is 1 V. The vertical distance between the plates is 1.5 mm, the horizontal distance is 3 mm.

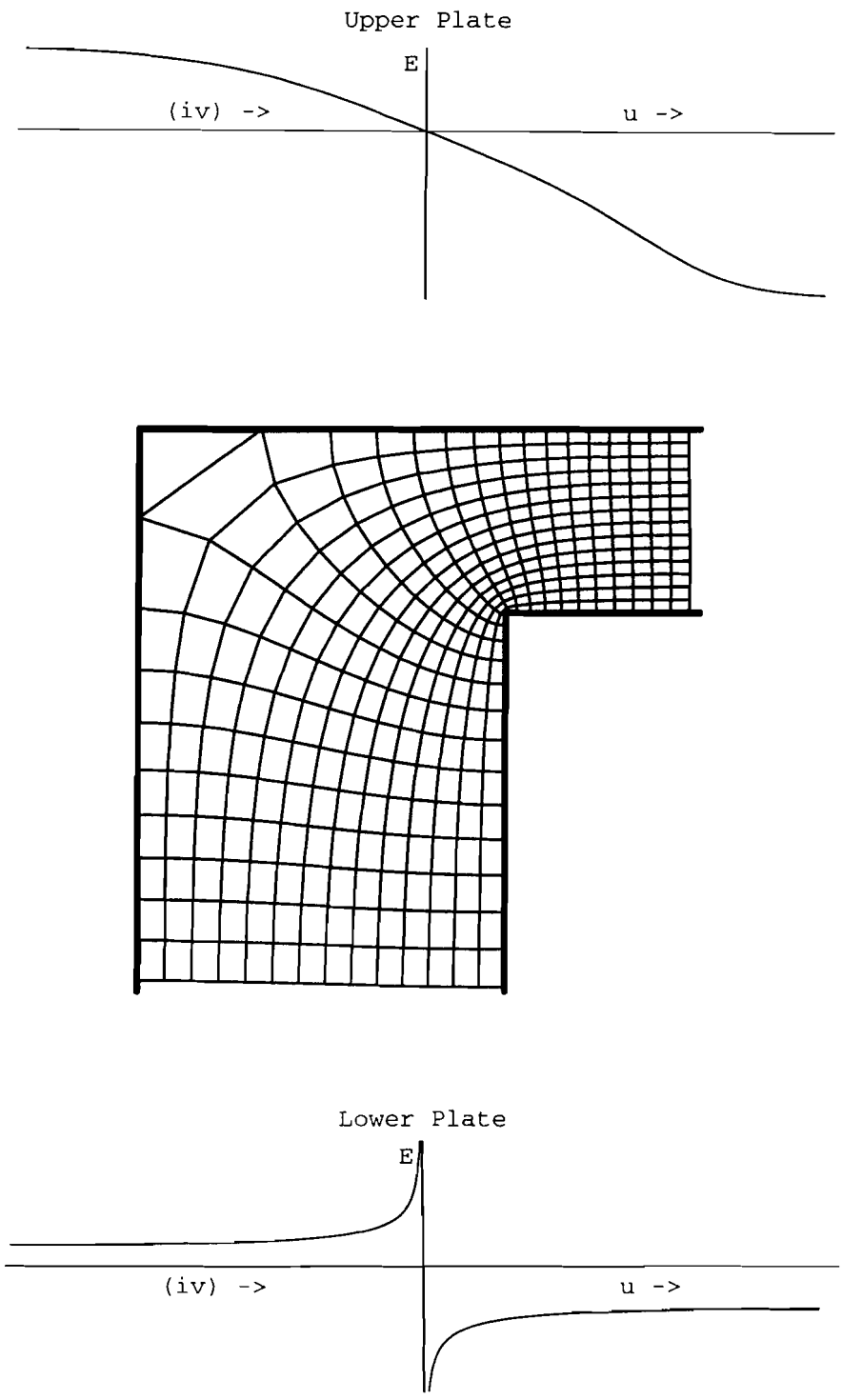


(d8)



(e8)

Figure 31: Graphics with ticks produced by the *Mathematica* program for the **Sharp Bend Condenser**. The labels indicate that part of the program which plotted the pictures.

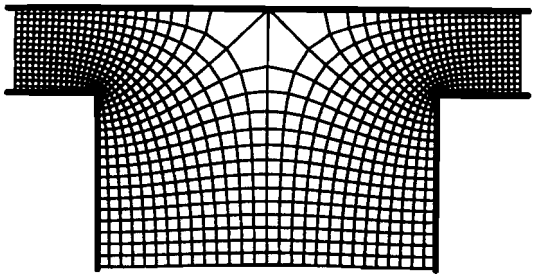


first **Show[ ]**

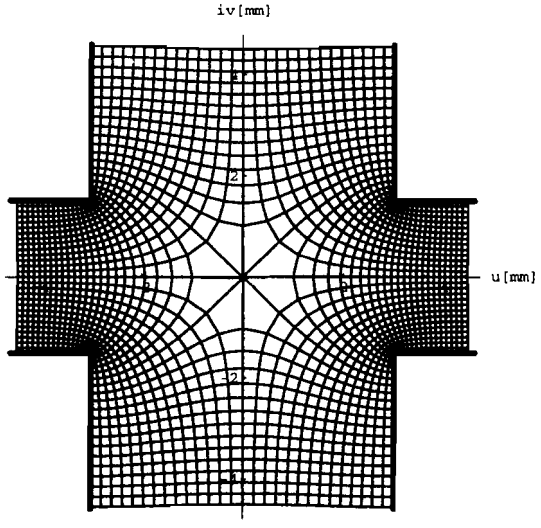
second **Show[ ]**

third **Show[ ]**

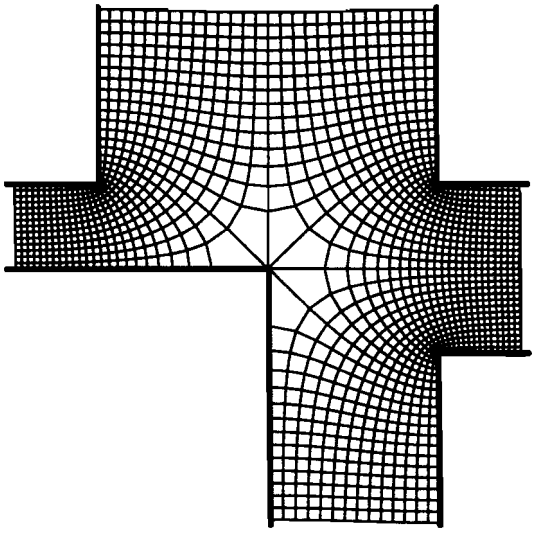
Figure 32: Graphics without ticks produced by the *Mathematica* program for the Sharp Bend Condenser. The labels indicate that part of the program which plotted the pictures.



Show[c8,c8left]



Show[c8,c8up,c8left,c8leftup]



Show[c8,c8up,c8leftup,
Axes -> None]

Figure 33: Graphics produced by the variants of the *Mathematica* program for the Sharp Bend Condenser.

2.11 *Mathematica* Program for the Infinitely Thin Aperture Condenser

$$w = 2e^{z/2} - \frac{a^2}{8}e^{-z/2}$$

This problem can be considered as a preliminary to the next one – the Thick Aperture Condenser – when I let the distance between the corners tend to zero. We get something that looks like a hole in a screen, in optics simply called an aperture. As in the problems before only the upper half of the aperture is constructed which can be continued holomorphically to the lower half by the Schwarz reflection principle as indicated in section 2.5.
The function

$$w = f(z) = 2e^{z/2} - \frac{a^2}{8}e^{-z/2}$$

maps the Infinite Plane Condenser to one whose one plate extends along the iv-axis with a hole between 0 and ia. The second plate is the u-axis.
The function is holomorphic in $\mathbb{C}$. For $a = 4$ the function $f(z)$ is simply 4 sinh(z/2).
The first derivative $f'(z) = e^{z/2} + \frac{a^2}{16}e^{-z/2}$ gets zero at $z = \ln(-\frac{a^2}{16}) = i\pi + 2\ln\frac{a}{4}$ and so the domain $\mathbf{D}_z$ is $\mathbf{D}_z = \mathbf{D}_{id}\backslash\{i\pi + 2\ln\frac{a}{4}\}$.

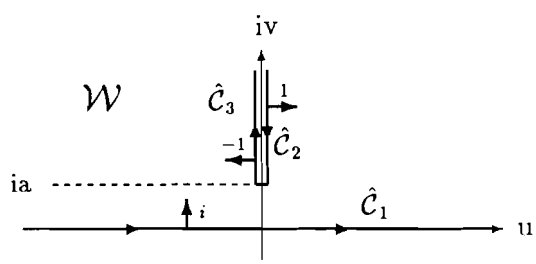


Figure 34: Curve $\hat{C}$ and normal vectors of the Infinitely Thin Aperture Condenser

The picture above shows the images of the condenser planes on the W-plane. The condenser consists of three parts which read

$$f(C_1) = 2e^{x/2} - \frac{a^2}{8}e^{-x/2} \implies \hat{C}_1(u) = f(C_1) = u \quad \text{for } u \in \mathbb{R} \ ;$$

$$f(C_2) = 2ie^{-x/2} + \frac{ia^2}{8}e^{x/2} \implies \begin{cases} \hat{C}_2(v) = f(C_2(x < 2\ln\frac{a}{4})) = -iv & \text{for } \frac{a}{4} < v < \infty \ , \\ \hat{C}_3(v) = f(C_2(x > 2\ln\frac{a}{4})) = iv & \text{for } \frac{a}{4} < v < \infty \ . \end{cases}$$

The unit normal vectors are calculated to $n_w(\hat{C}_1) = i, n_w(\hat{C}_2) = 1$ and $n_w(\hat{C}_3) = -1$.

So the surface charge density in terms of x is

$$\eta_{\hat{\mathcal{C}}_1}(x) = -\frac{\varepsilon_0 V_0}{\pi}\,\Im m\left\{ i\,\frac{1}{e^{x/2}+\frac{a^2}{16}e^{-x/2}}\right\} = -\frac{\varepsilon_0 V_0}{\pi}\,\frac{1}{e^{x/2}+\frac{a^2}{16}e^{-x/2}}$$

$$\text{for } \; x \in \mathbb{R}: \; \hat{C}_1(x) = 2e^{x/2} - \tfrac{a^2}{8}e^{-x/2} \;\; ;$$

$$\eta_{\hat{\mathcal{C}}_2}(x) = -\frac{\varepsilon_0 V_0}{\pi}\,\Im m\left\{ 1\,\frac{-i}{e^{-x/2}-\frac{a^2}{16}e^{x/2}}\right\} = \frac{\varepsilon_0 V_0}{\pi}\,\frac{1}{e^{-x/2}-\frac{a^2}{16}e^{x/2}}$$

$$\text{for } \; x < 2\ln\tfrac{a}{4}: \; \hat{C}_2(x) = f(C_2(x < 2\ln\tfrac{a}{4})) \;\; ;$$

$$\eta_{\hat{\mathcal{C}}_3}(x) = -\frac{\varepsilon_0 V_0}{\pi}\,\Im m\left\{ -1\,\frac{-i}{e^{-x/2}-\frac{a^2}{16}e^{x/2}}\right\} = -\frac{\varepsilon_0 V_0}{\pi}\,\frac{1}{e^{-x/2}-\frac{a^2}{16}e^{x/2}}$$

$$\text{for } \; x > 2\ln\tfrac{a}{4}: \; \hat{C}_3(x) = f(C_2(x > 2\ln\tfrac{a}{4})) \;\; .$$

Surface charge density and electric field of this example are very simple since the charge density for the left side of the upper plate is the same as for the right side. The field is of opposite sign on both sides. For $v \to \infty$ field and charge density tend to zero and for $v \to a$ they tend to infinity. The field and charge density on the lower plate are given by a bell shaped distribution.

Mathematica program 9: for the Infinitely Thin Aperture Condenser

```
Needs["Graphics`ComplexMap`"]

Clear[V,a,b,c,d,e,f,frb,fre,frf,chs,sgn9,decide]
V = 1 (* Voltage between the plates in Volt *) ;
a = 3 (* Shortest distance between the plates in mm :
         for 7 10^-5 < a < 20 (scd) and 7 10^-5 < a < 10^5 (ef) *) ;
decide = ef (* Decide whether you want the electric field (ef) or
                surface charge density (scd) of the plates *) ;

af90[z_, sgn9_] = 2 Exp[z/2] + sgn9 a^2/8 Exp[-z/2] ;
If[a < 1.7008433 10^2, chs = 1, chs = -1] ;

c = af90[b,1] ; d = af90[b,-1] ;
frb = FindRoot[d - 3 a, {b,1,10}]; b = b/.frb ;
fre = FindRoot[af90[e,-1] + 3 a,{e,-5,-15}]; e = e/.fre ;
frf = FindRoot[af90[-f,1] - c,{f,-10,-5}]; f = f/.frf ;

<< ~path/hilfsfile

a9 = ParametricPlot[{ af90[-x,1], 2 chs sgn mult/af90[-x,-1] },
      {x,f,-2 Log[a/4] - chs .1}, PlotRange -> {All,{0,sgn mult 6}},
      AxesOrigin -> {a,0}, AxesLabel -> {"iv[mm]",tex1},
      PlotLabel -> "Upper Plate: Left" ] ;

a9inv = ParametricPlot[{ -af90[-x,1], 2 chs sgn mult/af90[-x,-1] },
         {x,f,-2 Log[a/4] - chs .01},
         PlotRange -> {{-a,-c},mult 6{-branch ,1}},
         AxesOrigin -> {-a,0}, DisplayFunction -> Identity] ;

b9 = ParametricPlot[{ af90[-x,1], -2 chs mult/af90[-x,-1] },
      {x,-2 Log[a/4] + chs .01,-f - 4 Log[a/4]},
      PlotRange -> {{a,c},{0,mult 6}}, AxesOrigin -> {a,0} ,
      AxesLabel -> {"iv[mm]",tex1}, PlotLabel -> "Upper Plate: Right" ] ;

c9 = CartesianMap[af90[#,-1] &,{b,e},{0,Pi},
      PlotRange -> All, AxesLabel -> {"u[mm]","iv[mm]"},
      Epilog -> { Thickness[.011],{ Line[{{0,1.05 a},{0,1.02 c}}],
                                    Line[{{1.02 d,0},{-1.02 d,0}}] }} ] ;
```

```
d9 = ParametricPlot[{ af90[x,-1], -2 mult/af90[x,1] },{x,b,e},
      PlotRange -> {{d,-d},{0,-mult .7}},
      AxesLabel -> {"u[mm]","Lower Plate"},
      Epilog -> { Text[tex1,{d/15,-mult .4},{0,0},{0,1}] } ] ;

Show[GraphicsArray[{a9inv,b9}, GraphicsSpacing -> 0] /.
     (Ticks -> Automatic) -> (Ticks -> None) /.
     (AxesLabel -> {"iv[mm]",tex1}) -> (AxesLabel -> None) /.
     (PlotLabel -> "Upper Plate: Right") -> (PlotLabel -> None) /.
     (PlotRange -> {{a,c},{0,mult 6}}) ->
       (PlotRange -> {{a,c},mult 6{-branch ,1}}),
     Epilog -> { Text["<- (iv)                          (iv) ->",
                      {.5,If[decide == ef,.2,,.07]}],
                 Text[tex3,{.46,.29}] },
     DisplayFunction -> $DisplayFunction ] ;

Show[c9, Axes -> None] ;
Show[d9, Ticks -> None, AxesLabel -> {"u","Lower Plate"},
     Epilog -> { Text[tex2,{d/10,-mult .4},{0,0},{0,1}] } ] ;
```

Comments:

The program is structured like the other ones but I added a variable **chs** which does not occur in the other programs. For **a** = 170.08433 the lower limit ***xmin*** in **a9,a9inv,b9** becomes the upper limit ***xmax*** and vice versa. At this point the function changes sign which can be prevented by introduction of the variable **chs**. The pictures (**a9, b9, c9, d9**) are plotted with ticks. The following three **Show[ ]** commands produce these pictures without ticks. The first **Show[ ]** produces one picture of the left and right upper plate using **a9inv,b9**.

Variants:

- it is possible to change **ParametricPlot[ ]** and **CartesianMap[ ]** as described in section 2.1. The ranges of the *iv* co-ordinate and the location of the text plotted by the **Epilog** option of **a9, a9inv, b9**, the first and third**Show[ ]**as well as the starting values and options in **frb, fre, frf** vary for different values of **a** and have to be modified by hand (the location of the text is written in brackets behind the co-ordinates' value) :

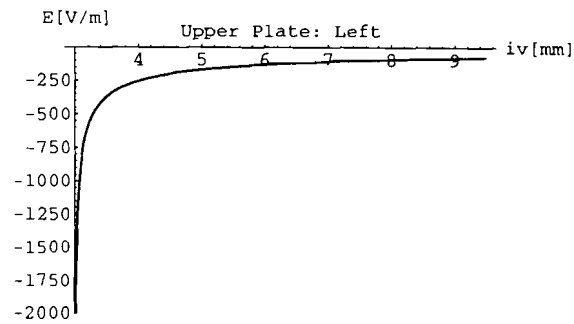
FindRoot[ ]	values of a	starting values	values of a	starting values
frb	**7 10^−5** to **1**	**{−1, −10 }**	**{ 1, 10^5 }**	**{1, 10 }**
fre	**7 10^−5** to **10^2**	**{−5, −15 }**	**{ 10^2, 10^5 }**	**{−5, −15 }**
frf	**7 10^−5** to **10^−1**	**{ 10, 5 }**	**{ 10^−1, 10^5 }**	**{−10, −5 }**

Graphics	*iv* coordinates for **a = 7 10^−5**	*iv* coordinates for **a = 1**	*iv* coordinates for **a = 10^5**
a9, a9inv,b9, first **Show**	**{... , ... mult 10^5 }**	**{... , ... mult 10 }**	**{... , ... mult 10^−4 }**
d9, third **Show[ ]**	**{... , ... mult 10^4 3 (2) }**	**{... , ... mult 2 (1) }**	**{... , ... mult 10^−5 2 (1) }**

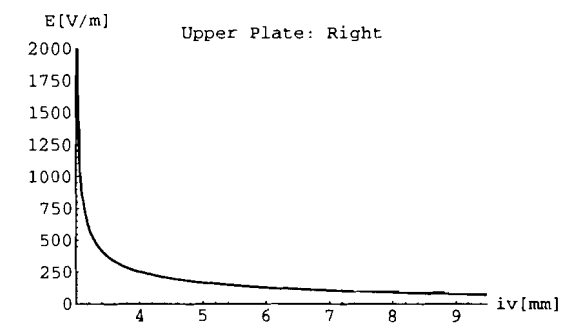
- since you will normally deal with condensers symmetric with the u-axis you can obtain that condenser by application of the Schwarz reflection principle. I have to replace the y co-ordinates **{0,Pi}** in the **Epilog** option of **c9** by **{−Pi,Pi}** and the "**Line[{{1.02 d,0},{−1.02 d,0}}]** " suboption by :

  Line[{{0,−1.05 a},{0,−1.02 c}}]

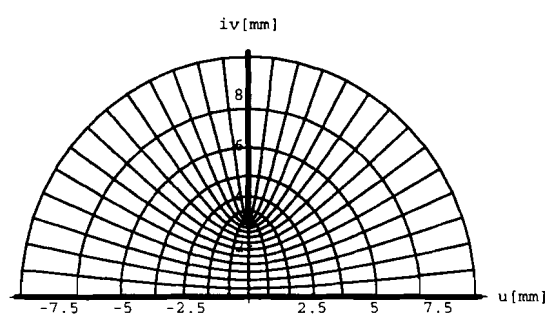
  The surface charge density of the lower plate will than be of oposite sign. In order to avoid confusion by the surface charge density of the middle plate that is not any more existent wrap (* *) around **d9 = ParametricPlot[...]** ; and **Show[d9, ...]** ; .



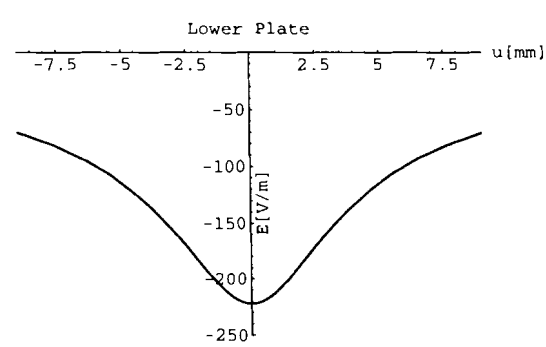
(a9)



(b9)

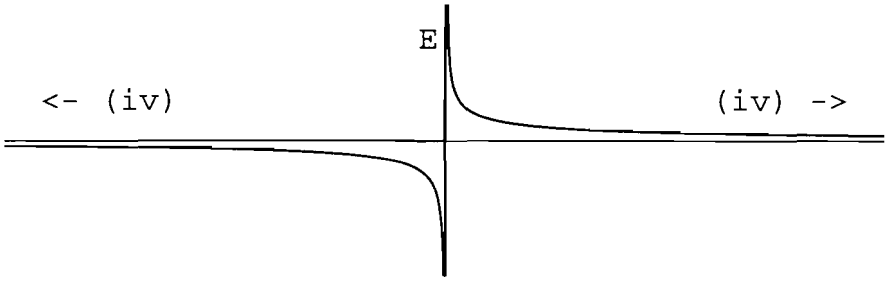


(c9)

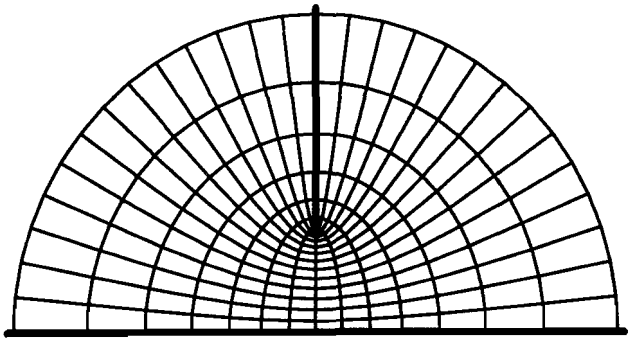


(d9)

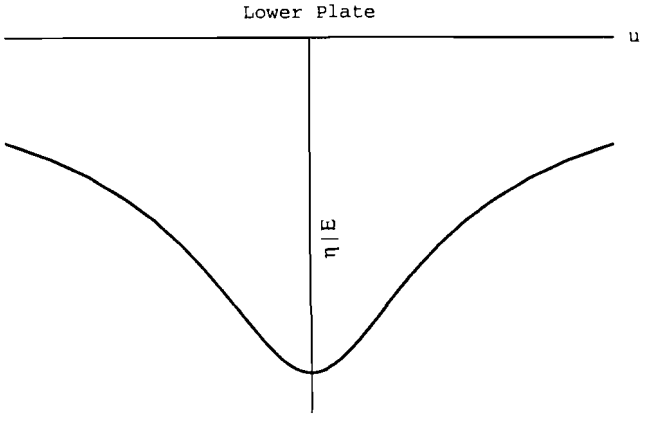
Figure 35: Graphics with ticks produced by the *Mathematica* program for the **Infinitely Thin Aperture Condenser**. The voltage between the plates is 1 V. The distance between the plates is 3 mm.



first **Show[ ]**



second **Show[ ]**



third **Show[ ]**

Figure 36: Graphics without ticks produced by the *Mathematica* program for the Infinitely Thin Aperture Condenser.

2.12 *Mathematica* Program for the Thick Aperture Condenser

$$w = ai\left[k'^2\tau - 2\mathrm{E}(\mathrm{am}(\tau,k),k) - \mathrm{cs}(\tau,k)\,\mathrm{dn}(\tau,k)\right]/(2\mathrm{E} - k'^2\mathrm{K}) - b$$
$$\text{with} \quad \tau = \mathrm{sn}^{-1}\sqrt{-\frac{e^z}{k}}$$

In this section half of an aperture with finite thickness is treated. The symmetry plane of this configuration is a metallic plate coinciding with the u-axis. The lower plate consist of three parts of which two are located at both sides of the iv-axis at a distance b. They both extend from $-\infty$ to $v = -a$. The third part connects these at $w = \pm b - ia$ making two corners.

The conformal function for this problem was calculated by N. Davy in the 1940's (cf. [da]). In this paper the whole problem is also treated numerically for the special thickness to gapwidth ratio $b/a = 1/2$ [da, p.824-833]. My *Mathematica* program can be applied to any ratio b/a between 0.0003 and 4.71.

The mathematic functions used in this section are not so familiar as those in the previous sections but they are still quite tractable from the mathematical point of view. In case you are not intimate with elliptic functions and integrals you should read part two of section 1.1 (anew).
The *Mathematica* programs, however, are of the same length. Unfortunately the time and memory space the computer needs for calculating the graphics will become quite long and large due to the complicated functions used.
Unlike the previous sections the symmetric condenser will be calculated, but *Mathematica* will only plot the upper or the lower half. The second half has to be constructed by use of the Schwarz refection principle.

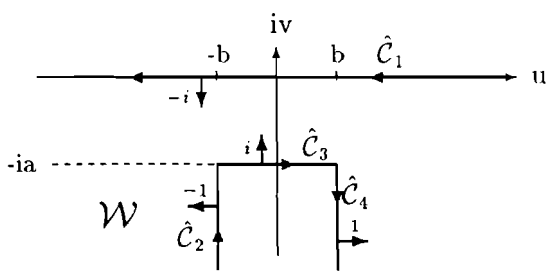


Figure 37: Curve $\hat{C}$ and normal vectors of the Thick Aperture Condenser.

For the convenience of the reader I show how the conformal map was constructed. Figure 38 shows the two problems that have to be solved: At first the mapping function $f_2(t)$ which maps the Flat Condenser to the Thick Aperture Condenser has to be found and integrated.

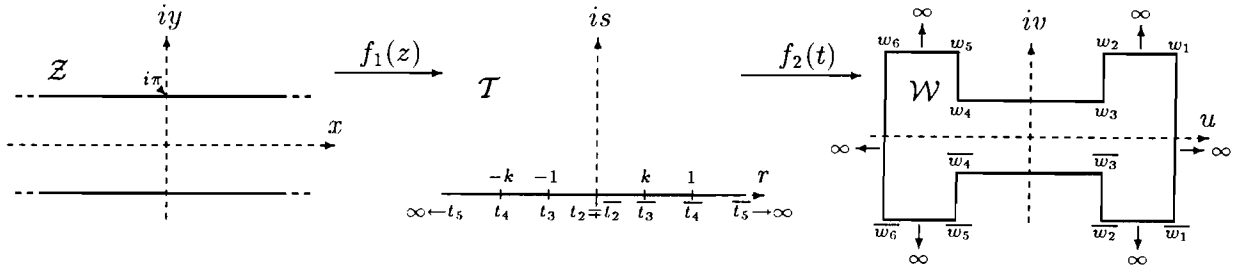


Figure 38: Image of the Infinite Plane Condenser in the Z-plane, that of the Flat Condenser in the T-plane and that of the Thick Aperture Condenser in the W-plane to demonstrate the application of the Schwarz-Christoffel transformation for this example.

Then the function $f_1^{-1}(t)$ must be constructed and inverted. The coupling of these two functions $f_1(t)$ and $f_2(t)$ makes up the conformal function $w = f(z) = f_2(t) \circ f_1(z)$ mapping the Infinite Plane Condenser to the Thick Aperture Condenser.

The application of the Schwarz-Christoffel transformation formula to the H-shape of the Thick Aperture Condenser leads to the equation

$$w = f_2(t) = A \int_{t_0}^{t} \frac{\sqrt{t+t_3}\,\sqrt{t-t_3}\,\sqrt{t+t_4}\,\sqrt{t-t_4}\,dt}{\sqrt{t+t_1}\,\sqrt{t-t_1}\,\sqrt{t+t_2}\,\sqrt{t-t_2}\,\sqrt{t+t_5}\,\sqrt{t-t_5}\,\sqrt{t+t_6}\,\sqrt{t-t_6}} + B$$

$$= A \int_{t_0}^{t} \frac{\sqrt{t^2-t_3^2}\,\sqrt{t^2-t_4^2}\,dt}{\sqrt{t^2-t_1^2}\,\sqrt{t^2-t_2^2}\,\sqrt{t^2-t_5^2}\,\sqrt{t^2-t_6^2}} + B \ . \tag{28}$$

I choose $t_5 = -\infty$ to remove the term $\sqrt{t^2 - t_5^2}$ from the product under the integral; moreover I let $t_2 = 0$ and $t_4 = -1$. In addition I rename $t_3 = -k$. k will turn out to be the modulus of the elliptic functions and integrals in $f_2(t)$. variable t_1 is given no special value.

To get the wanted form of the Thick Aperture Condenser I let all lines indicated by $\rightarrow\infty$ in figure 38 tend to infinity: $w_1, \overline{w_1}, w_2, \overline{w_2}, w_5, \overline{w_5}, w_6, \overline{w_6} \rightarrow\infty$. With that eq.(28) given above becomes

$$w = f_2(t) = A \int_{t_0}^{t} \frac{\sqrt{t^2-k^2}\,\sqrt{t^2-1}\,dt}{t} + B \ . \tag{29}$$

I substitute t for $t = 1/\mathrm{sn}(\tau, k)$ to get a form that can be integrated right away:

$$w = \tilde{f}_2(\tau) = -A \int_{\tau_0}^{\tau} \frac{\mathrm{cn}^2(\tau,k)\,\mathrm{dn}^2(\tau,k)\,d\tau}{\mathrm{sn}^2(\tau,k)} + B \ , \tag{30}$$

where I used eqs (731.01),(129.01) and (120.02) of [bf] (In the following I will always use formulas of [bf] without mentioning the source explicitely. All equations of the form (123.45)

are taken from [bf]). The use of (121.00) to rewrite eq.(30) in terms of $\mathrm{sn}^2(\tau,k)$ and integrating the terms using (311.02) and (310.02) of [bf] leads to the equation

$$f_2(t) = -A\,[k'^2\tau - 2\mathrm{E}(\mathrm{am}(\tau,k),k) - \mathrm{cn}(\tau,k)\,\mathrm{dn}(\tau,k)/\mathrm{sn}(\tau,k)\,] \;+\; B \qquad \text{with} \quad \tau = \mathrm{sn}^{-1}(1/t,k)\ . \tag{31}$$

To determine the constants A and B I calculate the values of $f_2(t)$ for $t = t_3 = -k$, $t = \overline{t_3} = k$, $t = t_4 = -1$ and $t = \overline{t_4} = 1$. Using eqs (122.00), (122.01), (130.02) and (111.09) I get the corresponding τ's:

$$\tau_3 = -(K + iK'),\ \ \overline{\tau_3} = K + iK' \quad \text{and} \quad \tau_4 = -K,\ \ \overline{\tau_4} = K\ .$$

After expressing the incomplete elliptic integral of the second kind E(am(τ,k),k) in terms of Jacobi's Zeta function using (140.01) as well as (141.01), (122.02), (122.18) and (122.01) I obtain

$$w_3 = \ b + ia = -A\left\{\left(-k'^2 + \frac{2\mathrm{E}}{\mathrm{K}}\right)(\mathrm{K} + i\mathrm{K}') - \frac{i\pi}{\mathrm{K}}\right\} \;+\; B\ , \tag{I}$$

$$\overline{w_3} = \ b - ia = -A\left\{\left(k'^2 - \frac{2\mathrm{E}}{\mathrm{K}}\right)(\mathrm{K} + i\mathrm{K}') + \frac{i\pi}{\mathrm{K}}\right\} \;+\; B\ , \tag{II}$$

$$w_4 = -b + ia = -A\left\{(-k'^2 + \frac{2\mathrm{E}}{\mathrm{K}})\mathrm{K}\right\} \;+\; B\ , \tag{III}$$

$$\overline{w_4} = -b - ia = -A\left\{(k'^2 - \frac{2\mathrm{E}}{\mathrm{K}})\mathrm{K}\right\} \;+\; B\ . \tag{IV}$$

This is a system of four equations for the unknowns A, B. Solving the system of the last two equations gives

$$III - IV \implies A = \frac{ia}{k'^2\mathrm{K} - 2\mathrm{E}} \quad \text{and}$$
$$III + IV \implies B = -b\ .$$

With that the function which maps the Flat Condenser to the Thick Aperture Condenser is

$$f_2(t) = \tilde{f}_2(\tau) = ia\,[k'^2\tau - 2\mathrm{E}(\mathrm{am}(\tau,k),k) - \mathrm{cs}(\tau,k)\,\mathrm{dn}(\tau,k)]/(2\mathrm{E} - k'^2\mathrm{K}) \;-b \qquad \text{with} \quad \tau = \mathrm{sn}^{-1}(t,k)\ . \tag{32}$$

Inserting A and B into eqs (I), (II), application of Legendre's relation (110.10) and solving for b/a gives the relation between a, b and the modulus k:

$$\frac{b}{a} = \pm\frac{(k'^2 - 2)\mathrm{K}' + 2\mathrm{E}'}{2(k'^2\mathrm{K} - 2\mathrm{E})}\ . \tag{33}$$

The plus sign belongs to the solution of (II) and the minus sign to that of (I). When I choose the plus sign the modulus $0 < k < 1$ leads to a ratio $0 < a/b < \infty$. Choosing the minus sign would demand a complex k to solve the equation for real ratios a/b. I choose the plus sign and therefore a given thickness and distance leads to a certain modulus $0 < k < 1$ which must be computed before I can insert this modulus into the conformal map.
The function f_1^{-1} can be obtained by the Schwarz-Christoffel fomula, too. It is

$$z = \alpha \ln t + \beta$$

with α and β still to be determined. Since z changes by $-2i\pi$ when t changes by $e^{i\pi}$ in going from t_2 to $\overline{t_2}$, $\alpha = -2$. I let the point $z = i\pi$ correspond to the point $t = -\sqrt{k}$ which becomes the centre of the line connecting t_3 with t_4. Therewith β becomes $\beta = i\pi + 2\ln(-\sqrt{k})$. Using equation $i\pi = \ln(-1) = 2\ln i$ I get

$$z = f_1^{-1}(t) = \ln(\frac{-k}{t^2}) \ .$$

Now I insert the transformation formula $t = 1/\mathrm{sn}(\tau, k)$ into the equation from above, invert the function obtained and get the modified function $\tilde{f}_1$

$$\tau = \tilde{f}_1(z) = \mathrm{sn}^{-1}(\sqrt{-e^z/k}, k) \ . \tag{34}$$

With that the relation $f(z) = \tilde{f}_2(\tau) \circ \tilde{f}_1(z)$ is

$$w = f(z) = ia\,[k'^2\tau - 2\mathrm{E}(\mathrm{am}(\tau,k),k) - \mathrm{cs}(\tau,k)\,\mathrm{dn}(\tau,k)]/(2\mathrm{E} - k'^2\mathrm{K}) \ - b$$
$$\text{with} \quad \tau = \mathrm{sn}^{-1}(\sqrt{-\tfrac{e^z}{k}}, k) \ .$$

This is exactly the equation given by Davy. It can be simplified a lot when the relation (131.01) in [bf] is used. I have to write down $\mathrm{sn}^{-1}(t,k)$ as the inverse functions of $\mathrm{am}(\tau,k)$, $\mathrm{cs}(\tau,k) = \mathrm{cn}(\tau,k)/\mathrm{dn}(u\tau,k)$ and $\mathrm{dn}(\tau,k)$ and insert these into the function $w = f(z)$. The result looks a lot nicer than before (I have also used (130.02)) :

$$f(z) = \frac{ia}{2\mathrm{E} - k'^2\mathrm{K}}\left[k'^2\mathrm{F}(\arcsin t,k)) - 2\mathrm{E}(\arcsin t,k)) - \frac{\sqrt{1-t^2}\,\sqrt{1-k^2t^2}}{t}\right] - b$$
$$\text{with} \quad t = \sqrt{-\tfrac{e^z}{k}} \tag{35}$$

For any ratio of b/a the modulus k is $0 < k < 1$; which bounds can be veryfied by a graph of the above equation. So the function $f(z)$ is holomorphic in $\mathbb{C}$ and the derivative

$$f'(z) = \frac{ia}{4\mathrm{E} - 2k'^2\mathrm{K}}\frac{\sqrt{1-t^2}\,\sqrt{1-k^2t^2}}{t} \qquad \text{is zero for } z = i\pi \pm \ln k.$$

With that the domain $\mathbf{D}_z$ is $\mathbf{D}_z = \mathbf{D}_{id} \backslash \{z = i\pi \pm \ln k\}$.

The curve $\hat{C}$ on the W-plane consists of four parts : $\hat{C}_1$ is the image of C_1 , $\hat{C}_2$ that of $C_2(x < -\ln k)$, $\hat{C}_3$ that of $C_2(-\ln k < x < \ln k)$ and $\hat{C}_4$ that of $C_2(x > \ln k)$.

$$f(C_1) = \frac{ia}{2\mathrm{E} - k'^2\,\mathrm{K}}\left[k'^2\,\mathrm{F}(\arcsin(i\tilde{t}),k) - 2\mathrm{E}(\arcsin(i\tilde{t}),k) - \frac{\sqrt{1+e^{-x}/k}\,\sqrt{1+ke^{-x}}}{i\tilde{t}}\right] - b$$

$$= \frac{-a}{2\mathrm{E} - k'^2\,\mathrm{K}}\left\{(1+k^2)\mathrm{F}(\beta,k') - 2\mathrm{E}(\beta,k') + 2\tan\left[\beta - \sqrt{1-k'^2\sin^2\beta}\right] - \frac{\sqrt{1+\tilde{t}^2}\,\sqrt{1+k^2\tilde{t}^2}}{\tilde{t}}\right\} - b$$

$$\tilde{t} = \sqrt{e^{-x}/k}\ , \qquad \sinh[\ln(-\tilde{t} + \sqrt{1+\tilde{t}^2}\,)] = \tan\beta$$

$$\Longrightarrow \quad \hat{C}_1(u) = f(C_1) = -u \quad \text{for} \quad u \in \mathbb{R} \ ;$$

$$f(C_2(x < -\ln k)) = \frac{ia}{2\mathrm{E} - k'^2\,\mathrm{K}}\left[k'^2\,\mathrm{F}(\arcsin\hat{t},k) - 2\mathrm{E}(\arcsin\hat{t},k) - \frac{\sqrt{1-e^x/k}\,\sqrt{1-ke^x}}{\hat{t}}\right] - b$$

$$\hat{t} = \sqrt{e^x/k}$$

$$\Longrightarrow \quad \hat{C}_2(v) = f(C_2(x < -\ln k)) = -b + iv \quad \text{for} \quad -\infty < v < -a \ ;$$

$$f(C_2(lnk < x < \ln k)) = \frac{ia}{2\mathrm{E} - k'^2\,\mathrm{K}}\left[k'^2\,\mathrm{F}(\arcsin\hat{t},k) - 2\mathrm{E}(\arcsin\hat{t},k) - \frac{i\sqrt{e^x/k-1}\,\sqrt{1-ke^x}}{\hat{t}}\right] - b$$

$$= -\frac{a}{2\mathrm{E} - k'^2\,\mathrm{K}}\left[(1+k^2)\mathrm{F}(A,k') - 2\mathrm{E}(A,k') + \frac{\sqrt{e^x/k-1}\,\sqrt{1-ke^x}}{\hat{t}}\right] - b - ia$$

$$A = \arcsin\left[\frac{\sqrt{\hat{t}^2-1}}{k'\hat{t}}\right]$$

$$\Longrightarrow \quad \hat{C}_3(u) = f(C_2(-\ln k < x < \ln k)) = u - ia \quad \text{for} \quad -b < u < b \ ;$$

$$f(C_2(x > \ln k)) = \frac{ia}{2\mathrm{E} - k'^2\,\mathrm{K}}\left[k'^2\,\mathrm{F}(\arcsin\hat{t},k) - 2\mathrm{E}(\arcsin\hat{t},k) - \frac{i\sqrt{e^x/k-1}\,\sqrt{ke^x-1}}{\hat{t}}\right] - b$$

$$= \frac{ia}{2\mathrm{E} - k'^2\,\mathrm{K}}\left[k'^2\,\mathrm{F}(\beta,k) - 2\mathrm{E}(\beta,k) + \frac{\sqrt{e^x/k-1}\,\sqrt{ke^x-1}}{\hat{t}}\right] + b$$

$$\beta = \arcsin\left[\frac{1}{k\hat{t}}\right]$$

$$\Longrightarrow \quad \hat{C}_4(v) = f(C_2(x > \ln k)) = -b - iv \quad \text{for} \quad a < v < \infty \ .$$

The unit normal vectors are calculated to $n_w(\hat{\mathcal{C}}_1) = -i, n_w(\hat{\mathcal{C}}_2) = -1, n_w(\hat{\mathcal{C}}_3) = i$ and $n_w(\hat{\mathcal{C}}_4) = 1$.

The surface charge density in terms of x is:

$$\begin{aligned}
\eta_{\hat{\mathcal{C}}_1}(x) &= -\frac{\varepsilon_0 V_0}{\pi}\, \Im m\left\{-i\,\frac{4\mathrm{E}-2k'^2\,\mathrm{K}}{ia}\,\frac{i\sqrt{e^{-x}/k}}{\sqrt{1+e^{-x}/k}\,\sqrt{1+ke^{-x}}}\right\} \\
&= \frac{\varepsilon_0 V_0}{a}\,\frac{4\mathrm{E}-2k'^2\,\mathrm{K}}{\pi}\,\frac{\sqrt{e^{-x}/k}}{\sqrt{1+e^{-x}/k}\,\sqrt{1+ke^{-x}}} \\
&\quad \text{for} \quad x \in \mathbb{R} \quad : \ \hat{C}_1(x) = f(C_1) \ ;
\end{aligned}$$

$$\begin{aligned}
\eta_{\hat{\mathcal{C}}_2}(x) &= -\frac{\varepsilon_0 V_0}{\pi}\, \Im m\left\{-1\,\frac{4\mathrm{E}-2k'^2\mathrm{K}}{ia}\,\frac{i\sqrt{e^{x}/k}}{\sqrt{1-e^{x}/k}\,\sqrt{1-ke^{x}}}\right\} \\
&= -\frac{\varepsilon_0 V_0}{a}\,\frac{4\mathrm{E}-2k'^2\mathrm{K}}{\pi}\,\frac{\sqrt{e^{x}/k}}{\sqrt{1-e^{x}/k}\,\sqrt{1-ke^{x}}} \\
&\quad \text{for} \quad x < -\ln k \quad : \ \hat{C}_2(x) = f(C_2(x < -\ln k)) \ ;
\end{aligned}$$

$$\begin{aligned}
\eta_{\hat{\mathcal{C}}_3}(x) &= -\frac{\varepsilon_0 V_0}{\pi}\, \Im m\left\{i\,\frac{4\mathrm{E}-2k'^2\,\mathrm{K}}{ia}\,\frac{\sqrt{e^{x}/k}}{i\sqrt{e^{x}/k-1}\,\sqrt{1-ke^{x}}}\right\} \\
&= \frac{\varepsilon_0 V_0}{a}\,\frac{4\mathrm{E}-2k'^2\,\mathrm{K}}{\pi}\,\frac{\sqrt{e^{x}/k}}{\sqrt{e^{x}/k-1}\,\sqrt{1-ke^{x}}} \\
&\quad \text{for} \quad -\ln k < x < \ln k \quad : \ \breve{C}_3(x) = f(C_2(-\ln k < x < \ln k)) \ ;
\end{aligned}$$

$$\begin{aligned}
\eta_{\hat{\mathcal{C}}_4}(x) &= -\frac{\varepsilon_0 V_0}{\pi}\, \Im m\left\{1\,\frac{4\mathrm{E}-2k'^2\,\mathrm{K}}{ia}\,\frac{\sqrt{e^{x}/k}}{-\sqrt{e^{x}/k-1}\,\sqrt{ke^{x}-1}}\right\} \\
&= -\frac{\varepsilon_0 V_0}{a}\,\frac{4\mathrm{E}-2k'^2\,\mathrm{K}}{\pi}\,\frac{\sqrt{e^{x}/k}}{\sqrt{e^{x}/k-1}\,\sqrt{ke^{x}-1}} \\
&\quad \text{for} \quad x > \ln k \quad : \ \hat{C}_4(x) = f(C_2(x > \ln k)) \ .
\end{aligned}$$

Mathematica program 10: for the Thick Aperture Condenser

```
Needs["Graphics`ComplexMap`"]

Clear[V,a,b,c,d,e,t,ksq,frc,frksq,sgn10a,sgn10b,decide]
V = 1 (* Voltage between the plates in Volt *) ;
a = 3 (* Half the distance of the plates in mm :
         for all 2 10^-5 < a < 25 (scd) and 2 10^-5 < a < 10^5 (ef) *);
b = a/2 (* Half the thickness of the plates in mm :
           for all .0003 a < b < 4.71 a *);
decide = ef (* Decide whether you want the electric field (ef) or
                surface charge density (scd) of the plates *) ;

af100 = .5 (2 EllipticE[1 - ksq] - (1 + ksq) EllipticK[1 - ksq])/
            (EllipticK[ksq] (1 - ksq) - 2 EllipticE[ksq]);

frksq = FindRoot[af100 - b/a, {ksq,.01,.05}]; ksq = ksq/.frksq ;

t[z_, sgn10a_] = Sqrt[sgn10a Exp[z]/Sqrt[ksq]] ;
af101[z_] = a I/(2 EllipticE[ksq] - (1 - ksq) EllipticK[ksq])*
          ((1 - ksq) EllipticF[ArcSin[t[z,-1]],ksq] -
           2 EllipticE[ArcSin[t[z,-1]],ksq] -
           Sqrt[1 - ksq t[z,-1]^2] Sqrt[1 - t[z,-1]^2]/t[z,-1]) - b ;
af102[z_, sgn10b_] = 2(2 EllipticE[ksq] - (1 - ksq) EllipticK[ksq])/a *
                 t[z,1]/(Sqrt[1 + sgn10b t[z,1]^2] Sqrt[1 + sgn10b ksq t[z,1]^2]) ;

d = 1.01 af101[c]; e = 1.01 Im[af101[c - I Pi]] ;
frc = FindRoot[af101[c] - 2.5 a - b,{c,1,10},
       AccuracyGoal -> 5 - Floor[N[Log[10,a]]]]; c = c/.frc ;

<< ~path/hilfsfile

a10 = ParametricPlot[{ af101[-x], sgn mult af102[-x,1] }, {x,-c,c},
      PlotRange -> {All,{0,sgn mult/a 3.2}}, AxesLabel -> {"u[mm]","Upper Plate"},
      Epilog -> { Text[tex1,{d/10,sgn mult/a},{0,0},{0,1}] } ] ;

b10 = CartesianMap[af101, {-c,c},{-Pi + 10^-10, -10^-10}, PlotRange -> All,
      AxesLabel -> {"u[mm]","iv[mm]"},
      Epilog -> { Thickness[.011],
                 { Line[{{-d,0},{d,0}}], Line[{{-b,e},{-b,-a},{b,-a},{b,e}}] }} ] ;
```

```
c10 = ParametricPlot[{ Im[af101[-Pi I + x]], -sgn mult Re[af102[x,-1]] },
      {x,-c, Log[Sqrt[ksq]] - .001}, PlotRange -> {{-a,e},{0,-sgn mult/a 7}},
      AxesOrigin -> {-a,0}, AxesLabel -> {"iv[mm]",tex1},
      PlotLabel -> "Lower Plate: Left"] ;

d10 = ParametricPlot[{ Re[af101[-Pi I + x]], mult Im[af102[x,-1]] },
      {x, Log[Sqrt[ksq]] + 10^-5, Log[1/Sqrt[ksq]] - 10^-5},
      PlotRange -> {All,{0,-mult/a 10}},
      AxesLabel -> {"u[mm]","Lower Plate: Horizontal"},
      Epilog -> { Text[tex1,{b/15,-mult/a 8},{0,0},{0,1}] } ] ;

e10 = ParametricPlot[{ Im[af101[-Pi I + x]], mult Re[af102[x,-1]] },
       {x, Log[1/Sqrt[ksq]] + .001,c}, PlotRange -> {{-a,e},{0,-mult/a 7}},
       AxesOrigin -> {-a,0}, AxesLabel -> {"iv[mm]",tex1},
       PlotLabel -> "Lower Plate: Right"] ;

e10inv = ParametricPlot[{ -Im[af101[-Pi I + x]], mult Re[af102[x,-1]] },
          {x, Log[1/Sqrt[ksq]] + .001,c},
          PlotRange -> {{a,-e},-mult/a 10{-branch,1}},
          DisplayFunction -> Identity] ;

Show[a10, Ticks -> None, AxesLabel -> {"u","Upper Plate"},
     Epilog -> {Text[tex2,{d/10,sgn mult/a},{0,0},{0,1}] } ] ;
Show[b10, Axes -> None] ;

Show[GraphicsArray[{c10, d10, e10inv}, GraphicsSpacing -> 0] /.
      (Ticks -> Automatic) -> (Ticks -> None) /.
      (AxesLabel -> {"iv[mm]",tex1}) -> (AxesLabel -> {None,None}) /.
      (AxesLabel -> {"u[mm]","Lower Plate: Horizontal"}) ->
        (AxesLabel -> {None,None}) /.
      (PlotLabel -> "Lower Plate: Left") -> (PlotLabel -> None) /.
      (PlotLabel -> "Lower Plate: Right") -> (PlotLabel -> None) /.
      (Epilog -> { Text[tex1,{b/15,-mult/a 8},{0,0},{0,1}] }) ->
        (Epilog -> { Text[tex3,{b/10,-mult/a 7}] }) /.
      (PlotRange -> {{-a,e},{0,-sgn mult/a 7}}) ->
        (PlotRange -> {{-a,e},-mult/a 10{1,-branch}}) /.
      (PlotRange -> {All,{0,-mult/a 10}}) ->
        (PlotRange -> {All,-mult/a 10{1,-branch}}),
      Epilog -> { Text["(iv) ->              u ->                    <- (iv)",
                      {.5,If[decide == ef,.13,,.16]}] },
      PlotLabel -> "Lower Plate", DisplayFunction -> $DisplayFunction ] ;
```

Comments:

The graphics commands named **a10,b10,c10,d10,e10** plot the pictures of the electric field or the surface charge density and the field distribution with ticks. The three **Show[ ]** commands draw the pictures **a10** to **e10** without ticks whereby the third **Show[ ]** unites **c10,d10,e10inv** in one picture.

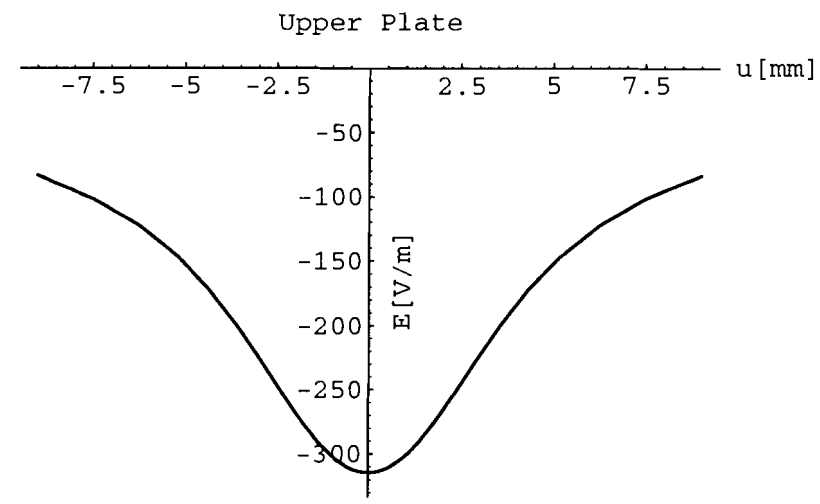
Variants:

▲ it is possible to change **ParametricPlot[ ]** and **CartesianMap[ ]** as described in section 2.1. The ranges of the *iv* co-ordinate and the location of the text in **Epilog** of **a10,c10,d10,e10,e10inv** and the first and third **Show[ ]** as well as the starting values and options in **frksq** vary for different values of **b/a** and have to be modified by hand (the location of the text is written in brackets behind the co-ordinates' value) :

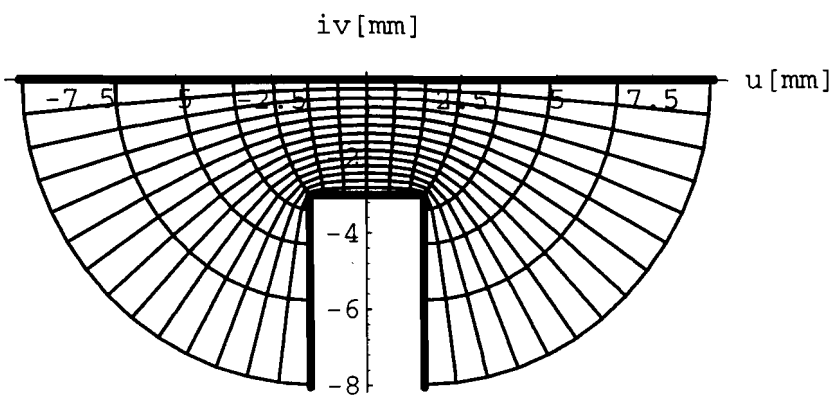
value of b	frksq starting values	WorkingPrecision	Max Iterations
.0003 a to a	{ .01, .05 }		
a to 4.71 a	{ 10^-14, 10^-15 }	25	30

Graphics	*iv* coordinates for b = .0003 a	*iv* coordinates for b = a	*iv* coordinates for b = 4.71 a
a10	{... , ... mult/a 2 (1) }	{... , ... mult/a 3.2 (1) }	{... , ... mult 3.2 (1) }
c10, e10	{... , ... mult/a 10 }	{... , ... mult/a 7 }	{... , ... mult 7 }
d10	{... , ... mult/a 1000 }	{... , ... mult/a 10 }	{... , ... mult 10 }
first Show[ ]	({..., ...mult/a })	({..., ...mult/a 2 })	({..., ...mult/a 2 }) }
e10inv,third Show[ ]	mult/a 200{...}	mult/a 10{...}	mult/a 10{...}

▲ The case of the Thick Aperture Condenser which is symmetric with the u-axis can be obtained by application of the Schwarz reflection principle. I have to delete the ", **Line[{{-d,0},{d,0}}]** " suboption (don't forget the comma !) in the **Epilog** option of **b10** and to add another picture:
b10up = CartesianMap[Conjugate[af101[#]] &, {-c,c},
{-Pi + 10^-10, -10^-10}, PlotRange -> All, Prolog -> {{ Thickness[.011],
Line[{{-b,-e},{-b,a},{b,a},{b,-e}}] }}] ;
The **Prolog** option does the same as **Epiolg**, however, not after but before the picture is plotted. That is why a second pair of curly brackets is wrapped around the **Thickness** and **Line** suboptions to prevent that all lines are drawn so thick.
To display both the upper (**b10up**) and lower (**b10**) part of the symmetric condenser in one picture you add:
Show[b10, b10up] ; .
This picture is drawn with ticks. To get the "pure" version add:
", **Axes -> None** " as an option to the **Show[ ]** command. The surface charge density of the upper plate will than be of opposite sign. In order to avoid confusion by the surface charge density of the middle plate that is not any more existent wrap (* *) around **a10 = ParametricPlot[...] ;** and **Show[a10, ...] ;** .

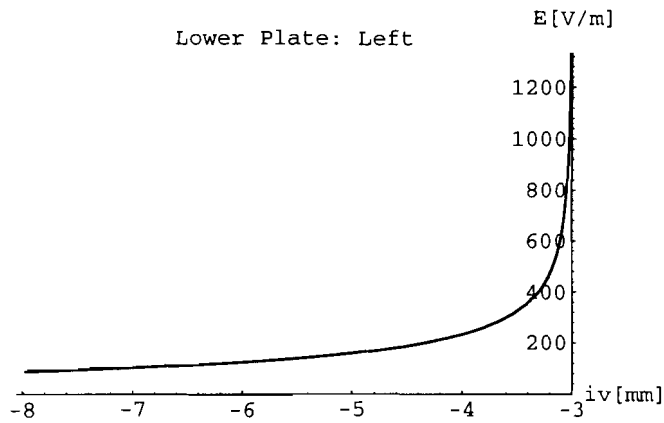


(a10)

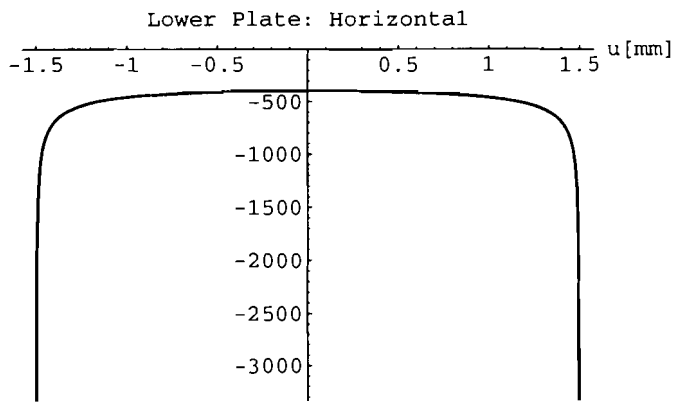


(b10)

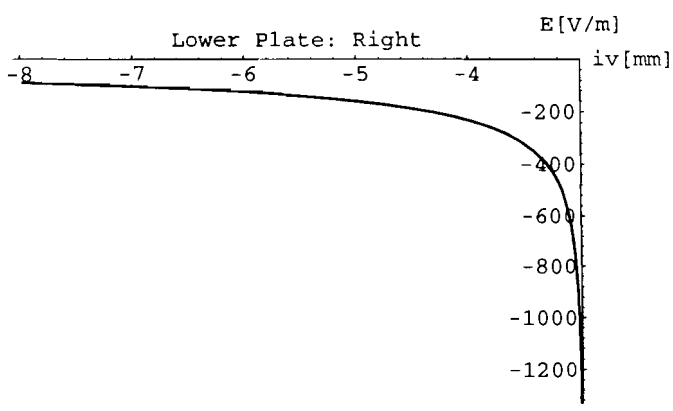
Figure 39: Graphics with ticks produced by the *Mathematica* program for the Thick Aperture Condenser. The potential difference between the plates is 1 V. The distance between the parallel branches of the plates is 3 mm. The thickness of the lower plate is 1.5 mm.



(c10)

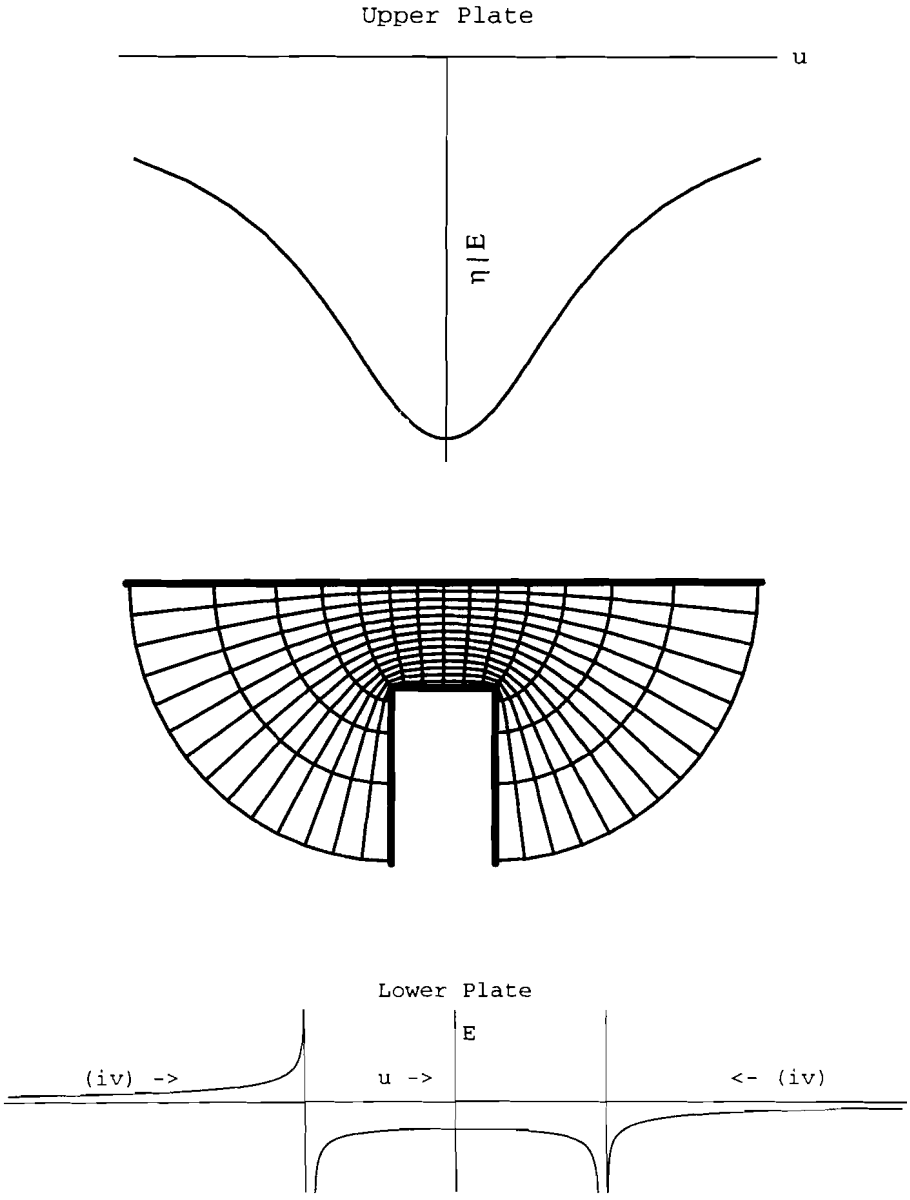


(d10)



(e10)

Figure 40: Graphics with ticks produced by the *Mathematica* program for the Infinitely Thin Aperture Condenser.



first **Show[ ]**

second **Show[ ]**

third **Show[ ]**

Figure 41: Graphics without ticks produced by the *Mathematica* program for the Infinitely Thin Aperture Condenser.

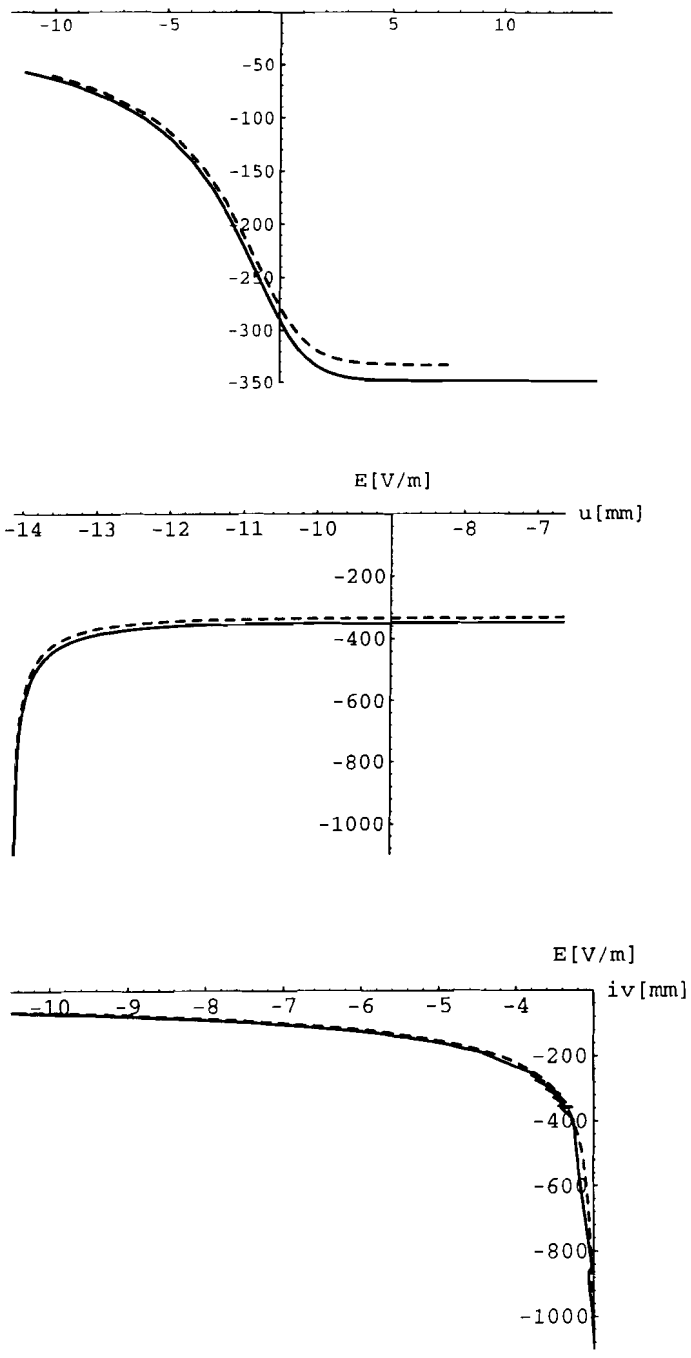


Figure 42: These graphics show that the electric field of the Thick Aperture Condenser for large values of the ratio: plate thickness to distance ($\mathbf{b/a = 4.71}$ for $\mathbf{a = 3}$ mm , $\mathbf{V = 1}$ Volt) can be approximated by use of the One Corner of 90° Condenser. The picture a) shows the fields of the "thick" upper and that of the "one corner" lower plate (dashed), where the origin of **d5** was shifted by **4.71 a**. The picture b) shows the fields of the left half of the "thick" horizontal lower and that of the "one corner" horizontal upper plate (dashed) (the origin of **b5** was shifted). The picture c) shows the fields of the right "thick" vertical lower and that of the "one corner" vertical upper plate (dashed) (the origin of **a5** was shifted).

2.13 *Mathematica* Program for the Finite Plane Condenser

$$w = f(z) = \frac{a}{\pi}\{(\mathrm{E'} - \mathrm{K'})\,\mathrm{F}(\arcsin t, k) + \mathrm{K'}\ \mathrm{E}(\arcsin t, k)\} \quad \text{with} \quad t = \mathrm{sn}(z, k)$$

In section 2.3 I treated the Infinite Plane Condenser, in section 2.5 the infinitely thin Semi-Infinite Plane Condenser followed by the finitely thick version in section 2.6 . In a line with these problems this chapter deals with the Finite Plane Condenser whose two plates are of zero thickness but have limited length: This corresponds to two equal and parallel strip lines[17].
The construction of this example's conformal map is published in [od] and very brievely in [bf, (119.01) and (119.03)].
The function

$$f(z) = \frac{a}{\pi}\{(\mathrm{E'} - \mathrm{K'})\,\mathrm{F}(\arcsin y, k) + \mathrm{K}\ \mathrm{E}(\arcsin y, k)\} \quad \text{with} \quad y(z) = \mathrm{sn}(z, k)$$

can again be simplified by the use of equations (130.02) of [bf] and $\mathrm{sn}(u, k) = \sin(\mathrm{am}(u, k))$ as indicated at the beginning of the previous section.
This yields

$$f(z) = \frac{a}{\pi}\{(\mathrm{E'} - \mathrm{K'})\,z + \mathrm{K}\ \mathrm{E}(\mathrm{am}(z, k), k)\}$$

a function with just one elliptic function as argument of an elliptic integral (the complete elliptic integrals are constant and therefore don't count). The function is holomorpic in $\mathbb{C}$ for $0 < k < 1$.
As in section 2.10 the ratio of distance a to length b of the planes is directly connected to the moduli k and k' by the equation

$$\frac{b}{a} = \frac{2}{\pi}(\mathrm{K'}\ \mathrm{E}[\arcsin(\frac{1}{k'}\sqrt{1 - \mathrm{E'}/\mathrm{K'}}), k'] - \mathrm{E'}\ \mathrm{F}[\arcsin(\frac{1}{k'}\sqrt{1 - \mathrm{E'}/\mathrm{K'}}), k']) \ .$$

The first derivative

$$f'(z) = \frac{a}{\pi}[\mathrm{E'} - k^2\mathrm{K'}\mathrm{sn}^2(z, k))]$$

has an infinite number of zeros at the points

$$z + 4m\,\mathrm{K} + 2n\,i\mathrm{K'} = \mathrm{nd}^{-1}(\pm\frac{1}{k}\sqrt{\mathrm{E'}/\mathrm{K'}}, k)$$

---

[17] The Schwartz-Christoffel formula for the finite condenser with finitely thick plates leads to an equation which is not even an elliptic integral but something really ugly since it is of the form $I = \int \sqrt{P}/\sqrt{Q}\ dt$ where P, Q are 4th order polynomials of the variable t with different coefficients.

where n and m are integers including zero.

Therefore the domain $\mathbf{D}_z$ becomes $\mathbb{C}\backslash\{\mathrm{nd}^{-1}(\pm\frac{1}{k}\sqrt{\mathrm{E'/K'}}, k) + 4m\,\mathrm{K} + 2n\,i\mathrm{K'}\}$.

The infinite number of zeros results from the periodicity $\omega = 4m\,\mathrm{K} + 4n\,i\mathrm{K'}$ of the elliptic function $\mathrm{sn}(u,k)$. Since corresponding points of different period parallelograms on the Z-plane are mapped to the same points on the W-plane and since $\mathrm{sn}(u+2\mathrm{K},k) = -\mathrm{sn}(u,k)$ [bf, (122.04)] the domain for the conformal map will be half[18] a period parallelogram. This implies the domain $\mathbf{D}_z$ used here is finite; so this differs from the domains used in the previous examples. The potential for the "Infinite" Plane Condenser has to be modified too, since its plates are now located parallel to the iv-axis. It reads $\Phi = \frac{V_0 z}{2\mathrm{K}} + \frac{V_0}{2}$ with V_0 the voltage between the plates. So the left plate is at zero potential and the right one is at the potential V_0 .
With this new potential the eq.(15) for the surface charge density is

$$\eta = -\frac{\varepsilon_0 V_0}{2\mathrm{K}}\,\Re e\left\{\frac{n_w}{f'(z)}\right\} \quad .$$

Equations (27), however, are still valid.

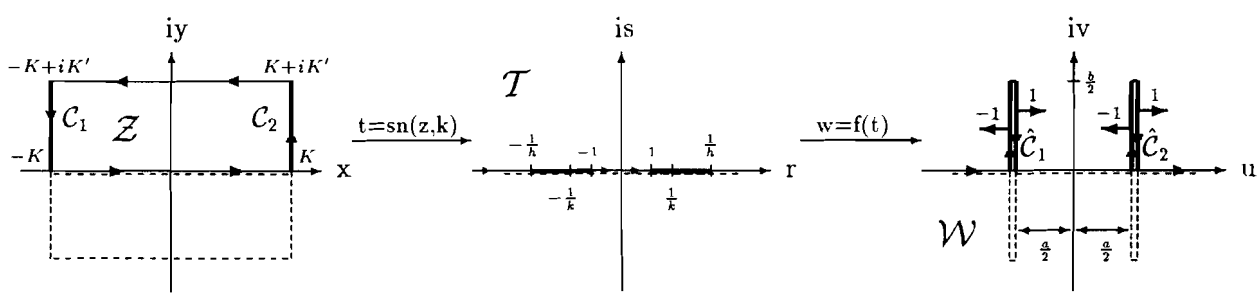


Figure 43: Domains and ranges for the Finite Plane Condenser.

The picture above shows in detail how the domain $\mathbf{D}_z \subset Z$-plane is mapped to the T-plain by the sine amplitude and from there to W-plane by the function $f(t)$. The thick lines denote the parts which become the condenser plates in the upper half of the W-plane. The dashed lines in the pictures are the boundaries of the lower half of the domain $\mathbf{D}_z$ and of its images. Although the domain $\mathbf{D}_z$ is finite its images are the whole T- and W-plane, which are both infinite.

Unlike curves $\mathcal{C}$ in the other sections the curve $\mathcal{C}$ here is finite and it consists of two

---

[18] Consider the sine with a period of 2π. Allthough the function's value after half the period is of opposite sign its range [-1,1] is already complete. Thus for symmetric problems like ours half the period is enough.

unconnected parts which are located parallel to both sides of the iv-axis at a distance K and have a length 2K'. The unconnected parts C_1 and C_2 in combination with the sine amplitude which maps the (finite) fundamental periodic parallelogram to the whole Gaussian plane is the trick which has to be used to get two unconnected plates as the image of a simply connected domain. The images of the upper half of the curve C in Z-plane are (the lower half might be gotten by the Schwarz reflection principle which, as mentioned, means to apply the conformal map to the lower part of the domain $\mathbf{D}_z$):

$$\begin{aligned} f(C1) &= \tfrac{a}{\pi}\{(\mathrm{E'}-\mathrm{K'})(-\mathrm{K}-iy)+\mathrm{K'}\ \mathrm{E}[\arcsin(\mathrm{sn}(-\mathrm{K}-iy,k)),k]\} \\ &= -\tfrac{a}{2}-\tfrac{ia}{\pi}\{\mathrm{E'}\,y-\mathrm{K'}\ \mathrm{E}[\mathrm{am}(y,k'),k']+k'^2\ \mathrm{K'}\,\mathrm{sd}(y,k')\,\mathrm{cn}(y,k')\} \end{aligned}$$

$$\Longrightarrow \hat{C}_1(v)=\begin{cases} f(C_1(-\mathrm{K'}<y<-\mathrm{nd}^{-1}(\tfrac{1}{k}\sqrt{\mathrm{E'}/\mathrm{K'}},k)\ )) = \ iv \ \text{ for } \ 0<v<\tfrac{b}{2}\ ; \\ f(C_1(-\mathrm{nd}^{-1}(\tfrac{1}{k}\sqrt{\mathrm{E'}/\mathrm{K'}},k)<y<0\ )) \ = -iv \ \text{ for } \ -\tfrac{b}{2}<v<0\ ; \end{cases}$$

$$\begin{aligned} f(C2) &= \tfrac{a}{\pi}\{(\mathrm{E'}-\mathrm{K'})(\mathrm{K}+iy)+\mathrm{K'}\ \mathrm{E}[\arcsin(\mathrm{sn}(\mathrm{K}+iy,k)),k]\} \\ &= \tfrac{a}{2}+\tfrac{ia}{\pi}\{\mathrm{E'}y-\mathrm{K'}\ \mathrm{E}[\mathrm{am}(y,k'),k']+k'^2\ \mathrm{K'}\,\mathrm{sd}(y,k')\,\mathrm{cn}(y,k')\} \end{aligned}$$

$$\Longrightarrow \hat{C}_2(v)=\begin{cases} f(C_2(\ 0<y<\mathrm{nd}^{-1}(\tfrac{1}{k}\sqrt{\mathrm{E'}/\mathrm{K'}},k)\ )) \ = \ iv \ \text{ for } \ 0<v<\tfrac{b}{2}\ ; \\ f(C_2(\ \mathrm{nd}^{-1}(\tfrac{1}{k}\sqrt{\mathrm{E'}/\mathrm{K'}},k)<y<\mathrm{K'}\ )) = -iv \ \text{ for } \ -\tfrac{b}{2}<v<0\ . \end{cases}$$

Since the left and right sides of the plates have the same co-ordinates it is particularly important not to mix up the directions of the curves.

The unit normal vectors are shown in the picture above. With these the surface charge density in terms of y is

$$\eta_{\hat{C}_1}(y)=-\frac{\varepsilon_0 V_0}{2\mathrm{K}}\ \Re e\left\{\mp 1\frac{\pi}{a}[\mathrm{E'}-k^2\ \mathrm{K'}\,\mathrm{sn}^2(-\mathrm{K}-iy,k))]^{-1}\right\}=\pm\frac{\varepsilon_0 V_0\ \pi}{2a\mathrm{K}}\ [\mathrm{E'}-k^2\ \mathrm{K'}\,\mathrm{nd}^2(y,k'))]^{-1}$$

for $-\mathrm{K'}<y<0$: $\hat{C}_1(y)=$ the expressions from above ;

$$\eta_{\hat{C}_2}(y)=-\frac{\varepsilon_0 V_0}{2\mathrm{K}}\ \Re e\left\{\mp 1\frac{\pi}{a}[\mathrm{E'}-k^2\ \mathrm{K'}\,\mathrm{sn}^2(\mathrm{K}+iy,k))]^{-1}\right\}\ =\pm\frac{\varepsilon_0 V_0\ \pi}{2a\mathrm{K}}\ [\mathrm{E'}-k^2\ \mathrm{K'}\,\mathrm{nd}^2(y,k'))]^{-1}$$

for $0<y<\mathrm{K'}$: $\hat{C}_2(y)=$ the expressions from above .

The plus sign of the surface charge density given above holds for the left side, the minus sign for the right side of the plates respectively.
The correct parametric representation $\hat{C}_i(y)$ corresponding to $\eta_i(y)$ can be found by looking at $f(C_i)\Longrightarrow\hat{C}_i(y)$: the left sides of the plates belong to $\hat{C}_i(y)=f(C_i(...))=-iv$, the dots "..." representing the range of y that has to be inserted into $\hat{C}_i(y)$.
Again the edges cause infinities, as those in the case of the Semi-Infinite Condenser. The electric field and surface charge density tend to constant values for the limit $v\to 0$. These

constants depend on the ratio length to distance. They are larger between the plates than on the outside.
For long plates the limits at the inside approach $\pm 2\frac{\epsilon_0 V}{a}$ and $\pm\frac{2V}{a}$; and at the outside they approach zero as in the case fo the **Semi-Infinite Condenser** which is the limit for the length b tending to infinity.

Mathematica program 11: for the Finite Plate Condenser

```
Needs["Graphics`ComplexMap`"]

Clear[V,a,b,m,sn,elk,mprime,frmprime,decide]
V = 1 (* Voltage between the plates in Volt *) ;
a = 3 (* Distance of the plates in mm :
       for all 2 10^-5 < a < 10^2 (scd) and 2 10^-5 < a < 10^5 (ef) *) ;
b = 2 (* Plate length in mm : for any  10^-2 a < b < 8.75 a *) ;

decide = ef (* Decide whether you want the electric field (ef) or
                surface charge density (scd) of the plates *) ;

af110 = 2/Pi (EllipticK[mprime] EllipticE[ArcSin[1/Sqrt[mprime]*
                                          Sqrt[1 - EllipticE[mprime]/
                                          EllipticK[mprime]]],mprime] -
      EllipticE[mprime] EllipticF[ArcSin[1/Sqrt[mprime]*
                                          Sqrt[1 - EllipticE[mprime]/
  EllipticK[mprime]]],mprime]);

frmprime = FindRoot[af110 - b/a ,{mprime,.99,.999}]; mprime = mprime/.frmprime ;
m = 1 - mprime ;

elk = Pi/(2 EllipticK[m]) ;
af111[z_] = a/Pi ((EllipticE[mprime] - EllipticK[mprime]) z +
                  EllipticK[mprime] EllipticE[JacobiAmplitude[z,m],m]) ;
af112[z_] = 1/(EllipticE[mprime] - m EllipticK[mprime] JacobiND[z,mprime]^2) ;
sn = InverseJacobiSN[Sqrt[(1 - m EllipticK[mprime]/EllipticE[mprime]) / mprime],
                     mprime] ;

<< ~path/hilfsfile

a11 = ParametricPlot[{{ Im[af111[EllipticK[m] - 10^-5 + I y]], -elk mult af112[y] },
       { -Im[af111[EllipticK[m] - 10^-5 + I y]], -elk mult af112[y] }},
       {y,sn + 10^-6, EllipticK[mprime] - 10^-6},
       PlotRange -> {{-1.02 b/2, 1.02 b/2},{0,elk mult 4}},
       AxesLabel -> {"u[mm]","Upper Plate's Outside"},
       Epilog -> { Text[tex1,{b/15,elk mult 3},{0,0},{0,1}] } ] ;
```

```
b11 = ParametricPlot[{{ Im[af111[EllipticK[m] - 10^-5 + I y]], elk sgn mult af112[y] }
      { -Im[af111[EllipticK[m] - 10^-5 + I y]], elk sgn mult af112[y] }},
      {y,0,sn - 10^-6}, PlotRange -> {{-1.02 b/2, 1.02 b/2},{0,elk sgn mult 4}},
      AxesLabel -> {"u[mm]","Upper Plate's Intside"},
      Epilog -> { Text[tex1,{b/15,elk sgn mult 3},{0,0},{0,1}] } ] ;

b11inv = ParametricPlot[{{ Im[af111[EllipticK[m] - 10^-5 + I y]], elk mult af112[y] },
         { -Im[af111[EllipticK[m] - 10^-5 + I y]], elk mult af112[y] }},
         {y,0,sn - 10^-6}, PlotRange -> {{-1.02 b/2, 1.02 b/2},{0,elk mult 4}},
         PlotStyle -> Dashing[{.02}], DisplayFunction -> Identity ] ;

c11 = CartesianMap[I af111[#] &, {-EllipticK[m] + .0001, EllipticK[m] - .0001},
      {-EllipticK[mprime] + .0001, EllipticK[mprime] - .0001},
      PlotPoints -> 25, PlotRange -> {{-2 b,2 b},{-2 a,2 a}},
      AxesLabel -> {"u[mm]","iv[mm]"},
      Epilog -> { Thickness[.011],
                 { Line[{{-b/2,a/2},{b/2,a/2}}],
                   Line[{{-b/2,-a/2},{b/2,-a/2}}] }} ] ;

Show[a11,b11inv, Ticks -> None, AxesLabel -> {"u","Upper Plate: In- & Outside"},
    Epilog -> { Text[tex2,{b/15,elk mult 3},{0,0},{0,1}] },
    DisplayFunction -> $DisplayFunction ] ;
Show[c11, Axes -> None ] ;
```

Comments:

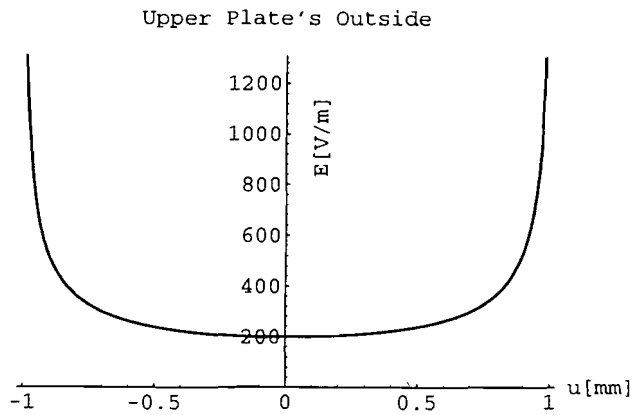
The field distribution is plotted by **c11** where the plates are rotated by 90° in multiplying the function $f(z)$ by the imaginary unit so that the field of the Infinite Plane Condenser, the Semi-Infinite Condenser, the Thick Semi-Infinite Condenser and the Finite Plane Condenser can be compared easier. Here the potential for the upper plate is positive for positive voltage between the plates and zero for the lower plate.
The graphics commands of **a11,b11** plot the pictures of the electric field or the surface charge density of the upper plate with ticks. The charge density of the lower plate is of opposite and the field of the same sign.
The two **Show[ ]** commands draw the pictures without ticks whereby the first **Show[ ]** unites **a11,b11inv** in one picture. To compare the electric field of the inside to that of the outside it is drawn with positive sign both times (the inside curve is dashed).

Variants:

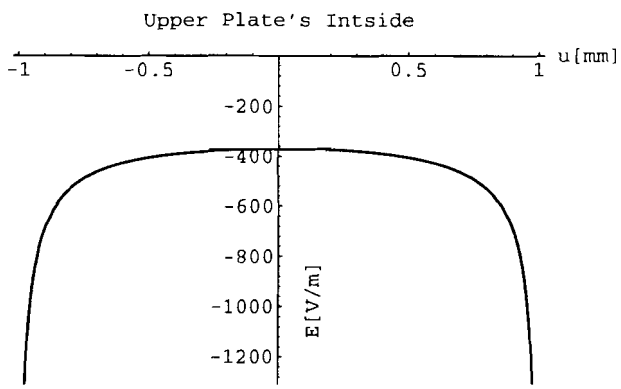
▲ it is possible to change **ParametricPlot[ ]** and **CartesianMap[ ]** as described in section 2.1. The ranges of the *iv* co-ordinate and the location of the text drawn by the **Epilog** option of **a11,b11,b11inv** as well as the starting values and options in **frmprime** vary for different values of **b/a** and have to be modified by hand (the lokation of the text is written in brackets behind the co-ordinates' value) :

value of b	**frmprime** starting values	**WorkingPrecision**	**Max Iterations**
10^-3 a to **.8 a**	{ **10^-9, 10^-11** }		
.8 a to **2.3 a**	{ **.99, .999** }		
2.3 a to **4.5 a**	{ **1 - 10^-7, 1 - 10^-9** }	**25**	**30**
4.5 a to **6.6 a**	{ **1 - 10^-10, 1 - 10^-13** }	**60**	**30**
6.6 a to **8.75 a**	{ **1 - 10^-13, 1 - 10^-15** }	**60**	**30**
Graphics	*iv* coordinates for **b = a/1000**	*iv* coordinates for **b = a**	*iv* coordinates for **b = 8.75 a**
a11, b11, b11inv	{... , ... **mult 2000 (1500)** }	{... , ... **mult 5 (3)** }	{... , ... **mult .7 (.5)** }

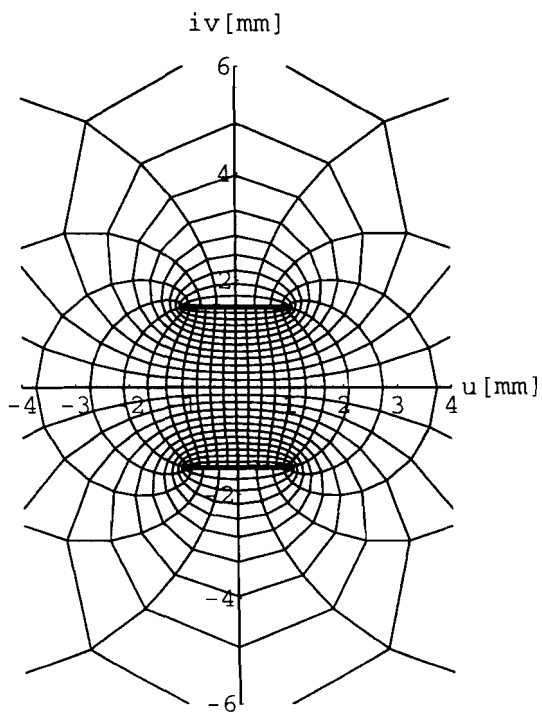
▲ If you want the correct sign for the electric field without ticks you have to replace the **b11inv** in the first **Show[ ]** by **b11** and replace **tex2** in **Epilog** by **tex3**.



(a11)

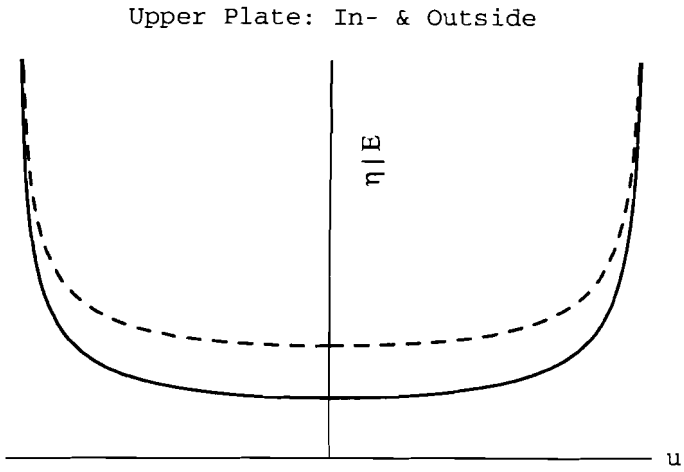


(b11)

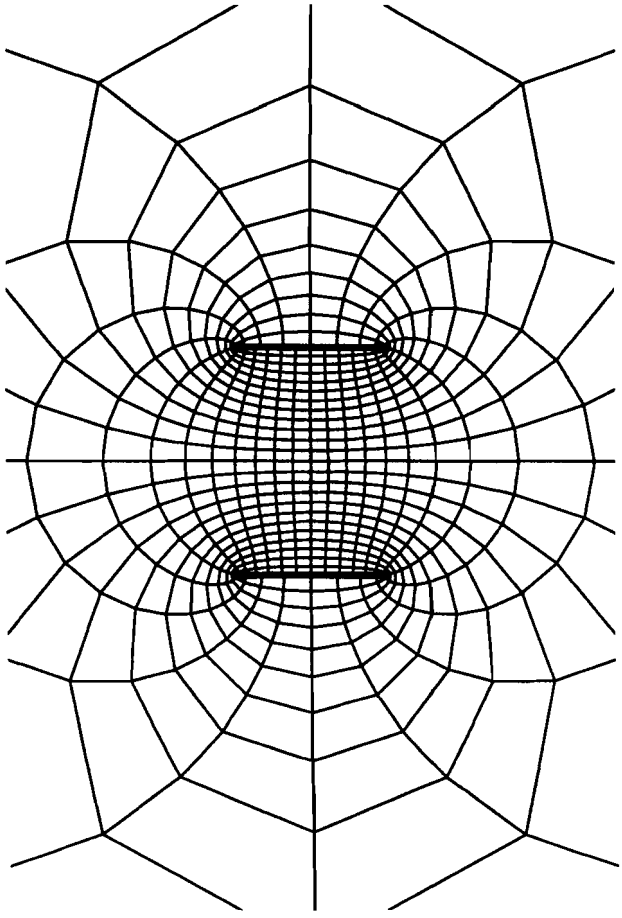


(c11)

Figure 44: Graphics with ticks produced by the *Mathematica* program for the Finite Plane Condenser. The potential difference between the plates is 1 V, their distance is 3 mm, their

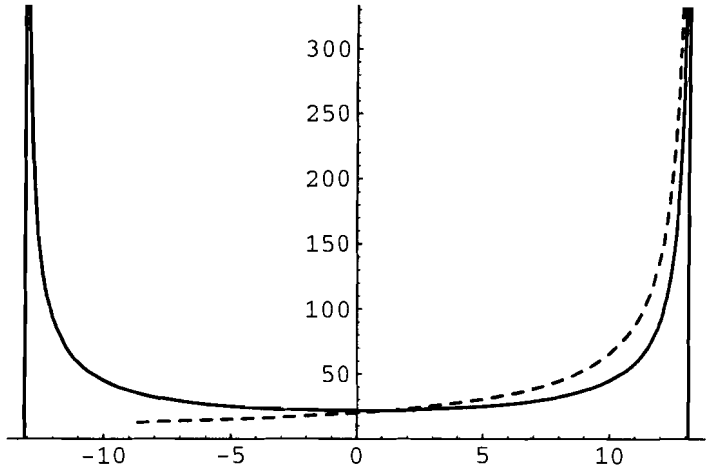


first **Show[ ]**

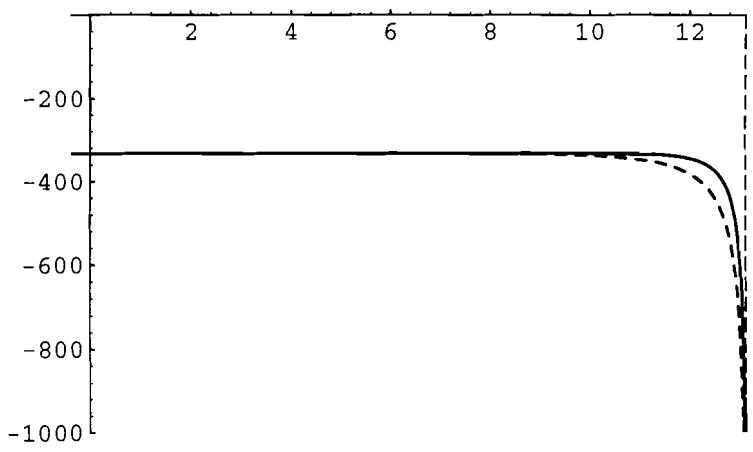


second **Show[ ]**

Figure 45: Graphics without ticks produced by the *Mathematica* program for the Finite Plane Condenser.



a)



b)

Figure 46: These graphics show that the electric field of the Finite Plane Condenser for large values of the ratio: plate length to distance ($\mathbf{b/a = 8.75}$ for $\mathbf{a = 3}$ mm , $\mathbf{V = 1}$ Volt) can be approximated by use of the Semi-Infinite Plane Condenser. The picture a) shows the fields of the outside of the upper finite and the top of the upper semi-infinite plate (dashed), where the origin of **a3** was shifted by $\mathbf{-8.75/2\ a}$. The picture b) shows the fields of the inside (just the right half) of the upper finite and the bottom of the upper semi-infinite plate (dashed) (the origin was again shifted).

2.14 *Mathematica* Program for the Three Plates in Line Condenser

$$w = \pm(a+b)\ \mathrm{dc}(izK', k) \quad \text{with} \quad k = \frac{a}{a+b}$$

The section for this condenser was suggested by Prof. Dr. St. Lindenmeier, FBH-Berlin. The condenser consists of three parts in a line : two semi-infinite plates are at both sides of a finite plate of the length $2a$. The distance from the central plate is b for both semi-infinite plates. The problem is symmetric with respect to the two axes.

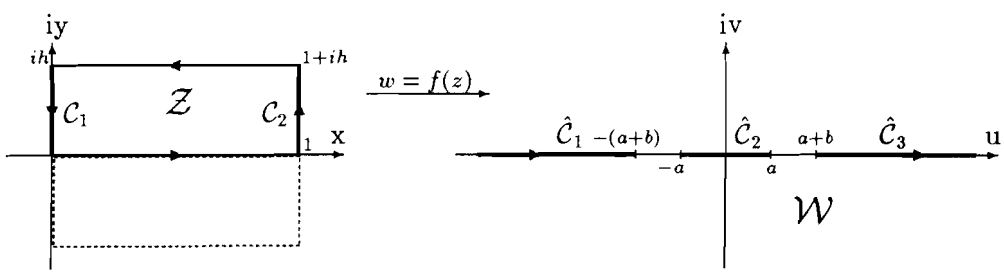


Figure 47: Mapping of the Three Plates in Line Condenser.

The inverse conformal map which maps the W- to the Z-plane is treated in [hm, p.2] (without loss of information I let the variable "a" of [hm] be 1 (the a of this thesis is "w" in [hm]).):

$$z = \frac{-ia}{k\,K'} \int \frac{dw}{\sqrt{w^2-(a+b)^2}\,\sqrt{w^2-a^2}} = \frac{-ia}{K'}\,\mathrm{F}\left[\arcsin\sqrt{\frac{w^2-(a+b)^2}{w^2-a^2}}, k\right]$$

This equation has to be inverted. Use of [bf, (130.02), (121.00)] gives

$$w = f(z) = \pm(a+b)\ \mathrm{dc}(izK', k) \quad \text{with} \quad k = \frac{a}{a+b} \text{ and } h = \frac{2K}{K'} \ .$$

I choose the positiv branch of the function.

This function f(z) is holomorphic in $\mathbb{C}\backslash\{\pm\frac{i4(m+1)\ K + 4in\ K'}{K'}\}$ with $n, m \in \mathbb{Z}$.
The singularities of the function are the left/right "ends" of the right/left semi-infinite plates.

The first derivative

$$f'(z) = -i(a+b)\,k'^2\,\mathrm{K'}\frac{\mathrm{sn}(iz\mathrm{K'})}{\mathrm{cn}^2(iz\mathrm{K'})} \qquad \text{has zeros at } z \in \{2n\,\mathrm{K} + im\,\mathrm{K'}\}.$$

This would lead to a very complicated domain $\mathbf{D}_z$, but as in the other cases I restrict $\mathbf{D}_z$ to $\mathbf{D}_z = \{z | z = x + iy, x \in [-1,1], y \in [0,h] \setminus \{h/2\}\}$.

The potential has be chosen as : $\Phi = V_0\ z$ so that the left plate on the Z-plane has zero potential and the right plate has the potential V_0 . The potentials on the W-plane are V_0 at the semi-infinite plates and zero at the central plate.

The curves on the W-plane in terms of y are:

$$f(C_1) = (a+b)\,\mathrm{dc}(y\mathrm{K'},k) \Longrightarrow \begin{cases} \hat{C}_1 = f(C_1(-h < y < -h/2)) = u & \text{for } -\infty < u < -(a+b) \ , \\ \hat{C}_3 = f(C_1(-h/2 < y < 0)) = u & \text{for } (a+b) < u < \infty \ ; \end{cases}$$

$$f(C_2) = (a+b)\,\mathrm{dc}(i\mathrm{K'} - y\mathrm{K'}) = a\,\mathrm{cd}(y\mathrm{K'}) \Longrightarrow \hat{C}_2 = f(C_2) = u \ \text{ for } \ -a < u < a \ .$$

For the evaluation of $f(C_2)$ I used [bf, (122.07), (122.00)]. The value of the normal unit vectors is the imaginary unit. So the surface charge density modified for the new potential is:

$$\begin{aligned} \eta_{\hat{C}_1} &= \frac{\varepsilon_0 V_0}{\pi}\,\frac{1}{(a+b)k'^2\mathrm{K'}}\left\{\frac{\mathrm{cn}^2(y\mathrm{K'},k)}{\mathrm{sn}(y\mathrm{K'})}\right\} \\ &\text{for } -h < y < -h/2 \quad : \hat{C}_1(y) = f(C_1(-h < y < -h/2)) \ ; \\ \eta_{\hat{C}_2} &= \frac{\varepsilon_0 V_0}{\pi}\,\frac{1}{(a+b)k'^2\mathrm{K'}}\,\Re e\left\{\frac{\mathrm{cn}^2(i\mathrm{K'} - y\mathrm{K'},k)}{\mathrm{sn}(i\mathrm{K'} - y\mathrm{K'})}\right\} = \frac{\varepsilon_0 V_0}{\pi}\,\frac{1}{k'^2\mathrm{K'}}\left\{\frac{\mathrm{dn}^2(y\mathrm{K'},k)}{\mathrm{sn}(y\mathrm{K'})}\right\} \\ &\text{for } -h < y < 0 \quad : \hat{C}_2(y) = f(C_2) \ ; \\ \eta_{\hat{C}_3} &= \frac{\varepsilon_0 V_0}{\pi}\,\frac{1}{(a+b)k'^2\mathrm{K'}}\left\{\frac{\mathrm{cn}^2(y\mathrm{K'},k)}{\mathrm{sn}(y\mathrm{K'})}\right\} \\ &\text{for } -h/2 < y < 0 \quad : \hat{C}_1(y) = f(C_1(-h/2 < y < 0)) \ ; \end{aligned}$$

The electric field and the suface charge density for the upper sides of the plate tend to zero for $u \to \pm\infty$ and to infinity for $u \to \pm a$ and to minus infinity for $u \to \pm(a+b)$.

Mathematica program 12: for the Three Plates in Line Condenser

```
Needs["Graphics`ComplexMap`"]

Clear[V,a,b,c,d,e,f,h,ksq,frc,frd,decide]
V = 1 (* Voltage between the plates in Volt *);
a = 1 (* Half the length of the central plate in mm :
         for any 10^-5 < a < 25 *);
b = 3 (* The gap between the central and the outer plates in mm :
         for any a/200 < b < 2.5 a *) ;
decide = ef (* Decide whether you want the electric field (ef) or
                surface charge density (scd) of the plates *) ;

ksq = (a/(a + b))^2; h = 2 EllipticK[ksq]/EllipticK[1 - ksq];
af120[z_] = (a + b) JacobiDC[I z EllipticK[1 - ksq], ksq] ;
af121[z_] = Sqrt[ksq]/((1 - ksq) Pi EllipticK[1 - ksq])*
            JacobiCN[I z EllipticK[1 - ksq], ksq]^2/
    JacobiSN[I z EllipticK[1 - ksq], ksq] ;

e = -af120[-I c]; f = -af120[-I d] ;
frc = FindRoot[e + 1.75(a + b),{c,-.5 h/2,-.7 h/2},
       AccuracyGoal -> 4 - Floor[N[Log[10,(a + b)]]],
       WorkingPrecision -> 40]; c = c/.frc ;
frd = FindRoot[f - 1.75(a + b),{d,-1.3 h/2,-1.5 h/2}]; d = d/.frd ;

<< ~path/hilfsfile

a12 = ParametricPlot[{ -af120[-I y], mult af121[-I y] }, {y,-.001,c},
       AxesOrigin -> {-a - b,0}, PlotRange -> {All,{10^-15,-.35 mult}},
       AxesLabel -> {"u[mm]",tex1}, PlotLabel -> "Left Plate: Top" ] ;

b12 = ParametricPlot[{ -af120[1 + I y], mult af121[1 + I y] }, {y,.0001,h - .001},
       PlotRange -> {All,{0,mult}}, AxesLabel -> {"u[mm]","Middle Plate: Top"},
       Epilog -> { Text[tex1,{a/10,.7 mult},{0,0},{0,1}] } ] ;

c12 = ParametricPlot[{ -af120[-I y], mult af121[-I y] }, {y,d,-h + .001},
       PlotRange -> {All,{10^-15,-.35 mult}}, AxesOrigin -> {a + b,0},
       AxesLabel -> {"u[mm]",tex1}, PlotLabel -> "Right Plate: Top"] ;
```

```
d12 = CartesianMap[af120, {-.9999,1},{0,h}, PlotPoints -> 30,
       AxesLabel -> {"u[mm]","iv[mm]"},
       PlotRange -> {{-1.75(a + b),1.75(a + b)},{-a - b,a + b}},
       Epilog -> { Thickness[.011], { Line[{{-1.8(a + b),0},{-a - b,0}}],
                   Line[{{-a,0},{a,0}}], Line[{{a + b,0},{1.8(a + b),0}}] }} ] ;

Show[a12,b12,c12, PlotRange -> {{-1.8(a + b), 1.8(a + b)},{-.35 mult,.6 mult}},
     Ticks -> None, AxesOrigin -> {0,0},
     AxesLabel -> {"u","Top & Bottom of All Plates"}, PlotLabel -> None,
     Epilog -> { Text[tex2,{.1(a + b),.4 mult},{0,0},{0,1}] } ] ;

Show[d12, Axes -> None] ;
```

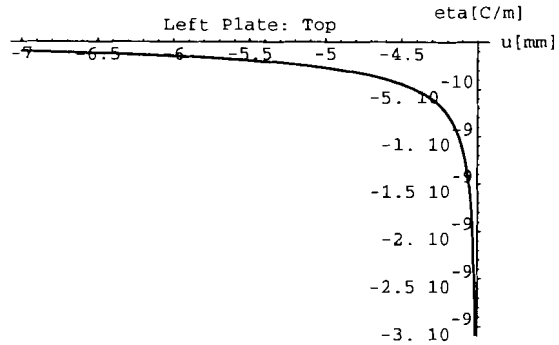
Comments:

The program is structured like the other ones. The pictures (**a12,b12,c12,d12**) are plotted with ticks. The following two **Show[ ]** commands produce these pictures without ticks. I do not have to use the combined commands **Show[GraphicsArray[ ]]** here to put the pictures **a12, b12, c12** together in one picture since the location of the plates is on one line.

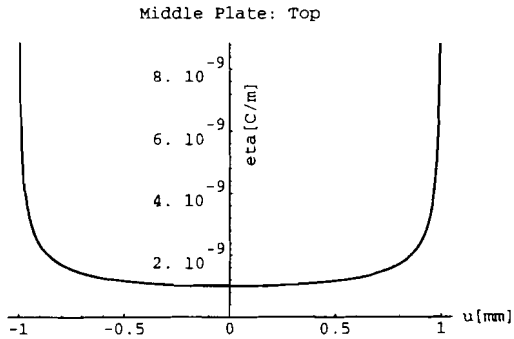
Variants:

▲ it is possible to change **ParametricPlot[ ]** and **CartesianMap[ ]** as described in section 2.1. The ranges of the *iv* co-ordinate in **a12,b12,c12** and the first **Show[ ]** vary a bit for different values of **b/a** and have to be modified by hand :

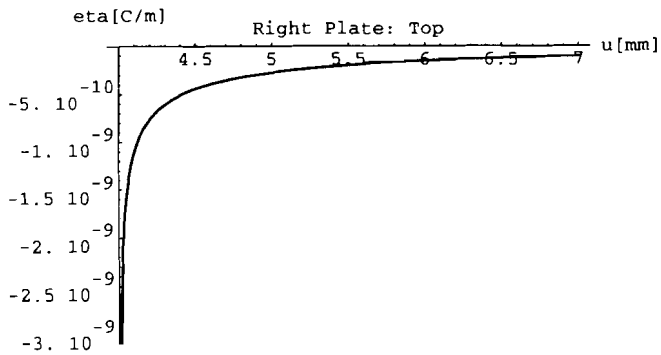
Graphics	*iv* coordinates for **b = a/200**	*iv* coordinates for **b = a**	*iv* coordinates for **b = 2.5 a**
a12, c12	{... , **−7 mult** }	{... , **−mult** }	{... , **−.2 mult** }
b12	{... , **2 mult** }	{... , **.8 mult** }	{... , **.6 mult** }
Show[ ]	{**−7 mult, 2 mult** }	{**−mult, .8 mult** }	{**−.2 mult, .6 mult** }



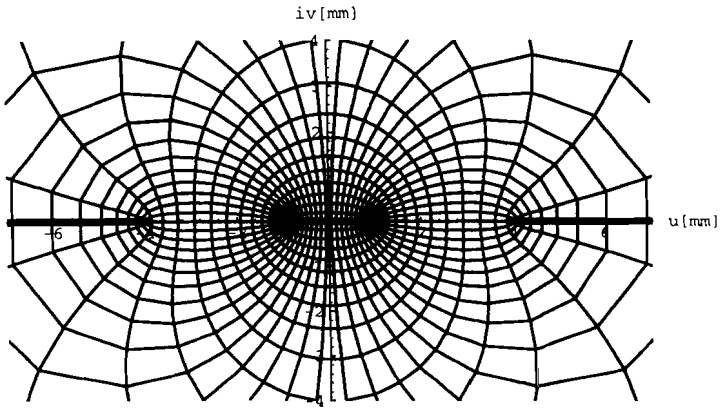
(a12)



(b12)

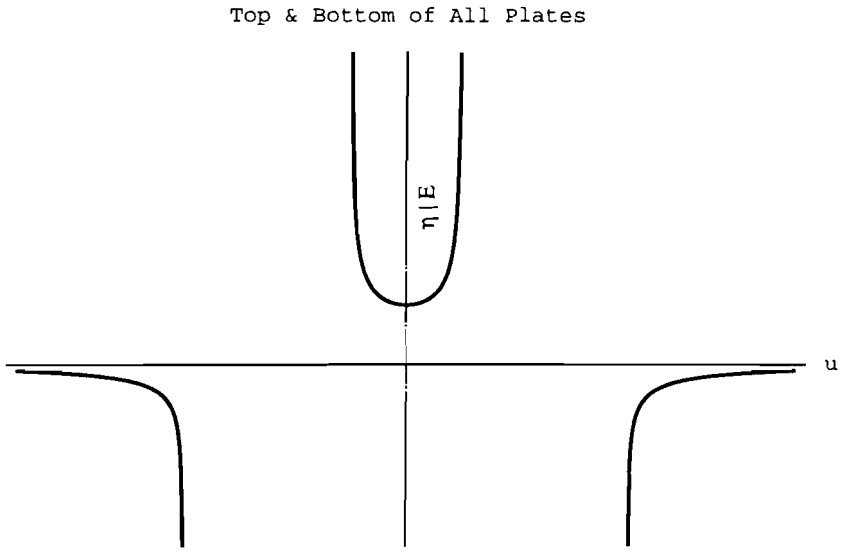


(c12)

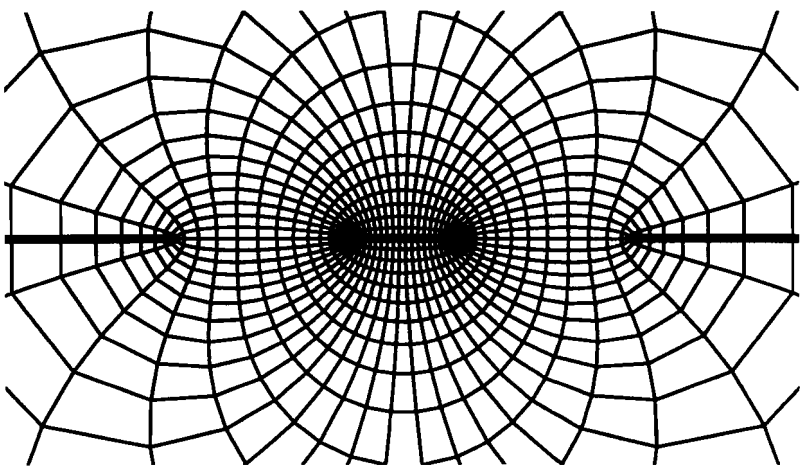


(d12)

Figure 48: Graphics with ticks produced by the *Mathematica* program for the Three Plates in Line Condenser. The potential difference between the plates is 1 V. The central plate has



first **Show[ ]**



second **Show[ ]**

Figure 49: Graphics without ticks produced by the *Mathematica* program for the Three Plates in Line Condenser.

2.15 Difficulties in Calculating Condensers with Rounded Edges

In the first section of this thesis I talked about the Schwarz-Christoffel formula to calculate conformal functions for condensers with corners whose angles α_i are $0 < |\alpha_i| \leq \pi$. Unfortunately the elecric field strength in the vicinity of convex corners tends to infinity. To avoid this singularity I introduced the modified Schwarz-Christoffel transformation.
I can always apply this modified Schwarz-Christoffel transformation to condensers with corners of angles: $|\alpha_i| < \pi$. (examples with $|\alpha_i| = \pi$ are the Semi-Infinite Condenser and Finite Plate Condenser; their mapping functions cannot be modified so that the field at the edge becomes finite.)
This application, however, causes serious trouble in calculating some new parameters that did not appear in the original Schwarz-Christoffel formula. Nevertheless these parameters have to be computed to get the conformal map for a wanted radius of the curve(s) representing the edges. Unfortunately *Mathematica* is no longer a good tool to calculate these parameters as it was to compute the elliptic moduli k with the **FindRoot[ ]** command (section 2.12 and 2.13) or the parameter p in section 2.8 .

Two examples will illuminate the difficulties :

- rounding the upper corner of the Sharp Bend Condenser
- rounding both corners of the Thick Aperture Condenser

Rounded Sharp Bend Condenser :

The conformal map for the Sharp Bend Condenser is

$$w = \frac{2}{\pi}\left(b \ln\left[\frac{\sqrt{t+1}+\sqrt{t-q}}{\sqrt{q+1}}\right] - ai \ln\left[\frac{\sqrt{t-q}+i\frac{a}{b}\sqrt{t+1}}{\sqrt{q+1}}\right]\right) + \frac{ia}{\pi}\ln t - ib$$

$$\text{with } t = e^z \text{and } q = \tfrac{a^2}{b^2} \ .$$

When I want to round the upper convex corner I have to apply the modified Schwarz-Christoffel transformation:
The start and end points of the curve on the W-plane are denoted by w_s and w_e, the origin by w_q and the cross point of the horizontal lines (crossing at infinity) by w_p (the cross point of the vertical lines is at infinity on the W-plane as well as on the T-plane which reduces the order of the product under the integral by one). The corresponding points on the T-plane are then $t_s = -1, t_e = 1, t_q$ and t_p, with t_p and $t_q > t_p > 1$ to be calculated later on.

The modified Schwarz-Christoffel formula gives :

$$\frac{dw}{dt} = A\frac{\sqrt{t+1} + \lambda\sqrt{t-1}}{(t-t_p)\sqrt{t-t_q}} \quad .$$

An integral of this form can be found in [gw, p.31, eq. (213.5b)]. This yields :

$$w = A\Bigg\{ i\sqrt{\frac{t_p+1}{t_q-t_p}}\ln\left[\frac{\sqrt{t-t_q}\sqrt{t_p+1} + i\sqrt{t_q-t_p}\sqrt{t+1}}{\sqrt{t-t_q}\sqrt{t_p+1} - i\sqrt{t_q-t_p}\sqrt{t+1}}\right] - \ln\left[\frac{\sqrt{t-t_q}+\sqrt{t+1}}{\sqrt{t-t_q}-\sqrt{t+1}}\right]$$
$$+\lambda\left(i\sqrt{\frac{t_p-1}{t_q-t_p}}\ln\left[\frac{\sqrt{t-t_q}\sqrt{t_p-1} + i\sqrt{t_q-t_p}\sqrt{t-1}}{\sqrt{t-t_q}\sqrt{t_p-1} - i\sqrt{t_q-t_p}\sqrt{t-1}}\right] - \ln\left[\frac{\sqrt{t-t_q}+\sqrt{t-1}}{\sqrt{t-t_q}-\sqrt{t-1}}\right]\right)\Bigg\} + B \ .$$

The constant A can be calculated by eq.(6) :

$$ib = -i\pi\left[A\prod_{i\neq p}(t_p-t_i)^{-\alpha_i}\right] = -\pi A\frac{\sqrt{t_p+1}+\lambda\sqrt{t_p-1}}{\sqrt{t_q-t_p}} \quad ,$$

$$A = -\frac{ib}{\pi}\frac{\sqrt{t_q-t_p}}{\sqrt{t_p+1}+\lambda\sqrt{t_p-1}} \quad .$$

When I choose the variable $t = t_q; (w_q = 0)$ and insert this symbolical value of t into the equation from above I get the value for B :

$$B = A\left[\frac{\pi}{\sqrt{t_q-t_p}}\left(\sqrt{t_p+1}+\lambda\sqrt{t_p+1}\right) - i\pi(1+\lambda)\right] = -ib + \frac{b\sqrt{t_q-t_p}(1+\lambda)}{\sqrt{t_p+1}+\lambda\sqrt{t_p+1}}$$

with the parameters t_p, t_q still to be calculated. [we] refers to an older paper that calculated a best $\lambda = \sqrt{(t_q+1)/(t_q-1)}$.

When I insert two specific values for $t = \pm 1$ into the equation this leads to two transcendental equations for t_p and t_q.

$$a - i(b+c) = \lambda A \left\{ i\sqrt{\frac{t_p+1}{t_q-t_p}} \ln\left[\frac{i\sqrt{t_p-1}\,\sqrt{1+t_q} - \sqrt{2}\,\sqrt{t_q-t_p}}{i\sqrt{t_p-1}\,\sqrt{1+t_q} + \sqrt{2}\,\sqrt{t_q-t_p}}\right] - \ln\left[\frac{\sqrt{1+t_q}+\sqrt{2}}{\sqrt{1+t_q}-\sqrt{2}}\right]\right\} + B$$

$$a + c - ib = A \left\{ i\sqrt{\frac{t_p+1}{t_q-t_p}} \ln\left[\frac{i\sqrt{t_p+1}\,\sqrt{t_q-1} + i\sqrt{2}\,\sqrt{t_q-t_p}}{i\sqrt{t_p+1}\,\sqrt{t_q-1} - i\sqrt{2}\,\sqrt{t_q-t_p}}\right] - \ln\left[\frac{i\sqrt{t_q-1}+\sqrt{2}}{i\sqrt{t_q-1}-\sqrt{2}}\right]\right\} + B$$

This sounds all very easy since the conformal function for this problem looks like the sum of two conformal functions of the Sharp Bend Condenser, but I have to solve these two coupled transcendental equations for t_p and t_q. That means I have to find solutions for these parameters at the same time. **FindRoot** finds solutions for two polynomials of low order but not for the equations mentioned above.
I suppose there are very good programs existing in some FORTRAN or C library which manage the problem in a rather comfortable way.

Rounded Thick Aperture Condenser :

Now I want to round both corners of the Thick Aperture Condenser. Its conformal function is :

$$f(z) = \frac{ia}{2\mathrm{E} - k'^2\mathrm{K}} \left[k'^2 \mathrm{F}(\arcsin t, k)) - 2\mathrm{E}(\arcsin t, k)) - \frac{\sqrt{1-t^2}\,\sqrt{1-k^2t^2}}{t} \right] - b$$
$$\text{with} \quad t = \sqrt{-\tfrac{e^z}{k}}$$

Application of the modified Schwarz-Christoffel formula with $t_2(w_2)$ and $t_5(w_5)$ as for the Thick Aperture Condenser and the start and end points of the left and right smooth

curve $w_{ls}, w_{le}, w_{rs}, w_{re}(t_{le} = -1, t_{rs} = -k, t_{le} < t_{ls} < t_{re} < t_{rs})$ gives :

$$
\begin{aligned}
w &= A \int_0^t \frac{\left(\sqrt{t^2-k^2} + \lambda_1\sqrt{t-t_{re}^2}\right)\left(\sqrt{t^2-t_{ls}^2} + \lambda_2\sqrt{t-1}\right)\, dt}{t^2} + B \\
&= A\Bigg[\int_0^t \frac{\sqrt{t^2-k^2}\,\sqrt{t^2-t_{ls}^2}\, dt}{t^2} + \lambda_2 \int_0^t \frac{\sqrt{t^2-k^2}\,\sqrt{t^2-1}\, dt}{t^2} \\
&\quad + \lambda_1 \int_0^t \frac{\sqrt{t^2-t_{re}^2}\,\sqrt{t^2-t_{ls}^2}\, dt}{t^2} + \lambda_1\lambda_2 \int_0^t \frac{\sqrt{t^2-t_{re}^2}\,\sqrt{t^2-1}\, dt}{t^2}\Bigg] + B \ .
\end{aligned}
$$

I can again substitute t by $t = 1/\mathrm{sn}(\tau_i, k_i)$ where $\tau_1 = \mathrm{sn}^{-1}(t_{ls}/t, k_1), k_1 = k/t_{ls}$; $\tau_2 = \mathrm{sn}^{-1}(1/t, k_2), k_2 = k; \tau_3 = \mathrm{sn}^{-1}(t_{ls}/t, k_3), k_3 = t_{re}/t_{ls}; \tau_4 = \mathrm{sn}^{-1}(1/t, k_4), k_4 = t_{re}$ are the substitutions for the ith integral.

With the help of [bf, (129.01), (121.00), (120.02)] I get :

$$
\begin{aligned}
w = -A\Bigg[&\int_0^{\tau_1} \frac{\mathrm{cn}^2(\tau_1,k_1)\,\mathrm{dn}^2(\tau_1,k_1)\, d\tau_1}{\mathrm{sn}^2(\tau_1,k_1)} + \lambda_2 \int_0^{\tau_2} \frac{\mathrm{cn}^2(\tau_2,k_2)\,\mathrm{dn}^2(\tau_2,k_2)\, d\tau_2}{\mathrm{sn}^2(\tau_2,k_2)} \\
&+ \lambda_1 \int_0^{\tau_3} \frac{\mathrm{cn}^2(\tau_3,k_3)\,\mathrm{dn}^2(\tau_3,k_3)\, d\tau_3}{\mathrm{sn}^2(\tau_3,k_3)} + \lambda_1\lambda_2 \int_0^{\tau_4} \frac{\mathrm{cn}^2(\tau_4,k_4)\,\mathrm{dn}^2(\tau_4,k_4)\, d\tau_4}{\mathrm{sn}^2(\tau_4,k_4)}\Bigg] + B \ .
\end{aligned}
$$

As for the original condenser these integrals can be integrated using [bf, (121.00), (310.02), (311.02)]. There are still three dependent moduli, two λ_i and the constants A and B to be calculated. Spending a week I can find values for A and B. Inserting these into the integrated equation and solving for different $t = (\pm 1, t_{le}, t_{rs})$ leads to four equations for the unknown t_{le}, t_{rs} (the λ_i have to be chosen so that the rounded corners can be approximated by circles). This is absolutely impossible for standard *Mathematica* commands.
I chose a very nasty example to demonstrate the difficulties to obtain the values of the parameters that determine the conformal function $f(z)$.

As soon as the values of the parameters t_{p_i}, t_{q_i} and λ_i are known I can inserte these into the general form and have the graphics plotted by *Mathematica* . One way would be to transfer the parameter values from the FORTRAN | C program via *MathLink*. For informations you have to study the *Mathematica* book [ma] or the *MathLink* Reference Guide of your *Mathematica* version.

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