ABSTRACT

Covariant representations of the Wigner "little group" are constructed for single particle states of definite momentum. The matrix elements of scalar operators between the covariant spin states are shown to be invariant amplitudes. The corresponding covariant Rarita-Schwinger wavefunctions are constructed as functionals of the momentum four-vectors. These wavefunctions are used with Green's functions of Rarita-Schwinger interpolating fields to study the kinematic singularities and crossing properties of the invariant amplitudes.

The covariant spin states and the infinite momentum limit are used to obtain a general representation of the local current algebra. A representation in terms of a set of single particle states having a mass - spin relation is considered. The current algebra implies a nonlinear relation between the invariant form factors. The analytic structure of the form factors as functions of the momentum transfer is shown to place a constraint on the spectrum of the single particle states used in the representation.

ACKNOWLEDGEMENTS

I would like to express my appreciation to Dr. S.G. Gasiorowicz for his guidance, interest, and patience during the course of this study.

I also wish to thank D.M. Brudnoy, C. Weil, and W.K. Tung for many useful discussions and suggestions.

I would like to express my gratitude to the University of Minnesota and to the National Aeronautics and Space Adminstration for support during the course of my graduate studies. I would also like to thank the members of the Institute for Theoretical Physics of the State University of New York at Stony Brook for their understanding during the preparation of this thesis.

The encouragement and understanding of my wife, Marge, has been indispensable.

PART I: Formalism for Arbitrary Spin Calculations

Relativistic calculations involving particles with spin have often proved difficult, as the transformation properties of the spin states under the Lorentz group are usually quite complicated. We will show that it is possible to parameterize the relativistic spin states so that the states have simple transformation properties under the Lorentz group. The method involves the introduction of two four-vector quantities. These four-vectors together with the four-momentum of the state will define the coordinate system in which the spin is quantized. This would appear to introduce a great deal of arbitrariness. We usually deal not with single particle states but with matrix elements involving two or more particles. The four momenta associated with these states are not arbitrary. We will, in fact, use these four-vectors to define the quantization coordinate system for each of the states.

In Section A, we define the covariant spin states, describe their properties, and show how they may be used to construct invariant matrix elements. In Section B, we define the wavefunctions that may be used to discuss the kinematic singularities and crossing properties of the invariant amplitudes.

Section A

We wish to consider a set of states which carry an irreducible representation of the Inhomogenous Lorentz Group (ILG).[1,2] The ten generators of the ILG satisfy the following commutation

relations.

$$[P_\mu, P_\nu] = 0 \qquad \mu = 0, 1, 2, 3$$

$$[M_{\mu\nu}, P_\sigma] = -i g_{\nu\sigma} P_\mu + i g_{\mu\sigma} P_\nu$$

(I-1)

$$[M_{\mu\nu}, M_{\sigma\tau}] = i g_{\nu\tau} M_{\mu\sigma} + i g_{\mu\sigma} M_{\nu\tau} - i g_{\mu\tau} M_{\nu\sigma} - i g_{\nu\sigma} M_{\mu\tau}$$

$$M_{\mu\nu} = - M_{\nu\mu}$$

where $g_{\mu\nu}$ is the metric tensor $g_{00} = 1$, $g_{0k} = g_{k0} = 0$ $g_{kl} = -\delta_{kl}$.

A set of states $\{|\psi\rangle\}$ will carry a representation of the ILG if

$$U(a,\Lambda)|\psi\rangle = |\psi'\rangle \qquad |\psi'\rangle \in \{|\psi\rangle\}$$

and (I-2)

$$U(a',\Lambda')U(a,\Lambda)|\psi\rangle = U(a' + \Lambda' a, \Lambda'\Lambda)|\psi\rangle$$

where $U(a,\Lambda)$ is a Lorentz transformation given by

$$U(a,\Lambda) = e^{-iP\cdot a} e^{-\frac{i}{2}\alpha^{\mu\nu} M_{\mu\nu}} \tag{I-3}$$

We follow Wigner[1] and label our states by the eigenvalue of the momentum operator, P_μ.

$$P_\mu |\mathring{p}\alpha\rangle = \mathring{p}_\mu |\mathring{p}\alpha\rangle \tag{I-4}$$

The subgroup of the homogenous Lorentz group which leaves the

eigenvalue of P_μ unchanged is called the little group. Its generators are given by

$$W_\mu(\mathring{p}) \equiv \frac{1}{2}\,\varepsilon_{\mu\nu\sigma\tau}\,\mathring{p}^\nu M^{\sigma\tau} \tag{I-5}$$

where $\varepsilon_{\mu\nu\sigma\tau}$ is the totally antisymmetric tensor with $\varepsilon_{0123} = 1$. The operators $W_\mu(\mathring{p})$ satisfy the commutation relations

$$[\,W_\mu(\mathring{p}),\, W_\nu(\mathring{p})\,] = -i\,\varepsilon_{\mu\nu\sigma\tau}\,\mathring{p}^\sigma W^\tau(\mathring{p}) \tag{I-6}$$

If $\mathring{p}$ is timelike, $\mathring{p}^2 > 0$, the little group is a compact group and is isomorphic to SU(2). This is most readily seen in the rest frame where $\mathring{p}_\mu = (m,0,0,0)$. In this frame, the generators may be written as

$$S_0 = W_0/m = 0 \qquad S_i = W_i/m \tag{I-7}$$

Equation (I-6) reduces to

$$[\,S_i,\, S_j\,] = i\,\varepsilon_{ijk}\,S_k \tag{I-8}$$

which are the commutation relations for SU(2).

The basis vectors for the representation may now be defined by the eigenvalue of the momentum and the eigenvalue of one of the spin operators. One usually starts in the rest frame with a representation of the spin operators, S_i. The state is given by

$$|\,m\,0\,0\,0 \quad s\,\lambda\,\rangle \tag{I-9}$$

A rotation transforms the state according to[3]

$$U(R)\,|m000\,s\lambda\rangle = \sum_{\lambda'} |m000\,s\lambda'\rangle\, D^{s}_{\lambda'\lambda}(R)$$

$$D^{s}_{\lambda'\lambda}(R) = \left(e^{-i\alpha\cdot S}\right)_{\lambda'\lambda} \tag{I-10}$$

where S_i is a (2s+1) x (2s+1) matrix representation of the SU(2) algebra and α_i represents the rotation.

A state of arbitrary momentum, P , is defined by

$$|ps\lambda\rangle = U(L(p))\,|m000\;s\lambda\rangle \tag{I-11}$$

where $L(p)$ is a given Lorentz transformation satisfying $L_\mu{}^0(p) = p_\mu/m$. An arbitrary Lorentz transformation is then given by

$$U(\Lambda)\,|ps\lambda\rangle = U(\Lambda)\,U(L(p))\,|m000\;s\lambda\rangle$$

$$= U(L(\Lambda p))\,U(L^{-1}(\Lambda p)\Lambda L(p))\,|m000\;s\lambda\rangle \tag{I-12}$$

$L^{-1}(\Lambda p)\Lambda L(p) = R(\Lambda p, p)$ is a rotation usually called a Wigner rotation so that

$$U(\Lambda)\,|ps\lambda\rangle = \sum_{\lambda'} |\Lambda p\;s\lambda'\rangle\, D^{s}_{\lambda'\lambda}(R(\Lambda p, p)) \tag{I-13}$$

These states have complicated transformation properties as the Wigner rotation is a complicated function of P and Λ . We will show it is possible to define a state by introducing two four-vectors q and K so that the Wigner rotation reduces to $\pm$ 1. We let the vector q represent the 3 axis and K represent the 2 axis. An arbitrary spin state will

be given by

$$|p s \lambda q k\rangle = U(L(p,q,k))\,|m\,0\,0\,0\;s\lambda\rangle \tag{I-14}$$

where $L(p,q,k)$ is a Lorentz transformation specified by

$$(L^{-1}(p,q,k)\,p)_\mu = (m,0,0,0)$$

$$(L^{-1}(p,q,k)\,q)_\mu = (L^{-1}(p,q,k)q_0,\ 0,0,\ L^{-1}(p,q,k)q_3) \tag{I-15}$$

$$(L^{-1}(p,q,k)\,k)_\mu = (L^{-1}(p,q,k)k_0,\ 0,\ L^{-1}(p,q,k)k_2,\ 0)$$

with $\quad L^{-1}(p,q,k)\,q_3 < 0 \qquad L^{-1}(p,q,k)\,k_2 < 0$

This requires that the vectors p, q, k satisfy the condition $p\cdot k\; p\cdot q - p^2\, q\cdot k = 0$. If p, q, k are linearly independent then the conditions in (I-15) specify $L(p,q,k)$ to within a rotation of 2π.

A Lorentz transformation of this state yields

$$U(\Lambda)\,|p s \lambda q k\rangle = U(\Lambda)\,U(L^{-1}(p,q,k))\,|m\,0\,0\,0\;s\lambda\rangle$$

$$= U(L(\Lambda p,q',k')\,U(L^{-1}(\Lambda p,q',k')\Lambda L(p,q,k))\,|m\,s\lambda\rangle \tag{I-16}$$

$$= \sum_{\lambda'} |\Lambda p\; s\lambda'\; q' k'\rangle\, D^{s}_{\lambda'\lambda}(L^{-1}(\Lambda p,q',k')\Lambda L(p,q,k))$$

If we depart from the conventional use of a fixed coordinate system, but instead allow the coordinate system to change with the physical system, so that $q' = \Lambda q$ $k' = \Lambda k$, then the Wigner rotation $L^{-1}(\Lambda p,\Lambda q,\Lambda k)\Lambda L(p q k)$ is 1 or a rotation by 2π. Equation (I-16) becomes

$$U(\Lambda)\,|p\,s\,\lambda\,q\,k\rangle = |\Lambda p\,s\,\lambda\,\Lambda q\,\Lambda k\rangle\,(\pm 1)^{2s} \tag{I-17}$$

To check that all we have done is specify the quantization coordinate system in a covariant way, we note that

$$\begin{aligned} q\cdot W(p)\,|p\,s\,\lambda\,q\,k\rangle &= q\cdot W(p)\,U(L(p,q,k))\,|m\,0\,0\,0\,s\,\lambda\rangle \\ &= U(L(p,q,k))\,L^{-1}(p,q,k)\,q\cdot W(L^{-1}(p,q,k)\,p)\,|m\,0\,0\,0\,s\,\lambda\rangle \qquad \text{(I-18)} \\ &= U(L(p,q,k))\,(-L^{-1}(p,q,k)\,q_3\,m\,S_3)\,|m\,0\,0\,0\,s\,\lambda\rangle \end{aligned}$$

Hence,

$$q\cdot W(p)\,|p\,s\,\lambda\,q\,k\rangle = [(q\cdot p)^2 - q^2p^2]^{1/2}\sum_{\lambda'}|p\,s\,\lambda'\,q\,k\rangle\langle\lambda'|S_3|\lambda\rangle .$$

Similarly, (I-19)

$$k\cdot W(p)\,|p\,s\,\lambda\,q\,k\rangle = [(k\cdot p)^2 - k^2p^2]^{1/2}\sum_{\lambda'}|p\,s\,\lambda'\,q\,k\rangle\langle\lambda'|S_2|\lambda\rangle$$

These equations show that the states $|p\,s\,\lambda\,q\,k\rangle$ are defined such that the operators $[(q\cdot p)^2 - q^2p^2]^{-\frac{1}{2}}\,q\cdot W(p)$ and $[(k\cdot p)^2 - k^2p^2]^{-\frac{1}{2}}\,k\cdot W(p)$ are just the spin operators S_3 and S_2 in an arbitrary Lorentz frame. We could, in fact, use the equations in (I-19) to define the covariant spin states.

The normalization of the single particle states may be taken to be

$$\begin{aligned} &\langle p'\,s'\,\lambda'\,q'\,k'|\,p\,s\,\lambda\,q\,k\rangle \\ &= 2p_0\,\delta^3(p'-p)\,\langle p\,s'\,\lambda'\,q'\,k\,||\,p\,s\,\lambda\,q\,k\rangle \\ &= 2p_0\,\delta^3(p'-p)\,\langle m\,s'\,\lambda'||\,U(L^{-1}(p\,q'\,k')\,U(L(p\,q\,k))\,|m\,s\,\lambda\rangle \qquad \text{(I-20)} \\ &= 2p_0\,\delta^3(p'-p)\,\delta_{s's}\,D^{s}_{\lambda'\lambda}(R) \end{aligned}$$

where $R = L^{-1}(P,q',K')L(P,q,K)$ is a rotation. If $K'=K$, then R is a rotation about the 2 axis; if $q'=q$, then R is a rotation about the 3 axis; if $q'=q$ and $K'=K$, then $R = 1$. Thus

$$\langle P's'\lambda' qK | Ps\lambda qK\rangle = 2P_0\,\delta^3(P'-P)\,\delta_{s's}\,\delta_{\lambda'\lambda} \qquad \text{(I-21)}$$

The properties of these states under time reversal and space inversion may be found by examining the equations of (I-19). We use the relations for $\omega_\mu(P)$

$$\begin{aligned} \mathcal{P}\,\omega_\mu(P)\,\mathcal{P}^{-1} &= -i_s\,\omega(i_sP) \\ T\,\omega_\mu(P)\,T^{-1} &= i_s\,\omega(i_sP) \end{aligned} \qquad \text{(I-22)}$$

where $\mathcal{P}$ is the parity operator, T is the time reversal operator, and $i_s(P_0,\vec{P}) = (P_0,-\vec{P})$

From these relations follow

$$\begin{aligned} \mathcal{P}\,q\cdot\omega(P)|Ps\lambda qK\rangle &= \sum_{\lambda'} [(q\cdot P)^2 - q^2P^2]^{\frac{1}{2}}\,\mathcal{P}\,|Ps\lambda' qK\rangle\langle\lambda'|S_3|\lambda\rangle \\ &= -i_s\,q\cdot\omega(i_sP)\,\mathcal{P}\,|Ps\lambda qK\rangle \end{aligned} \qquad \text{(I-23)}$$

$$\begin{aligned} \mathcal{P}\,K\cdot\omega(P)|Ps\lambda qK\rangle &= \sum_{\lambda'} [(K\cdot P)^2 - K^2P^2]^{\frac{1}{2}}\,T\,|Ps\lambda' qK\rangle\langle\lambda'|S_2|\lambda\rangle \\ &= -i_sK\cdot\omega(i_sP)\,\mathcal{P}\,|Ps\lambda qK\rangle \end{aligned}$$

and

$$T\, q\cdot W(p)\,|p s \lambda q k\rangle = \sum_{\lambda'} \left[(q\cdot p)^2 - q^2 p^2\right]^{\frac{1}{2}} T\,|p s \lambda' q k\rangle \langle \lambda' | S_3 | \lambda\rangle^*$$

$$= i_s\, q\cdot W(i_s p)\, T\,|p s \lambda q k\rangle \qquad \text{(I-24)}$$

$$T\, k\cdot W(p)\,|p s \lambda q k\rangle = \sum_{\lambda'} \left[(k\cdot p)^2 - k^2 p^2\right]^{\frac{1}{2}} T\,|p s \lambda' q k\rangle \langle \lambda' | S_2 | \lambda\rangle^*$$

$$= i_s\, k\cdot W(i_s p)\, T\,|p s \lambda q k\rangle$$

If we specify the transformation properties of the four-vectors q and k and take S_3 to be a real, diagonal matrix and S_2 to be an imaginary, hermitian matrix, the transformation properties of the states are determined.

For example,

$$P\,|p s \lambda q k\rangle = |i_s p\; s\; -\lambda\; i_s q\; -i_s k\rangle (-1)^{s+\lambda} \eta_P$$
$$T\,|p s \lambda q k\rangle = |i_s p\; s\; \lambda\; i_s q\; -i_s k\rangle \eta_T \qquad \text{(I-25)}$$

where η_P and η_T are phase factors, represents an acceptable choice.

Two particle states may be constructed using these single particle states. In particular, the covariant spin state which corresponds to the helicity state of Jacob and Wick[4] in the center of mass frame is given by

$$|p s \lambda q k;\; q\, T\sigma\, p\, -k\rangle \qquad \text{(I-26)}$$

and satisfies

$$q\cdot W(p)|\;\rangle = -\,p\cdot W(q)|\;\rangle = \left[(q\cdot p)^2 - q^2 p^2\right]^{\frac{1}{2}} (\lambda - \sigma)|\;\rangle$$

These two particle states may be expanded in terms of eigenstates of the total angular momentum.

$$|p s \lambda q k ; q \tau \sigma p k\rangle = \sum_{J} |p+q\ J\ \lambda-\sigma\ q k\ (\lambda\sigma)\rangle \qquad \text{(I-27)}$$

The covariant spin states may be used to discuss the partial wave expansions of the covariant scattering matrix elements. We consider first the decay of a particle into two others. The matrix element may be written

$$\langle p s \lambda k N | T | q \tau \sigma k N ; k R \varphi\ q -N\rangle \qquad \text{(I-28)}$$

with $p = k+q$ $\qquad k\cdot N = q\cdot N = p\cdot N = 0$

This matrix element is an invariant function as

$$\begin{aligned} &\langle p s \lambda k N | T | q \tau \sigma k N ; k R \varphi\ q -N\rangle \\ &= \langle p s \lambda k N | U^{+}(\Lambda) U(\Lambda) T U^{-1}(\Lambda) | q \tau \sigma k N ; k R \varphi\ q -N\rangle \qquad \text{(I-29)} \\ &= \langle \Lambda p\ s \lambda\ \Lambda k\ \Lambda N | T | \Lambda q\ \tau \sigma\ \Lambda k\ \Lambda N ;\ \Lambda k\ R \varphi\ \Lambda q -\Lambda N\rangle \end{aligned}$$

We may use the operator $k\cdot W(p) = k\cdot W(q) = -q\cdot W(k)$ to show that $\lambda = \sigma - \varphi$. Therefore,

$$\langle p s \lambda k N | T | q \tau \sigma k N ; k R \varphi\ q -N\rangle = \delta(\lambda-\sigma+\varphi)\, T_{\lambda\sigma}(p^2, q^2, k^2) \qquad \text{(I-30)}$$

If the T-matrix is invariant under space inversion, we have

$$\begin{aligned} &\langle p s \lambda k N | T | q \tau \sigma k N ; k R \varphi\ q -N\rangle \\ &= \langle p s \lambda k N | P^{+} P T P^{-1} P | q \tau \sigma k N ; k R \varphi\ q -N\rangle \qquad \text{(I-31)} \\ &= (-1)^{s+\lambda} \eta_s \langle i_s p\ s -\lambda\ i_s k\ -i_s N | T | i_s q\ \tau -\sigma\ i_s k\ -i_s N ; \\ &\qquad i_s k\ R -\varphi\ i_s q\ i_s N\rangle (-1)^{\tau+\sigma} \eta_\tau (-1)^{R+\varphi} \eta_R \end{aligned}$$

which implies

$$F_{\lambda\sigma}(p^2, q^2, k^2) = (-1)^{s+\lambda+T+\sigma+R+\varphi} \eta_s \eta_T \eta_R \; F_{-\lambda,-\sigma}(p^2, q^2, k^2) \tag{I-32}$$

A two body amplitude may be written as

$$\langle p's'\lambda' q'N; q'T'\sigma' p'-N | T | p s \lambda q N; q T \sigma p -N \rangle \tag{I-33}$$

This amplitude is also an invariant function depending only upon the invariants $s = (p+q)^2$ and $t = (p'-p)^2$. The amplitude may be expanded in partial waves using (I-27).

$$\begin{aligned}
&\langle p's'\lambda' q'N; q'T'\sigma' p'-N | T | p s \lambda q N; q T \sigma p -N \rangle \\
&= \sum_{J'J} \langle p'+q' J' \lambda'-\sigma' q'N (\lambda'\sigma') | T | p+q J \lambda-\sigma q N (\lambda\sigma) \rangle \\
&= \sum_{J'J m'} \langle p'+q' J' \lambda'-\sigma' q'N \| p'+q' J' m' q N \rangle \\
&\qquad \langle p+q J' m' q N (\lambda'\sigma') | T | p+q J \lambda-\sigma q N (\lambda\sigma) \rangle \\
&= \sum_J d^J_{\lambda'-\sigma', \lambda-\sigma}(\chi) \; \langle \lambda'\sigma' | T^J(s) | \lambda\sigma \rangle
\end{aligned} \tag{I-34}$$

where χ is the center of mass scattering angle and the reduced matrix element of (I-20) is used.

We conclude this section with a review of the properties of the covariant spin states.

a) The defining equations are

$$|p s \lambda q k\rangle = U(L(p,q,k)) | m 0 0 0 \; s\lambda \rangle$$

with (I-35a)

$$L^{-1}(p,q,k)\, p = (m, 0, 0, 0)$$

$$L^{-1}(p,q,k)q = (|L^{-1}q|_0,\ 0,\ 0,\ -|\overrightarrow{L^{-1}q}|)$$

$$L^{-1}(p\,q\,k)k = (\,(L^{-1}k)_0,\ 0,\ -|\overrightarrow{L^{-1}k}|,\ 0\,)$$

and $\qquad k\cdot p\; q\cdot p - k\cdot q\; p^2 = 0$

b) The Lorentz transformation properties of these states are given by

$$U(\Lambda)\,|p\,s\,\lambda\,q\,k\rangle = |\Lambda p\ \ s\,\lambda\ \ \Lambda q\ \ \Lambda k\rangle \tag{I-35b}$$

c) The transformation properties under the discrete symmetries of space inversion and time reversal are

$$\begin{aligned} \mathcal{P}\,|p\,s\,\lambda\,q\,k\rangle &= |i_s p\ \ s\ -\lambda\ \ i_s q\ -i_s k\rangle\,(-1)^{s-\lambda}\,\eta_P \\ T\,|p\,s\,\lambda\,q\,k\rangle &= |\,i_s p\ \ s\ \ \lambda\ \ i_s q\ -i_s k\rangle\,\eta_T \end{aligned} \tag{I-35c}$$

d) The spin equations are

$$\begin{aligned} q\cdot W(p)\,|p\,s\,\lambda\,q\,k\rangle &= [(q\cdot p)^2 - q^2p^2]^{\frac{1}{2}} \sum_{\lambda'} |p\,s\,\lambda'\,q\,k\rangle\,\langle\lambda'|S_3|\lambda\rangle \\ k\cdot W(p)\,|p\,s\,\lambda\,q\,k\rangle &= [(k\cdot p)^2 - k^2p^2] \sum_{\lambda'} |p\,s\,\lambda'\,q\,k\rangle\,\langle\lambda'|S_3|\lambda\rangle \end{aligned} \tag{I-35d}$$

e) The helicity states are given by

$$|p\,s\,\lambda\,q\,k;\,q\,T\,\sigma\,p\,-k\rangle = \sum_J |p+q\ \ J\ \ \lambda-\sigma\ \ q\,k\,(\lambda\sigma)\rangle \tag{I-35e}$$

f) A change in the quantization axes is given by

$$\begin{aligned} |p\,s\,\lambda\,q\,k\rangle &= \sum_{\lambda'} |p\,s\,\lambda'\,q'k'\rangle\,\langle p\,s\,\lambda'\,q'k'||p\,s\,\lambda\,q\,k\rangle \\ &= \sum_{\lambda'} |p\,s\,\lambda\,q'k'\rangle\,D^{s}_{\lambda'\lambda}(R) \end{aligned} \tag{I-35f}$$

$$R = L^{-1}(p\,q'k')\,L(p\,q\,k)$$

Section B

In the previous section, we have shown that it is possible to define single particle states having covariant transformation properties. These states may be used to express matrix elements of various tensor operators. An important property of these states is that the matrix element of a scalar operator will always be an invariant function.

The matrix element of a scalar operator between two single particle states has the form

$$\langle p's'\lambda' - p\,k \,|\, J(0) \,|\, p\,s\,\lambda\; p'k\rangle = \delta(\lambda'-\lambda)\, F_\lambda\left(p'^2, (p'\cdot p)^2, p^2\right) \qquad \text{(I-36)}$$

The diagonalization of the spin dependence results from the operator $-p\cdot w(p') = p'\cdot w(p)$ being diagonal on both states. We have explicitly, using the defining equations for the spin polarization in (I-35d).

$$\begin{aligned}
&\left[(p'\cdot p)^2 - p^2 p'^2\right]^{\frac{1}{2}} (\lambda'-\lambda) \langle p's'\lambda' - p\,k \,|\, J(0) \,|\, p\,s\,\lambda\; p'k\rangle \\
&= \langle p's'\lambda' - p\,k \,|\, \{-p\cdot w(p')\, J(0) - J(0)\, p'\cdot w(p)\} \,|\, p\,s\,\lambda\; p'k\rangle \\
&= \tfrac{1}{2}\, \varepsilon^{\mu\nu\sigma\tau} p'_\mu p_\nu \langle p's'\lambda' - p\,k \,|\, [M_{\sigma\tau}, J(0)] \,|\, p\,s\,\lambda\; p'k\rangle \\
&= 0
\end{aligned} \qquad \text{(I-37)}$$

This implies the equation written in (I-36).

A similar analysis can be made for a vector density.[5] It is useful, at this point, to introduce a complete set of vectors

which have simple transformations under the little group.

We define the following set of four-vectors.

$$U^\mu_\alpha(pqk) \qquad \alpha = 0,1,2,3$$

$$U^\mu_0(pqk) = p^\mu$$

$$U^\mu_1(pqk) = \left[\frac{p^2}{-k^2}\right]^{\frac{1}{2}} \left[(q\cdot p)^2 - q^2p^2\right]^{-\frac{1}{2}} \varepsilon^\mu{}_{qpk} \tag{I-38}$$

$$U^\mu_2(pqk) = \left[\frac{p^2}{-k^2}\right]^{\frac{1}{2}} k^\mu$$

$$U^\mu_3(pqk) = \left[(q\cdot p)^2 - q^2p^2\right]^{-\frac{1}{2}} \left[p^2 q^\mu - q\cdot p\, p^\mu\right]$$

where we have taken $k\cdot p = k\cdot q = 0$. These vectors are normalized such that

$$g_{\mu\nu} U^\mu_\alpha U^\nu_\beta = g_{\alpha\beta}\, p^2$$
$$g^{\alpha\beta} U^\mu_\alpha U^\nu_\beta = g^{\mu\nu}\, p^2 \tag{I-39}$$

We may use these vectors to define invariant spin operators.

$$\bar{W}_0(pqk) = U^\mu_0(pqk)\, W_\mu(p) = 0$$
$$\bar{W}_i(pqk) = U^\mu_i(pqk)\, W_\mu(p) \tag{I-40}$$

These operators may be shown to satisfy the commutation relations

$$[\bar{W}_i, \bar{W}_j] = p^2\, i\, \varepsilon_{ijk}\, \bar{W}_k \tag{I-41}$$

If $p^2 > 0$, the vectors in (I-38) are real and the operators

defined by

$$J_i = \bar{W}_i / p^2 \tag{I-42a}$$

are hermitian and satisfy the commutation relations

$$[J_i, J_j] = i\varepsilon_{ijk} J_k$$

The little group is therefore isomorphic to SU(2).

If $p^2 < 0$, the vectors U_0 and U_3 are real, but U_1 and U_2 are imaginary. The operators defined by

$$J_3 = \bar{W}_3 / p^2, \quad K_1 = -i\bar{W}_1 / p^2, \quad K_2 = -i\bar{W}_2 / p^2 \tag{I-42b}$$

are hermitian and satisfy the commutation relations

$$[J_3, K_1] = iK_2, \quad [J_3, K_2] = -iK_1, \quad [K_1, K_2] = -iJ_3$$

The little group is therefore isomorphic to the group O(2,1).

In the case $p^2 = 0$, the little group operators are given by

$$J_3 = \bar{W}_3 / p^2, \quad L_1 = \bar{W}_1 [p^2]^{-\frac{1}{2}}, \quad L_2 = \bar{W}_2 [p^2]^{-\frac{1}{2}} \tag{I-42c}$$

These operators satisfy the commutation relations, with $p^2 = 0$,

$$[J_3, L_1] = iL_2, \quad [J_3, L_2] = -iL_1, \quad [L_1, L_2] = 0$$

The little group is therefore isomorphic to the Euclidian group in two dimensions.

We consider the transformation properties of a vector density under the little group operators. Under Lorentz transformations, the vector density is rotated according to

$$U(\Lambda)\, J_\mu(0)\, U^{-1}(\Lambda) = \Lambda^{-1}{}_\mu{}^\nu\, J_\nu(0) \tag{I-43}$$

We use the vectors defined in (I-38) to define a set of covariant vector densities

$$J_\alpha(pqk;0) = U^\mu_\alpha(pqk)\, J_\mu(0)\,. \tag{I-44}$$

Under Lorentz transformations, we have

$$U(\Lambda)\, J_\alpha(pqk;0)\, U^{-1}(\Lambda) = J_\alpha(\Lambda p, \Lambda q, \Lambda k\,;0) \tag{I-45}$$

The commutation relations of the covariant vector densities with the covariant spin operators of (I-40) are given by

$$\begin{aligned} [\,\bar{W}_i(pqk),\ J_0(pqk;0)\,] &= 0 \\ [\,\bar{W}_i(pqk),\ J_j(pqk;0)\,] &= p^2\, i\,\epsilon_{ijk}\, J_k(pqk,0) \end{aligned} \tag{I-46}$$

As the covariant vector densities are scalars under Lorentz transformations, their matrix elements between covariant spin states are invariant functions. We could use the completeness relation given in (I-39) to expand the matrix elements of the vector density in terms of these functions and the U^μ_α vectors.

It is more convenient, however, to introduce linear combinations of the U^μ_α vectors which transform as a spin basis under the little group operators. These vectors are the covariant polarization vectors. We label the polarization vectors by $\varepsilon^\mu_{s\lambda}(pqk)$ where

$$\varepsilon^{\mu}_{00}(pqk) = U^{\mu}_{0}(pqk) = p^{\mu}$$

$$\varepsilon^{\mu}_{10}(pqk) = U^{\mu}_{3} = [(q\cdot p)^2 - q^2p^2]^{-\frac{1}{2}}\,[p^2q^{\mu} - q\cdot p\,p^{\mu}]$$

$$\varepsilon^{\mu}_{11}(pqk) = -\frac{1}{\sqrt{2}}\,(U^{\mu}_{1} + iU^{\mu}_{2})$$

$$= -\frac{1}{\sqrt{2}}\left[\frac{p^2}{-k^2}\right]^{\frac{1}{2}}\left\{[(q\cdot p)^2 - q^2p^2]^{-\frac{1}{2}}\,\varepsilon^{\mu}\,qpk + ik^{\mu}\right\} \tag{I-47}$$

$$\varepsilon^{\mu}_{1-1}(pqk) = \frac{1}{\sqrt{2}}\,(U^{\mu}_{1} - iU^{\mu}_{2})$$

$$= \frac{1}{\sqrt{2}}\left[\frac{p^2}{-k^2}\right]^{\frac{1}{2}}\left\{[(q\cdot p)^2 - q^2p^2]^{-\frac{1}{2}}\,\varepsilon^{\mu}\,qpk - ik^{\mu}\right\}$$

with $\quad p\cdot k = q\cdot k = 0$

These polarization vectors are normalized so that

$$\varepsilon_{s'\lambda'}(pqk)\cdot\varepsilon_{s\lambda}(pqk) = p^2(-1)^{s+\lambda}\,\delta_{s's}\,\delta_{\lambda',-\lambda}$$

$$p^2 g^{\mu\nu} = \sum_{s\lambda}\varepsilon^{\mu}_{s\lambda}(pqk)\,(-1)^{s+\lambda}\,\varepsilon^{\nu}_{s-\lambda}(pqk) \tag{I-48}$$

A change in the quantization axes may be calculated using the completeness relation of (I-48).

$$p^2\varepsilon^{\mu}_{s\lambda}(pq'k') = \sum_{s'\lambda'}\varepsilon^{\mu}_{s'\lambda'}(pqk)\,(-1)^{s+\lambda'}\,\varepsilon_{s'-\lambda'}(pqk)\,\varepsilon_{s\lambda}(pq'k') \tag{I-49}$$

A transformation in which the q' vector is unchanged and $p\cdot k' = p\cdot k = q\cdot k' = q\cdot k = 0$ is given by

$$\varepsilon_{s\lambda}(pqk') = \varepsilon^{\mu}_{s\lambda}(pqk)\,e^{-i\lambda\varphi} \tag{I-50}$$

where

$$\cos\varphi = -K'\cdot K\,[K^2K'^2]^{-1/2}$$

$$\sin\varphi = \varepsilon\, K'qpK\,[K^2K'^2]^{-\frac{1}{2}}\,[(q\cdot p)^2 - q^2p^2]^{-\frac{1}{2}}$$

In general, we have

$$\varepsilon_{s\lambda}(pq'K') = \sum_{\lambda'} \varepsilon_{s\lambda'}(pqK)\, e^{-i\lambda'\varphi_1}\, d^{s}_{\lambda'\lambda}(\theta)\, e^{-i\lambda\varphi_2} \qquad \text{(I-51)}$$

where

$$\cos\varphi_1 = -N\cdot K\,[N^2K^2]^{-\frac{1}{2}}$$

$$\sin\varphi_1 = \varepsilon\, NqpK\,[N^2K^2]^{-\frac{1}{2}}\,[(q\cdot p)^2 - q^2p^2]^{-\frac{1}{2}}$$

$$\cos\varphi_2 = -K'N\,[N^2K'^2]^{-\frac{1}{2}}$$

$$\sin\varphi_2 = \varepsilon\, K'qpN\,[N^2K'^2]^{-\frac{1}{2}}\,[(q\cdot p)^2 - q^2p^2]^{-\frac{1}{2}}$$

$$\cos\theta = (q\cdot p\, q'\cdot p - q\cdot q'\, p^2)\,[(q\cdot p)^2 - q'^2p^2]^{-\frac{1}{2}}\,[(q\cdot p)^2 - q^2p^2]^{-\frac{1}{2}}$$

$$\sin\theta = \varepsilon\, q'qpN\,[(q\cdot p)^2 - q^2p^2]^{-\frac{1}{2}}\,[(q\cdot p)^2 - q'^2p^2]^{-\frac{1}{2}}\,[-N^2]^{-\frac{1}{2}}$$

and $N_\mu = \varepsilon_\mu\, pqq'$

The function $d^{s}_{\lambda'\lambda}(\theta)$ is the spin rotation matrix.[3] If $p^2 > 0$, the angle is real; if $p^2 < 0$, the angle θ is imaginary.

We may now define a covariant spin density by

$$J_{s\lambda}(pqK;0) = \varepsilon^{\mu}_{s\lambda}(pqK)\, J_\mu(0) \qquad \text{(I-52)}$$

The commutation relations of the spin density with the invariant spin operators is given by

$$[\bar{\omega}_i(pqk), J_{s\lambda}(pqk;0)] = p^2 \sum_{\lambda'} J_{s\lambda'}(pqk;0) \langle \lambda' | S_i | \lambda \rangle \qquad \text{(I-53)}$$

where S_i is a spin s, hermitian matrix representation of the SU(2) algebra in which S_3 is diagonal, S_1 real, S_2 imaginary.

The covariant spin density is a scalar under Lorentz transformations.

$$U(\Lambda)\, J_{s\lambda}(pqk;0)\, U^{-1}(\Lambda) = J_{s\lambda}(pqk;0) \qquad \text{(I-54)}$$

We use the completeness relation of (I-48) to expand the vector densities in terms of the spin densities. We have

$$\begin{aligned} J^\mu(0) &= \frac{1}{p^2} \sum_{s\lambda} (-1)^{s+\lambda} \cdot \varepsilon^\mu_{s-\lambda}(pqk)\, \varepsilon^\nu_{s\lambda}(pqk)\, J_\nu(0) \\ &= \frac{1}{p^2} \sum_{s\lambda} (-1)^{s+\lambda}\, \varepsilon^\mu_{s-\lambda}(pqk)\, J_{s\lambda}(pqk,0) \end{aligned} \qquad \text{(I-55)}$$

Since the spin densities are scalar under Lorentz transformations, their matrix elements between covariant spin states are invariant functions.

$$\begin{aligned} &\langle p's'\lambda' - p\, k | J_{r\sigma}(p'-p, p'+p, k;0) | p s \lambda\, p'k \rangle \\ &= \delta(\lambda' - \lambda + \sigma)\, F_{\lambda'\lambda}(p'^2 s, (p'-p)^2 T, p^2 s) \end{aligned} \qquad \text{(I-56)}$$

The spin delta function follows from the little group 3 axis spin operators for the three states being related.

$$\begin{aligned} &p' \cdot W(p)\, [(p'-p)^2 - p'^2 p^2]^{-\frac{1}{2}} \\ &= J_3(pp'k) = J_3(p'-p, k) = -J_3(p'-p, p'+p, k) \\ &J_3(pp'k) = \bar{\omega}_3(pp'k)/p^2 \end{aligned} \qquad \text{(I-57)}$$

Therefore, we have

$$\langle P'S'\lambda'-PK|\,[-J_3(P'-P,P'+P,K),\,J_{T\sigma}(P'-P,P'+P,K)]\,|PS\lambda P'K\rangle$$

$$= -\sigma\,\langle P'S'\lambda'-PK|\,J_{T\sigma}(P'-P,P'+P,K)\,|PS\lambda P'K\rangle \qquad \text{(I-58)}$$

$$= (\lambda'-\lambda)\,\langle P'S'\lambda'-PK|\,J_{T\sigma}(P'-P,P'+P,K;0)\,|PS\lambda P'K\rangle$$

This implies the result of (I-56).

We use these polarization vectors and the results of (I-56) to expand the matrix element of a vector density.

$$\langle P'S'\lambda'-PK|\,J_\mu(0)\,|PS\lambda P'K\rangle$$

$$= \frac{1}{(P'-P)^2}\sum_{T\sigma}(-1)^{T+\sigma}\,\varepsilon^{\mu}_{T-\sigma}(P'-P,P'+P,K)$$

$$\langle P'S'\lambda'-PK|\,J_{T\sigma}(P'-P,P'+P,K;0)\,|PS\lambda P'K\rangle \qquad \text{(I-59)}$$

$$= \frac{1}{(P'-P)^2}\sum_{T}(-1)^{T+\lambda-\lambda'}\,\varepsilon^{\mu}_{T\,\lambda'-\lambda}(P'-P,P'+P,K)$$

$$F_{\lambda'\lambda}(P'^2S',(P'-P)^2T,P^2S)$$

The transformation properties of the spin densities under space inversion, P , and time reversal, T , may be determined from the transformation properties of the vector densities. The polarization vectors satisfy the relations

$$\varepsilon_{S\lambda}(i_SP,i_SQ,-i_SK) = (-1)^{\lambda}\,i_S\,\varepsilon_{S-\lambda}(PQK)$$

$$\varepsilon^{*}_{S\lambda}(PQK) = (-1)^{\lambda}\,\varepsilon_{S-\lambda}(PQK) \qquad P^2>0 \qquad \text{(I-60)}$$

$$= \varepsilon_{S-\lambda}(PQK) \qquad P^2<0$$

with $\quad K^2 < 0 \;, \quad (q\cdot p)^2 - q^2 p^2 > 0$

Under the discrete symmetries, the vector densities transform according to

$$
\begin{aligned}
P\, J_\mu(0)\, P^{-1} &= i_s\, J(0)\, \eta_P \\
T\, J_\mu(0)\, T^{-1} &= i_s\, J(0)\, \eta_T
\end{aligned}
\tag{I-61}
$$

Hence, the spin densities transform according to

$$
\begin{aligned}
P\, J_{s\lambda}(pqK;0)\, P^{-1} &= \varepsilon_{s\lambda}(pqK)\; i_s\, J(0)\; \eta_P \\
&= (-1)^\lambda\, \varepsilon_{s-\lambda}(i_s p, i_s q, -i_s K)\cdot J(0)\, \eta_P \\
&= (-1)^\lambda\, \eta_P\; J_{s-\lambda}(i_s p, i_s q, -i_s K;0)
\end{aligned}
$$

$$
\begin{aligned}
T\, J_{s\lambda}(pqK;0)\, T^{-1} &= \varepsilon^*_{s\lambda}(pqK;0)\; i_s\, J(0)\; \eta_T \\
&= \eta_T\; J_{s\lambda}(i_s p, i_s q, -i_s K;0) \quad p^2 > 0 \\
&= \eta_T (-1)^\lambda\, J_{s\lambda}(i_s p, i_s q, -i_s K;0) \quad p^2 < 0
\end{aligned}
\tag{I-62}
$$

We are able to discuss the kinematic singularities and crossing properties of matrix elements of the covariant spin states. We consider, for example, a two body scattering amplitude. We use the reduction formulas to express the amplitude in terms of a Green's function of Rarita-Schwinger interpolating fields. We assume the Green's function has

no kinematic singularities as a function of the four-vector components. Therefore, the kinematic singularities can only come from the polarization tensors used to define the states. The effect of the kinematic singularities can be found using the explicit representation of the polarization vectors in terms of the four-vectors in (I-47).

We consider the amplitude

$$\langle p's'\lambda' q'N; q'T'\sigma' p'-N | T | p s\lambda q N; q T\sigma p-N\rangle$$

$$N^\mu = \varepsilon^\mu p'pq = \varepsilon^{\mu\nu\sigma\tau} p'_\nu p_\sigma q_\tau \tag{I-63}$$

The reduction formulas are used to obtain

$$\langle p's'\lambda' q'N; q'T'\sigma' p'-N | T | p s\lambda q N; q T\sigma p-N\rangle$$

$$= Z \; \varepsilon^{*\{\mu_1\}}_{s'\lambda'}(p'q'N)\, \varepsilon^{*\{\mu_2\}}_{T'\sigma'}(q'p'-N)\, \varepsilon^{\{\mu_3\}}_{s\lambda}(pqN)$$

$$\varepsilon^{\{\mu_4\}}_{T\sigma}(qp-N) \left\{ (p'^2-m^2)(q'^2-\mu^2)(p^2-m^2)(q^2-\mu^2) \right.$$

$$\int dx_1 \int dx_2 \int dx_3 \; e^{ip'x_1} e^{iq'x_2} e^{-ipx_3}$$

$$\left. \langle 0 | T\{ A_{\{\mu_1\}}(x_1) A_{\{\mu_2\}}(x_2) A_{\{\mu_3\}}(x_3) A_{\{\mu_4\}}(0)\} | 0\rangle \right\} \tag{I-64}$$

$$= \varepsilon^{*\{\mu_1\}}_{s'\lambda'}(p'q'N)\, \varepsilon^{\{\mu_2\}}_{T'\sigma'}(q'p'-N)\, \varepsilon^{\{\mu_3\}}_{s\lambda}(pqN)\, \varepsilon^{\{\mu_4\}}_{T\sigma}(qp-N)$$

$$G(p'\{\mu_1\}, q'\{\mu_2\}; p\{\mu_3\}, q\{\mu_4\})$$

where $p'+q' = p+q$ and Z is a normalization factor.

The fields, $A_{\{\mu\}}(x) = A_{\mu_1 \cdots \mu_s}(x)$, are Rarita-Schwinger[6] fields, and

$$\begin{aligned} \varepsilon_{s\lambda}^{\{\mu\}}(pqN) &= \varepsilon_{s\lambda}^{\mu_1 \cdots \mu_s}(pqN) \\ &= \sum_{\varphi_1 \cdots \varphi_s} C(s\lambda, 1\varphi_1; \cdots; 1\varphi_s)\, \varepsilon_{1\varphi_1}^{\mu_1}(pqN) \cdots \varepsilon_{1\varphi_s}^{\mu_s}(pqN) \end{aligned} \tag{I-65}$$

are the polarization tensors for a spin s particle: $C(s\lambda, 1\varphi)$ are the usual SU(2) Clebsch-Gordon coefficients. The set of functions

$$G(p'\{\mu_1\}, q'\{\mu_2\}; p\{\mu_3\}, q\{\mu_4\})$$

are the Green's functions (four point functions, M - functions) for the process.

We assume the Green's functions have no kinematic singularities as functions of the four-vector components.[7] The polarization tensors do have kinematic singularities. To determine the effect of these singularities, we use the explicit representation of the polarization vectors given in (I-47). If the polarization vectors are denoted by $\varepsilon_{s\lambda}^{\mu}(pqk)$, the kinematic singularities are square root singularities and occur at the points

$$p^2 = 0 \quad , \quad (q \cdot p)^2 - q^2 p^2 = 0 \, , \quad k^2 = 0$$

The effect of circling the branch point may be determined directly as

a) branch point at $p^2 = 0$

$$\varepsilon_{s\lambda}^{\{\mu\}}(pqk) \longrightarrow (-1)^{\lambda}\, \varepsilon_{s\lambda}^{\{\mu\}}(pqk) \tag{I-66}$$

b) branch point at $(q\cdot p)^2 - q^2 p^2 = 0$

$$\mathcal{E}^{\{\mu\}}_{s\lambda}(pqk) \longrightarrow (-1)^{s+\lambda}\, \mathcal{E}^{\{\mu\}}_{s-\lambda}(pqk)$$

c) branch point at $k^2 = 0$

$$\mathcal{E}^{\{\mu\}}_{s\lambda}(pqk) \longrightarrow (-1)^{\lambda}\, \mathcal{E}^{\{\mu\}}_{s\lambda}(pqk)$$

Therefore the 'on the mass shell' amplitude, as defined in (I-63), has kinematic singularities at the three points[8]

$$N^2 = 0, \quad (q'\cdot p')^2 - q'^2 p'^2 = 0, \quad (q\cdot p)^2 - q^2 p^2 = 0 \qquad \text{(I-67)}$$

The effect of circling these branch points is given by

a) branch point at $N^2 = 0$

$$\langle p's'\lambda' q'N;\, q'T'\sigma' p'-N \,|\, T \,|\, p s\lambda q N;\, qT\sigma p -N\rangle$$
$$\longrightarrow (-1)^{\lambda'+\sigma'-\lambda-\sigma} \langle p's'\lambda' q'N;\, q'T'\sigma' p'-N \,|\, T \,|\, p s\lambda q N;\, qT\sigma p -N\rangle \qquad \text{(I-68)}$$

b) branch point at $(q'\cdot p')^2 - q'^2 p'^2 = 0$

$$\langle s'\lambda';\, T'\sigma' \,|\, T \,|\, s\lambda;\, T\sigma\rangle$$
$$\longrightarrow (-1)^{s'+\lambda'} (-1)^{T'+\sigma'} \langle s'-\lambda';\, T'-\sigma' \,|\, T \,|\, s\lambda;\, T\sigma\rangle$$

c) branch point at $(q\cdot p)^2 - q^2 p^2 = 0$

$$\langle s'\lambda';\, T'\sigma' \,|\, T \,|\, s\lambda;\, T\sigma\rangle$$

$$\longrightarrow (-1)^{s+\lambda} (-1)^{T+\sigma} \langle s'\lambda';\, T'\sigma' \,|\, T \,|\, s-\lambda;\, T-\sigma\rangle$$

The crossing relations may also be obtained with the use of the representation in (I-64). We assume the Green's function may be continued in the four-vectors to the crossed channel. We consider a continuation in which

$$p' \to p' \ , \quad p \to p \ , \quad q' \to -\bar{q}' \ , \quad q \to -\bar{q}$$

The Green's function in the crossed channel is then given by

$$\bar{G}\left(p'\{\mu_1\}, \bar{q}\{\mu_4\}; p\{\mu_3\}, \bar{q}'\{\mu_2\}\right)$$

$$= \text{ continuation of } \quad G\left(p'\{\mu_1\}, q'\{\mu_2\}; p\{\mu_3\}, q\{\mu_4\}\right) \quad \text{(I-69)}$$

$$= G\left(p'\{\mu_1\}, -\bar{q}'\{\mu_2\}; p\{\mu_3\}, -\bar{q}\{\mu_4\}\right)$$

We consider the continuation of the amplitude

$$\langle p' s' \lambda' q' N; q' T' \sigma' p' - N | T | p s \lambda q N; q T \sigma p - N \rangle$$

$$= \varepsilon^{*\{\mu_1\}}_{s\lambda'}(p'q'N)\, \varepsilon^{*}_{T'\lambda'}(q'p'-N)\, \varepsilon^{\{\mu_3\}}_{s\lambda}(pqN)\, \varepsilon^{\{\mu_4\}}_{T\sigma}(qp-N) \quad \text{(I-70)}$$

$$G\left(p'\{\mu_1\}, q'\{\mu_2\}; p\{\mu_3\}, q\{\mu_4\}\right)$$

To make this continuation, we must continue the polarization tensors. We again inspect the explicit representation of the polarization vectors given in (I-47). If the kinematic singularities are avoided, ie. the square roots do not change sign, the continuation is given directly. For example,

$$\begin{aligned} \varepsilon^{\{\mu\}}_{s\lambda}(pqN) &\longrightarrow \varepsilon^{\{\mu\}}_{s\lambda}(p, -\bar{q}, -\bar{N}) \\ \varepsilon^{\{\mu\}}_{s\lambda}(qp-N) &\longrightarrow \varepsilon^{*\{\mu\}}_{s\lambda}(\bar{q}, p, -\bar{N}) \end{aligned} \qquad \text{(I-71)}$$

if $N_\mu = \varepsilon_\mu p'pq$ and $\bar{N}_\mu = \varepsilon_\mu p'p\bar{q}$

Therefore, for this continuation we have

$$\langle p's'\lambda' q' N \,;\, q'T'\sigma' p' - N \,|\, T \,|\, p s \lambda q N \,;\, q T \sigma p - N \rangle_c$$

$$= \mathcal{E}^{*\{\mu_1\}}_{s'\lambda'}(p'q'N)_c \; \mathcal{E}^{*\{\mu_2\}}_{T'\sigma'}(q'p'-N) \; \mathcal{E}^{\{\mu_3\}}_{s\lambda}(pqN) \; \mathcal{E}^{\{\mu_4\}}_{T\sigma}(qp-N)$$

$$G(p'\{\mu_1\}, q'\{\mu_2\} \,;\, p\{\mu_3\}, q\{\mu_4\})$$

$$= \mathcal{E}^{*\{\mu_1\}}_{s'\lambda'}(p'-\bar{q}'-\bar{N}) \; \mathcal{E}^{\{\mu_2\}}_{T'\sigma'}(\bar{q}'p'-N) \; \mathcal{E}^{\{\mu_3\}}_{s\lambda}(p-\bar{q}-N) \qquad \text{(I-72)}$$

$$\mathcal{E}^{*\{\mu_4\}}_{T\sigma}(\bar{q}p-N) \; \bar{G}(p'\{\mu_1\}, \bar{q}\{\mu_4\} \,;\, p\{\mu_3\}, \bar{q}\{\mu_4\})$$

$$= \langle p's'\lambda' - \bar{q}' - \bar{N} \,;\, \bar{q}T\sigma p - N \,|\, T \,|\, p s \lambda - \bar{q} - \bar{N} \,;\, \bar{q}'T'\sigma' p' - \bar{N} \rangle$$

where $\quad N_\mu = \varepsilon_\mu p'pq \qquad \bar{N}_\mu = \varepsilon_\mu p'p\bar{q}$

$$p'_c = p' \,,\quad p_c = p \,,\quad q'_c = -\bar{q}' \,,\quad q_c = -\bar{q}$$

and A_c means the continuation of A.

The crossing relation derived in (I-72) may be related to that discussed by Trueman and Wick[9] for the helicity amplitudes. The amplitude on the L.H.S.(I-72) may be identified as the invariant helicity amplitude, $G_{\lambda'\sigma'\lambda\sigma}(s,t,u)$ of Trueman and Wick. The continued amplitude on the R.H.S.(I-72) is related to the crossed channel helicity amplitude by a change of basis in the spin quantization frames. This transformation is given by

$$\langle p's'\lambda' - \bar{q}' - \bar{N} \,;\, \bar{q}T\sigma p - \bar{N} \,|\, T \,|\, p s \lambda - \bar{q} - \bar{N} \,;\, q'T'\sigma' p' - \bar{N} \rangle =$$

$$= \sum_{\varphi'\varphi+\psi} \langle p's'\chi' - \bar{q}' - \bar{N} \| p s \varphi' \bar{q} N \rangle \langle \bar{q} T \sigma p - \bar{N} \| \bar{q} T \psi p' - \bar{N} \rangle$$

$$\langle p's'\varphi' \bar{q} \bar{N} ; \bar{q} T \psi p' - \bar{N} | T | p s \varphi \bar{q}' \bar{N} ; \bar{q}' T' \psi' p - \bar{N} \rangle$$

$$\langle p s \varphi \bar{q}' \bar{N} \| p s \chi \bar{q} - \bar{N} \rangle \langle \bar{q}' T' \psi' p - \bar{N} \| \bar{q}' T' \sigma' p' - \bar{N} \rangle \tag{I-73}$$

$$= \sum_{\varphi'\varphi+\psi} (-1)^{\eta} d^{s'}_{\chi\varphi'}(\chi_1) d^{T}_{\sigma\psi}(\chi_2) d^{s}_{\varphi\chi}(\chi_3) d^{T'}_{\psi\sigma'}(\chi_4)$$

$$\langle p's'\varphi' \bar{q} \bar{N} ; \bar{q} T \psi p' - \bar{N} | T | p s \varphi \bar{q}' \bar{N} ; \bar{q}' T' \psi' p - \bar{N} \rangle$$

The amplitude on the R.H.S.(I-73) is the helicity amplitude in the crossed channel. The rotation angles χ and the phase η may be determined from the reduced matrix elements with the use of (I-35f). The continuation of the invariant helicity amplitude expressed in terms of the crossed channel helicity amplitude is found by the insertion of (I-73) into (I-72). We have

$$\langle p's'\chi' q' N ; q' T' \sigma' p' - N | T | p s \chi q N ; q T \sigma p - N \rangle_c$$

$$= \sum_{\varphi'\varphi+\psi} (-1)^{\eta} d^{s'}_{\chi\varphi'}(\chi_1) d^{T}_{\sigma\psi}(\chi_2) d^{s}_{\varphi\chi}(\chi_3) d^{T'}_{\psi\sigma'}(\chi_4) \tag{I-74}$$

$$\langle p's'\varphi' \bar{q} \bar{N} ; \bar{q} T \psi p' - \bar{N} | T | p s \varphi \bar{q}' \bar{N} ; \bar{q}' T' \sigma' p - \bar{N} \rangle$$

This relation corresponds to the relation of Eq.(41) of Trueman and Wick.[9]

The above analysis for the wavefunctions and the associated kinematic singularities has been done for integral spin and non-zero mass. A similar analysis can be made for half-

integral spin through the introduction of a set of covariant spinor wavefunctions. We need only consider the spin one-half case as arbitrary half-integral spin wavefunctions may be constructed by vector coupling the spin one-half wavefunctions to the integral spin polarization tensors.

The spinor wavefunctions will be defined in terms of a set of three linearly vectors as in the case of integral spin. We write the wavefunction as

$$U_n^{\sigma}(pqk) \tag{I-75}$$

The vectors are again to represent the quantization frame. However, these vectors well only specify the frame to within a rotation of . Therefore, under Lorentz transformations we have

$$U^{\sigma}(\Lambda p\ \Lambda q\ \Lambda k) = S(\Lambda)\, U^{\sigma}(pqk)\ (\pm 1) \tag{I-76}$$

where $S(\Lambda)$ is a spinor representation of the homogenous Lorentz group.

The covariant spinor may be written as

$$U^{\sigma}(pqk) = S(L(pqk))\, U^{\sigma}(m,0,0,0,z,y) \tag{I-77}$$

where $L(pqk)$ is a given Lorentz transformation from the frame (m,z,y) to the frame (pqk). As we have said before, the vectors (pqk) will only specify the transformation to within a rotation of 2π. However, we are usually interested in matrix elements involving an even number of half-integral spin

fields and states. We may choose to define their wavefunctions so that under Lorentz transformations the spinors change sign at the same time.

The spinors will satisfy the following equations for the little group operators.

$$q\cdot W(p)\, U^{\sigma}(pqk) = [(q\cdot p)^2 - p^2 q^2]^{\frac{1}{2}}\, U^{\sigma'}(pqk)\, \langle \sigma' | S_3 | \sigma \rangle$$
$$k\cdot W(p)\, U^{\sigma}(pqk) = [(k\cdot p)^2 - p^2 k^2]^{\frac{1}{2}}\, U^{\sigma'}(pqk)\, \langle \sigma' | S_3 | \sigma \rangle \tag{I-78}$$

where $W_\mu(p) = \frac{1}{2}\, \varepsilon_{\mu\nu\sigma\tau}\, p^\nu\, \Sigma^{\sigma\tau}$

and $\Sigma^{\sigma\tau}$ is a spinor matrix representation of the homogenous Lorentz group.

A covariant spinor field may be defined by

$$\bar{\psi}^{\sigma}(pqk;0) = \bar{\psi}(0)\, U^{\sigma}(pqk) \tag{I-79}$$

If the half-integral spin wavefunctions are properly matched, the matrix element of covariant spinors between covariant spin states will be invariant functions. These wavefunctions will contain kinematic singularities as in the case on integral spin. With a specific choice of representation for the wavefunctions, these singularities may be analysed.

We next consider the wavefunctions which have smooth limits to zero mass. The polarization vectors defined in (I-47) are not correctly normalized for this limit, and the little group operators must be those given in (I-42c).

The covariant little group operators defined in (I-40) satisfy the commutation relations

$$[\bar{\omega}_i(pqk), \bar{\omega}_j(pqk)] = p^2 i \varepsilon_{ijk} \bar{\omega}_k$$

If we take $p\cdot k = q\cdot k = 0 \quad p\cdot q \neq 0$, then we see, in the limit $p^2 \to 0$, all the $\bar{\omega}_i$ vanish. If we define the little group operators according to (I-42c), then these operators become the correct little group operators in the limit . These operators are given by

$$J_3 = \bar{\omega}_3 [p^2]^{-1}, \quad L_2 = \bar{\omega}_2(pqk) [p^2]^{-\frac{1}{2}}, \quad L_1 = \bar{\omega}_1 [p^2]^{-\frac{1}{2}} \tag{I-80}$$

and satisfy the commutation relations

$$\begin{aligned} [J_3, L_1] &= i L_2 \\ [J_3, L_2] &= -i L_1 \\ [L_1, L_2] &= i p^2 J_3 \end{aligned} \tag{I-81}$$

The polarization vectors are to be constructed from vectors which also have smooth limits as $p^2 \to 0$. We may define a set of vectors V_i^μ by

$$\begin{aligned} V_0^\mu(pqk) &= U_0^\mu(pqk) = p^\mu \\ V_1^\mu &= U_1^\mu [p^2]^{-\frac{1}{2}} = [-k^2]^{-\frac{1}{2}} [(q\cdot p)^2 - p^2 q^2]^{-\frac{1}{2}} \varepsilon^\mu qpk \\ V_2^\mu &= U_2^\mu [p^2]^{-\frac{1}{2}} = [-k^2]^{-\frac{1}{2}} k^\mu \\ V_3^\mu &= U_3^\mu = [(q\cdot p)^2 - p^2 q^2]^{-\frac{1}{2}} [p^2 q^\mu - q\cdot p\, p^\mu] \end{aligned} \tag{I-82}$$

These vectors are normalized such that

$$V_0^2 = -V_3^2 = p^2$$
$$V_1^2 = V_2^2 = -1 \tag{I-83}$$

We consider an arbitrary vector field $A_\mu(0)$ and the covariant fields given by

$$A_i(pqk;0) = V_i^\mu(pqk)\, A_\mu(0) \tag{I-84}$$

The commutation relations of the covariant fields with the little group operators are given by

$$[J_3, A_1] = i A_2 \qquad [J_3, A_2] = -i A_1$$
$$[L_1, A_2] = i A_3 \qquad [L_1, A_3] = -i p^2 A_2 \tag{I-85}$$
$$[L_2, A_1] = -i A_3 \qquad [L_2, A_3] = i p^2 A_2$$

with all other combinations commuting.

If the vector field contains a spin one, zero mass excitation, then the matrix elements of the fields will, in general, have a singularity at $p^2 = 0$. If there is no spin zero, zero mass excitation, then the divergence of the field will not have the singularity.

We consider the amplitude given by

$$T_r(p) = \int dx\, e^{-ipx} \langle W' | r^\mu A_\mu(x) | W \rangle \tag{I-86}$$

If the vector $r^\mu = p^\mu$, then the amplitude $T_p(p)$ will have no singularity at $p^2 = 0$. If $r^\mu \neq p^\mu$, then the amplitude

should have a singularity of the form

$$T_r(p) \longrightarrow \bar{T}_r(p)/p^2 \qquad p^2 \to 0 \tag{I-87}$$

The matrix elements of the covariant field may be calculated in this limit. We define the amplitude

$$\begin{aligned} T_i(p) &= \int dx\, e^{-ipx}\, p^2 \langle n' | A_i(pqk;x) | n \rangle \\ &= \int dx\, e^{-ipx}\, p^2 \langle n' | V_i^{\mu}(pqk)\, A_\mu(x) | n \rangle \end{aligned} \tag{I-88}$$

Using (I-82) and the definition (I-84), the amplitudes of (I-88) may be evaluated in the limit $p^2 \to 0$.

$$\begin{aligned} &\lim_{p^2\to 0} T_0(p^2) = \lim_{p^2\to 0} T_3(p^2) = 0 \\ &\lim_{p^2\to 0} T_i(p^2) = T_i(p, p^2=0) \qquad i=1,2 \end{aligned} \tag{I-89}$$

The vectors V_1 and V_2 may therefore be used to construct the polarization vectors for massless particles. In the zero mass limit, the polarization vectors become

$$\begin{aligned} \varepsilon_+^{\mu}(pqk) &= -\frac{1}{\sqrt{2}}\,[V_1^{\mu} + iV_2^{\mu}] \\ &= -\frac{1}{\sqrt{2}}\,[-k^2]^{-\frac{1}{2}} \left\{ [q\cdot p]^{-1}\, \varepsilon_\mu qpk + ik_\mu \right\} \\ \varepsilon_-^{\mu}(pqk) &= \frac{1}{\sqrt{2}}\,[-k^2]^{-\frac{1}{2}} \left\{ [q\cdot p]^{-1}\, \varepsilon_\mu qpk - ik_\mu \right\} \end{aligned} \tag{I-90}$$

In the massless limit, the little group operators obey the commutation relations

$$[J_3, L_1] = iL_2\,, \quad [J_3, L_2] = -iL_1\,, \quad [L_1, L_2] = 0 \tag{I-91}$$

The field operators defined by

$$A_{\pm}(pqk) = \lim_{p^2 \to 0} \mathcal{E}^{\mu}_{\pm}(pqk)\, p^2 \int dx\, e^{-ipx} A_{\mu}(x) \qquad \text{(I-92)}$$

satisfy the commutation relations

$$[J_3, A_+] = A_+ \quad [J_3, A_-] = -A_- \quad [L_{1,2}, A_{\pm}] = 0 \qquad \text{(I-93)}$$

The kinematic singularities of amplitudes with states defined in terms of these polarization vectors may be determined. We consider the amplitude for the emission of a photon. We use the reduction formulas to obtain

$$\langle n' | p \pm qk\, n \rangle$$
$$= Z \lim_{p^2 \to 0} \int dx\, e^{-ipx} p^2 \mathcal{E}^{\mu}_{\pm}(pqk) \langle n' | A_{\mu}(x) | n \rangle \qquad \text{(I-94)}$$

where Z is a normalization factor and $A_{\mu}(x)$ is an arbitrary interpolating containing the zero mass excitation. The matrix element $\langle n' | A_{\mu}(0) | n \rangle$ will have kinematic singularities due to the polarization vectors used in defining the states in $\langle n'|, |n\rangle$. However, the only additional kinematic singularities in the amplitude will come from normalization factors in the polarization vectors. There are two of these:

a) a square root singularity at $k^2 = 0$. The result of circling the branch point is a change in sign of the polarization vectors.

b) a pole at $q \cdot p = 0$.

The crossing properties of these amplitudes may be easily determined. If the kinematic singularities are avoided, the continuation of the polarization vectors is given by

$$\varepsilon^{\mu}_{\pm}(pqk) \longrightarrow \varepsilon^{\mu}_{\pm}(\bar{p}q\bar{k}) \quad p \to -\bar{p} \quad q \to q \quad k \to -\bar{k}$$
$$\varepsilon^{\mu}_{\pm}(pqk) \longrightarrow \varepsilon^{\mu}_{\pm}(p\bar{q}\bar{k}) \quad p \to p \quad q \to -\bar{q} \quad k \to -\bar{k} \tag{I-95}$$

Summary

We have tried to present a comprehensive study of spin in the context of special relativity. We use covariant little group operators to generate states and wavefunctions which transform as irreducible representations under the little group, but which are covariant under the full Lorentz group. While the single particle states and wavefunctions appear to be more complicated than the usual formulation, their use in discussing any physical question, as in the analysis of matrix elements, avoids the introduction of extraneous parameters as 3 axis quantization. The covariant wavefunctions also provide a framework in which one may investigate and remove the kinematic singularities of the invariant amplitudes.

PART II: Representations of the Local Current Algebra

Gell-Mann[10] has proposed that the current densities involved in the electromagnetic and weak interactions form a closed algebra under equal time commutation. We consider a representation of the algebra in terms of a set of states of definite space momentum and total angular momentum. The matrix elements of the current densities may be expressed in terms of invariant functions. The current algebra generates relations which must be satisfied by these functions.

In Section A, we consider a representation in terms of an arbitrary set of states. In Section B, we consider a representation in terms of an infinite set of single particle states. We show how it is possible for these states to generate the known analyticity properties of the form factors.

Section A

The assumption that the current densities form a closed algebra under equal time commutation may be expressed as

$$[\, J_0^\alpha(\vec{x},t)\,,\; J_0^\beta(\vec{y},t)\,] = \delta(\vec{x}-\vec{y})\, i f^{\alpha\beta\gamma} J_0^\gamma(\vec{x},t) \qquad \text{(II-1)}$$

where $\{ J_0^\alpha(\vec{x},t) \}$ are the time components of the currents and $\{ f^{\alpha\beta\gamma} \}$ are the structure constants of the algebra.

The isotopic spin currents satisfy (II-1) with $I_0^\alpha(x) = J_0^\alpha(x)$ $\alpha = 1,2,3$. The structure constants are given by $f^{\alpha\beta\gamma} = \varepsilon^{\alpha\beta\gamma}$; $\varepsilon^{\alpha\beta\gamma}$

is the totally antisymmetric tensor with $\varepsilon^{123} = 1$.

If we define an integrated charge by

$$F^{\alpha}(\vec{k},t) = \int d^3x \; e^{-i\vec{k}\vec{x}} \; J_0^{\alpha}(\vec{x},t) \tag{II-2}$$

These operators satisfy relations corresponding to (II-1) as

$$[F^{\alpha}(\vec{k},t), F^{\beta}(\vec{k}',t)] = i f^{\alpha\beta\gamma} F^{\gamma}(\vec{k}'+\vec{k},t) \tag{II-3}$$

The charge operators are given by

$$Q^{\alpha}(t) = F^{\alpha}(0,t) \tag{II-4}$$

In the following discussion, we will be interested in an extension of equation of equation (II-1). We assume the equal time commutator of the charge density with the current density satisfies

$$[J_0^{\alpha}(\vec{x},t), J_{\mu}^{\beta}(\vec{y},t)] = \delta(\vec{x}-\vec{y}) \, i f^{\alpha\beta\gamma} J_{\mu}^{\gamma}(\vec{x},t) + C_{\mu}^{\alpha\beta}(\vec{x},\vec{y},t) \tag{II-5}$$

It has been generally believed and proved by Meyer and Suura[11] that in a sufficiently convergent theory $C_{\mu}^{\alpha\beta}(\vec{x},\vec{y},t)$ is symmetric in α and β . We will thus ignore this term when studying the antisymmetric part.

We consider a representation of the algebra, defined in (II-5), by a set of states of definite momentum and total angular momentum. The states will be labeled by

$$|m p s \lambda q N\rangle \tag{II-6}$$

where p is the four-momentum of the state, s is its total

angular momentum, λ is the spin projection in the frame (pqN) as defined in (I-35), and n is a degeneracy parameter.

We use equations (II-2) and (II-5) to obtain

$$\langle n' p' s' \lambda' - p N | [F^{\alpha}(\vec{k},0), J^{\beta}_{\mu}(0)] | n p s \lambda p' N \rangle$$
$$= i f^{\alpha\beta\gamma} \langle n' p' s' \lambda' - p N | J^{\gamma}_{\mu}(0) | n p s \lambda p' N \rangle \qquad \text{(II-7)}$$

where $N_{\mu} = \varepsilon_{\mu\nu\rho\sigma} p^{\nu} p^{\sigma} k^{\rho}$, $k_{\mu} = (0, \vec{k})$

We use the polarization vectors as defined in (I-47) to express equation (II-7) in terms of the form factors of (I-56). Equation (II-7) becomes

$$\delta(\lambda' - \lambda + \varphi)\, i f^{\alpha\beta\gamma}\, F^{\gamma}_{\lambda'\lambda}(n' p'^2 s', (p'-p)^2 T, n p^2 s)$$
$$= i f^{\alpha\beta\gamma}\, \varepsilon^{\mu}_{T\varphi}(p'-p, p'+p, N) \langle p' s' \lambda' - p N | J^{\gamma}_{\mu}(0) | p s \lambda p' N \rangle \qquad \text{(II-8)}$$
$$= \varepsilon^{\mu}_{T\varphi}(p'-p, p'+p, N) \langle p' s' \lambda' - p N | [F^{\alpha}(\vec{k},0), J^{\beta}_{\mu}(0)] | p s \lambda p' N \rangle$$

We introduce a complete set of intermediate states in the commutator of (II-8). The integral $\int \frac{d^3q}{2q_0}$ is over the four-momentum of the intermediate state. The summation $\sum_{m}$ is over all other degrees of freedom. We have

$$\delta(\lambda' - \lambda + \varphi)\, i f^{\alpha\beta\gamma}\, F^{\gamma}_{\lambda'\lambda}(n' p'^2 s', (p'-p)^2 T, n p^2 s)$$
$$= \varepsilon^{\mu}_{T\varphi}(p'-p, p'+p, N) \sum_{m} \int \frac{d^3q}{2q_0} \frac{1}{(2\pi)^3} \sum_{\sigma\tau}$$
$$\Big\{ \langle p' s' \lambda' - p N | F^{\alpha}(\vec{k},0) | m q \sigma \tau \rangle \langle m q \sigma \tau | J^{\beta}_{\mu}(0) | n p s \lambda p' N \rangle$$
$$- \langle p' s' \lambda' - p N | J^{\beta}_{\mu}(0) | m q \sigma \tau \rangle \langle m q \sigma \tau | F^{\alpha}(\vec{k},0) | p s \lambda p' N \rangle \Big\} \qquad \text{(II-9)}$$

$$= \mathcal{E}^{\mu}_{T\varphi}(p'-p, p'+p, N) \sum_{\lambda m} \sum_{J\pi\pi'} \sum_{\sigma'\sigma} \frac{1}{2q_0'}$$

$$\Big\{ \langle p's'\chi'-p\,N \| p's'\sigma'-q'\,N_1' \rangle \langle \lambda' p's'\sigma'-q'\,N_1' | J_0^{\nu}(0) | \lambda m q' J \pi' p' N_1' \rangle$$

$$\langle q' J \pi' p' N_1' \| q' J \pi - p\,N_1 \rangle$$

$$\langle \lambda m q' J \pi - p\,N_1 | J_\mu^{\rho}(0) | \lambda p s \sigma q' N_1 \rangle \langle p s \sigma q' N_1 \| p s \lambda p' N \rangle$$

$$- \text{cross term } [\, J_0^{\nu}(0) \leftrightarrow J_\mu^{\rho}(0) \,,\; q' \leftrightarrow q \,,\; N_1' \leftrightarrow N_1 \,] \Big\}$$

where

$$q' = (q_0', \vec{p}'+\vec{k}) \qquad q_0' = [\, (\vec{p}'+\vec{k})^2 + m_J^2 \,]^{1/2}$$

$$q = (q_0, \vec{p}-\vec{k}) \qquad q_0 = [\, (\vec{p}-\vec{k})^2 + m_J^2 \,]^{1/2}$$

$$N_1' = \mathcal{E}_\mu p' p q' \qquad N_1 = \mathcal{E}_\mu p' p q$$

The reduced matrix elements $\langle \quad \| \quad \rangle$ represent a change in the quantization axes for the covariant spin states. The transformation is given in (I-35f). For example, we have

$$\langle p's'\chi'-p\,N \| p's'\sigma'-q'\,N \rangle$$
$$= D^{s'}_{\chi'\sigma'}(R) = e^{-i\chi'\varphi_1'}\, d^{s'}_{\chi'\sigma'}(\chi_1') \qquad \text{(II-10)}$$

where

$$\cos\varphi_1' = -N N_1' \,[\, N^2 N_1'^2 \,]^{-1/2}$$

$$\cos\chi_1' = \cos\chi(p_1'-p_1-q_1')$$

$$= [\, p'\cdot p\; p'\cdot q' - p'^2\, p\cdot q' \,][\, (p'\cdot p)^2 - p'^2 p^2 \,]^{-\frac{1}{2}} [\, (p'\cdot q')^2 - p'^2 q^2 \,]^{-\frac{1}{2}}$$

We use the relation of (I-59) to expand the matrix elements of the currents in terms of the form factors. We also use the relation of (I-50) to give

$$\mathcal{E}^{M}_{T\sigma}(P'-P, P'+P, N) = \mathcal{E}^{M}_{T\sigma}(P'-P, P'+P, N_1)\, e^{-i\sigma\varphi'}$$

$$\cos\varphi' = -N\cdot N_1' \,[N^2 N_1'^2]^{-\frac{1}{2}} \qquad \text{(II-11)}$$

The relation in (II-8) becomes

$$\delta(X-\lambda+\varphi)\; i f^{\alpha\beta\gamma}\; F^{\gamma}_{X\lambda}(M'P'^2S', (P'-P)^2T, MP^2S)$$

$$= \sum_{M}\sum_{J\pi'\pi}\sum_{R\sigma'\sigma} \Big\{ e^{-i(X'-\lambda+\varphi)\varphi'}\, d^{S'}_{X'\sigma'}(\chi_1')$$

$$\langle M'P'S'\sigma' -q'N_1' | J^{\alpha}_0(0) | Mq'J\pi'P'N_1'\rangle \frac{1}{2q_0'}\, d^{J}_{\pi'\pi}(\chi_2')$$

$$F^{\beta}_{\pi\sigma}(Mq'^2J, (q'-P)^2R, MP^2S)\, d^{S}_{\sigma\lambda}(\chi_3')\, \frac{1}{(q'-P)^2}$$

$$(-1)^{R+\sigma-\pi}\, \mathcal{E}_{R\pi-\sigma}(q'-P, q'+P, N_1')\, \mathcal{E}_{T\varphi}(P'-P, P'+P, N_1') \qquad \text{(II-12)}$$

$$- e^{-i(X'-\lambda+\varphi)\varphi}\, d^{S'}_{X'\sigma'}(\chi_1)\, F^{\beta}_{\sigma'\pi'}(M'P'^2S', (P'-q)^2R, Mq^2J)$$

$$d^{J}_{\pi'\pi}(\chi_2)\, \frac{1}{2q_0}\, \langle Mq J\pi - PN_1 | J^{\alpha}_0(0) | PSJqN_1\rangle$$

$$d^{S}_{\sigma\lambda}(\chi_3)\; 1/(P'-q)^2 \quad (-1)^{R+\pi'-\sigma'}$$

$$\mathcal{E}_{R\sigma'-\pi'}(P'-q, P'+q, N_1)\cdot \mathcal{E}_{T\varphi}(P'-P, P'+P, N_1)\Big\}$$

where $\chi_1' = \chi(P', -P, -q')$, $\chi_2' = \chi(q', P', -P)$, $\chi_3' = \chi(P, q', P')$

$\chi_1 = \chi(P', -P, -q)$, $\chi_2 = \chi(q, P', -P)$, $\chi_3 = \chi(P, q, P')$

$\cos\varphi' = -NN_1' [N^2 N_1'^2]^{-\frac{1}{2}}$, $\cos\varphi = -NN_1 [N^2 N_1^2]^{-\frac{1}{2}}$

The left hand side of (II-12) is an invariant function of $p'^2, p^2, (p'-p)^2$. The right hand side must also be an invariant function. However, the R.H.S.(II-12) is a sum over contributions from various intermediate states. Each term in the sum is not necessarily invariant under Lorentz transformations. Gell-Mann[12] has shown that if we evaluate the sum in the frame in which the component of the momentum of the external states perpendicular to the vector $\vec{K}$ is infinite, then each term in the sum is, in fact, invariant.

In this frame, all the products of the four-vectors p', p, N, q', q, K take on finite limits. To specify the limit more precisely, we write

$$\vec{p}' = \vec{p}_\perp + \vec{p}'_\parallel \quad , \quad \vec{p} = \vec{p}_\perp + \vec{p}_\parallel \qquad \vec{K} = \vec{K} \tag{II-13}$$

where $\vec{p}_\perp \cdot \vec{p}'_\parallel = \vec{p}_\perp \cdot \vec{p}_\parallel = \vec{p}_\perp \cdot \vec{K} = 0$

The frame of interest is the one in which $|\vec{p}_\perp| \to \infty$. For example, the four-vector p_μ becomes

$$p_\mu = (p_0, \vec{p}) \qquad \vec{p} = \vec{p}_\perp + \vec{p}_\parallel$$
$$p_0 \to p_\perp + (p_\parallel^2 + m^2)/2p_\perp \tag{II-14}$$

The four-vector products of q' and q with the external vectors may be expressed in terms of the products of the external vectors and q'^2, q^2 in the infinite momentum limit. We have

$$p'\cdot q' = \tfrac{1}{2}\left[q'^2 + p'^2 - K^2\right]$$

$$p\cdot q' = \tfrac{1}{2}\left[q'^2 + p^2 - (p'+K-p)^2\right]$$

$$K\cdot q' = p'\cdot K + K^2$$

$$N\cdot q' = \tfrac{1}{2}\left[(p'-p)^2 K^2 - ((p'-p)\cdot K)^2\right]^{\frac{1}{2}}\left[q'^2 - (p'+K)^2\right]$$

$$p'\cdot q = \tfrac{1}{2}\left[q^2 + p'^2 - (p'+K-p)^2\right] \qquad \text{(II-15)}$$

$$p\cdot q = \tfrac{1}{2}\left[q^2 + p^2 - K^2\right]$$

$$K\cdot q = p\cdot K - K^2$$

$$N\cdot q = \tfrac{1}{2}\left[(p'-p)^2 K^2 - ((p'-p)\cdot K)^2\right]^{\frac{1}{2}}\left[q^2 - (p-K)^2\right]$$

All the terms in (II-12) involving four-vector products may be evaluated directly with the use of the limiting values in (II-15). We note that the angles χ and φ, the form factors, and the products of the polarization vectors are all of this form. The remaining terms may be evaluated with the use of the vectors $q'-p'-K$ and $q-p+K$ which have only time components. We have

$$\frac{1}{2q_0'}\langle n'p's'\sigma'-q'N_1'|\,J_0^{\alpha}(0)\,|m\,q'J\,n'p'N_1'\rangle$$

$$= (q'-p'-K)^{\mu}\langle n'p's'\sigma'-q'N_1'|\,J_{\mu}^{\alpha}(0)\,|m\,q'J\,n'p'N_1'\rangle$$

$$\left[2q'(q'-p'-K)\right]^{-1} \qquad \text{(II-16)}$$

$$= \sum_R F^{\alpha}_{\sigma'n'}(n'p'^2s', (p'-q')^2 R, m\,q'J)\frac{1}{(p'-q')^2}(-1)^{R+n'-\sigma'}$$

$$\varepsilon_{R\,\sigma'-n'}(p'-q', p'+q', N_1')\cdot(q'-p'-K)\left[2q'(q'-p'-K)\right]^{-1}$$

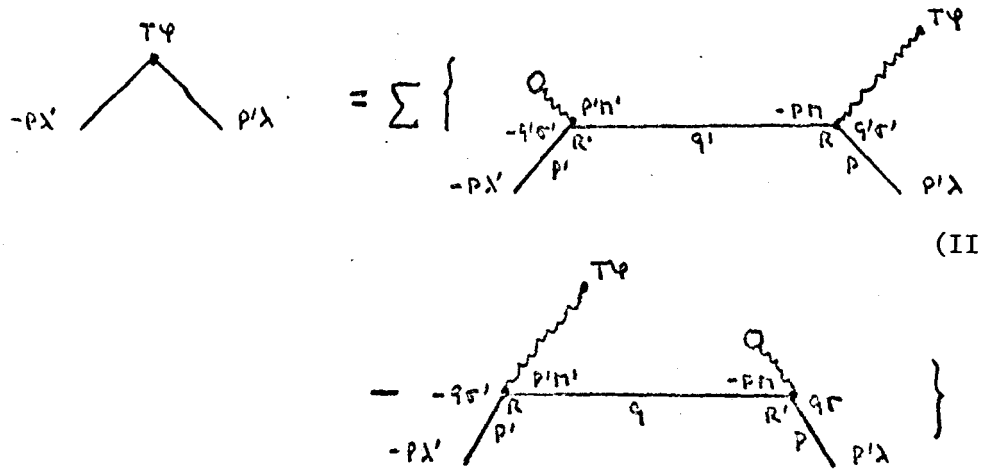
In the infinite momentum frame, we have from (II-15),

$$
\begin{aligned}
2q'(q'-p-k) &\longrightarrow q'^2-(p'+k)^2 \\
(q'-p')^2 &\longrightarrow k^2 \\
(q'-p)^2 &\longrightarrow (p'+k-p)^2 \\
2q(q-p+k) &\longrightarrow q^2-(p-k)^2 \\
(q-p')^2 &\longrightarrow (p'+k-p)^2 \\
(q-p)^2 &\longrightarrow k^2
\end{aligned}
\qquad \text{(II-17)}
$$

In the infinite momentum frame, the relation in (II-12) becomes

$$
\begin{aligned}
&\delta(\chi'-\lambda+\varphi)\, i f^{\alpha\beta\gamma} F^{\gamma}_{\chi\lambda}(\mu' p'^2 s', (p'-p)^2 T, \mu p^2 s) \\
&= \sum_{\mu}\sum_{R'R\sigma'\sigma}\sum_{J n' n} \Big\{ e^{-i(\chi'-\lambda+\varphi)\varphi'} d^{s'}_{\chi'\sigma'}(\chi_1') F^{\alpha}_{\sigma' n'}(\mu' p'^2 s', \\
&R'k^2, \mu q^2 s)(k^2)^{-1}(-1)^{R'+n-\chi'} \mathcal{E}_{R'\chi'-n'}(p'-q', p'+q', N_1')\cdot(q'-p'-k) \\
&[q'^2-(p'+k)^2]^{-1} d^{J}_{n'n}(\chi_2') F^{\beta}_{n\sigma}(\mu q^2 J, (p'+k-p)^2 R, \mu p^2 s) \\
&d^{s}_{\sigma\lambda}(\chi_3)[(p'+k-p)^2]^{-1}(-1)^{R+\sigma-n} \mathcal{E}_{Rn-\sigma}(q'-p, q'+p, N_1')\cdot \mathcal{E}_{T\varphi}(p'-p, p'+p, N_1') \qquad \text{(II-18)} \\
&- e^{-i(\chi'-\lambda+\varphi)\varphi} d^{s'}_{\chi'\sigma'}(\chi_1) F^{\beta}_{\sigma' n'}(\mu' p'^2 s', (p'+k-p)^2 R, \mu q^2 s) \\
&[(p'+k-p)^2]^{-1}(-1)^{R+n-\sigma'} \mathcal{E}_{R\sigma' n'}(p'-q', p'+q', N_1')\cdot \mathcal{E}_{T\varphi}(p'-p, p'+p, N_1) \\
&[q^2-(p-k)^2]^{-1} d^{J}_{n'n}(\chi_2) F^{\alpha}_{n\sigma}(\mu q^2 J, k^2 R, \mu p^2 s)\, d^{s}_{\sigma\lambda}(\chi_3) \\
&(k^2)^{-1}(-1)^{R'+\sigma-n} \mathcal{E}_{Rn-\sigma}(q-p, q+p, N_1)\cdot(q-p+k) \Big\}
\end{aligned}
$$

The relation in (II-18) may be written schematically as



(II-19)

where the external line propagators have the form

$$D^{s'}_{\chi\sigma'}\left(R(\varphi',\chi')\right)$$

the internal line propagator

$$d^{J}_{n'n}(\chi_2)\quad\left[q'^2-(p'+k)^2\right]^{-1}$$

the vertex functions

$$\delta(\chi'-n'+\sigma')\;F^{\alpha}_{\chi'n'}(w'p'^2s',\ k^2T,\ wp^2s)$$

the tadpole

$$\frac{1}{k^2}\;(-1)^{R+\sigma}\;\varepsilon_{R-\sigma}(q'-p',q'+p',N')\,(q'-p'-k)$$

the vector line propagator

$$\frac{1}{(p'+k-p)^2}\;(-1)^{R+\sigma}\;\varepsilon_{T\sigma}(q'-p,q'+p,N')\,\varepsilon_{T\varphi}(p'-p,p'+p,N)$$

We note that the form factors depend only on the mass variables q'^2, p'^2, p^2, k^2, $(p'+k-p)^2$ and not on the energy variable $\nu = \frac{1}{4}(p'+p)(2k+p'-p)$ or the momentum transfer $t = (p'-p)^2$. Further, the rotation angles χ, as given in (II-12), depend only upon the products of thus are also independent of the energy ν.

Section B

In this section, we wish to consider a representation of the isotopic spin currents in terms of a set of single particle states having a mass-spin relationship. In particular, we will be interested in the dependence of the form factors on the momentum transfer.

The isotopic spin current contributes as the isovector part of the source of the electromagnetic field, but it may also be used as an interpolating field for the rho meson. For this to be the case, the form factors given in Section A must contain a pole when continued in the momentum transfer to .

We will take the isotopic spin to be conserved. Therefore, the spin zero form factor will be zero. For the form factors to be consistent with the existance of a rho meson, we must have

$$\delta(x-\lambda+\varphi)\, F^{\mu}_{x'\lambda}(s'p'^2, 1t, sp^2)$$

(II-20)

$$= \varepsilon^{\mu}_{1\varphi}(p'-p, p'+p, N)\, \langle p's'\lambda'-p'N|\, J^{s}_{\mu}(0)\, |ps\lambda p'N\rangle$$

$$= \varepsilon^{\mu}_{1\varphi}(p'-p, p'+p, N) \langle p's'\chi - pN | J^{\nu}_{\rho\mu}(0) | ps\lambda p'N \rangle [m_\rho^2 - t]^{-1}$$

$$= \delta(\chi - \lambda + \varphi) \; G^{\nu}_{\chi'\lambda}(sp^2, t, sp^2) \; [m_\rho^2 - t]^{-1}$$

where $J^{\nu}_{\rho\mu}(0)$ is the source of the rho meson and $G^{\nu}_{\chi'\lambda}$ is the corresponding form factor. The form factors will also have other kinematic and dynamic singularities.

We noted at the end of the last chapter that the R.H.S. (II-18) had dependence on the momentum transfer only through the kinematic functions. These functions have kinematic singularities but have no singularities corresponding to the rho pole. The form factor on the L.H.S. (II-18) should have a singularity due to the rho meson. If we assume that the relation in (II-18) may be continued in the momentum transfer variable, t , then the rho pole must arise as a divergence in the sum over states.

To see how this may happen, we consider a representation in terms of single particle states. We take the mass to be related to the spin in the form $p^2 = m_J^2$ and let the mass go to infinity with the angular momentum.

$$m_J^2 \longrightarrow \infty \qquad J \longrightarrow \infty$$

The relation in (II-18) becomes

$$\delta(\chi' - \lambda + \varphi) \; i\varepsilon^{\alpha\beta\gamma} F^{\gamma}_{\chi'\lambda}(s', t, s) =$$

$$= \sum_{J n' n \sigma' \sigma} \Big\{ e^{-i(\lambda'-\lambda+\varphi)\varphi'} d^{s'}_{\lambda'\sigma'}(\chi_1') F^{\alpha}_{\sigma' n'}(s', \mu'^2, J)$$

$$(\mu'^2)^{-1} (-1)^{1+n'-\sigma'} \varepsilon_{\sigma'-n'}(P'-q', P'+q', N_1') \cdot (q'-P'-K)$$

$$[M_J^2 - (P'+K)^2]^{-1} d^{J}_{n'n}(\chi_2') F^{\beta}_{n\sigma}(J, \mu^2, s) \qquad \text{(II-21)}$$

$$d^{s}_{\sigma\lambda}(\chi_1') (\mu^2)^{-1} (-1)^{1+\sigma-n} \varepsilon_{n-\sigma}(q-P, q+P, N_1) \cdot \varepsilon_{\tau p}(P'-P, P'+P, N_1')$$

$$- \text{ cross term} \Big\}$$

We have used current conservation to eliminate the scalar form factors and have surpressed the index $R = 1$ on the polarization vectors. The mass dependence has been replaced by the spin dependence, and the vector line masses are given by $\mu^2 = (P'+K-P)^2$ and $\mu'^2 = K^2$.

We are interested in the contributions of the high J terms to the sum on the R.H.S.(II-21). For these terms, the mass M_J becomes large. In this case, the angles may be approximated by

$$\cos\chi_1' = \cos\chi_1 = [P'P - P'^2][(P'P)^2 - P'^2P^2]^{-1/2}$$

$$\cos\chi_3' = \cos\chi_3 = [P'P - P^2][(P'P)^2 - P'^2P^2]^{-1/2} \qquad \text{(II-22)}$$

$$\cos\chi_2' = \cos\chi_2 = -1 - 2t/M_J^2 + \cdots$$

$$\cos\varphi' = \cos\varphi = \tfrac{1}{2}[(P'^2-P^2)K(P'-P) - (P'-P)^2 K(P'+P)][N^2(P'-P)^2]^{-1/2}$$

The tadpole contributions are given by

$$(T\sigma) \quad \varepsilon_{T\sigma}(p'-q', p'+q', N_i') \cdot (q'-p'-k)$$

$$00 \qquad (p'-q')(q'-p'-k) = 0$$

$$10 \qquad \tfrac{1}{2}\left[(p'q')^2 - p'^2 q'^2\right]^{-1/2} k^2 \left[q'^2 - (p'+k)^2\right]$$

$$\longrightarrow \frac{k^2}{q'^2}\left[q'^2 - (p'+k)^2\right]$$

$$1\pm1 \qquad \mp \frac{1}{\sqrt{2}}\left[\frac{k^2}{-N^2}\right]^{\frac{1}{2}} \Big\{ \tfrac{1}{2}\left[(p'q')^2 - p'^2 q'^2\right]^{-1/2} \tag{II-23}$$

$$\cdot \left[k^2 p'(p'-p) + \tfrac{1}{2}\left[q'^2 - p'^2 - k^2\right](p-p')\cdot k\right] \pm i\left[(p'-p)^2 k^2 - ((p'-p)\cdot k)^2\right]^{\frac{1}{2}} \Big\}$$

$$\longrightarrow \mp \left[\frac{k^2}{-(p'-p)^2}\right]^{1/2} \Big\{ (p-p')\cdot k$$

$$\pm i\left[(p'-p)^2 k^2 - ((p'-p)\cdot k)^2\right]^{\frac{1}{2}} \Big\} \frac{1}{q'^2}\left[q'^2 - (p'+k)^2\right]$$

The full tadpole contribution, in the large mass limit, may be written

$$(\mu'^2)^{-1} (-1)^{1-\sigma} \varepsilon_{T\sigma}(p'-q', p'+q'; N_i')$$

$$= V_{T\sigma}(p'pk) \frac{1}{q'^2}\left[q'^2 - (p'+k)^2\right] \tag{II-24}$$

where

$$V_{00} = 0$$

$$V_{10} = -1$$

$$V_{1\pm1} = \mp \frac{1}{\sqrt{2}}\left[-t\mu'^2\right]^{-\frac{1}{2}} \Big\{ \left[\mu^2 - t - \mu'^2\right]$$

$$\pm i\left[t\mu'^2 - (t + \mu'^2 - \mu^2)\right]^{1/2} \Big\}$$

The vector line propagator may also be calculated in the large mass limit. We have

$$(\mu^2)^{-1}(-1)^{R-\sigma}\,\varepsilon_{R\sigma}(q'-p,\,q'+p,\,N_1')\cdot\varepsilon_{T\varphi}(p'-p,\,p'+p,\,N_1')$$

$$\simeq D_{R\sigma;T\varphi}(p',p,k,q'^2)$$

$$D_{00;00} = D_{10;00} = \tfrac{1}{2}\,[t-\mu'^2+\mu^2]/\mu^2$$

$$D_{1\pm1;00} = \pm\frac{1}{\sqrt{2}}\,[-t/\mu^2]^{1/2}$$

$$D_{00;10} = D_{10;10} = \tfrac{1}{2}\,q'^2\,[t/\mu^2]\,[(p'\cdot p)^2-p'^2p^2]^{-1/2}$$

(II-25)

$$D_{1\pm1;10} = \mp\frac{1}{\sqrt{2}}\,[-t/\mu^2]^{1/2}\,[p^2-p\cdot p']\,[(p'\cdot p)^2-p'^2p^2]^{-1/2}$$

$$D_{00;1\pm1} = D_{10;1\pm1} = \pm\frac{1}{2\sqrt{2}}\,q'^2\,[t/\mu^2]\,[p'^2p^2-(p'\cdot p)^2]^{-1/2}$$

$$D_{1\pm1;1\pm1} = \tfrac{1}{2}\,[t/\mu^2]^{\frac{1}{2}}\left\{1+(\pm)(\pm)\,[p'\cdot p-p^2][p'^2p^2-(p'\cdot p)^2]^{-\frac{1}{2}}\right\}$$

The rotation matrices for the external line propagators depend upon (J, m_J^2) only through their angle dependence. Therefore, we use the limiting values of (II-22). The internal line propagators depend on 'J' through the representation matrix as well. We must evaluate this limit more carefully. We have, in the limit $J\to\infty$ $m_J\to\infty$

$$[q'^2-(p'+k)^2]^{-1}\,d^J_{m'm}(\chi_2')$$

(II-26)

$$\longrightarrow\ [m_J^2-(p'+k)^2]^{-1}\,d^J_{m'm}(\chi_2)$$

$$\cos\chi_2 \longrightarrow -1 - 2t/m_J^2$$

$$d^J_{n'n}(\chi_2) = (-1)^{J-n'}\, d^J_{-n'n}(\chi_2-\pi)$$

$$\cos(\chi_2-\pi) = 1 + 2t/m_J^2$$

The rotation angle $\chi_2-\pi$ becomes small as $m_J^2 \to \infty$. For large 'J' and small angle, the rotation matrix may be evaluated in terms of Bessel functions of the first kind. We have

$$d^J_{n'n}(\chi) \longrightarrow e^{-|n'-n|}\, J_{|n'-n|}(J\chi) \qquad \text{(II-27)}$$

for $\quad J \to \infty \qquad \chi \to 0$

Expression (II-26) becomes

$$[q'^2 - (p'+k)^2]\, d^J_{n'n}(\chi_2')$$

$$\longrightarrow [m_J^2 - (p'+k)^2]^{-1} (-1)^{J-n'}\, d^J_{-n'n}(\chi_2-\pi) \qquad \text{(II-28)}$$

$$\longrightarrow [m_J^2 - (p'+k)^2]^{-1}\, e^{-|n'+n|}\, J_{|n'+n|}\left(J\sqrt{\frac{-4t}{m_J^2}}\right)$$

The contribution of the large mass, large 'J' states to the relation in (II-21) becomes,

$$\delta_{\tau 1}\, \delta(\lambda'-\lambda+\varphi)\; i\varepsilon^{\alpha\beta\gamma}\, F^{\gamma}_{\lambda'\lambda}(s',t,s)$$

$$\cong e^{-i(\lambda'-\lambda+\varphi)\varphi'} \sum_{\sigma'\sigma} \sum_{n'n} d^{s'}_{\lambda'\sigma'}(\chi_1')\, d^{s}_{\sigma\lambda}(\chi_3')$$

$$\Big\{ \sum_J e^{-|n'+n|}\, J_{|n'+n|}\left(J\sqrt{\frac{-4t}{m_J^2}}\right) \frac{1}{m_J^2} \qquad \text{(II-29)}$$

$$[\, F^{\alpha}_{\sigma'n'}(s',\mu'^2,J)\, V_{|\sigma'-n'|}\, F^{\beta}_{n\sigma}(J,\mu^2,s)\, D_{|n-\sigma|;\tau\varphi}$$

$$- F^{\beta}_{\sigma'n'}(s',\mu^2,J)\, D_{|\sigma'-n'|;\tau\varphi}\, F^{\alpha}_{n\sigma}(J,\mu'^2,s)\, V_{|n-\sigma|}\, \Big\}$$

where

$$\cos\chi_1' = [p'\cdot p - p'^2][(p'\cdot p)^2 - p'^2 p^2]^{-1/2}$$

$$\cos\chi_3' = [p'\cdot p - p^2][(p'\cdot p)^2 - p'^2 p^2]^{-1/2}$$

$$\cos\varphi' = -\tfrac{1}{2}[(p'\cdot p)^2 K\cdot(p'+p) - (p'^2 - p^2)K\cdot(p'-p)][N^2(p'-p)^2]^{-1/2}$$

and $V_{1\sigma}$ is given in (II-24)

$D_{R\sigma,T\varphi}$ is given in (II-25).

As we have noted before, the form factor on the L.H.S. (II-29) contains dynamic singularities in addition to the kinematic singularities. On the R.H.S.(II-29), the dynamic singularities may arise only through the divergence of the sum for $t > t_0$; t_0 representing the beginning of the dynamic cuts. The only dynamic dependence upon t on the R.H.S(II-29) is in the argument of the Bessel function. Thus, the analyticity properties of the L.H.S.(II-29) place restriction on the mass spectrum on the intermediate states.

If $J/m_J \to 0$ as $J \to \infty$, the argument of the Bessel function goes to zero for all values of t . Hence, no dynamic singularities can arise. Therefore, we must have

$$m_J / J \longrightarrow 0 \quad , \quad J \longrightarrow \infty$$

If we define a new variable,

$$x = J m_1 / m_J \tag{II-30}$$

we may convert the sum in (II-29) to an integral over X . The R.H.S.(II-29) becomes a Bessel transform.

$$\delta_{\tau 1}\ \delta(\lambda'-\lambda+\varphi)\ i\,\varepsilon^{\alpha\beta\gamma}\ F_{X\lambda}(s',t,s)$$
$$= \sum_{\sigma'\sigma} d^{s'}_{X\sigma'}(\chi_1)\ R^{\alpha\beta}_{\sigma'\sigma,\tau\varphi}(t)\ d^{s}_{\sigma\lambda}(\chi_3') \qquad \text{(II-31)}$$

where $R^{\alpha\beta}_{\sigma'\sigma,\tau\varphi}(t)$ is given by

$$R^{\alpha\beta}_{\sigma'\sigma,\tau\varphi}(t) = \sum_{n'n} \int_{x_0}^{\infty} dx\ e^{-|n'+n|}\ J_{|n'+n|}\left(x\sqrt{\frac{-4t}{\Lambda m_1^2}}\right)$$
$$\frac{dJ}{dx}\frac{1}{\Lambda m_J^2}\ \Big\{ F^{\alpha}_{\sigma'n'}(s',\mu^2,J)\ V_{1\sigma'-n'}\ F^{\beta}_{n\sigma}(J,\mu^2,s) \qquad \text{(II-32)}$$
$$D_{1n-\sigma;\tau\varphi} - F^{\beta}_{\sigma'n'}(s',\mu^2,J)\ D_{1\sigma'n';\tau\varphi}\ F^{\alpha}_{n\sigma}(J,\mu^2,s)\ V_{1n-\sigma}\Big\}.$$

and $\quad x = J\, m_1/m_J$.

The Bessel transform may be continued to $t>0$. For large values of the argument, the Bessel function becomes

$$J_{|n'+n|}\left(x\sqrt{\frac{-t}{\Lambda m_1^2}}\right) = J_{|n'+n|}\left(ix\sqrt{\frac{4t}{\Lambda m_1^2}}\right) \longrightarrow e^{x\sqrt{4t/\Lambda m_1^2}} \qquad \text{(II-33)}$$

If the cut is to begin at $t=t_0$, the integral must converge for $t<t_0$ and diverge for $t>t_0$. This will be the case if the argument of the Bessel transform behaves, for large 'J' as

$$e^{-x\sqrt{4t_0/\Lambda m_1^2}} = e^{-J\sqrt{4t_0/\Lambda m_J^2}}\ ,\quad J\longrightarrow\infty \qquad \text{(II-34)}$$

The only significant 'J' dependence of the argument of the Bessel transform resides in the form factors. Therefore, for large 'J', we have

$$F^{\alpha}(J,\mu^2,s) \longrightarrow e^{-J\sqrt{t_0/m_J^2}} \tag{II-35}$$

and $\quad m_J/J \to 0\,, \quad m_J \to \infty\,, \quad J \to \infty$

Summary

In Section A, we derived a representation of the equal time current algebra in terms of the invariant form factors of the currents and invariant kinematic factors. The method involves the introduction of covariant states of definite total angular momentum and the use of the infinite momentum limit to evaluate the commutators.

In Section B, we discussed the possibility that a set of single particle states having a mass - spin constraint be responsible for the dynamic singularities of the form factors as functions of the momentum transfer. The mass spectrum and the form factor dependence on the spin were shown to be restricted by the conditions in (II-35) in order that the current algebra generate the appropriate analyticity properties.

BIBLIOGRAPHY

1. E.P. Wigner, Ann. Math. 40, 149(1939).
2. H. Joos, Fortschr. Physik 10, 65(1962).
   A. MacFarlane, Rev. Mod. Phys. 34, 41(1962).
3. M.E. Rose, "Elementary Theory of Angular Momentum", John Wiley & Sons, Inc.(1957).
4. M. Jacob and G.C. Wick, Ann. Phys.(N.Y.) 7, 404(1959).
5. W.R. Theis, Nuovo Cimento 45A, 124(1966).
6. W. Rarita and J. Schwinger, Phys. Rev. 60, 61(1941).
7. K. Hepp, Helv. Phys. Acta 36, 355(1963).
8. David N. Williams, UCRL - 11113 (1963), unpublished.
   Y. Hara, Phys. Rev. 136, B507(1964).
   Ling-Lie Chau Wang, Phys. Rev. 142, 1187(1966).
9. T.L. Trueman and G.C. Wick, Ann. Phys. 26, 322(1964).
10. M. Gell-Mann, Phys. Rev. 125, 1067(1962).
11. H. Suura and J. Meyer, Phys. Rev., to be published.
    K. Johnson and F.E. Low, Prog. Theoret. Phys. Suppl. No. 37, 74(1966).
12. R. Dashen and M. Gell-Mann, Phys. Rev. Lett. 17, 340(1966).