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WIGNER ANALYSIS AND CASIMIR OPERATORS OF $\overline{SA}(4,R)$

Jürgen Lemke*§, Yuval Ne'eman†‡ and Jose Pecina-Cruz

Center for Particle Physics, University of Texas, Austin, USA

ABSTRACT

$\overline{SA}(4,R)$ plays a role in theories involving gravity, including QCD–generated gravity–like effects in hadrons. We evaluate its single Casimir invariant and that of its $\overline{SA}(2,R)$ and $\overline{SA}(3,R)$ subgroups. We study the group orbits and classify the unitary irreducible representations.

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§ Permanent address: Institute for Theoretical Physics, University of Cologne, D–5000 Köln 41, Germany.

†Wolfson Chair Extraordinary in Theoretical Physics, Tel-Aviv University, Israel.

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1. The Role of $GA(4,R)$ and $SA(4,R)$ in Gravity and Gravity-related Physics

The Affine Groups, both the general affine $GA(4,R)$ and its unimodular ("special") subgroup $SA(4,R)$, with their double-covering groups $\overline{GL}(4,R)$ and $\overline{SL}(4,R)$ appear as symmetries of the spectrum of particle states in various gravity-related theories. The following list is not exhaustive:

a) Theories in which spacetime is no more Riemannian, above Planck energies [1]. In such theories, the primordial local symmetry is either the Conformal group or its Homothecy subgroup, i.e. the Poincaré group combined with dilations, or alternatively, $GL(4,R)$ (which also includes dilations) or its $SL(4,R)$ subgroup (excluding the dilations). Here, we are interested in the latter case. The fields then carry non-unitary representations of $\overline{SL}(4,R)$ and the particle Hilbert space is that of $\overline{SA}(4,R)$. Under spontaneous symmetry breakdown, the local gauge group reduces to the Lorentz group and the Hilbert space becomes that of the Poincaré group. Similar situations arise in Metric-Affine theories of gravity [2].

b) Einsteinian gravity, when interacting with hadron matter, in a phenomenological description in which quarks and gluons are replaced by baryons and their excitations. Such a description [3] involves *manifields*, i.e. de-unitarized [4] infinite-dimensional representations of $\overline{SL}(4,R)$, the double covering of the special linear group. The Hilbert space here is then that of $\overline{SA}(4,R)$.

c) This formalism can be extended (for any fields) to fit a semi-quantized description for particles under the effect of gravity, i.e. particles in a curved space. The Hilbert space group is then defined by the group of Diffeomorphisms, induced over $\overline{SA}(4,R)$.

d) An approximation to QCD in the (confinement) IR region [5] which emulates (Riemannian) gravity, with applications in particle [6] and nuclear [7] physics. Here again matter is represented by $\overline{SL}(4,R)$ manifields, with states classified by $\overline{SA}(4,R)$.

The field-particle algebraic relationship follows the prescriptions of Relativistic Quantum Field Theory, which at the classical level, at least, contains the tools for a smooth transition to General Relativity. The Principle of Covariance, for one, requires the fields to carry the action of the group of Diffeomorphisms. This action will generally be represented non-linearly, over the linear subgroup $SL(4,R)$ or over its double-covering group $\overline{SL}(4,R)$. Therefore, even in Special Relativity, before the introduction of the gravitational field or of curved space, the fields carry non-unitary representations of $SL(4,R) \supset SO(1,3)$ or (for spinors) $\overline{SO}(1,3) = SL(2,C)$. The Hilbert particle space symmetry, on the other hand, is determined by the Principle of Equivalence, i.e. it is that of the Special Theory of

Relativity, i.e. the Poincaré group $\mathcal{P} = SL(2,C) \times_s \mathcal{R}^4$. Similarly, in the affine situation, when $\overline{SL}(4,R)$ replaces $SL(2,C)$, we obtain, as Hilbert space of particle states that of

$$\overline{SA}(4,R) = \overline{SL}(4,R) \times_s \mathcal{R}^4. \tag{1}$$

The elements of $\overline{SA}(4,R)$ are given by 5×5 matrices (the Möbius representation)

$$A = \begin{pmatrix} L & p \\ o & 1 \end{pmatrix}, \quad L \in \overline{SL}(4,R), \; p \in \mathcal{R}^4 . \tag{2}$$

2. The Casimir Invariant of $SA(N,R)$ and the Group Orbits

In a work treating the invariants of real low-dimensional Lie algebras, Patera et al. [8] evaluated the Casimir invariant of SA(2,R) (named $A_{5,40}$ in their list). Defining the elements of the Lie *algebra* in the 3×3 matrix form

$$a = \begin{pmatrix} l & p \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} r & s & p^0 \\ t & -r & p^1 \\ 0 & 0 & 0 \end{pmatrix} \tag{3}$$

where, as displayed in the 3×3 matrix, $l \in sl(2,R)$ is a traceless 2×2 matrix and p is a column vector (the momenta). The Casimir invariant is quadratic in the translations p^i and is altogether of cubic order (we denote the dimensions by numbering them from 0 to $N-1$, in analogy with Minkowski space),

$$C(2) = t(p^0)^2 - rp^0p^1 - s(p^1)^2 . \tag{4}$$

The basic advance in the study of the Casimir invariants of the affine and related groups followed the work of Sternberg [9]. M. Rais [10], M. Perroud [11] and Demichev et al. [12] showed finally that the $SA(N,R)$ have a single such operator (and the $GA(N,R)$ have none) which, using the Cartan-Weyl basis in the related $gl(N,R)$,

$$(E_i^j)_v^u = \delta_i^u \delta_v^j \tag{5a}$$

is given by

$$C(N) = Sym\left[\epsilon^{r_0..r_{N-1}} p_{r_0} (E_{r_1}^{i_0} p_{i_0})(E_{r_2}^{l_0} E_{l_0}^{k_1} p_{k_1})...(E_{r_{N-1}}^{m_0} E_{m_0}^{m_1}..E_{m_{N-3}}^{m_{N-2}} p_{m_{N-2}})\right] \tag{5b}$$

which is equivalent to the determinant

$$C(N) = \det(p, Ep, (E)^2 p, .., (E)^{N-1} p)\,. \tag{5c}$$

This involves powers of E (the basis in the related $gl(N,R)$ algebra) going from 0 to $N-1$. $C(N)$ is thus a polynomial of degree N in the translations p and of degree $N(N-1)/2$ in the $sl(N,R)$ generators; altogether it is thus of degree $N(N+1)/2$. The expression (5b) or (5c) automatically takes care of the tracelessness of the $sl(N,R)$ generators, i.e. the diagonal generators appear in combinations $E^j_j - E^1_1$.

The group $SA(N,R)$, acting on the space of momenta, has two orbits:

$$Orb_1 = \{0\}, \qquad Orb_2 = \mathcal{R}^N - \{0\}. \tag{6}$$

For the null orbit, i.e. when we select states for which all N components of the momenta vanish, the Casimir invariant vanishes, since it is a homogeneous symmetric polynomial of degree N in the momenta. In the second orbit (which, incidentally is invariant under the entire $GL(N,R)$), for small values of the momenta, the invariance of the Casimir operator implies that the eigenvalues of the $SL(N,R)$ homogeneous operators must grow fast. In an example treating $SA(2,R)$ and due to S. Sternberg, putting the non-vanishing N-vector p at rest (i.e. $p^0 \neq 0, p^1 = 0$), the "little group" (the stability subgroup) consists of matrices

$$\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \tag{7a}$$

¿From (4) we have $C(2) = q(p^0)^2$, the product of the squared "energy" by q, a fake translation momentum of $SA(1,R)$ and really a component of the $SL(2,R)$ shear, pointing in the 0 direction. If we now use the coadjoint matrix

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \tag{7b}$$

to rescale the p^0 momentum by a factor λ, we see that the q will be rescaled by a factor λ^{-2}, thus preserving the invariance of the Casimir operator.

3. The Projective Representations and Cohomology

The basic construction follows Wigner's [13] classical treatment of the Poincaré group's Hilbert space and projective representations (for Quantum Mechanics). Let H be a Hilbert space with scalar product $<,>$ and $\hat{H}$ the corresponding projective Hilbert space, i.e.

$$\hat{H} := [\alpha\Psi \mid \Psi \in H, \alpha \in \mathcal{C}^*]\,. \tag{8}$$

Measurable quantities, in QM, are invariant under phase transformations. Physical systems are therefore described by elements $\hat{\Psi} \in \hat{H}$, the "ray" representations. For the same reason, should the group acting on spacetime possess a double-covering group, the latter may act as an isomorphism (up to a phase) on the representations (e.g. $Spin(N)$ instead of just $SO(N)$, the geometrical orthogonal group).

An $\overline{SA}(4,R)$ transformation A on spacetime (in the above sense) induces a transformation $\hat{\rho}(A) : \hat{H} \longrightarrow \hat{H}$. This has to be an element of the set $U(\hat{H})$ of unitary operators on $\hat{H}$, as long as $\overline{SA}(4,R)$ is assumed to be a symmetry of the system. Each homomorphism ρ from $\overline{SA}(4,R)$ to $U(H)$ gives rise to a projective representation $\pi(\rho) = \hat{\rho}$.

In the opposite sense, according to Wigner's theorem, each projective representation $\hat{\rho}$ of $SA(N,R)$ can be obtained from a representation ρ of a group G, i.e. we can find a group G and homomorphisms μ and ρ such that the following diagram is commutative and that both sequences are exact sequences.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & K & \longrightarrow & G & \xrightarrow{\mu} & SA(N,R) & \longrightarrow & 1 \\
 & & & & \rho\downarrow & & \downarrow\hat{\rho} & & \\
1 & \longrightarrow & U(1) & \longrightarrow & U(H) & \xrightarrow{\pi} & U(\hat{H}) & \longrightarrow & 1
\end{array} \tag{9}$$

The groups G and K are determined by the second cohomology of the Lie algebra of SA(N,R), i.e. $\mathcal{H}^2(sa(N,R))$ [14].

For $N = 1$, we have $\mathcal{H}^2(sa(N,R)) = \{0\}$ and G is the covering group of $SA(1,R)$, which is $SA(1,R) \simeq (R,+)$ itself. Hence $G = SA(1,R)$ and $K = 1$.

For $N = 2$, the 2-form $dt_1 dt_2$ is left-invariant and closed, but the 1–forms $t_1 dt_2$ or $-t_2 dt_1$ are not left–invariant. Thus $\dim \mathcal{H}^2(sa(2,R)) = 1$ and G is the central extension of the universal covering group of $SA(2,R)$ (an infinite covering [15]) by $\mathcal{R}$. Moreover, the lift of a certain projective representation is uniquely determined, since $\mathcal{H}^1(sa(2,R)) = 0$.

For $N = 3,4$ we obtain $\mathcal{H}^2(sa(N,R)) = 0$ and G is equal to the universal covering group [16,17] of the group $SA(N,R)$. For $N = 4$ we have $K = -1, 1$, i.e. G is the double-covering group $\overline{SA}(4,R)$ [4].

4. Induced Representations.

The two orbits of (6) provide for a classification of the unitary irreducible representations of $\overline{SA}(4,R)$. We have a hierarchy of stability subgroups over which the unirrep is constructed as an induced representation a la Wigner and Mackey. The 4-vector p either vanishes, $p=0$ (case I) and $C(4)=0$ or it doesn't, $p \neq 0$ (case II) and $C(4) \sim (p^0)^4 = m^4$.

Case I: physically, it is useful to think of this case as the very-low frequency limit of a massless particle, with its Regge excitations. The little group is $\overline{SL}(4,R)$. The unirreps of this group have been classified [4,18]. They are rather unphysical in that the Lorentz subgroup will appear in unitary infinite representations, the unirreps of Gelfand and Yaglom [19]. These contain all spins, and the action of the Lorentz boost on a state with spin j connects it with the $j+1$ and $j-1$ spins. Particles here are thus not characterized by definite spins, as phenomenologically required. These representations are also known as "infinite spin" representations. Still, there are problems in physics in which the $SO(1,3) \subset SL(4,R)$ is not the physical Lorentz group, and these unirreps may then prove useful. Note also that we do not encounter this difficulty with the fields and manifields, since these are constructed with the *deunitarizing automorphism* $\mathcal{A}$ [4]. In a non-unitary and finite representation, the Lorentz boosts stay anti-Hermitean and cancel.

Case II: the little group is $\overline{SA}(3,R)'$. This affine group consists of the semi-direct product of the spatial $\overline{SL}(3,R)$ with a "fake" set of three "translation" momenta p', in fact representing contributions of the spatial shears to the 0 direction. We now have two subcases:

Case IIA: all three components $p'=0$. The effective little group is then $\overline{SL}(3,R)$. The unirreps are induced over this subgroup, they can be reduced to infinite discrete sums of spins, fitting the hadron situation and also providing an interesting model for primordial fermion fields (in fact manifields). This picture has been studied in [3,20]. It fits all applications mentioned in our introductory comments. Note that $C(3')=0$, and as a result $C(4)=0$ as well, since the multiplyer of $(p^0)^4$ is precisely the $C(3')$ Casimir invariant of the stability subgroup defined by p^0.

Case IIB: the fake momenta $p' \neq 0$. We can select a frame in which only p'^0 does not vanish, a fake energy-like component. $C(3') \sim (p^{0'})^3 = (m')^3$, m' a mass-like eigenvalue. The new little group is $\overline{SA}(2,R)''$. Again, the "translations" are fake momenta p''. We can have two cases.

Case IIB1: all components of $p''=0$ and $C(2'')=0$. In that case, we get again both $C(3")=0$ and $C(4)=0$. The effective little group is $\overline{SL}(2,R)$ (i.e. the double-covering,

in an infinitely covered group). The unirreps have been classified by Bargmann [15] and are useful in a variety of physical contexts.

Case IIB2: $p'' \neq 0$, $C(2'') \sim (p'')^2 = (m'')^2$. The little group is $SA(1,R)$ as in (7a), with one fake momentum p'''. Again we have two possibilities;

Case IIB2a: $p''' = 0$, $C(1''') = 0$. This is a scalar representation. As a result, $C(2'') = C(3') = C(4) = 0$.

Case IIB2b: $p''' \neq 0$, $C(1''') = q = m'''$ (see (7a)). Note that here $C(2'') = (m'')^2 m'''$, $C(3') = (m')^3 (m'')^2 m'''$ and $C(4) = (m)^4 (m')^3 (m'')^2 m'''$.

To summarize, we have 5 classes of representations: I, IIA, IIB1, IIB2a, IIB2b; which are illustrated in the following diagram:

$$\begin{array}{ccccc}
\overline{SA}(4,R) & \longleftarrow & \overline{SL}(4,R) & : & I \\
\uparrow & & & & \\
\overline{SA}(3,R)' & \longleftarrow & \overline{SL}(3,R) & : & IIA \\
\uparrow & & & & \\
\overline{SA}(2,R)'' & \longleftarrow & \overline{SL}(2,R) & : & IIB1 \\
\uparrow & & & & \\
SA(1,R) & & & : & IIB2a\,,\ IIB2b
\end{array} \tag{10}$$

Moreover

$$C(4) = 0, \quad \text{for I, IIA, IIB1, IIB2a;} \quad C(4) = (m)^4 (m')^3 (m'')^2 m''', \quad \text{for IIB2b.} \tag{11}$$

5. Dynamical Considerations

At first sight, the Casimir invariant (5b) appears to constrain the masses and spins in a wrong manner, as in the Majorana [19] infinite equation: the higher the spin, the lower the mass; this is the opposite of what we observe in hadron phenomenology and of what is assumed in the Chew-Frautschi plot for a Regge trajectory. However, considering that in the general case (including the most useful case IIA the invariant vanishes (as seen in (11)), the value of $(m)^4$ stays unconstrained in all but case IIB2b. Instead, constraints on the value of the masses may be derived dynamically [5], rather than kinematically as in (11). It is remarkable that an evaluation based on the pseudo–gravity approximation for QCD in the IR region does reproduce the linear correlation between $(m)^2$ and the spin j.

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