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REMARKS ON MULTIPARAMETRIC QUANTUM (SUPER) ALGEBRAS[1]

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ABSTRACT

A simple ansatz,which is more general than that of Reshetikhin is proposed for twisting of Hopf algebras .It is also shown that the quantum super algebra $sl(n|m)$ can be multiparametrized consistently.

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[1]This is the revised form of a previous preprint with the same title

1.INTRODUCTION

There has been many attempts and many methods to multiparametrize quantum groups [1-7] The basic methods are, constructing the transformation groups on multiparametric quatum planes[1] , applying the FRT formalism to multiparametric R matrices[4] , and finally the abstract twisting of Reshetikhin [5].In this article we want to clarify the basic idea of multiparametrization , an idea which has come about by a close examination of the above works . We begin by multiparametrizing a classical Lie algebra and then extend the method to quantum groups where we generalize the method of and (correct the explicit results) of Reshetikhin [5].

Let g be any simple Lie algebra of rank r. In the Chevalley basis, to each simple root $\alpha_i, i = 1...r$ corresponds a triple of generators H_i and $X_i^{\pm}$ with the following commutation relations.

$$[H_i, H_j] = 0 \tag{1}$$

$$[H_i, X_j^{\pm}] = \pm a_{ij} X_j^{\pm} \tag{2}$$

$$[X_i^{\pm}, X_j^{\pm}] = 0 \qquad\qquad a_{ij} = 0 \tag{3}$$

$$[X_i^{+}, X_j^{-}] = \delta_{ij} H_i \tag{4}$$

To the above relations one must also add the Serre' relations which read:

$$ad(X_i)^{1-a_{ij}}(X_j) = 0 \tag{5}$$

where $ad(X_i)(X_j) = [X_i, X_j]$.

Any simple Lie algebra is equipped with the following standard co-commutative co-product, which is used in defining tensor products of representations.

$$\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i \tag{6}$$

$$\Delta(X_i^{\pm}) = X_i^{\pm} \otimes 1 + 1 \otimes X_i^{\pm} \tag{7}$$

By co-commutativity we mean that $\sigma \circ \Delta(a) = \Delta(a) \quad \forall a \in g$ where $\sigma : U(g) \otimes U(g) \rightarrow U(g) \otimes U(g)$ is the flip homomorphism of the algebra: $\sigma(a \otimes b) = b \otimes a$. With the addition of two other operations ,namely the antipode S : $U(g) \rightarrow U(g); S(a) = -a$ and the co-unit $\epsilon : U(g) \rightarrow C$ ($\epsilon(X_i^{\pm} = \epsilon(H_i) = 0, and \epsilon(1) = 1$), and by extending Δ and ϵ homomorphically and S anti-homomorphicallly to all of $U(g)$ one can consider the universal enveloping of any Lie algebra as a Hopf algebra.The above operations satisfy a set axioms [8].

As is well known, non-co-commutativity is the essential property that characterizes the quantum groups [8]. In fact the simplest definition of a quantum group is a noncommutative and non-cocommutative Hopf algebra [10]. In the standard quantization of the simple Lie algebras [8-9] , the co-products of H_i ,and $X_i^{\pm}$ are deformed as follows:

$$\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i \tag{8}$$

$$\Delta(X_i^{\pm}) = q^{\frac{-H_i}{2}} \otimes X_i^{\pm} + X_i^{\pm} \otimes q^{\frac{H_i}{2}} \tag{9}$$

But then in order that the coproduct be a homomorphism of the algebra ,thats for: $\Delta(ab) = \Delta(a)\Delta(b)$ the commutation relations are also deformed to the effect of changing eq. (4) to

$$[X_i^+, X_j^-] = \delta_{ij} \frac{q^{H_i} - q^{-H_i}}{q - q^{-1}} \tag{10}$$

and the serre relations (eqs. (5)) to :

$$\sum_{k=1}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} X_i^{\pm k} X_j^{\pm} X_i^{\pm(1-a_{ij}-k)} = 0 \tag{11}$$

In this paper we want to deform the coproduct (6,7) without deforming the structure of the algebra itself.

Let $Q_i \quad ; i = 1, 2, ...r$ be operators in the Cartan subgroup , with the following commutation relations:

$$Q_i X_j^{\pm} = s_{ij}^{\pm 1} X_j^{\pm} Q_i \tag{12}$$

where s_{ij} are arbitrary complex numbers . We define a new coproduct Δ_o as follows:

$$\Delta_o(H_i) = H_i \otimes 1 + 1 \otimes H_i \tag{13}$$

$$\Delta_o(X_i^+) = Q_i^{-1} \otimes X_i^+ + X_i^+ \otimes Q_i \tag{14}$$

$$\Delta_o(X_i^-) = Q_i \otimes X_i^- + X_i^- \otimes Q_i^{-1} \tag{15}$$

Now it is easily verified that all the commutation relations ,eqs.(1-4) are compatible with the new coproduct. Thats $\Delta_o[a, b] = [\Delta_o(a), \Delta_o(b)]$), provided that we set

$$s_{ij} s_{ji} = 1$$

for all i and j. In particular $s_{ii} = \pm 1$. from which it follows that

$$Q_i X_i^{\pm} = \pm X_i^{\pm} Q_i$$

. The above constraint immediately reduces the number of parameters to $r(\frac{r-1}{2})$. To check that the coproduct is also compatible with the Serre' relations we note that from eq.(14) we have:

$$ad\Delta_o X_i^+(\Delta_o X_j^+) \equiv [Q_i^{-1} \otimes X_i^+ + X_i^+ \otimes Q_i, Q_j^{-1} \otimes X_j^+ + X_j^+ \otimes Q_j] \tag{16}$$

$$= Q_i^{-1} Q_j^{-1} \otimes adX_i^+(X_j^+) + adX_i^+(X_j^+) \otimes Q_i Q_j \tag{17}$$

Repeating the above process, we find :

$ad\Delta_o(X_i^+)^n(\Delta_o X_j^+) =$

$$Q_i^{-n} Q_j^{-1} \otimes ad(X_i^+)^n(X_j^+) + ad(X_i^+)^n(X_j^+) \otimes Q_i^n Q_j = 0 \tag{18}$$

where we have have used eq.(5).

Therefore the new coproduct, while being non-cocommutative is completely consistent with the structure of the undeformed algebra. It is very easy to determine the operators Q_i : For the present we restrict ourselves to the solution $s_{ii} = 1$ As we will see for this particular choice our ansatz (eq. 12) exactly reproduces the results of Reshetikhin in [5].

We have:

$$Q_i = e^{\sum t_{ij} a_{jk}^{-1} H_k} \tag{19}$$

where $s_{ij} = e^{t_{ij}}$ and a sum over repeated indices is understood. Note that due to (16) the matrix t_{ij} is antisymmetric. Eq.(12) is easily verified by using Hausdorf formula.In order to complete the structure of the Hopf algebra , we should also give the definition of co-unit and antipode.It is trivial to check that with the usual definition of co-unit and antipode all the axioms of Hopf algebra are satisfied for the new coproduct. Especially for the antipode , we have:

$$m(S \otimes 1)\Delta(X_i^{\pm}) = m(S \otimes 1)(Q_i^{-1} \otimes X_i^+ + X_i^+ \otimes Q_i)$$

$$= m(Q_i \otimes X_i - X_i \otimes Q_i) = Q_i X_i - X_i Q_i = 0 \tag{20}$$

where we have used eq.(12),with $s_{ii} = 1$.

2-QUASI-TRIANGULAR STRUCTURE

The reader may well ask if the above Hopf algebra is quasitriangular [8]

. Thats if there exists an invertible map $R : U(g) \otimes U(g) \rightarrow U(g) \otimes U(g)$ such that :

$$R\Delta(a)R^{-1} = \sigma \circ \Delta(a) \tag{21}$$

Fortunately the answer is positive.Let

$$R = exp(u_{lk} H_k \otimes H_l) \tag{22}$$

We want to have

$$e^{(u_{lk} H_k \otimes H_l)}(Q_i^{-1} \otimes X_i^+ + X_i^+ \otimes Q_i)e^{(-u_{lk} H_k \otimes H_l)} =$$

$$(Q_i \otimes X_i^+ + X_i^+ \otimes Q_i^{-1}) \tag{23}$$

A simple use of Baker-Hausdorf formula shows that : Left hand side of eq.(23) is equal to :

$$X_i \otimes e^{a_{ki} u_{lk} H_l} Q_i + e^{u_{lk} a_{li} H_k} Q_i^{-1} \otimes X_i$$

Comparision with the right hand side shows that

$$e^{a_{ki} u_{lk} H_l} = Q_i^{-2}$$

and

$$e^{a_{li} u_{lk} H_k} = Q_i^2$$

using the fact that $Q_i = e^{t_{il} H_l}$ and the antisymmetry of the matrix of parameters $\hat{t}$ we find the matrix u:

$$u = 2a^{-1^t} \hat{t} a^{-1} \tag{25}$$

Lets consider the simplest example which allows a deformation of this type:

EXAMPLE Q-deformation of SL(3)

For Sl(3) r = 2 and there is only one parameter $s_{12} = s$.

The comultiplicaion is

$$\Delta(X_1^{\pm}) = s^{\mp H_1 + 2H_2} \otimes X_1^{\pm} + X_1^{\pm} \otimes s^{\pm H_1 + 2H_2} \tag{26}$$

$$\Delta(X_2^{\pm}) = s^{\mp 2H_1 + H_2} \otimes X_2^{\pm} + X_2^{\pm} \otimes s^{\pm 2H_1 + H_2} \tag{27}$$

where we have rescaled $s \to s^3$

REMARKS

1)Clearly eq. (16) has two solutions : $s_{ii} = +1$ and $s_{ii} = -1$ for each individual i . Up to now we have used the solution $s_{ii} = 1$ With this choise, the matrix t_{ij} becomes antisymmetric.If for some index set J we set $s_{jj} = -1$ for $j \in J$ then Q_j no longer commutes with $X_j^{\pm}$ but anticommutes with it and solution of the eqs. (12) lead to a new matrix $\hat{t}_{ij}$ which is related to t_{ij} by :

$$\hat{t}_{ij} = t_{ij} + i\pi\epsilon_i\delta_{ij} \tag{28}$$

where

$$\epsilon_i = \begin{cases} 0 & if\ s_{ii}=1 \\ 1 & if\ s_{ii}=-1 \end{cases} \tag{29}$$

Let us denote the explicit form of the operators Q_i corresponding to the mixed choices of s_{ii} by $\hat{Q}_i$. A simple calculation leads to

$$\hat{Q}_i = Q_i\eta_i \tag{30}$$

where

$$\eta_i = \begin{cases} 1 & if\ s_{ii}=1 \\ (-1)^{a_{ik}^{-1}H_k} & if\ s_{ii}=-1 \end{cases} \tag{31}$$

Although the co-product induced by $\hat{Q}_i$ is a homomorphism of the algebra,it does not introduce a quazitriangular structure,since the matrix $\hat{t}$ is not antisymmetric in this general case. This rather exotic co-product has the new feature that when all the parameters s_{ij} approach 1 they do not approch the classical co-product but take the following form which is still non-co-commutative:

$$\Delta(X_i^{\pm}) = (-1)^{\mp a_{ik}^{-1}H_k} \otimes X_i^{\pm} + X_i^{\pm} \otimes (-1)^{\pm a_{ik}^{-1}H_k} \tag{32}$$

2) The above construction works also for multiparametrization of one parametric quantum groups.One can still use eqs.(12) to show that insertion of the operators Q_i in the usual co-products of any simple quantum Lie algebra (eqs. (8-9)) in the form

$$\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i \tag{33}$$

$$\Delta(X_i^{\pm}) = q^{\frac{-H_i}{2}} Q_i^{\mp} \otimes X_i^{\pm} + X_i^{\pm} \otimes q^{\frac{H_i}{2}} Q_i^{\pm} \tag{34}$$

leads to a multiparametric co-product compatible with the commutation and Serre'relations.

In fact the q-deformation and the Q-deformation are independent, and the previous analysis can be regarded as consistency checks of the process of multiparametrization of the quantum group, in the limit $q \longrightarrow 1$.

Many authors have tried to multiparametrize various quantum groups by using one method or another [2-7.11]. In all of these works the extra

parameters (besides q) can be completely transfered to the coprodut by a simple redifinition of generators which makes the commution relations one parametric.

In fact given any co-product of the form

$$\Delta X_i = a_i \otimes X_i + X_i \otimes b_i \tag{35}$$

,in these works where X_i is either X_i^+ or X_i^- , we propose that the reader redefine the generators as follows :

$$X_i \longrightarrow X_i' = (a_i b_i)^{\frac{-1}{2}} X_i \tag{36}$$

Then he or she will see that with this redefinition , all the commution relations of the algebra become 1 parametric and the co-product will take the form given in the present article.Recasting the co-product into the canonical form $\Delta X_i = A_i \otimes X_i + X_i \otimes A_i^{-1}$ is a very effective way of getting rid of any non-genuine extra parameters which has been introduced in the algebra . This remark also applies to the work of Reshetikhin in [5] . Although the starting point of Reshetikhin is quite correct, his final equations for the multiparametric deformed co-product need modification,(In the notation of [5], he makes mistakes in calculating $\Delta^F = F\Delta F^{-1}$) after which his results will be equivalent to that of us modulo a redifinition of parameters.

4) Lets see if this procedure works also for twisting of finite dimensional super algebras with symmetrizable Cartan matrix.

a) For superalgebras of rank < 3 ,where the relations in the Chevalley basis adequately represent the superalgebra , the above procedure (which is essentially based on the symmetry $s_{ij}s_{ji} = 1$)clearly works,provided that one takes care about the graded nature of the tensor products.

b) For super algebras of higher rank the naive Serre' presentation may not be adequate [11]. For the super algebra $sl(n|m)$, it is shown in [12]

that the naive Serre presentation defines a strictly larger algebra $s(n|m)$ of which $sl(n|m)$ is a quotient. The quotients is determined by vanishing of the following elements :

$$C^{\pm} \equiv [X_n^{\pm}, [X_{n-1}^{\pm}, [X_n^{\pm}, X_{n+1}^{\pm}]]] \tag{36}$$

where $X^{\pm}{}_n$ are the odd elements of the algebra. Now what we have to do is to show that the addition of these relations to the relations of the Chevalley basis does not spoil the homomorphism property of the multiparametric coproduct. From eq. (16) we have

$$\Delta_o C^{\pm} = (Q_n Q_{n-1} Q_n Q_{n+1})^{\mp 1} \otimes C^{\pm} + C^{\pm} \otimes (Q_n Q_{n-1} Q_n Q_{n+1})^{\pm} = 0 \tag{37}$$

which shows that $\Delta_o(C^{\pm}) = 0$ whenever $C^{\pm} = 0$. If we denote the above projection by $\pi : U(s(n|m)) \longrightarrow U(sl(n|m))$ then the above discussion can be restated as the commutativity of the following diagram:

$$\begin{array}{ccc} U(s(n|m)) & \xrightarrow{\Delta_o} & U(s(m|n) \otimes U(s(m|n) \\ \Big\downarrow \pi & & \Big\downarrow \pi \otimes \pi \\ U(sl(n|m)) & \xrightarrow{\Delta_o} & U(sl(m|n) \otimes U(sl(m|n) \end{array} \tag{38}$$

How can we go beyond $sl(n|m)$?

Denote the enveloping algebra of g by U(g) and the associative algebra presented by the naive Serre presentation by $U(g)^*$ .If as in the case of $sl(n|m)$,a relation like $U(g) = \frac{U(g)^*}{\{C_i = 0\}}$ holds ,then the above diagram is still commutative provided that the elements $C_i \in g$,(i.e: can be expressed in terms of Lie superbrakets see eqs(16,36,37)and this is true for Kac-Moody superalgebras. In fact the additional 4-th degree relations looks as follows. For any subdiagram

of Dynkin diagram, if $(\alpha_i, \alpha_i) = (\alpha_{i+1}, \alpha_{i-1}) = (\alpha_i, \alpha_{i-1} + \alpha_{i+1}) = 0$ We have

$$[[e_{\pm\alpha_i}, e_{\pm\alpha_{i-1}}, [e_{\pm\alpha_i}, e_{\pm\alpha_{i+1}}]] = 0$$

4) Let us see how the mixing of Cartan generators in the co-products affects the reduction of representations of the classical Lie algebra in a simple example. Consider the quark and antiquark representaions of SU(3),which are denoted as usual by 3 and $\bar{3}$.

In the quark representation the cartan generators (in the Cartan-Weyl basis) are $h_1 = diag(\frac{1}{2}, \frac{-1}{2}, 0)$ and $h_2 = diag(\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{-1}{\sqrt{3}})$. The Cartan generators H_1 and H_2 (in the Chevalley basis) are $H_1 = 2h_1$ and $H_2 = -h_1 + \sqrt{3}h_2$. For constructing the representation $3 \otimes 3$ according to eqs. (32) we have:

$$\Delta_o X_1^- = \begin{pmatrix} s & & \\ & s & \\ & & s^{-2} \end{pmatrix} \otimes X_1^- + X_1^- \otimes \begin{pmatrix} s^{-1} & & \\ & s^{-1} & \\ & & s^2 \end{pmatrix} \tag{39}$$

$$\Delta_o X_2^- = \begin{pmatrix} s^{-2} & & \\ & s & \\ & & s \end{pmatrix} \otimes X_2^- + X_2^- \otimes \begin{pmatrix} s^2 & & \\ & s^{-1} & \\ & & s^{-1} \end{pmatrix} \tag{40}$$

and

$$\Delta_o X_3^- = \begin{pmatrix} s^{-1} & & \\ & s^2 & \\ & & s^{-1} \end{pmatrix} \otimes X_3^- + X_3^- \otimes \begin{pmatrix} s & & \\ & s^{-2} & \\ & & s \end{pmatrix} \tag{41}$$

where $X_3 = [X_1, X_2]$. For the positive roots , one must change s to s^{-1} in all the above formulas . Denoting the basis vectors of 3 by $|1>, |2>$ and $|3>$, It is easy to establish that the following states form the 6 of $SU(3)$:

$$|\mu_1> = |11> \qquad |\mu_2> = \frac{1}{\sqrt{|s|^2 + |s|^{-2}}}(s|12> + s^{-1}|21>) \tag{42}$$

$$|\mu_3> = |22> \qquad |\mu_4> = \frac{1}{\sqrt{|s|^2 + |s|^{-2}}}(s|23> + s^{-1}|32>) \tag{43}$$

$$|\mu_5> = |33> \qquad |\mu_6> = \frac{1}{\sqrt{|s|^2 + |s|^{-2}}}(s|31> + s^{-1}|13>) \tag{44}$$

The other three states which form the $\bar{3}$ of SU(3) are:

$$|\nu_2 >= \frac{1}{\sqrt{|s|^2 + |s|^{-2}}}(s|12 > -s^{-1}|21 >) \tag{45}$$

$$|\nu_4 >= \frac{1}{\sqrt{|s|^2 + |s|^{-2}}}(s|23 > -s^{-1}|32 >) \tag{46}$$

$$|\nu_6 >= \frac{1}{\sqrt{|s|^2 + |s|^{-2}}}(s|31 > -s^{-1}|13 >) \tag{47}$$

Note that the symmetry of the multiplets is dependent on the the value of s . Also note that the two invariant subspaces carrying the representations $\bar{3}$ and 6 are no longer orthogonal. In fact we have :

$$< \nu_{2i}, \mu_{2i} >= \frac{|s|^2 - |s|^{-2}}{|s|^2 + |s|^{-2}} \tag{48}$$

showing that only when s is a pure phase ,the two multiplets will be orthogonal.

It is interesting to note that for $s = \pm i$ the $\bar{3}$ representation is symmetric and the 6 is no longer symmetric.

As the simplest example , concerning the co-product (32), consider the product of two spin $\frac{1}{2}$ representations of SU(2). The generators are :

$$J_o = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{49}$$

with basis vectors $|+>$ and $|->$

For su(2), in the fundamental representation , we have $(-1)^{a_{ik}^{-1}H_k} = (-1)^{\frac{1}{2}J_o} = iJ_o$ Consequently we will have from eq.(32):

$$\Delta J_+ = i(J_o \odot J_+ - J_+ \odot J_o) \tag{50}$$

$$\Delta J_- = -i(J_o \odot J_- - J_- \odot J_o) \tag{51}$$

With the above co-product , the tensor product of two spin $\frac{1}{2}$ representation reduces to the following ireps:

$$\text{Triplet} \begin{cases} |++\rangle \\ \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle) \\ |--\rangle \end{cases} \qquad \text{Singlet} \left\{ \frac{1}{\sqrt{2}} (|+-\rangle + |-+\rangle) \right.$$

The particular thing is that , the singlet state is now symmetric and the triplet has a mixed symmetry.

DISCUSSION

It is interesting to see how considerations of continous s-dependent symmetry in multiplets of SU(3), can affect our present models of elementary particles.

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