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Cylindrically Symmetric Thin Walls in General Relativity[1]

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Absract

Assuming a continuous ansatz for the metric we solve the Einstein equations for a thin wall with cylindrical symmetry. The Solution can correspond to a collapsing matter shell leading to a naked, without any horizon, singularity.

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The study of (2+1) dimensional thin walls is now very common; this is partly because of the role domain walls may play in cosmology [1-8]. Usually the authors are used to handle the problem in the Gauss-Codazzi formalism [9,10]. The work of Geroch and Traschen [11], however encourages one to tackle the problem directly using distribution-valued tensors. This has already been successfully done for plane- [12] and spherically symmetric [13] thin walls, with the same results as had been previously obtained by Gauss-Codazzi formalism.

The axially symmetric case has not yet been considered in the literature. It should be noted that even domain walls could be in the shape of an infinite cylinder. Here we report on a new solution which we have found for the Einstein equations representing a thin infinite cylinder using our direct method [12,13].

Using coordinates $\rho > 0$, $0 < \varphi \leq 2\pi$, and $-\infty < z, t < \infty$ we write the metric in the form

$$ds^2 = -f dt^2 + d\rho^2 + h d\varphi^2 + w dz^2 \tag{1}$$

Where f, h, and w are functions of ρ and t . The components of the Ricci tensor are then calculated to be

$$\begin{aligned}
R_{00} &= \frac{\ddot{h}}{2h} - \frac{\dot{h}^2}{4h^2} + \frac{\ddot{w}}{2w} - \frac{\dot{w}^2}{4w^2} - \frac{f''}{2} + \frac{f'^2}{4f} - \frac{h'f'}{4h} - \frac{w'f'}{4w} - \frac{\dot{h}\dot{f}}{4hf} - \frac{\dot{w}\dot{f}}{4wf} \\
R_{01} &= \frac{\dot{h}'}{2h} - \frac{\dot{h}h'}{4h^2} + \frac{\dot{w}'}{2w} - \frac{\dot{w}w'}{4w^2} - \frac{\dot{h}f'}{4hf} - \frac{\dot{w}f'}{4wf} \\
R_{11} &= -\frac{f''}{2f} - \frac{f'^2}{4f^2} + \frac{h^2}{2h} - \frac{h'^2}{4h^2} + \frac{w''}{2w} - \frac{w'^2}{4w^2}
\end{aligned} \tag{2}$$

$$R_{22} = -\frac{\ddot{h}}{2f} + \frac{\dot{h}\dot{f}}{4f^2} + \frac{h''}{2} + \frac{1}{4}\frac{\dot{h}^2}{hf} - \frac{1}{4}\frac{\dot{w}\dot{h}}{wf} + \frac{h'f'}{4f} - \frac{h'^2}{4h} + \frac{h'w'}{4w}$$

$$R_{33} = -\frac{\ddot{w}}{2f} + \frac{\dot{w}\dot{f}}{4f^2} + \frac{w''}{2} + \frac{1}{4}\frac{\dot{w}^2}{wf} - \frac{1}{4}\frac{\dot{w}\dot{h}}{hf} + \frac{w'f'}{4f} - \frac{w'^2}{4w} + \frac{h'w'}{4h}$$

Where dot and prime mean derivative with respect to t and ρ respectively. The corresponding field equations for this general form is too difficult to be solved. Therefore we make the simplifing assumption that h and w are independent of time:

$$\dot{h} = \dot{w} = 0 \tag{3}$$

The 01 component of the Ricci tensor then vanishes identically. The other components are

$$\begin{aligned} R_{00} &= -\frac{f''}{2} + \frac{f'^2}{4f} - \frac{h'f'}{4h} - \frac{w'f'}{4w} \\ R_{11} &= \frac{f''}{2f} - \frac{f'^2}{4f^2} + \frac{h''}{2h} - \frac{h'^2}{4h^2} + \frac{w''}{2w} - \frac{w'^2}{4w^2} \\ R_{22} &= \frac{h''}{2} + \frac{h'f'}{4f} - \frac{h'^2}{4h} + \frac{h'w'}{4w} \\ R_{33} &= \frac{w'}{2} + \frac{w'f'}{4f} - \frac{w'^2}{4w} + \frac{h'w'}{4h} \end{aligned} \tag{4}$$

Assuming the vacuum Einstein field equations, we obtain the following relations from $R_{22} = 0$ and $R_{33} = 0$:

$$h'^2 f \frac{w}{h} = c_1(t) \tag{5}$$

$$w'^2 f \frac{h}{w} = c_2(t) \tag{6}$$

Where c_1 and c_2 are functions of time. Relation (5) shows that f can be made independent of time through a suitable time transformation. That is we can choose $c_1(t)$ to be a

constant. In this new time coordinate c_2 must also be a constant since the left hand side of (6) is independent of time. Therefore we may write

$$h'^2 f \frac{w}{h} = c_1 \tag{7}$$

$$w'^2 f \frac{h}{w} = c_2 \tag{8}$$

From the above equations we obtain

$$f = \frac{c_2}{h'^2 h^{b-1}} \tag{9}$$

$$w = c' h^b \tag{10}$$

where $b = \frac{c_2}{c_1}$ and c' is an integration constant. Simple manipulations using the other field equations leads to

$$h = c^2 \rho^{[2(b+1)/(b^2+b+1)]} \tag{11}$$

$$w = \rho^{[2b(b+1)/(b^2+b+1)]} \tag{12}$$

$$f = \rho^{[-2b/(b^2+b+1)]} \tag{13}$$

We are looking for a solution representing the field in- and out-side of an infinite cylindrical thin mass shell. For the metric inside the shell to be nonsingular we have to require

$$c = 1 \quad , \quad b = 0 \tag{14}$$

which leads to

$$h = \rho^2 \tag{15}$$

$$w = f = 1 \tag{16}$$

Hence, assuming the shell to be at $\rho = R(t)$ at the time t, we can finally write the metric in- and out-side the shell in the form

$$ds^2 = -\rho^{[-2b/(b^2+b+1)]}\, dt^2 + d\rho^2 + c^2 \rho^{[2(b+1)/(b^2+b+1)]}\, d\varphi^2$$

$$+\rho^{[2b(b+1)/(b^2+b+1)]}\, dz^2 \qquad \rho > R(t) \tag{17}$$

$$ds^2 = -dt'^2 + d\rho'^2 + \rho'^2 d\phi + dz'2 \qquad \rho < R(t) \tag{18}$$

To avoid terms quadratic in distributions and derivative of distributions in the components of the Ricci tensor we consider the following transformations to make the metric continuous at the shell:

$$\begin{aligned} t' &= A(t, \rho, z) \\ \rho' &= B(t, \rho, z) \\ z' &= D(t, \rho, z) \end{aligned} \tag{19}$$

carrying out the transformation and requiring the continuity at $\rho = R(t)$, we obtain the following conditions:

$$\dot{A}^2 - \dot{B}^2 - \dot{D}^2 = f$$

$$\dot{A}A' - \dot{B}B' - \dot{D}D' = 0$$

$$A'^2 - B'^2 - D'^2 = -1$$

$$\dot{A}\hat{A} - \dot{B}\hat{B} - \dot{D}\hat{D} = 0 \tag{20}$$

$$A'\hat{A} - B'\hat{B} - D'\hat{D} = 0$$

$$\hat{A}^2 - \hat{B}^2 - \hat{D}^2 = -w$$

$$B = \sqrt{h}$$

where equalities hold at $\rho = R(t)$; w, h, and f are given by (11)-(13), and hat means derivative with respect to z. Here we are faced with 7 equations for the unknown functions A, B, and D. However, there is no overdetermination and the equations can be solved. It can be shown that w must be constant.

Now, from (17), (18), and (19) it is easily seen that in order to have continuity at $\rho = R(t)$ we only need to make the transformations:

$$t' = A(t,\rho) \quad , \quad \rho' = B(t,\rho) \tag{21}$$

which lead to

$$\dot{A}^2 - \dot{B}^2 = 1$$

$$\dot{A}A' - \dot{B}B' = 0 \tag{22}$$

$$B = cR \quad \Longrightarrow \quad \dot{B} + \dot{R}B' = c\dot{R}$$

where again the equations hold at $\rho = R(t)$. These equations can be solved to obtain:

$$B' = \dot{A} = \frac{1}{1-\dot{R}^2}\left[\sqrt{1+(c^2-1)\dot{R}^2} - c\dot{R}^2\right]$$

$$\dot{B} = A' = \frac{\dot{R}}{1-\dot{R}^2}\left[c - \sqrt{1+(1+(c^2-1)\dot{R}^2}\right] \tag{23}$$

Defining the functions

$$U = \dot{A}^2 - \dot{B}^2 \qquad Q = A'^2 - B'^2$$

$$V = \dot{A}A' - \dot{B}B' \qquad X = -B^2 \tag{24}$$

We can write the metric (17) and (18) for the case $w = 1$ in the following closed and continuous form

$$ds^2 = -[U\theta(R-\rho) + \theta(\rho - R)]dt^2 - 2V\theta(R-\rho)dtd\rho$$

$$+[-Q\theta(R-\rho) + \theta(\rho-R)]d\rho^2 + [-X\theta(R-\rho) + c^2\rho^2\theta(\rho-R)]d\phi^2 + dz^2 \tag{25}$$

This metric is now regular [11] and can be differentiated formally. The first derivative of the metric is proportional to the step function. Its second derivative enters linearly in the expressions for curvature and Ricci tensor, which are therefore just proportional to $\delta(\rho - R)$. Hence it suffices to calculate just terms in the second derivative of the metric which are proportional to δ. This leads to:

$$R_{00} = \left[\frac{(UQX)\dot{R}}{2c^2R^2} - \frac{1}{2}\dot{U}\dot{R} + \frac{1}{2}U' - \dot{V}\right]\delta(\rho - R)$$

$$R_{01} = \Big[-\frac{(UQX)^{\cdot}}{2c^2R^2} + \frac{1}{2}\dot{U} - \frac{1}{2}\dot{W}\Big]\,\delta(\rho - R)$$

$$R_{11} = \Big[\frac{2c^2R - (UQX)'}{2c^2R^2} + \frac{1}{2}\dot{Q}\dot{R} + \dot{V} - \frac{1}{2}Q'\Big]\,\delta(\rho - R) \tag{26}$$

$$R_{22} = [\frac{1}{2}(\dot{X}\dot{R} + X') + c^2R]\delta(R - \rho)$$

$$R_{33} = 0$$

To specify the energy-momentum tensor, we proceed as in our previous paper [13]. The unit spacelike normal to the shell n^μ and the unit timelike vector field u^μ representing the fluid motion (shell matter) are given by

$$u^\mu = \Big(\frac{1}{\sqrt{1-\dot{R}^2}}\ ,\ \frac{\dot{R}}{\sqrt{1-\dot{R}^2}}\ ,\ 0\ ,\ 0\Big)$$

$$n^\mu = \Big(\frac{\dot{R}}{\sqrt{1-\dot{R}^2}}\ ,\ \frac{1}{\sqrt{1-\dot{R}^2}}\ ,\ 0\ ,\ 0\Big) \tag{27}$$

Now, the surface energy-momentum tensor of the shell is given by

$$S^{\mu\nu} = \sigma u^\mu u^\nu + \tau^\mu_\beta (h^{\beta\nu} - u^\beta u^\nu) \tag{28}$$

Where

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \tag{29}$$

is the induced three metric, and τ^μ_ν is the stress tensor, which would be proportional to δ^μ_ν if the wall were isotropic. $S^{\mu\nu}$ is related to the energy momentum tensor $T^{\mu\nu}$ by

$$S^{\mu\nu} = \int dl\ T^{\mu\nu} \tag{30}$$

Where l is the proper distance through Σ ,the support of the source , in the direction of the normal n^ν. Now from (27) and (30)

$$T_{\mu\nu} = \sqrt{1-\dot{R}^2}\; S_{\mu\nu}\,\delta(\rho - R) \tag{31}$$

Substituting (25) - (29) into (31) , we get

$$T_{\mu\nu} = \sqrt{1-\dot{R}^2}\begin{pmatrix} \frac{\sigma}{1-\dot{R}^2} & \frac{-\dot{R}\sigma}{1-\dot{R}^2} & 0 & 0 \\ \frac{-\dot{R}\sigma}{1-\dot{R}^2} & \frac{\dot{R}^2\sigma}{1-\dot{R}^2} & 0 & 0 \\ 0 & 0 & -c^2R^2\tau_2 & 0 \\ 0 & 0 & 0 & -\tau_3 \end{pmatrix}\delta(\rho - R) \tag{32}$$

The Einstein field equations

$$R_{\mu\nu} = -\kappa(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}) \tag{33}$$

combined with the continuity conditions (23) then lead to

$$\begin{gathered} -\frac{(UQX)^{\cdot}\dot{R}}{2c^2R^2} + \frac{1}{2}\dot{U}\dot{R} - \frac{1}{2}U' + \dot{V} = \kappa\frac{\sigma(1+\dot{R}^2) + (\dot{R}^2-1)(\tau_2+\tau_3)}{2\sqrt{1-\dot{R}^2}} \\ \frac{(UQX)^{\cdot}}{2c^2R^2} - \frac{1}{2}\dot{U} + \frac{1}{2}\dot{Q} = -\kappa\frac{\dot{R}\sigma}{\sqrt{1-\dot{R}^2}} \\ \frac{-2c^2R + (UQX)'}{2c^2R^2} - \frac{1}{2}\dot{Q}\dot{R} - \dot{V} + \frac{1}{2}\dot{Q} = \kappa\frac{\sigma(1+\dot{R}^2) + (1-\dot{R}^2)(\tau_2-\tau_3)}{2\sqrt{1-\dot{R}^2}} \\ \frac{1}{2}(\dot{X}\dot{R} + X') + c^2R = -\kappa\frac{\sigma+\tau_3-\tau_2}{2}\sqrt{1-\dot{R}^2}\,c^2R^2 \\ 0 = \sigma + \tau_2 - \tau_3 \end{gathered} \tag{34}$$

From the last equation we get

$$\tau_3 = \sigma + \tau_2 \tag{35}$$

The remaining equations are not independent. Using (35), after some manipulations , we obtain two independent equations from (34)

$$\dot{Q}\dot{R} - U' + 2\dot{V} = 2\kappa\tau_2\sqrt{1-\dot{R}^2}$$

$$\sqrt{1+(c^2-1)\dot{R}^2} - c = \kappa\sigma\sqrt{1-\dot{R}^2}cR \tag{36}$$

Substituting from (24) in (36) we obtain

$$\frac{d}{ds}\Big[c\sqrt{1+(\frac{dR}{ds})^2} - \sqrt{1+c^2(\frac{dR}{ds})^2}\Big] = -\kappa\tau_2 c\frac{dR}{ds}$$

and

$$\sqrt{1+c^2(\frac{dR}{ds})^2} - c\sqrt{1+(\frac{dR}{ds})^2} = \kappa\sigma cR \tag{37}$$

where s is the proper time of the observer u^μ. From (27) we know that

$$\frac{ds}{dt} = \sqrt{1-\dot{R}^2} \tag{38}$$

Assuming an equation of state , $\sigma = f(\tau)$, we can integrate (37) to get

$$R = R_0 \exp\Big[-\int_{\sigma_0}^{\sigma}\frac{d\sigma'}{\sigma' - f(\sigma')}\Big] \tag{39}$$

$$c = \frac{1}{\sqrt{1+2\sqrt{1+(\frac{dR}{ds})^2}\kappa\sigma R + \kappa^2\sigma^2 R^2}} \tag{40}$$

Where R_0 is a constant of integration and $R_0 = R(\sigma_0)$. When the wall is stationary, $\frac{dR}{ds} = 0$, we have

$$c = \frac{1}{1+\kappa\sigma R} \tag{41}$$

In the case the energy density and stresses are proportional, $\tau_2 = \alpha\sigma$, we obtain from (39)

$$\frac{\sigma}{\sigma_0} = \left(\frac{R}{R_0}\right)^{\alpha-1} \tag{42}$$

The stability of the shell is easily checked by (37). From this equation one gets

$$\frac{d^2R}{ds^2} = -\kappa\tau_2\left[\frac{1}{\sqrt{1+(dR/ds)^2}} - \frac{1}{\sqrt{1/c^2+(dR/ds)^2}}\right] \tag{43}$$

Now, three cases can be differentiated:

i) $\tau_2 = 0$, $\frac{d^2R}{ds^2} = 0$, which corresponds to an unstale stationary configuration.

ii) $\tau_2 < 0$, $\frac{d^2R}{ds^2} > 0$. In this case the final configuration of the thin wall will be expanding.

iii) $\tau_2 > 0$, $\frac{d^2R}{ds^2} < 0$. This case corresponds to a (finally) collapsing configuration.

Outside the shell the space is (locally) flat. Therefore free observers will feel no acceleration. Therefore the acceleration of the observer with respect to the wall is equal to $-\frac{d^2R}{ds^2}$, which has the same sign as τ_2. This means that the wall acts as an attractive (a repulsive) source for $\tau_2 < 0$ ($\tau_2 > 0$). This is in agreement with the sign of the effective density of source, $\sigma - \tau_2 - \tau_3 = -2\tau_2$.

The defect parameter c can be obtained from the linear density μ defined by

$$\mu := \int (cRd\varphi)\sigma = 2\pi cR\sigma \tag{44}$$

calculated at the moment when $\frac{dR}{ds} = 0$. Therefore, using (41), one obtains

$$c = 1 - 4\mu \tag{45}$$

which holds for $\mu < \frac{1}{4}$. If $\mu \geq \frac{1}{4}$, the outer metric will aquire the form

$$ds^2 = -dt^2 + d\rho^2 + (\rho_0 + c\rho)^2 d\varphi^2 + dz^2 \tag{46}$$

Now, equation (45) also holds for $\mu \geq \frac{1}{4}$. Note that for $\mu > \frac{1}{4}$ there is a singularity out of the wall. This singularity can be filled in by a suitable source having the linear density

$$\mu' = \frac{1}{2} - \mu \tag{47}$$

Similar results have been obtained by Gott[14] for cosmic strings. Gott's results concerning gravitational lensing effects also hold for our solution of cylindrical shells.

Note that the collapsing solution in the case of $\tau_2 > 0$ leads to a naked singularity without any horizon.

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