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FISHER INFORMATION AND PERIODIC STRUCTURES IN COSMOLOGY

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Fisher Information and Periodic Structures in Cosmology

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ABSTRACT

We use the principle of minimum Fisher information to develop a preliminary theory for extragalactic redshift periodicities. We show that the principle leads to periodic distributions in phase space, where the velocity part of the space is the velocity of a galaxy relative to the local Hubble flow. The results seem applicable to a variety of kinematic problems in cosmology.

1. INTRODUCTION

There continues to be strong evidence for periodicities in extragalactic redshifts. Most recently, Cocke & Tifft (1993) have discovered global 36 and 72 km s^{-1} periodicities in several radial velocity data sets in which velocities were corrected for the solar motion relative to the cosmic background radiation. These periodicities are so strong that they cannot plausibly be explained as random fluctuations in the data. For a general review of the status of redshift periodicities, see Tifft (1993). On a much larger scale, Broadhurst et al (1990) have observed redshift periodicities corresponding to a spacing of about $130h^{-1}$ Mpc, where h is the Hubble constant in units of 100 km s^{-1} Mpc^{-1}.

The theory developed here uses the the principle of minimum Fisher information (MFI), which has recently been exploited by B. R. Frieden (1990) as a basis for understanding physical laws. MFI is particularly useful in statistical problems, and here we consider the positions and velocities of galaxies in phase space as random variables described by a probability density $p(\mathrm{x}, \mathrm{v})$, where v is the velocity relative to the local Hubble flow.

In Section 2 we discuss MFI and review its application to the Boltzmann distribution. In Sections 3 and 4, we apply the principle to redshift periodicities, in the context of an observer–centered cosmology, and in Section 5 we broaden the discussion to include a general distribution $p(\mathrm{x}, \mathrm{v})$ in Cartesian coordinates.

The paper deals primarily with the small–scale periodicities of 36 and 72 km s^{-1}, but our results should have observable consequences for more general studies of the kinematics, distances, and angular distributions of galaxies.

2. MINIMUM FISHER INFORMATION AS A GENERAL PHYSICAL PRINCIPLE

MFI is a useful tool in studying the statistical properties of physical systems. The concept has been used by B. R. Frieden to derive such diverse results as the time-independent Schrödinger equation and Maxwell's equations. See Frieden (1990, 1992).

The idea may be summarized as follows: Consider a random variable x with probability density $p(x)$. The Fisher information associated with $p(x)$ is defined as

$$I \equiv \int \left(\frac{dp}{dx}\right)^2 \frac{dx}{p}. \tag{1}$$

Minimizing I by varying $p(x)$ leads to physically interesting results if the constraints in the problem are chosen appropriately. One of the constraints is that the average of a kinetic energy be held constant during the variation of $p(x)$. Another constraint often used is normalization, $\int p(x)\,dx = 1$. For example, if the random variable is one of the velocity components v of a molecule in a gas, the constraint $< E_{\mathrm{kin}} >= \frac{1}{2}m \int v^2 p(v) dv = \frac{1}{2}kT$, together with the normalization constraint, leads to the variational problem

$$\begin{aligned} S &= \int [p'(v)^2/p(v)]dv + \lambda\Big(\int \frac{1}{2}mv^2 p(v)dv - \frac{1}{2}kT\Big) + \mu\Big(\int p(v)dv - 1\Big) \\ &\equiv \int L(p,p')dv = \min\ , \end{aligned} \tag{2}$$

where λ and μ are Lagrange multipliers and $p'(v) = dp/dv$.

The Lagrangian for the problem is

$$L = \frac{p'^2}{p} + \frac{\lambda}{2} m v^2 p + \mu p \ . \tag{3}$$

The usual Lagrange equation for $p(v)$, $d(\partial L/\partial p')/dv - \partial L/\partial p = 0$, then gives the Boltzmann distribution $p(v) \propto \exp(-av^2)$. One easily finds the Lagrange multipliers to be $\lambda = 8a^2/m$ and $\mu = -4a < 0$, where of course $a = m/2kT$. See Frieden (1990) for further details.

In this problem, as well as others, Lagrange's equation is easily linearized by the substitution $p = q^2$. This substitution assures $p \geq 0$.

Frieden (1990) has shown that the MFI principle (typified by equation (2)) is equivalent to maximizing the mean-square-error of the best possible estimator of the random variable in question.

3. THE MFI PRINCIPLE IN AN OBSERVER-CENTERED COSMOLOGY

In this section we consider the probability density $p(\mathbf{x}, \mathbf{v})$ for the location of a galaxy in a reduced 4-dimensional phase space. As random variables, we use the galaxy's position in spherical coordinates $\mathbf{x} = (r, \theta, \phi)$ and the radial (line-of-sight) peculiar velocity v. By this we mean the galaxy's velocity relative to the local Hubble flow, projected along our line of sight. The MFI principle then leads directly to periodicities in the observed (total) radial velocity $v_{\rm obs}$.

The treatment here is nonrelativistic and does not predict a value for the periodicity, but shows only that the observed redshift of a galaxy has preferred values and that these values may be equally spaced. This is in contrast with earlier work (Cocke 1985), which showed that redshifts might be proportional to the square root of a quantum number.

Let $p = p(r, \theta, \phi, v) = q^2$ be the probability density for the position and radial peculiar velocity v of a galaxy as seen by an observer at the origin, $r = 0$. Frieden (1990) has pointed out that Fisher information is additive for densities that are functions of several random variables, provided that the variables may be scaled to have the same dimensions. In our case, the Hubble constant h_0 with dimensions $\sec^{-1}$ gives a natural scaling between position and velocity.

As in the Boltzmann distribution problem, we use the constraints of fixed kinetic energy and normalization. Here, it is appropriate to use the kinetic energy $\frac{1}{2}Mv^2$ associated with statistical deviations from the highly ordered Hubble flow. We write the Lagrangian in a way similar to equation (3) as

$$L = \frac{1}{h_0^2 p} \nabla p \cdot \nabla p + \frac{1}{p}\left(\frac{\partial p}{\partial v}\right)^2 + \frac{\lambda}{2} M v^2 p + \mu p \ , \tag{4}$$

where $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$. Note the similarity between the first and second terms on the RHS of this equation.

Defining $\lambda_1 = \lambda/4$ and $\lambda_2 = -\mu/4$ and using the substitution $p = q^2$ leads to the Lagrange equation

$$\frac{1}{h_0^2}\nabla^2 q + \frac{\partial^2 q}{\partial v^2} - \left(\frac{\lambda_1}{2} M v^2 - \lambda_2\right) q = 0 \ . \tag{5}$$

Here, $p = q^2$ is the probability density in units of sec cm^{-4}, in rectangular coordinates. We wish to use spherical polar coordinates in x-space, which are more germane to our perspective as observers. A separable solution $q = R(r)Y(\theta, \phi)V(v)$ of equation (5) is easily found, with the usual $Y = Y_{lm}(\theta, \phi) = P_l^m(\cos\theta)e^{im\phi}$, P_l^m being the associated Legendre functions for $l \geq 0$ and $|m| \leq l$. The MFI principle is not normally used for complex functions, but one may clearly use $\cos(m\phi)$ and $\sin(m\phi)$ instead of $e^{im\phi}$.

The equation for $R(r)$ is

$$\frac{d^2 R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - \frac{l(l+1)}{r^2} R = -\zeta h_0^2 R \ , \tag{6}$$

where ζ is a separation constant. If ζ is positive, the solutions of this equation are spherical Bessel functions $j_l(\rho)$, where $\rho = \sqrt{\zeta} h_0 r$. The $j_l(\rho)$ are sines and cosines of ρ divided by powers of ρ, but they are all regualar at the origin. For large ρ, they have the asymptotic expansion $j_l(\rho) \sim \cos[\rho - \frac{1}{2}(l+1)\pi]/\rho$. The spherical

Neumann functions are also solutions of equation (6), but they are not integrable at the origin. See, for example, Abramowitz & Stegun (1972).

If the separation constant ζ is negative, the solutions of equation (6) involve the modified Bessel functions $I_{l+1/2}(\rho)$ and the Macdonald functions $K_{l+1/2}(\rho)$. The former diverge exponentially at infinity, and the latter are not integrable at the origin. See, for example, Lebedev (1972). Therefore, the $j_l(\rho)$ solutions are the only physically reasonable ones. This implies that ζ is positive.

The equation for $V(v)$ becomes

$$\frac{d^2V}{dv^2} - \left(\frac{\lambda_1}{2}Mv^2 - \lambda_2 + \zeta\right)V = 0 \ , \tag{7}$$

the solutions of which involve Hermite functions $H_\nu(\eta)$, where $\eta = (\frac{\lambda_1}{2}M)^{1/4}v$ and $\lambda_2 = \zeta + (\frac{\lambda_1}{2}M)^{1/2}(2\nu+1)$. The Hermite functions diverge exponentially at either $+\infty$ or $-\infty$ unless ν is an integer, n. Therefore the only physically admissible solutions to equation (7) are the harmonic oscillator wavefunctions $U_n(\eta) = e^{-\eta^2/2}H_n(\eta)$. In this case

$$\lambda_2 = \zeta + \left(\frac{\lambda_1}{2}M\right)^{1/2}(2n+1) \ . \tag{8}$$

4. APPLICATION TO REDSHIFT PERIODICITIES

The above solution to equation (5) yields periodicities in the total radial velocity $v_{\rm obs}$: The spherical Bessel functions are combinations of $\sin(\sqrt{\zeta}h_0r)$ and $\cos(\sqrt{\zeta}h_0r)$, and the node spacing in h_0r is $h_0\delta r = \pi/\sqrt{\zeta}$. The harmonic oscillator wavefunctions are roughly periodic in $\eta = (\frac{\lambda_1}{2}M)^{1/4}v$. In fact, for n even and large, $n = 2k$, we have the limiting relation (Bateman 1953) $\lim_{k\to\infty}[(-)^k\sqrt{k}H_{2k}(u/2\sqrt{k})/(2^{2k}k!)] = \pi^{-1/2}\cos u$. There is a similar expression for odd n. The spacing between the nodes is therefore $\delta u \approx \pi$, and it follows that the node spacing in η is $\delta\eta \approx \pi/\sqrt{2n} \approx 2n^{-1/2}$. The node spacing in v is then $\delta v \approx 2(\frac{\lambda_1}{2}M)^{-1/4}n^{-1/2}$.

Since $v_{\rm obs} = v + h_0r$, the periodicity $\delta v_{\rm obs}$ is most visible when $h_0\delta r$ and δv are equal, in which case $\delta v_{\rm obs} = h_0\delta r = \delta v$. An observed redshift periodicity $\delta v_{\rm obs}$ then fixes the parameter ζ. For example, if we take $\delta v_{\rm obs} \approx 36$ km s^{-1}(Tifft & Cocke 1984), we get $\zeta \approx (\pi/36)^2 \approx 0.0076$ s^2 km^{-2}.

Note that the probability density in spherical coordinates is $q^2r^2\sin\theta$. By virtue of the asymptotic relation for the spherical Bessel functions quoted in Section 3, this density is not integrable at $r \to \infty$. In fact, q^2r^2 is $\propto \cos^2[\rho - \frac{1}{2}(l+1)\pi]$ in the limit of large r. Therefore it is necessary to introduce a cutoff at large r; otherwise, the Fisher information would not converge. It might be possible to use MFI with delta–function normalization, but in any case the cosmological event horizon would be a suitable physical cutoff.

We may determine the parameters n and λ_1 by invoking a velocity dispersion Δv relative to the local Hubble flow. One might choose, say, $\Delta v \approx 250$ km s^{-1}. The Hermite polynomials have n nodes, and therefore we have $2\Delta v \approx n\delta v \approx 36n$. It follows that $n \approx 14$; and since $\delta v \approx 2(\frac{\lambda_1}{2}M)^{-1/4}n^{-1/2}$, we find $(\frac{\lambda_1}{2}M)^{-1/4} \approx 67$ km s^{-1}. A suitable value for the galaxy mass M would then give a value for λ_1. From equation (8) one finds $\lambda_2 \approx 0.0141$.

The solutions to equation (5) are cosmologically reasonable only if, given a measured $v_{\rm obs}$, the expectation of r is $\approx v_{\rm obs}/h_0$. It is easy to see that this is the case for the radial solution investigated here. Let $p(r|v_{\rm obs})$ be the probability density for r for a given value of $v_{\rm obs}$. By Bayes' theorem, this is equal to $p(r, v_{\rm obs})/\int p(r', v_{\rm obs})\,dr'$. The density $p(r, v_{\rm obs})$ may be obtained formally by the transformation $r = \bar{r}$, $v = v_{\rm obs} - h_0/\bar{r}$, the Jacobian for which is unity.

In the limit of large $\rho = \sqrt{\zeta}h_0r$, one gets, replacing $\bar{r} \to r$, the result $p(r, v_{\rm obs})r^2 \propto \cos^2[\rho - \pi(l+1)/2]U_n^2[\alpha(v_{\rm obs} - h_0r)]$, where we abbreviate $\alpha = (\frac{\lambda_1}{2}M)^{1/4}$. Both cos and U_n have roughly 14 nodes over the entire width of U_n, and the latter function has exponential cutoffs at both ends of its width. And since U_n^2 is symmetric in its argument $\alpha(v_{\rm obs} - h_0r)$, we conclude that $p(r|v_{\rm obs})$ is, as function of r, roughly centered at $v_{\rm obs}/h_0$ and that the expectation of r (given $v_{\rm obs}$) is $\approx v_{\rm obs}/h_0$, as desired.

We must now discuss the role of the angular quantum numbers l and m, which would relate to the angular structure of a distribution of galaxies, as seen from the origin. The $j_l(\rho)$ have their largest extrema at $\rho \approx l$. (These are also their first extrema. See Figure 1 for a plot of $j_l(\rho)^2$ for $l = 40$.) Therefore, if a distribution of galaxies has observed angular detail $1/l$ radians, the distribution is most likely located at a distance $r \gtrsim l/(\sqrt{\zeta}h_0) \approx 0.11\,h^{-1}l$ Mpc. Thus the characteristic physical size of the detail is approximately $r/l \gtrsim 1/(\sqrt{\zeta}h_0) \approx 0.11\,h^{-1}$ Mpc. This characteristic size is independent of l and is roughly the diameter of

a large spiral galaxy. This is a satisfying interpretation for the parameter ζ, especially since $\pi/(\sqrt{\zeta}h_0)$ is the node spacing in r.

The separated solution $j_l(\rho)Y_{lm}(\theta,\phi)U_n(\alpha v)$ describes a situation in which the galaxies are distributed in spherical shells centered about the origin. More general solutions may be built up as series in l, n, and m, with the stipulation that $\lambda_1 = 2\alpha^4/M$ and λ_2 be constant. ζ and n would continue to be related by equation (8).

Since $\zeta > 0$ and $n \geq 0$, we have maximum ζ and n given by

$$\zeta_{\max} = \lambda_2 - \alpha^2 \text{ and } n_{\max} < \frac{1}{2}\Big(\frac{\lambda_2}{\alpha^2} - 1\Big) . \tag{9}$$

The numbers used above imply $\zeta_{\max} \approx 0.0139\ \mathrm{s}^2\ \mathrm{km}^{-2}$ and $n_{\max} \approx 31$.

Equations (9) show that for fixed λ_2 and α, it is not possible to fit an arbitrary function of velocity with these solutions. Using Cartesian coordinates, one may easily show that it is likewise not possible to fit an arbitrary function of position, and we demonstrate this in the next section. Since $p = q^2$, these more general solutions would show interesting and complicated interference phenomena. The period of $j_l(\rho)$ depends only very weakly on l; therefore, terms in different values of l could easily reinforce each other.

5. GENERAL PERIODIC STRUCTURES IN GALAXY DISTRIBUTIONS

In the above section, we focused on the subject of redshift periodicities. One may, however, also ask how one might apply the MFI method to the broader problem of the distribution of galaxies in 6-dimensional phase space, with all three peculiar velocity components considered.

In this case it is convenient to use Cartesian coordinates, in which equation (5) is generalized to

$$\frac{1}{h_0^2}\nabla^2 q + \nabla_v^2 q - \Big(\frac{\lambda_1}{2}Mv^2 - \lambda_2\Big)q = 0 , \tag{10}$$

where ∇_v^2 is the Laplacian in peculiar velocity space, which we now refer to simply as "velocity space." The general solution to equation (10) is

$$q(\mathbf{x},\mathbf{v}) = \sum_{\mathbf{n}} \int d^3\mathbf{k}\, C(\mathbf{k},\mathbf{n}) e^{i\mathbf{k}\cdot\mathbf{x}} U_{n_x}(\alpha v_x) U_{n_y}(\alpha v_y) U_{n_z}(\alpha v_z) , \tag{11}$$

where α is defined above, $\mathbf{k} = (k_x, k_y, k_z)$, and $\mathbf{n} = (n_x, n_y, n_z)$. The discussion about Hermite functions in Section 3 implies that the harmonic oscillator wavefunctions, with non-negative integers n_x, n_y, and n_z, must be used.

Equations (10) and (11) show that $C(\mathbf{k},\mathbf{n})$ is arbitrary except for the reality condition $\overline{C(\mathbf{k},\mathbf{n})} = C(-\mathbf{k},\mathbf{n})$ and for the restriction

$$k^2 = h_0^2\{\lambda_2 - \alpha^2[2(n_x + n_y + n_z) + 3]\} , \tag{12}$$

where $k = |\mathbf{k}|$.

Since λ_2 and α are constants, this equation restricts $\mathbf{k}$ to discrete spherical shells in $\mathbf{k}$-space, and hence $C(\mathbf{k},\mathbf{n})$ contains delta–functions in k. Since the integers in equation (12) are non-negative, there is a maximum value for k given by $k_{\max} = h_0\sqrt{\lambda_2 - 3\alpha^2}$. This proves that one may not expand an arbitrary function of $\mathbf{x}$ in terms of the solutions to an MFI problem. The fact that one must use discrete shells in $\mathbf{k}$-space is an additional restriction.

This also means that there is a kind of classical uncertainty relation for a typical position variable x. Starting from the well known Fourier transform uncertainty relation $\delta x\,\delta k \approx 1$, one can substitute $\delta k \approx k_{\max}$ to get $h_0\sqrt{\lambda_2 - 3\alpha^2}\delta x \approx 1$.

This equation is hard to interpret. Note, however, that one may rewrite equation (12) as

$$k_{\max}^2 = k^2 + 2h_0^2\alpha^2(n_x + n_y + n_z) . \tag{13}$$

Setting $k \approx 0$ in equation (13) gives us a maximum value $n_{\max}$ for the sum of the n's. One finds $k_{\max}^2 \approx 2h_0^2\alpha^2 n_{\max}$. But the minimum velocity spacing is $\delta v_{\min} \approx 2/(\alpha\sqrt{n})$. It follows that $k_{\max}\delta v_{\min} \approx 2h_0$. Since $k_{\max}\delta x \approx 1$, we see that

$$\delta x \approx \frac{\delta v_{\min}}{2h_0} . \tag{14}$$

Thus the scale of the smallest possible detail in configuration space is equal to the scale of the minimum detail in velocity space divided by twice the Hubble constant. One may use equation (9) to derive the corresponding relation in spherical coordinates, which is $\delta r_{\rm min} \approx \delta v_{\rm min}/h_0$.

The fact that we have discrete shells in k-space means that $q(\mathbf{x}, \mathbf{v})$ consists (for each $\boldsymbol{k}$) of a superposition of waves of constant wave-length. These waves include both plane waves and waves of the form $j_0(\boldsymbol{kr}) = \sin(\boldsymbol{kr})/\boldsymbol{kr}$ with arbitrary centers. These different components could reinforce and cancel each other at different positions.

It is also interesting to consider interference effects in peculiar velocity space. Since equation (11) is the general solution to equation (10), we can without loss of generality orient the coordinate system in velocity space so that the v_z axis points toward the observer. This is independent of the orientation of the spatial coordinate axes.

Let us consider the case for which there is only a single value of $\boldsymbol{k}$ involved. The distribution of the radial peculiar velocity $v_{\rm pec} = v_z$ is then proportional to

$$\begin{aligned}\tilde{p}(v_{\rm pec}) &= \int\int dv_x dv_y \Big(\sum_{\mathbf{n}} C_{\mathbf{n}} U_{n_x}(\alpha v_x) U_{n_y}(\alpha v_y) U_{n_z}(\alpha v_{\rm pec})\Big)^2 \\ &= \sum_{\mathbf{n},\mathbf{n}'} C_{\mathbf{n}} C_{\mathbf{n}'} \int\int dv_x dv_y U_{n_x}(\alpha v_x) U_{n_y}(\alpha v_y) U_{n_z}(\alpha v_{\rm pec}) \\ &\quad \times U_{n'_x}(\alpha v_x) U_{n'_y}(\alpha v_y) U_{n'_z}(\alpha v_{\rm pec}) \end{aligned} \tag{15}$$

The functions U_n are orthogonal, and therefore the only surviving terms in this equation are for $n'_x = n_x$ and $n'_y = n_y$. Since we are dealing with only a single value of $\boldsymbol{k}$, equation (12) implies that $n'_z = n_z$. Equation (15) then reduces to

$$\tilde{p}(v_{\rm pec}) = \sum_{\mathbf{n}} C'^2_{\mathbf{n}} U^2_{n_z}(\alpha v_{\rm pec}) \ , \tag{16}$$

where $C'_{\mathbf{n}}$ contains the normalization factors of the U's. The different terms in this equation have different periods $2/(\alpha n_z^{1/2})$, but they could sum together to produce beat periods in a way similar to the identity $\sum_{m=1}^{M} \sin^2 a_m = M/2 - \sum_{m<m'} \cos(a_m + a_{m'}) \cos(a_m - a_{m'})/(M-1)$.

These effects should help to produce observable redshift periodicities at various scales. A more comprehensive analysis of possible periodicities in total redshift $v_{\rm obs}$ would be very complicated and is beyond the scope of this paper.

6. DISCUSSION

Minimum Fisher information is a principle of wide applicability in statistical physics, and it would be surprising if it did not have something to say about the distribution of galaxies in phase space. Thus one would expect equations (9) and (12) – (16) to be helpful in interpreting our observations of the positions and radial velocities of extragalactic objects.

This seems especially true of the small-scale redshift periodicities, with their intriguing multiples and sub-multiples of 72 km s^{-1}. There are also the much larger periodicities and voids with scales as large as 5000 km s^{-1}. See Broadhurst et al (1990) and da Costa (1991). MFI should be useful in interpreting these structures, although the principle should certainly be broadened to include gravitational interactions between galaxies. Its effects should then be apparent in numerical simulations as well as in observational data.

Note that MFI does not predict the actual values of the periodicities, which are set by the free parameters in the theory, as is the case with the temperature in the simple Boltzmann distribution problem in Section 2.

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REFERENCES

Abramowitz, M., & Stegun, I. A. 1972, Handbook of Mathematical Functions (New York: Dover Publications)

Bateman, H. 1953, Higher Transcendental Functions, Vol 2, ed. A. Erdélyi,

W. Magnus, F. Oberhettinger, & F. G. Tricomi (New York: McGraw-Hill)
Broadhurst, T. J., Ellis, R. S., Koo, D. C., & Szalay, A. S. 1990, Nature 343, 726
Cocke, W. J. 1985, ApJ 288, 22
Cocke, W. J., & Tifft, W. G. 1993, ApJ (submitted)
da Costa, L. N. 1991, in The Distribution of Matter in the Universe, ed. D. Gerbal & G. A. Mamon (Meudon: Obs Paris)
Frieden, B. R. 1990, Phys Rev A 41, 4265
Frieden, B. R. 1992, Physica A (Netherlands) 169, 123
Lebedev, N. N. 1972, Special Functions and their Applications (New York: Dover Publications)
Tifft, W. G. 1993, Proc 2nd International IEEE Conference on Plasma Astrophysics and Cosmology
Tifft, W. G. & Cocke, W. J. 1984, ApJ 287, 492

FIGURE CAPTION

1. The square of the spherical Bessel function of order $l = 40$. Galaxies described by such a distribution are concentrated around $\rho = \sqrt{\zeta} h_0 r \gtrsim l$.
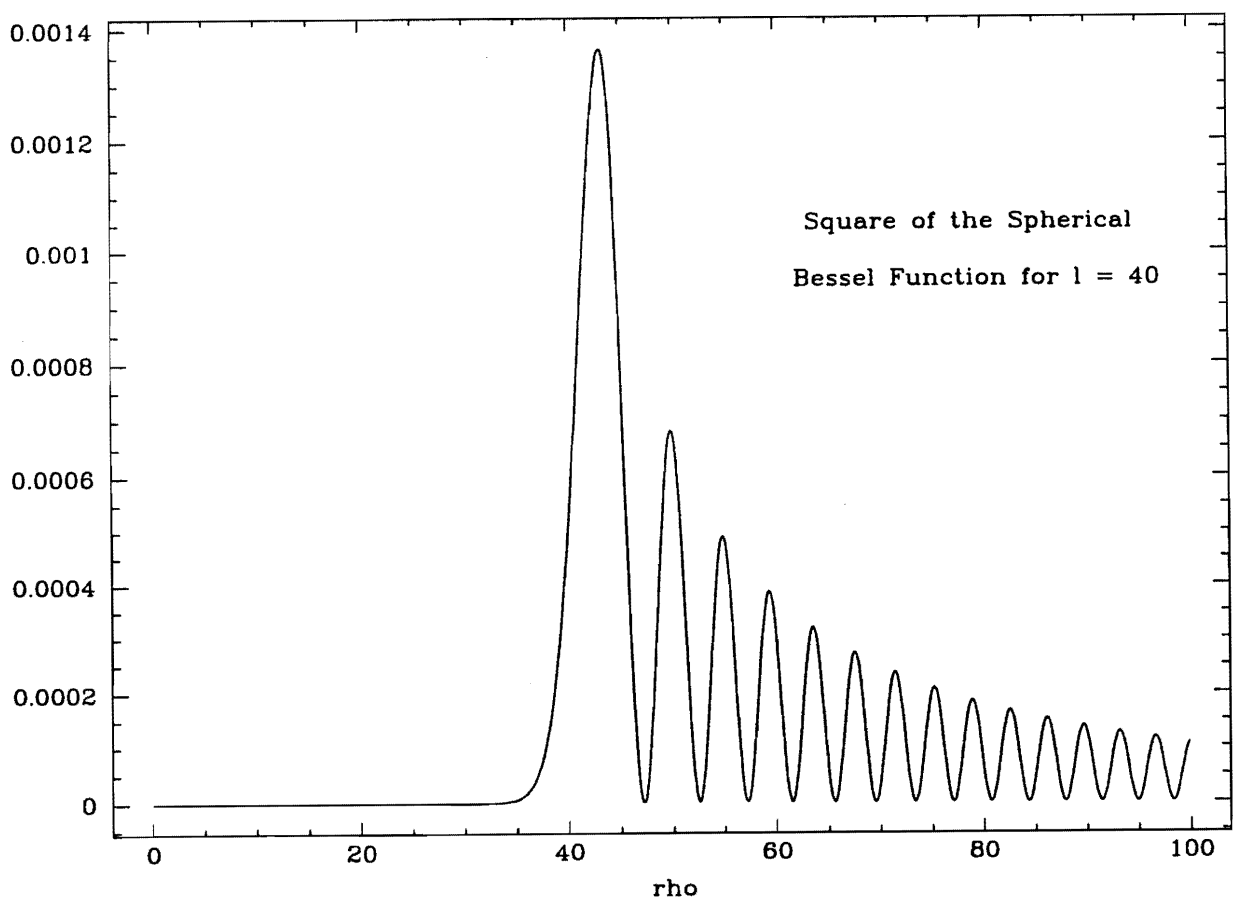
Figure 1 - Cocke