SSCL-Preprint-438

SSCL-Preprint-438
May 1993
Distribution Category: 414


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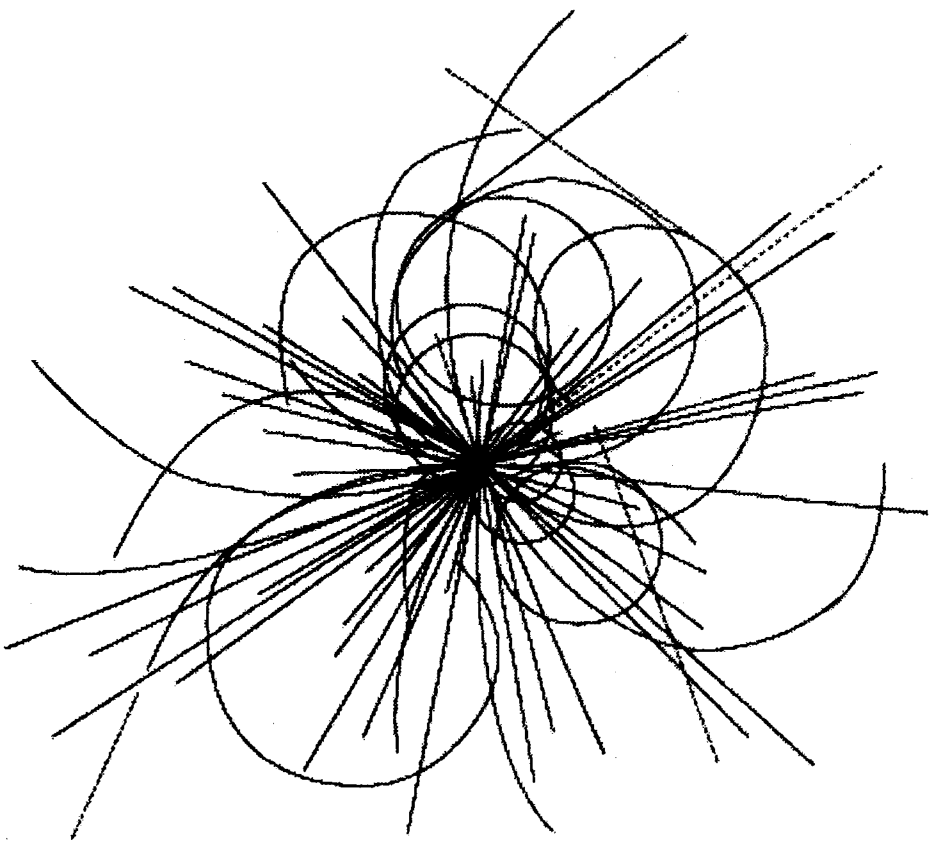

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# Amplitude Function Mismatch*

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May 1993

*Presented at the 1993 IEEE Particle Accelerator Conference on May 17–20, Washington, D.C.
†Operated by the Universities Research Association, Inc., for the U.S. Department of Energy under Contract No. DE-AC35-89ER40486.

# Amplitude Function Mismatch

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*Abstract*

We develop the general equation of motion of an amplitude function mismatch in an accelerator lattice and look at its solution for some interesting cases. For a free $\beta$-wave oscillation the amplitude of the mismatch is written in terms of the determinant of a single matrix made up of the difference between the new Courant-Snyder parameters and their ideal values. Using this result, once one calculates the mismatch of the amplitude function and its slope at one point in the lattice (at the end of a *nearly* matched insertion, for example), then the maximum mismatch downstream can be easily computed. The formalism is also used to describe emittance growth in a hadron synchrotron caused by amplitude function mismatches at injection.

While most of the content of this paper is not new to the accelerator physics community, we thought it would be useful to place this important, basic information all in one place. Besides the classic work of Courant and Snyder, our sources include other papers, internal reports, and numerous discussions with our colleagues. More details may be found in a related paper.[1]

## I. A STARTING POINT

The general solution for linear betatron oscillations in one transverse degree of freedom can be written as[2] $x(s) = A\sqrt{\beta(s)}\cos[\psi(s)+\delta]$ where $A$ and $\delta$ are constants given by the particle's initial conditions. The phase advance $\psi(s)$ and the amplitude function $\beta(s)$ satisfy the differential equations $\psi' = \frac{1}{\beta}$, $2\beta\beta'' - \beta'^2 + 4\beta^2 K = 4$, where $K = e(\partial B_y/\partial x)/p$, with $e$ = charge, $p$ = momentum, $\partial B_y/\partial x$ = magnetic field gradient, and $\beta' = d\beta/ds$, etc. When one considers the periodic solution of the amplitude function, the motion through a single repeat period can be described in terms of the Courant-Snyder parameters $\beta(s)$, $\alpha(s) \equiv -(d\beta(s)/ds)/2$, and $\gamma(s) \equiv (1+\alpha^2)/\beta$, using the matrix

$$\begin{pmatrix} \cos\psi_C + \alpha\sin\psi_C & \beta\sin\psi_C \\ -\gamma\sin\psi_C & \cos\psi_C - \alpha\sin\psi_C \end{pmatrix} \quad (1)$$

which operates on the state vector $X$, with $X = (x, x')^T$. Here, the phase advance is $\psi_C = 2\pi\nu = \int_{s_0}^{s_0+C} \frac{ds}{\beta(s)}$, where $C$ is the repeat distance of the hardware, which may be the circumference of the accelerator, and $\nu$ is the *tune* of the synchrotron.

The matrix of Equation 1 is often written in compact form as $M = I\cos\psi_C + J\sin\psi_C$ where

$$J \equiv \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}. \quad (2)$$

The amplitude function and its slope propagate through an accelerator section according to

$$J_2 = M(s_1 \to s_2) J_1 M(s_1 \to s_2)^{-1}, \quad (3)$$

where $J_1$ and $J_2$ contain Courant-Snyder parameters corresponding to points 1 and 2, and $M(s_1 \to s_2)$ is the transport matrix between these two points.

## II. PROPAGATION OF A THIN GRADIENT ERROR

We wish to see how the amplitude function downstream of a thin gradient error is altered. If $J_0(s_0)$ is the matrix of unperturbed Courant-Snyder parameters at the location of the error and $J_0(s)$ contains the unperturbed parameters at a point downstream, then, using Equation 3,

$$\Delta J(s) = M(s_0 \to s)\Delta J(s_0) M(s_0 \to s)^{-1}, \quad (4)$$

where

$$\Delta J(s) = J(s) - J_0(s) = \begin{pmatrix} \alpha - \alpha_0 & \beta - \beta_0 \\ -(\gamma - \gamma_0) & -(\alpha - \alpha_0) \end{pmatrix}; \quad (5)$$

$\beta$ is the new value of the amplitude function at $s$, $\beta_0$ is the unperturbed value, etc. Through a thin quad, $\Delta\alpha = q\beta_0$, $\Delta\beta = 0$, and $\Delta\gamma = 2\alpha q + \beta q^2$ and so

$$\frac{\Delta\beta(s)}{\beta_0(s)} = -(\beta_i q)\sin 2\psi_0(s-s_0) + \frac{1}{2}(\beta_i q)^2\left[1 - \cos 2\psi_0(s-s_0)\right] \quad (6)$$

where $\psi_0(s-s_0)$ is the unperturbed phase advance between points $s_0$ and $s$ and $\beta_i \equiv \beta_0(s_0)$. The amplitude function perturbation oscillates at twice the betatron frequency and for $(\beta_i q)$ sufficiently small, the perturbation describes simple harmonic motion. The change in $\alpha$ also propagates at

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twice the betatron frequency, it being given by

$$\begin{aligned}\Delta\alpha(s) &= \beta_i q\left[\cos 2\psi_0(s-s_0) - \alpha_0(s)\sin 2\psi_0(s-s_0)\right] \\ &\quad -\frac{1}{2}(\beta_i q)^2[\sin 2\psi_0(s-s_0) \\ &\quad -\alpha_0(s)(1-\cos 2\psi_0(s-s_0))] \ . \end{aligned} \tag{7}$$

Introducing this quad error also changes the phase advance across the lattice. The new phase advance $\psi(s-s_0)$ across this section may be calculated using $\sin\psi(s-s_0) = \bar{M}(s_0 \to s)_{12}/\sqrt{\beta_i\beta(s)}$ where $\bar{M}(s_0 \to s)_{12}$ is the (1,2) element of the new ring matrix and $\beta(s)$ is the new amplitude function at $s$. Using Equation 6, we obtain

$$\sin\psi(s-s_0) = [1-\beta_i q \sin 2\psi_0(s-s_0)$$

$$+(\beta_i q)^2 \sin^2\psi_0(s-s_0)]^{-1/2}\sin\psi_0(s-s_0) \ . \tag{8}$$

An explicit result for the change in the phase advance may be obtained perturbatively in orders of the quad error $q$ from the above exact expression. To second order in $q$, we find that the change $\Delta\psi \equiv \psi(s-s_0) - \psi_0(s-s_0)$ is

$$\Delta\psi = \beta_i q \sin^2\psi_0(s-s_0)$$

$$-(\beta_i q)^2 \sin 2\psi_0(s-s_0)\sin^2\psi_0(s-s_0) + O(q^3) \ . \tag{9}$$

To first order in $q$, at a point $\pi/2$ away from the location of the error, there is no change in the $\beta$ function while the change in phase advance is at its maximum value of $\beta_i q$.

## III. EQUATION OF MOTION OF $\beta$-WAVE

The equation of motion for an amplitude function mismatch is nonlinear when $s$ is taken as the independent variable. A more congenial equation can be developed by using the the reduced phase $\phi \equiv \psi/\nu$ as the independent variable. For betatron oscillations the Floquet transformation, where the other variable is $\zeta = x/\sqrt{\beta}$, produces the equation of motion $\frac{d^2\zeta}{d\phi^2} + \nu^2\zeta = 0$ which is pure simple harmonic motion with frequency (tune) $\nu$. For the amplitude function mismatch, we need to define the reduced phase in terms of the unperturbed functions. That is, let $\phi \equiv \psi_0/\nu_0$, where $d\psi_0/ds = 1/\beta_0$, and $\nu_0$ is the unperturbed tune. The equation of motion for $[\beta(\phi) - \beta_0(\phi)]/\beta_0(\phi) \equiv \Delta\beta/\beta_0$ in the absence of gradient errors is then

$$\begin{aligned}\frac{d^2}{d\phi^2}\frac{\Delta\beta}{\beta_0} + (2\nu_0)^2\frac{\Delta\beta}{\beta_0} &= -2\nu_0^2 \det\Delta J \\ &= 2\nu_0^2[\Delta\alpha^2 - \Delta\beta\Delta\gamma] \end{aligned} \tag{10}$$

where $\Delta\alpha = \alpha(\phi) - \alpha_0(\phi)$, etc. The quantity $\det\Delta J$ is an invariant in portions of the lattice without gradient perturbations as can be seen with the aid of Equation 3.

So, the free amplitude function distortion oscillates with twice the betatron tune and with a constant offset given by the determinant of the $\Delta J$ matrix at any point. This offset must be there since $\beta > 0$ and hence $\Delta\beta/\beta$ must always be greater than $-1$.

Rewritten in terms of the Courant-Snyder parameters,

$$\det\Delta J = -\frac{\left(\frac{\Delta\beta}{\beta_0}\right)^2 + \left(\Delta\alpha - \alpha_0\frac{\Delta\beta}{\beta_0}\right)^2}{1+\Delta\beta/\beta_0} < 0. \tag{11}$$

Thus, $|\det\Delta J|^{1/2}$ can be interpreted as the amplitude of the $\beta$ mismatch for small perturbations.

The solution to Equation 10 is just simple harmonic motion with a constant term added:

$$\frac{\Delta\beta}{\beta_0}(\phi) = A\cos 2\nu_0\phi + B\sin 2\nu_0\phi + \frac{1}{2}|\det\Delta J|. \tag{12}$$

The constants $A$ and $B$ are found from the initial conditions:

$$A = \frac{\Delta\beta}{\beta_0}(0) - \frac{1}{2}|\det\Delta J|, \tag{13}$$

$$B = \alpha_0\frac{\Delta\beta}{\beta_0}(0) - \Delta\alpha(0). \tag{14}$$

Thus, the maximum value of $\Delta\beta/\beta_0$ downstream of our starting point $\phi = 0$ is given by

$$\begin{aligned}\left(\frac{\Delta\beta}{\beta_0}\right)_{max} &= \sqrt{A^2+B^2} + \frac{1}{2}|\det\Delta J| \\ &\approx \frac{|\det\Delta J|}{2} + \sqrt{|\det\Delta J| + \left(\frac{|\det\Delta J|}{2}\right)^2} \end{aligned} \tag{15}$$

where use has been made of Equation 11. The maxima occur at phases where

$$\tan 2\nu_0\phi = \left(\frac{\alpha_0\frac{\Delta\beta}{\beta_0} - \Delta\alpha}{\frac{\Delta\beta}{\beta_0} - |\det\Delta J|/2}\right)_0 . \tag{16}$$

The usefulness of the above result is, of course, that once one calculates the mismatch of the amplitude function and its slope at one point in the lattice (at the end of a *nearly* matched insertion, for example), then the maximum mismatch downstream can be computed immediately.

If we look once again at the perturbation downstream of a thin quadrupole error, we see that just after the quad,

$$\det\Delta J = \begin{vmatrix} q\beta_i & 0 \\ -\Delta\gamma & -q\beta_i \end{vmatrix} = -(q\beta_i)^2 \tag{17}$$

where $\beta_i = \beta_0$ at the location of the quadrupole. Then,

$$\left(\frac{\Delta\beta}{\beta_0}\right)_{max} = q\beta_i\sqrt{1+(q\beta_i)^2/4} + \frac{1}{2}(q\beta_i)^2 \tag{18}$$

$$\approx q\beta_i = \sqrt{|\det\Delta J|} \tag{19}$$

where the last line is valid for small perturbations.

## IV. GENERAL EQUATION OF MOTION

To include the driving terms due to gradient errors in the equation of motion for $\Delta\beta/\beta_0$, we let $\beta_0$ satisfy the differential equation $K\beta_0 = \gamma_0 + \alpha_0'$, and let $\beta$ satisfy $(K + k)\beta = \gamma + \alpha'$, where $\beta = \beta_0 + \Delta\beta$, etc. Then, the relative $\beta$ error satisfies

$$\frac{d^2}{d\phi^2}\frac{\Delta\beta}{\beta_0}(\phi) + (2\nu_0)^2\frac{\Delta\beta}{\beta_0}(\phi)$$
$$= -2\nu_0^2\left[\beta_0^2(\phi)\,k(\phi)\left(1 + \frac{\Delta\beta}{\beta_0}(\phi)\right) + \det\Delta J(\phi)\right]. \quad (20)$$

Here, in general, $\det\Delta J(\phi)$ is not invariant as it is altered by gradient perturbations:

$$\frac{d}{d\phi}\det\Delta J(\phi) = \beta_0^2 k\frac{d}{d\phi}\frac{\Delta\beta}{\beta_0} \quad (21)$$

For small perturbations we can drop quantities which are second order in the small quantities, e.g. $k\Delta\beta$. This reduces Equation 20 to

$$\frac{d^2}{d\phi^2}\frac{\Delta\beta}{\beta_0}(\phi) + (2\nu_0)^2\frac{\Delta\beta}{\beta_0}(\phi) = -2\nu_0^2\,\beta_0^2\,k(\phi) \quad (22)$$

as appears in Courant and Snyder.[2]

Noting that $\Delta\alpha - \alpha_0(\Delta\beta/\beta_0) = -(1/2\nu_0)d(\Delta\beta/\beta_0)/d\phi$, one can easily exhibit Equation 20 entirely in terms of $\Delta\beta/\beta_0$ and its derivatives with respect to $\phi$. Differentiating this resulting equation one obtains a *linear* differential equation for $\Delta\beta/\beta_0$:

$$\frac{d^3}{d\phi^3}\frac{\Delta\beta}{\beta_0} + (2\nu_0)^2(1 + \beta_0^2 k)\frac{d}{d\phi}\frac{\Delta\beta}{\beta_0}$$
$$+ 2\nu_0^2\frac{d}{d\phi}[\beta_0^2 k](1 + \frac{\Delta\beta}{\beta_0}) = 0. \quad (23)$$

## V. INJECTION MISMATCH

It is also of interest to look at the effects of mismatches of amplitude functions upon entrance to an accelerator. The treatment below may be followed in more detail in [3] and [4]. A beam which is described by Courant-Snyder parameters that are not the periodic parameters of the accelerator into which it is injected will tend to filament due to nonlinearities and hence have its emittance increased. Suppose $\beta$ and $\alpha$ are the Courant-Snyder parameters as delivered by the beamline to a particular point in an accelerator, and $\beta_o$, $\alpha_o$ are the periodic lattice functions of the ring at that point. A particle with trajectory $(x, x')$ can be viewed in the $(x, \beta x' + \alpha x) \equiv (x, \eta)$ phase space corresponding to the beamline functions, or in the $(x, \beta_0 x' + \alpha_0 x) \equiv (x, \eta_0)$ phase space corresponding to the lattice functions of the ring. If the phase space motion lies on a circle in the beamline view, then the phase space motion will lie on an ellipse in the ring view. The equation of the ellipse in the "ring" system will be

$$\frac{(1 + \Delta\alpha_r^2)}{\beta_r}x^2 + 2\Delta\alpha_r\,x\eta_0 + \beta_r\eta_0^2 = \beta_0 A^2. \quad (24)$$

where $\beta_r \equiv \beta/\beta_0$ and $\Delta\alpha_r \equiv \alpha - \alpha_0(\beta/\beta_0)$.

If the phase space coordinate system were rotated so that the cross-term in the equation of the ellipse were eliminated, the ellipse would have the form $x_e^2/b_r + b_r\eta_{oe}^2 = \beta_0 A^2$ where $b_r \equiv F + \sqrt{F^2 - 1}$ and $F$ is given by

$$F \equiv \frac{1}{2}\left[\beta_0\gamma + \gamma_0\beta - 2\alpha_0\alpha\right]. \quad (25)$$

Note that if $\Delta\alpha_r = 0$, then $b_r = \beta_r$.

There is a physical significance to the quantity $b_r$; it is the ratio of the areas of two circumscribed ellipses which have shapes and orientations given by the two sets of Courant-Snyder parameters found in the matrices $J$ and $J_0$. This might suggest that a beam contained within the smaller ellipse upon injection into the synchrotron (whose periodic functions give ellipses similar to the larger one) will have its emittance increased by a factor $b_r$. However, this would be an over-estimate of the increase of the average of the emittances of all the particles.

If in the beamline view the new phase space trajectory is $x^2 + \eta^2 = b_r R^2$, then in the synchrotron view, the equation of the ellipse would be $\frac{x^2}{b_r R^2} + \frac{\eta_0^2}{R^2/b_r} = 1$. A particle with initial phase space coordinates $x_i$ and $\eta_{oi}$ will commence describing a circular trajectory of radius $a$ in phase space upon subsequent revolutions about the ring. The equilibrium distribution will have variance in the $x$ coordinate

$$\sigma^2 = \langle x^2\rangle = \frac{\langle a^2\rangle}{2} = \frac{b_r^2 + 1}{2b_r}\sigma_0^2 = F\,\sigma_0^2, \quad (26)$$

where $\sigma_0^2$ is the variance in the absence of a mismatch. This expression can be rewritten in terms of $\det\Delta J$ which we found in Section III.:

$$\frac{\sigma^2}{\sigma_0^2} = 1 + \frac{1}{2}\left|\det(\Delta J)\right|. \quad (27)$$

For the case where the slope of the amplitude function is matched and equal to zero, we have

$$\frac{\sigma^2}{\sigma_0^2} = 1 + \frac{1}{2}\left(\frac{\Delta\beta/\beta_0}{\sqrt{1 + \Delta\beta/\beta_0}}\right)^2. \quad (28)$$

This says that a 20% $\beta$ mismatch at injection, for example, would cause only a 2% increase in the rms emittance.

## VI. REFERENCES


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