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G. Stupakov
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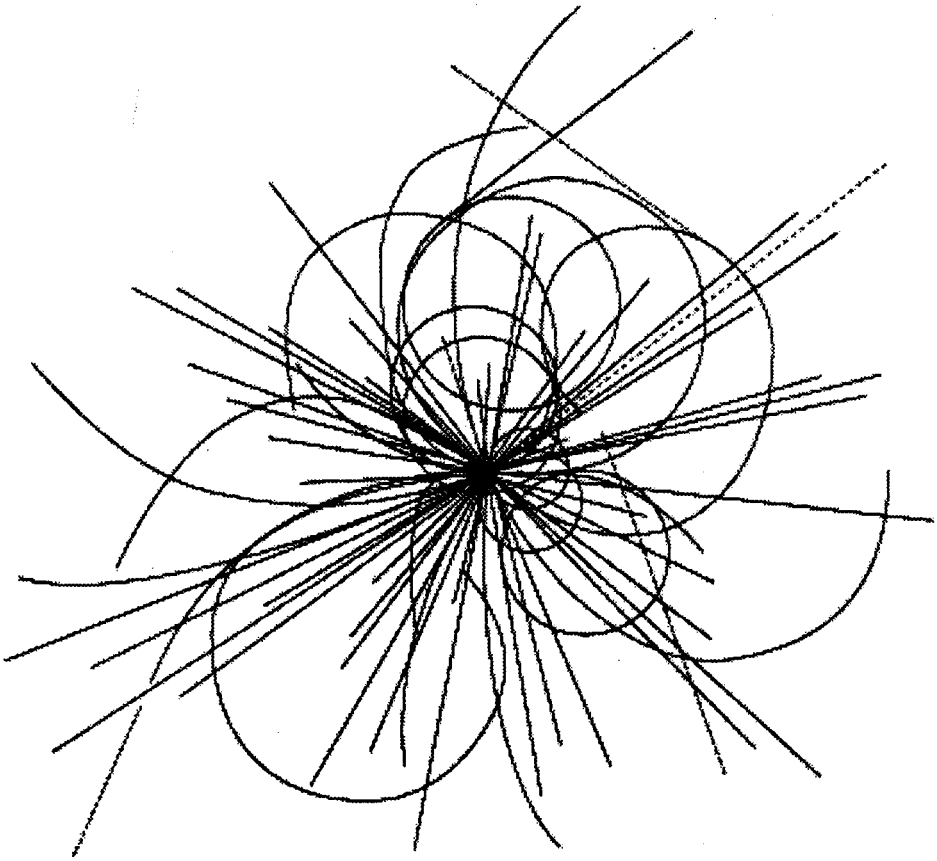

Superconducting Super Collider Laboratory

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Echo Effect in Accelerators*

G. Stupakov and K. Kauffmann

Superconducting Super Collider Laboratory†
2550 Beckleymeade Ave.
Dallas, TX 75237

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Echo Effect in Accelerators

G. V. Stupakov and S. K. Kauffmann
Superconducting Super Collider Laboratory*
2550 Beckleymeade Ave., Dallas, TX 75237

I. INTRODUCTION

The echo effect has been known for many years in different fields of physics. Examples are the spin echo in solids [1], photon echo in solids and gases [2], plasma wave echo [3], and the echo in a liquid with gas bubbles [4]. The media that exhibit the echo characteristically consist of (or contain in them) an ensemble of oscillators with different eigenfrequencies and negligibly small dissipation. An initial perturbation applied to such a medium excites oscillations (or waves) that slowly damp due to dephasing. This kind of damping differs essentially from that which would result from a dissipative mechanism; in particular, dephasing does not increase the entropy of the medium. It has the remarkable feature that even after the oscillations are completely damped out, the system keeps a "recollection" about them, and a special kind of a disturbance applied to the medium can transiently restore the oscillations in the form of an echo signal.

The betatron echo in a hadron accelerator [5,6] can be observed in a situation where the beam is injected off-center into the ring at time $n = 0$ (n is the time measured in the number of turns), causing its centroid to undergo betatron oscillations. After these oscillations have completely damped out due to beam decoherence, the beam is excited by a quadrupole kick at time $n = n_1$. This kick does not produce any visible displacement of the beam at that time, but it turns out that close to time $n = 2n_1$ the beam centroid undergoes transient betatron oscillations with an amplitude which is a fraction of the initial beam offset.

A manifestation of echo effect is illustrated by Figures 1 and 2. Figure 1 shows the amplitude of betatron oscillation of an offset beam that damps to extinction due to decoherence. If, however, a quadrupole kick is performed at n = 20, one observes an echo that recoheres back (and then decoheres again) with the maximum around n = 40, as shown in Figure 2.

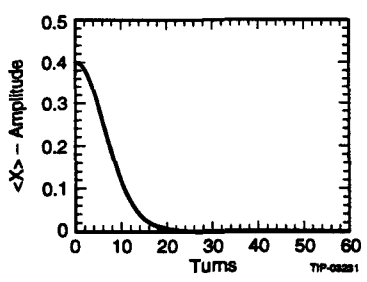

Figure 1. Damping of amplitude of the betatron oscillations of an offset Gaussian beam.

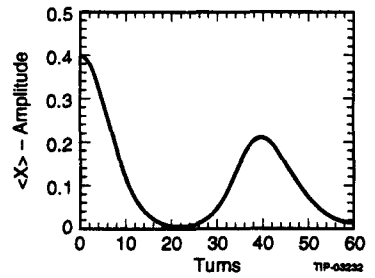

Figure 2. Echo response of the betatron amplitude after a quadrupole kick at $n = 20$.

II. THEORY OF ECHO

To describe one-degree-of-freedom transverse motion of a beam particle in an accelerator ring we choose to work with the normalized phase space coordinates

$$x = \frac{X}{\sqrt{\beta}}, \quad p = \beta \frac{d}{s} \frac{X}{\sqrt{\beta}}, \tag{1}$$

where X is the particle deviation with respect to the closed orbit, β is the beta function and s is the path along the orbit. We also define the vector $\mathbf{z}$,

$$\mathbf{z} = \begin{pmatrix} x \\ p \end{pmatrix}. \tag{2}$$

and the action J and the angle ϕ according to

$$J = \tfrac{1}{2}|\mathbf{z}|^2 = \tfrac{1}{2}\left(p^2 + x^2\right), \quad \phi = -\arctan\tfrac{p}{x}. \tag{2.1}$$

According to Courant-Snyder theory, in a perfectly linear accelerator, particles are simply rotated clockwise in this normalized phase space at a fixed radius through angle $2\pi\nu$ on each turn, where ν is the accelerator tune. Thus,

$$\mathbf{z}_n = R_n(\nu)\mathbf{z}_{n=0}, \tag{3}$$

where the subscript n indexes the turn number, and $R_n(\nu)$ is represented as the clockwise rotation matrix,

$$R_n(\nu) = \begin{pmatrix} \cos 2\pi n\nu & \sin 2\pi n\nu \\ -\sin 2\pi n\nu & \cos 2\pi n\nu \end{pmatrix}. \tag{4}$$

Offset particle beams decohere, if the tune itself depends on the particle amplitude $|\mathbf{z}|$, due to systematic non-linearities present in the accelerator. For a monochromatic beam, the presence of systematic sextupole or octapole non-linearities generates a quadratic dependence of tune on amplitude $|\mathbf{z}|$, i.e.,

$$\nu(|\mathbf{z}|) = \nu_0 + \Delta\nu \frac{|\mathbf{z}|^2}{\sigma^2}, \tag{5}$$

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where ν_0 is the nominal tune, $\Delta\nu$ has the interpretation of a tune spread and σ is the rms beam radius. In realistic situations, $\Delta\nu \ll \nu_0$.

As stated previously, the beam is assumed to be injected off center into the ring at $n = 0$. Let $\psi(\mathbf{z}) \equiv \psi(x, p)$ denote its initial particle distribution function. After the injection, the beam experiences free betatron oscillations and at $n = n_1$, just before the quadrupole kick, the vector $\mathbf{z}_{n=n_1-\varepsilon}$ can be obtained by the linear transformation of the initial $\mathbf{z}$,

$$\mathbf{z}_{n=n_1-\varepsilon} = R_{n_1}(\nu_1)\mathbf{z}, \tag{6}$$

where $\nu_1 = \nu(|\mathbf{z}|)$ is given by Eq. (5) with $\mathbf{z}$ being given by the initial values of the particle's coordinate and momentum.

The quadrupole kick produces a transformation of $\mathbf{z}$ that is characterized by the following matrix:

$$Q = \begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix}, \tag{7}$$

where q is equal to the ratio of β at the quad location to the focal length of the quad. After the quadrupole kick, betatron oscillations, proceed with a different frequency, because the kick changes the amplitude $|\mathbf{z}|$ and therefore the tune. Before writing down the matrix that describes oscillations after the kick, we have to express the new value of $|\mathbf{z}_{n=n_1+\varepsilon}|$ just after the quad kick in terms of the old one $|\mathbf{z}|$. Straightforward calculations yield,

$$|\mathbf{z}_1|^2 \equiv |\mathbf{z}_{n=n_1+\varepsilon}|^2 = |QR_{n_1}(\nu_1)\mathbf{z}|^2 = \\ = |\mathbf{z}|^2\left(1 + q\sin 2(\phi_1 - \theta) + q^2\cos^2(\phi_1 - \theta)\right), \tag{8}$$

where θ is the polar angle in the phase space, $\theta = \arctan(p/x)$, and

$$\phi_1 = 2\pi n_1 \nu_1. \tag{9}$$

Now, free oscillations after the quadrupole kick, $n > n_1$, generate a transformation given by $R_{n-n_1}(\nu_2)$, with the tune,

$$\nu_2 = \nu_0 + \Delta\nu\frac{|\mathbf{z}_1|^2}{d^2}, \tag{10}$$

so that a complete transformation, casting the initial $\mathbf{z}$ into the final $\mathbf{z}_n$ $(n > n_1)$, is the product of the three matrices,

$$\mathbf{z}_n = R_{n-n_1}(\nu_2)QR_{n_1}(\nu_1)\mathbf{z} = \\ = \begin{pmatrix} \cos(\phi_1+\phi_2) & \sin(\phi_1+\phi_2) \\ -\sin(\phi_1+\phi_2) & \cos(\phi_1+\phi_2) \end{pmatrix}\begin{pmatrix} x \\ p \end{pmatrix} - \\ -q\begin{pmatrix} \cos\phi_1\sin\phi_2 & \sin\phi_1\sin\phi_2 \\ \cos\phi_1\cos\phi_2 & \sin\phi_1\cos\phi_2 \end{pmatrix}\begin{pmatrix} x \\ p \end{pmatrix}, \tag{11}$$

where,

$$\phi_2 = 2\pi(n - n_1)\nu_2. \tag{12}$$

Thus, after n turns $(n > n_1)$, the averaged displacement of the beam is,

$$\bar{\mathbf{z}}_n = \iint \mathbf{z}_n \psi(\mathbf{z}) d^2\mathbf{z} = \\ = \iint dxdp\,\psi(x,p)x\begin{pmatrix} \cos(\phi_1+\phi_2) & -q\cos\phi_1\sin\phi_2 \\ -\sin(\phi_1+\phi_2) & -q\cos\phi_1\cos\phi_2 \end{pmatrix} + \\ + \iint dxdp\,\psi(x,p)p\begin{pmatrix} \sin(\phi_1+\phi_2) & -q\sin\phi_1\sin\phi_2 \\ \cos(\phi_1+\phi_2) & -q\sin\phi_1\cos\phi_2 \end{pmatrix}. \tag{13}$$

Based on this formula, further analysis developed in Reference 6 shows that an echo signal appears around the time $n = 2n_1$ with an amplitude that depends on the initial beam offset a, the strength of the quadrupole kick q and time n_1 of this kick. Moreover, in principle, multiple echoes with diminishing amplitudes can be observed at times equal to even multiples of n_1.

A simplified perturbation theory of the echo, based on a slightly different approach, can be found in Reference 5.

Using Eq. (13), we performed the integration for a Gaussian distribution function,

$$\psi(\mathbf{z}) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{|\mathbf{z} - a\hat{\mathbf{x}}|^2}{2\sigma^2}\right), \tag{14}$$

where $\hat{\mathbf{x}} = (1,0)$ is the unit vector in the x-direction, in which the initial Gaussian beam has been offset by a. The following values of parameters have been chosen, $\nu_0 = 0.285$, $\Delta\nu = 2.18\cdot10^{-3}$, $a/\sigma = 5.84$ and q=–0.16. The result is shown in Figure 2.

III. PHASE SPACE PORTRAITS AND THE OPTIMAL KICK

A deeper insight into the physical nature of the echo can be obtained by examining successive phase space portraits of the distribution of particles. We performed computer simulations by tracking 32,000 particles from an initial Gaussian distribution, each of which was advanced in time in accordance with the equations of the previous section. The results are shown in Figure 3.

An initially offset Gaussian beam is displayed in Figure 3a. It decoheres into a spiral-like structure (Figure 3b) until, at n=20, a quadrupole kick is applied to the beam. This kick produces elliptical flattening of decoherence spiral, as shown in Figure 3c. The subsequent evolution of the beam (see Figures 3d and 3e) shows development of sharp tips which interrupt the smooth shape of the spiral. These tips come into confluence near where our original off-center beam was launched in the first instance, as is seen in Figure 3f. This tends to occur at approximately $2n_1$ turns, where n_1 is the number of turns from the launch of the offset beam to the quad kick. This confluence of the phase reversal tips is what causes the "echo" of the original beam offset.

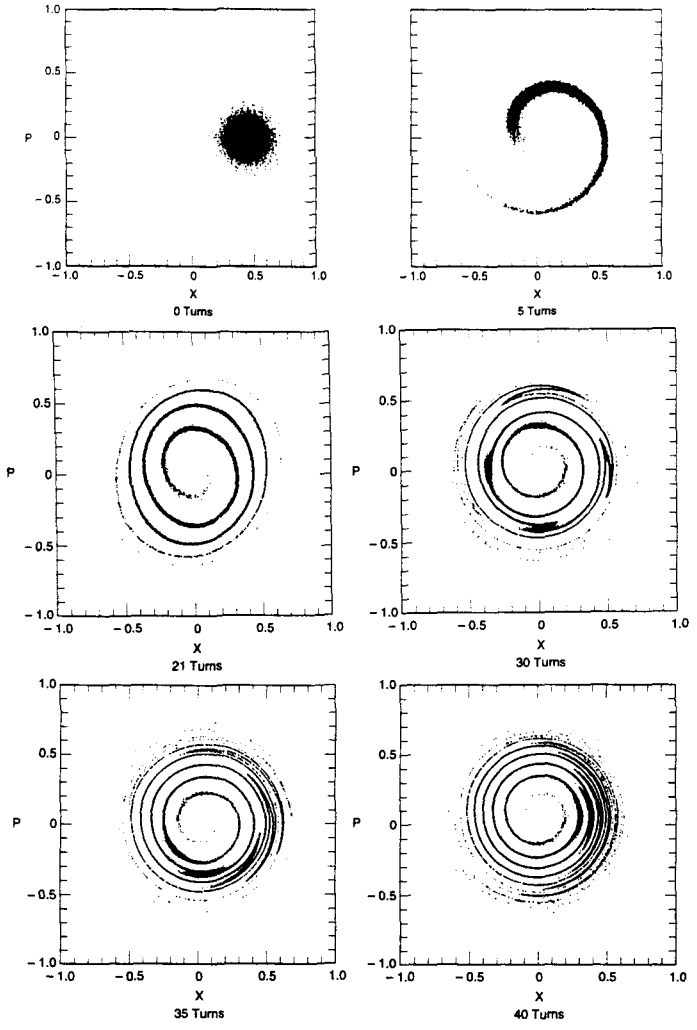

Figure 3. Phase portraits of an evolving Gaussian beam, showing the echo effect.

An important question, concerning the echo, is what values of parameters generate the maximal echo. Based on Eq. (13), a perturbation theory can be developed, assuming that the initial offset is small, $a \ll \sigma$, and the kick is weak, $q \ll 1$. It predicts the following form of the echo signal,

$$\left|\bar{z}_n\right| = aq\left|\frac{4\pi\Delta\nu(n-n_1)+\frac{1}{2}\left[1+4\pi i\Delta\nu(n-2n_1)\right]}{\left\{\left[1+4\pi i\Delta\nu(n-2n_1)\right]^2+\left[4\pi q\Delta\nu(n-n_1)\right]^2\right\}^{3/2}}\right| \tag{15}$$

(this formula is valid in the vicinity of the echo peak, $n = 2n_1$). A simple investigation shows that the maximum echo is attained for

$$q = 0.056(\Delta\nu n_1)^{-1} \tag{16}$$

and is equal to $\left|\bar{z}_n\right|_{max} = 0.38a$. As shown in Figure 2, for a relatively large initial offset (for which Eq. (15) is not applicable), more exact model calculations show an even larger echo, which can reach about 50% of the initial displacement.

IV. SUMMARY

The transverse echo effect is a consequence of reversibility of the particle motion in an accelerator. Investigation of the echo requires special hardware in the form of a pulsed quadrupole, that has to be able to produce a kick, having duration less than the revolution period.

Any dissipative mechanism that breaks the reversibility (such as synchrotron radiation, intrabeam scattering or collisions with the residual gas) will attenuate the echo, eventually destroying it completely. A more detailed theory should account for these additional effects. On the other hand, sensitivity of the echo to these kinds of effects could possibly be used as a diagnostic tool.

Note that the echo which we are discussing should not be confused with the recoherence due to machine chromaticity and RF-induced synchrotron motion studied in Reference 7.

The result of the present paper can be also applied to longitudinal dynamics to demonstrate the existence of an analogous echo in synchrotron oscillations [8].

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