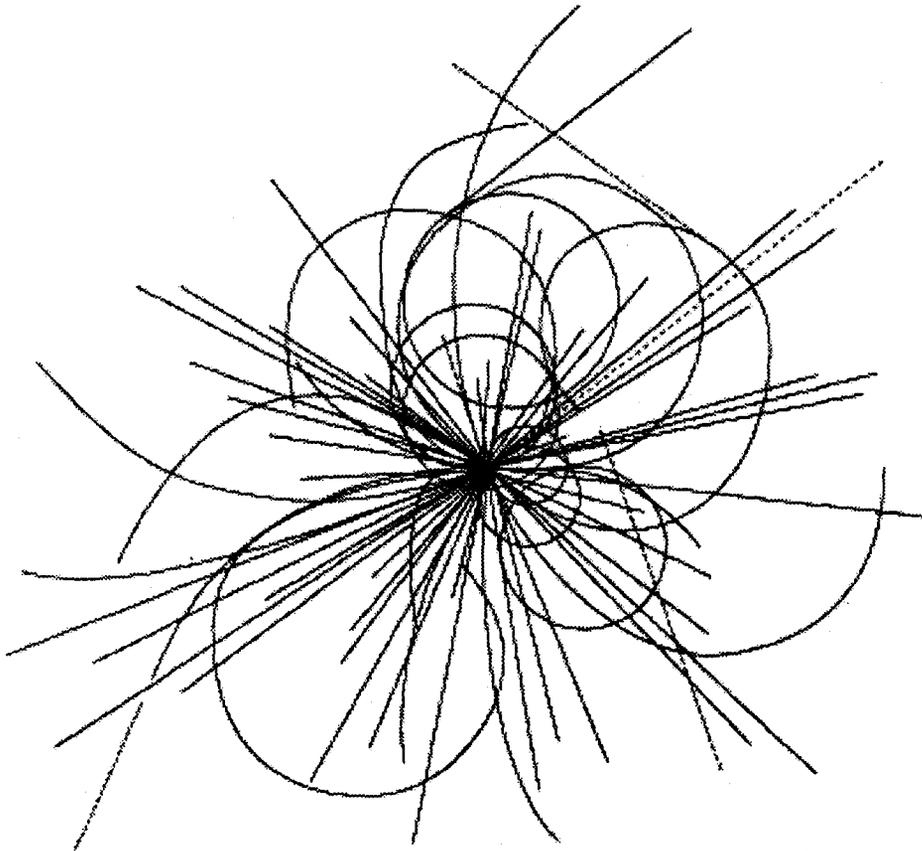


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**Superconducting Super Collider
Laboratory**

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ABSTRACT

Strictly one-dimensional theories of particle loss due to longitudinal diffusion model the loss by an absorbing boundary condition at the separatrix of the underlying unperturbed motion. Particle loss always occurs at a physical aperture and the loss is always coupled to the betatron motion. A theory of particle loss which includes the effect of betatron motion is presented. Results are compared with Monte Carlo simulations.

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It has long been understood that noise in the rf systems of storage rings leads to a slow increase of the longitudinal emittance of the stored bunches.¹⁻³ Especially, in the design of new hadron storage rings, intrinsic sources of noise must be identified and investigated to insure that they do not represent sources of unacceptable loss of beam lifetime.⁴⁻⁵ Conversely, the issue of controlled injection of noise into the rf system of these machines for the purpose of extracting beam has recently become of considerable interest.⁶⁻⁷ Here the idea is to “diffuse” particles in momentum until due to dispersion they reach a physical extraction septum.

Previous treatments of the loss process have been strictly one-dimensional, taking no account of the betatron motion. The losses are described in terms of longitudinal action and are assumed to occur at the separatrix of the longitudinal motion. This does not correspond to a real aperture stop and the quantitative results from these theories exhibit significant discrepancies when compared, for example, to Monte Carlo tracking studies.⁷ Knowledge of these loss rates is essential in the design of high-luminosity hadron colliders. In this paper, we obtain a statistical model of the transverse x – coordinates of an ensemble of particles in a bunch, calculate the loss at an x – septum and find excellent agreement with the tracking results.

It is conventional⁸ to describe the transverse motion in asynchrotron as a superposition of a closed orbit, x_{co} , and a (betatron) oscillation, x_β . The closed orbit, as the name implies, is periodic with the machine period; the betatron motion, in general, is not periodic. At a fixed lattice position, the closed orbit of a synchronous particle is the machine axis while an off-momentum particle executes an oscillation about the closed orbit position of the synchronous particle. This is the synchrotron oscillation, of frequency Ω for small amplitudes. The motion is on an energy oval in the longitudinal phase space of a simple pendulum. The position of the closed orbit is related to the relative momentum deviation, δp by $x_{co} = \delta p D$. Here, D is the dispersion which is a lattice function. Under certain assumptions, the effect of the noise can be described as a diffusion process in which there is a slow increase in amplitude of the synchrotron oscillation on a time scale long compared to a synchrotron period. Consequently, there is an increase in the closed orbit amplitude of these off-momentum particles. The position $x = x_{co} + x_\beta$ determines when a particle hits the stop. This could be an extraction septum, an array of collimators or even a “dynamical” aperture which results from nonlinearities in the system.

The theory⁹⁻¹¹ of the longitudinal dynamics in a noisy rf system leads to a description of the evolution of an ensemble by a diffusion in the action, J which is a constant of the unperturbed motion. The time scales of the diffusion in action, the synchrotron period and the betatron oscillation period are disparate with $t_d \gg t_s \gg t_\beta$. Thus the collimation process sweeps a shell in the transverse phase space ($A, A + dA$), where A characterizes a Courant-Snyder invariant, and a shell ($J, J + dJ$) in the longitudinal phase space in a time t , $t_s < t < t_d$, (see Fig. 1). The maximum betatron displacement is related to A by $|x_\beta|_{max} = \sqrt{\beta} A$ where β is the usual betatron function.⁸

In the presence of an aperture stop, a particle slowly diffuses toward the periphery of the beam pipe until it strikes the stop. The horizontal position of the stop is x_c and we assume the “image” of the stop in momentum, x_c/D is inside the bucket. This is appropriate for both the proposed superslow extraction⁷ or momentum scrapers.¹² The time it takes until the particle is lost, or conversely, the loss rate, depends on both the closed orbit position (equivalently, action) and the betatron displacement.

Consider particles which, at a given time, t , have not yet reached the stop. Referring to Fig. 2, we see that the protons which are inside the stop lie in the domain \mathcal{S} in J, x_β space.

The entire space is the quadrant $\mathcal{Q} : J, x_\beta \geq 0$. We have simply written x_β for $|x_\beta|_{max}$ in order to keep the notation from becoming too cumbersome. The domain \mathcal{S} is bounded by segments of the coordinate axes and the curve $\partial\mathcal{S} : \{x_\beta, J = J_b(x_\beta) | 0 \leq x_\beta \leq x_c\}$; $J_b(x_\beta)$ is the action for a particle on an energy surface, $k \equiv (\delta p)_{max} / \delta p_s$, defined by $k = k_b(x_\beta) \equiv (x_c - x_\beta) / (D\delta p_s)$. Here δp_s is the relative bucket half-height. For the pendulum, $k_b(x_\beta)$ can be found in terms of elliptic integrals. It might be expected that for these particles, the two random variables, x_β and J might be statistically independent for all time because initially they are. That this is not so is the crucial observation in the construction of the theory. However, the calculation of the joint distribution is deceptively simple.

Consider the joint density $p(J, x_\beta)$ and the action density conditioned on x_β , $p(J|x_\beta)$. Then,

$$p(J, x_\beta, t) = p(J, t|x_\beta) p(x_\beta) \quad (1)$$

where $p(x_\beta)$ is the marginal density of x_β . These must be probability densities and must be defined throughout \mathcal{Q} . In the domain \mathcal{S} , the protons have a marginal x_β - distribution which is just the initial distribution; they are Rayleigh distributed with parameter $\sqrt{\epsilon\beta}$, where ϵ is the emittance. (The transverse phase space variables are bi-Gaussian.) We must now find the conditional density, $p(J, t|x_\beta)$. This satisfies the same diffusion equation considered in earlier work:^{7,9-11}

$$\frac{\partial}{\partial t} p(J, t|x_\beta) = \frac{\partial}{\partial J} \left[\mathbf{D}(J) \frac{\partial}{\partial J} p(J, t|x_\beta) \right], \quad (2a)$$

but with boundary condition:

$$p(J_b, t|x_\beta) = 0. \quad (2b)$$

The action dependent diffusion coefficient is $\mathbf{D}(J)$. The absorbing boundary now is dependent on x_β and is just the curve $\partial\mathcal{S}$ defined earlier. In $\mathcal{Q} - \mathcal{S}$, the joint density is a delta function. Furthermore, consistency requires that the identity:

$$p(x_\beta) \equiv \int_0^\infty p(J, t|x_\beta) dJ p(x_\beta) \quad (3)$$

be satisfied. The joint density in $\mathcal{Q} - \mathcal{S}$ is constructed so this consistency condition is satisfied. It is never needed to evaluate any quantity of interest although it is implicit

in the required conservation of probability. It should be noted that the same arguments apply if the marginal density $p(x_\beta)$ has time dependence provided the evolution is due to a process which is independent of the particle's longitudinal action, J . Finally, the fraction of protons which reach the stop is given by a quadrature of the joint density over the domain \mathcal{S} :

$$N(t) = 1. - \iint_{\mathcal{S}} p(J, t|x_\beta) p(x_\beta) dJ dx_\beta. \quad (4)$$

It is also instructive to derive Eq. (4) in the context of a first passage problem. Suppose T is the (random) time a particle strikes the stop. This time depends on the particle's betatron motion and is a random variable. Conditioning on x_β , the law of total probability gives

$$P\{T > t\} = \int_0^{x_c} P\{T > t|x_\beta\} p(x_\beta) dx_\beta \quad (5)$$

where the lhs is the probability of not reaching the stop in $[0, t]$. Now,

$$P\{T > t|x_\beta\} = P\{J(s) < J_b(x_\beta), 0 \leq s \leq t|x_\beta\} \quad (6)$$

where $J(t)$ is the action stochastic process and the probability on the rhs is $\int_0^{J_b} p(J, t|x_\beta) dJ$ where the integrand is defined by Eq. (2). Therefore,

$$P\{T > t\} = \int_0^{x_c} \int_0^{J_b} P(J, t|x_\beta) dJ dx_\beta, \quad (7)$$

and the lhs can be interpreted as the fraction of particles not reaching the stop in the time interval $[0, t]$. This is the same as $1. - N(t)$, Eq. (4).

In general, the diffusion coefficient is not expected to have a simple dependence on the action and Eq. 2 must be solved numerically.^{7,11} If the diffusion coefficient is linear in the action, the diffusion equation is easily solved by separation of variables⁹ in terms of a Fourier-Bessel series. A quadratic diffusion coefficient, $\mathbf{D} = \mathbf{D}_o J^2$, leads to a continuous spectrum and an integral representation of the solution obtains by means of Fourier transforms.¹³ Neither of these exactly correspond to simple noise spectral densities. However, the diffusion coefficient for small action is linear in the case of white phase noise and quadratic for white amplitude noise¹¹. Furthermore, we will see that the diffusion coefficient for white phase and for white amplitude noise can reasonably be fit over most

of the bucket by linear and quadratic approximations respectively; the fit is very good in the latter case.

It is straightforward to apply the solutions of Ref. 9 and Ref. 13 to our problem. We reiterate that the solution in terms of the absorbing boundary condition at $J_b(x_\beta)$ is the required conditional probability density. The solutions are functionals of the initial density. It is convenient to express the initial density distribution in terms of $k(0 \leq k \leq 1)$ rather than the action J . There is a one-one mapping between the two: $J = (8\Omega/\pi)k^2 B(k)$ where B is related to the complete elliptic integrals of the first and second kinds, K and E respectively; $B(k) = 1/k^2[E(k) - (1 - k^2)K(k)]$ Then, the initial density is

$$\rho_{0l}(k) = k/\sigma_k^2 \exp(-k^2/2\sigma_k^2) \quad (8)$$

where $\sigma_k = \sigma_{\delta p}/\delta p_s$ and $\sigma_{\delta p}$ is the rms momentum spread. We have neglected exponentially small corrections to the normalization in Eq. 8.

In the case of a linear diffusion coefficient Eq. (4) gives:

$$N = 1 - 2 \int_0^{x_c} dx_\beta \rho_{0\perp}(x_\beta) \sum_n \tilde{a}_n T_n(x_\beta) \quad (9)$$

where $\tilde{a}_n = (J_1(j_n)X_b/j_n)a_n$, $x_b = J_b/J_s$, $J_s = 8\Omega/\pi$ and $\rho_{0\perp}$ is the initial probability density of x_β . Here T_n is the temporal part of the separated solution and a_n the Fourier-Bessel coefficient:

$$T_n = \exp[-(\pi^3/16)(Q_s^2\sigma_\phi^2 j_n^2/X_b)t], \quad (10a)$$

$$\tilde{a}_n = \frac{1}{j_n J_1(j_n) \sigma_k^2} \int_0^{k_b} k \exp(-k^2/2\sigma_k^2) J_0(j_n k \sqrt{B(k)/X_b}) dk \quad (10b)$$

where, j_n is the n 'th zero of the Bessel function J_0 , t is the time measured in machine periods, T_0 , $Q_s = (\Omega/2\pi)T_0$ and σ_ϕ (rad) is the phase noise standard deviation.

Likewise, in the case of a quadratic diffusion coefficient, we find:

$$N = 1 - \frac{1}{2} \int_0^{x_c} dx_\beta \rho_{0\perp}(x_\beta) \int_0^{k_b(x_\beta)} dk \rho_{0l}(k) \\ \times \left\{ \operatorname{erfc} \left[\left(\pi Q_s \sigma_a \sqrt{t} \right) / 2 + \left(\ln(k^2 B(K)/X_b) \right) / \left(2\pi Q_s \sigma_a \sqrt{t} \right) \right] \right. \\ \left. - \left(X_b / k^2 B(k) \right) \operatorname{erfc} \left[\left(\pi Q_s \sigma_a \sqrt{t} \right) / 2 - \left(\ln(k^2 B(K)/X_b) \right) / \left(2\pi Q_s \sigma_a \sqrt{t} \right) \right] \right\}. \quad (11)$$

where σ_a is the standard deviation of the amplitude noise (relative to applied voltage) and $\operatorname{erfc}(z)$ is the complementary error function. Recall $\rho_{0\perp}$ is the Rayleigh distribution:

$$\rho_{0\perp}(x_\beta) = \frac{x_\beta}{\epsilon\beta} \exp(-x_\beta^2/2\epsilon\beta). \quad (12)$$

We will use these results to make a comparison with Monte Carlo tracking studies^{7,11}. However, two points need to be discussed. The first is to account for what we call transient particles. These are particles whose initial phase coordinates are such that they will strike the aperture stop in a synchrotron period independent of the diffusion. They contribute to the count in Eqs. (9) and (11) as well as in the simulations. We then will briefly discuss the use of the linear and quadratic approximations in treating the diffusion in the white noise case.

The number of transient hits are simply found by integrating the initial joint distribution in x_{co} , x_β over the appropriate domain in the space of the random variables. Initially, the random variables x_{co} and x_β are independent and the initial joint distribution is just the product of the initial distributions of the individual random variables. We do not present the details; the calculation is completely straightforward.

If one simply uses the small amplitude expressions for the diffusion coefficients all the way to $J_b(x_\beta)|_{x_\beta=0} \equiv J_\beta(0)$, these diffusion coefficients overestimate the actual diffusion. It is better to fit the approximations by forcing them to be equal to the actual diffusion coefficient at $J_\beta(0)$. This is illustrated in Fig. 3.

We have been using Monte Carlo tracking studies in the linear SSC lattice to simulate the effect of noise in the rf system for the purpose of extracting particles^{7,11} as well as understanding its impact on collider performance^{5,11}. Typically, the simulations follow 1000 particles for about a million turns around the ring. Comparison of the numbers of tracks reaching the aperture at the end of the run with the value of N from the appropriate expression above is summarized the following table:

The simulation results are from a single realization of the random process. The machine parameters are the nominal values for the SSC collider rings, $\sigma_{\delta p} = 5 \times 10^{-5}$, $\epsilon_x = 1\pi\text{mm} - \text{mrad}$. The value of β_x at the aperture stop was 1385 m in all cases but one. The exceptional case is indicated. The dispersion $D = 4\text{ m}$. For both simulation and theory, thirty-one transient particles have been taken into account. The noise variances are larger than would

Phase Noise			Amplitude Noise		
σ_ϕ	$N(sim)$	$N(theory)$	σ_a	$N(sim)$	$N(theory)$
0.2	584	732/595	0.5	745	772/718
0.2*	343	542/595	0.2	182	211/178
0.1	187	261/179	0.1	67	56/48
0.05	51	69/49	0.05	22	19/17
0.02	13	14/14	0.02	7	7/6

* ($\beta_x = 346m$)

Table 1: Values of N_h . The two theoretical values are for the case of small amplitude extrapolation/monomial fit.

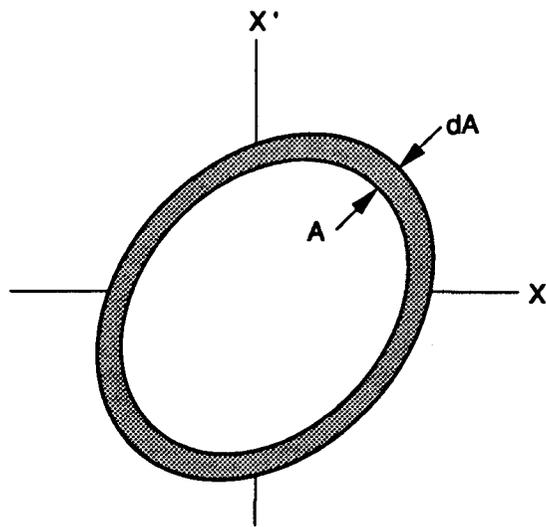
be encountered in practice. This is necessary to get a measurable loss rate in a reasonable computing time. The first figure in the column of theoretical values is the result using the extrapolated small amplitude diffusion coefficient and the second value was obtained by using the diffusion coefficient fit as just discussed. The agreement between the simulation and theory is generally very good for the latter. The largest discrepancy, about 30%, occurs for the amplitude noise case with $\sigma_a = 0.1$. There is reason to believe it might be due to statistical fluctuations in the single realization of initial conditions.

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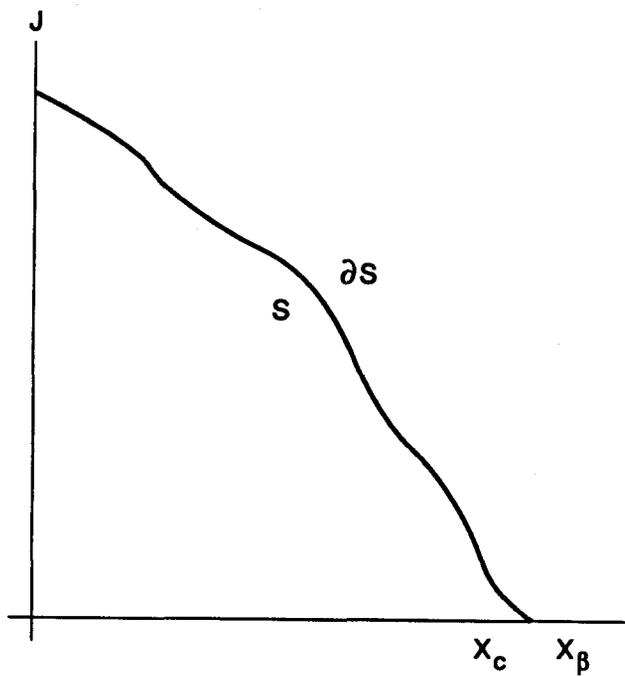
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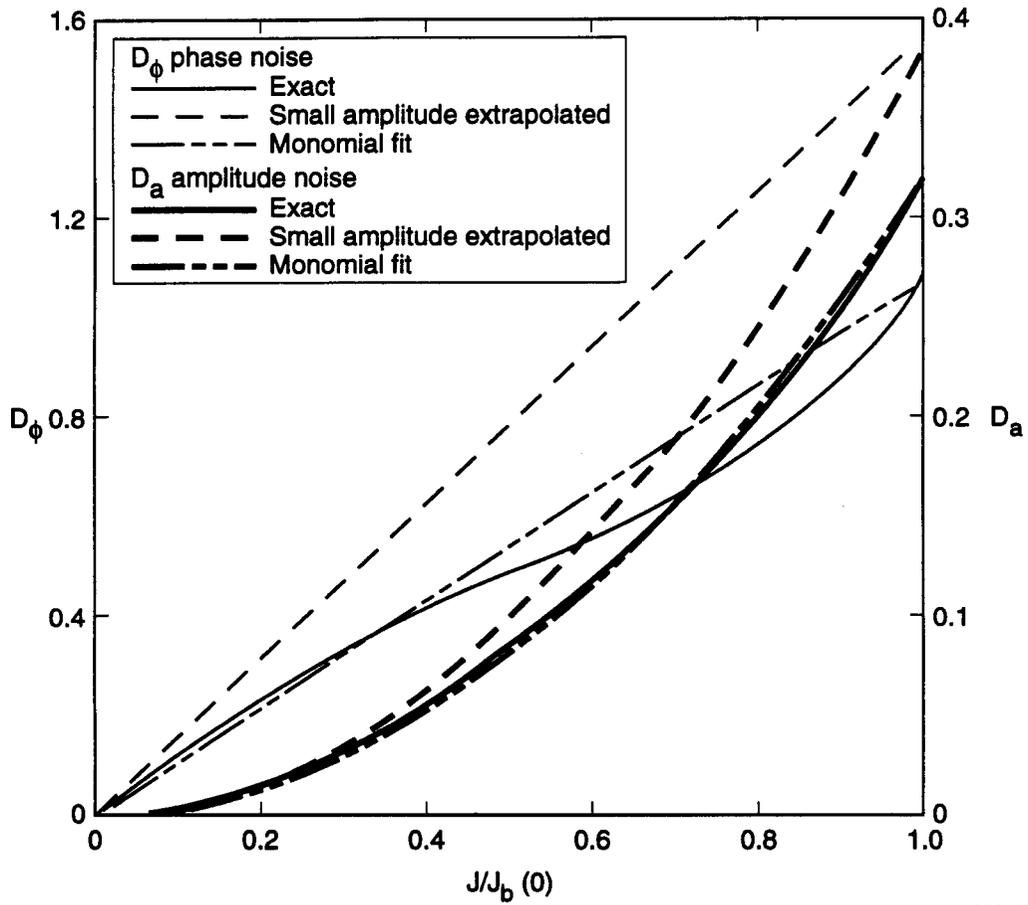
TIP-03906

FIG. 1. Transverse phase space domain (schematic) swept by collimation process. The longitudinal phase space is similar.



TIP-03907

FIG. 2. Domain of random variables, J , x_β for particles inside the aperture.



TIP-03908

FIG. 3. Diffusion coefficients for amplitude and phase noise.

