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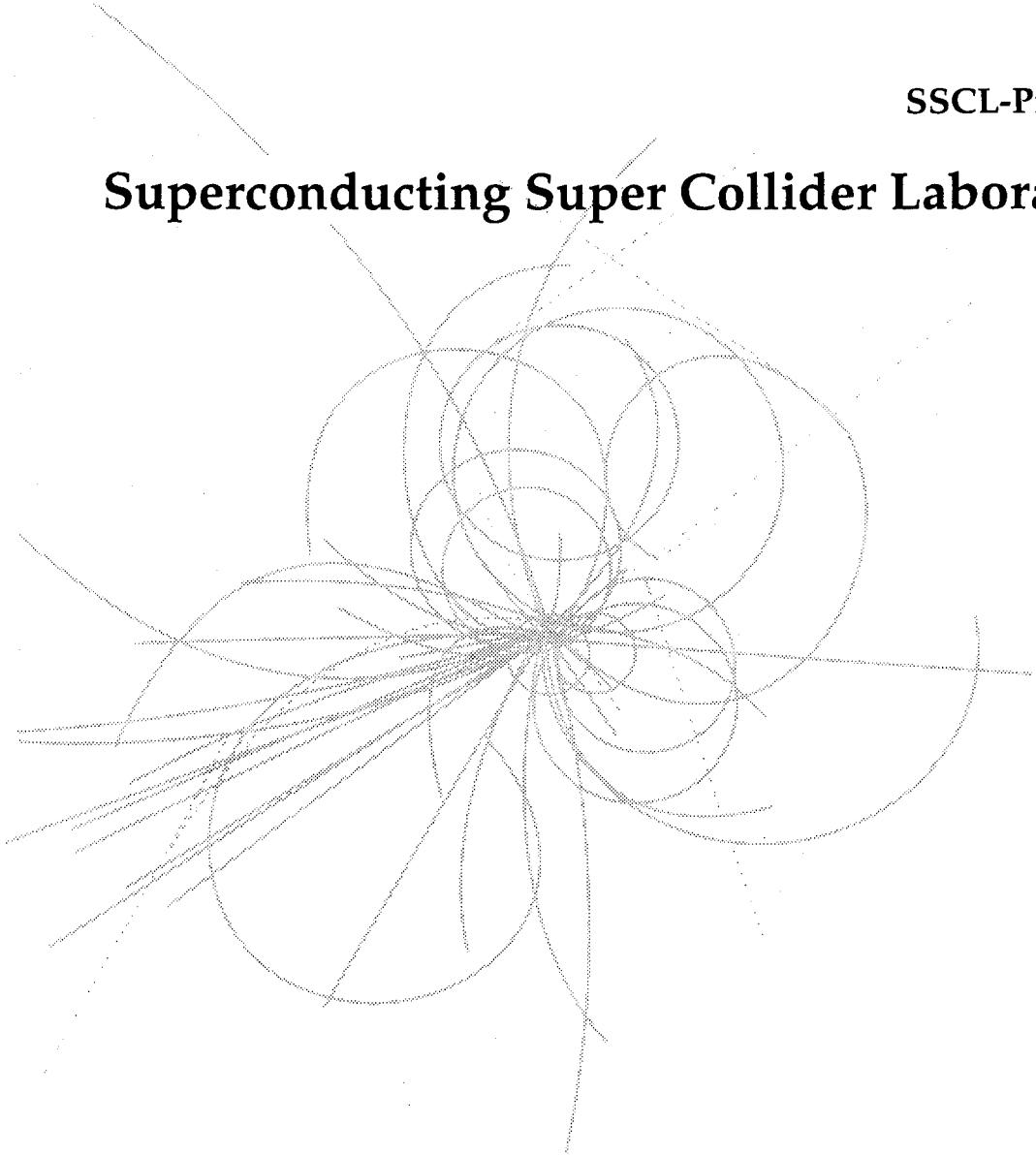

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AN APPROXIMATE INVARIANT USING LIE ALGEBRA

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Abstract

An approximate invariant is found for sextupole transverse dynamics. It is represented in terms of the elements of a Lie algebra associated with a sextupole contribution to the time-dependent Hamiltonian for transverse dynamics.

I. INTRODUCTION

It is well known that the Courant-Snyder [1] invariant is particularly useful for studying the phase-space pattern associated with the transverse dynamics of a particle storage ring. Using the Hamiltonian for a time-dependent simple harmonic oscillator, one can obtain the relevant invariant. When the Hamiltonian applies to a storage ring, time, usually denoted by s, has the dimensions of length and corresponds to distance along the ideal orbit. When there are non-linear contributions to the Hamiltonian, little success has been achieved in finding generalizations of the Courant-Snyder invariant. In order to better understand the behaviour of a particle beam, it is useful to find an approximate invariant, which is associated with a non-linear time-dependent Hamiltonian. A method, based on using the Lie algebra associated with elements obtained from the products of the position and conjugate momentum coordinates, is applied to a non-linear system. To illustrate the method, an example is given for the linear system, where the invariant is exact, and the relevant algebra is SU(1,1) [2]. The method used for the linear system can be easily generalized to study a non-linear one-dimensional system.

For classical one-dimensional transverse dynamics, an approximate invariant, associated with a time-dependent Hamiltonian containing a non-linear sextupole term [3], is found. It is obtained using the elements of a Lie algebra constructed from the products of the transverse position and momentum coordinates. Both the Hamiltonian and the approximate invariant can be represented as linear sums of the elements of the Lie algebra. For the approximate invariant the time-dependent coefficients of the elements satisfy a system of linear first-order differential equations. The invariant is approximate in the sense that terms of order greater than three, resulting from the Poisson bracket of elements of the algebra, are neglected. This results in a closed Lie algebra.

II. THE COURANT SNYDER INVARIANT AND SU(1,1)

The time-dependent Hamiltonian for one-dimensional transverse dynamics is written in terms of the position coordinate q and the conjugate momentum p as

$$H(t) = \frac{p^2}{2} + K(t)\frac{q^2}{2}. \tag{2.1}$$

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The invariant associated with this Hamiltonian is

$$I(t) = \frac{\beta(t)p^2 + 2\alpha(t)pq + \gamma(t)q^2}{2}, \qquad (2.2)$$

which satisfies the partial differential equation

$$\frac{dI(t)}{dt} = \frac{\partial I(t)}{\partial t} + \{H(t), I(t)\} = 0, \qquad (2.3)$$

with Poisson bracket

$$\{H(t), I(t)\} = \frac{\partial H}{\partial p}\frac{\partial I}{\partial q} - \frac{\partial I}{\partial q}\frac{\partial H}{\partial p}. \qquad (2.4)$$

The functions $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ satisfy the equations

$$\frac{d\alpha(t)}{dt} = K(t)\beta(t) - \gamma(t), \qquad (2.5a)$$

$$\frac{d\beta(t)}{dt} = -2\alpha(t), \qquad (2.5b)$$

$$\frac{d\gamma(t)}{dt} = 2K(t)\alpha(t), \qquad (2.5c)$$

where $\gamma(t) = (1 + \alpha^2(t))/\beta(t)$.

Both the Hamiltonian (2.1) and the invariant (2.2) may be expressed in terms of the elements of the Lie algebra SU(1,1). If one introduces the coordinates

$$a = \frac{q + ip}{\sqrt{2}}, \quad a^* = \frac{q - ip}{\sqrt{2}}, \qquad (2.6)$$

with Poisson bracket $\{a, a^*\} = i$, then the functions

$$A_1 = a^2, \quad A_2 = a^{*2}, \quad A_3 = a^* a, \qquad (2.7)$$

satisfy the Lie algebra of SU(1,1). Namely,

$$\{A_1, A_2\} = 4iA_3, \ \{A_1, A_3\} = 2iA_1,$$

$$\{A_2, A_3\} = -2iA_2. \qquad (2.8)$$

This form of the algebra is useful for determining the quantum uncertainties of q, p, and the Courant-Snyder invariant [4].

In terms of the elements of the algebra (2.7), the Hamiltonian and the invariant become

$$H(t) = \alpha_1(t)A_1 + \alpha_2(t)A_2 + \alpha_3(t)A_3$$

$$I(t) = \beta_1(t)A_1 + \beta_2(t)A_2 + \beta_3(t)A_3. \qquad (2.9a)$$

Requiring $I(t)$ to be real gives the relations

$$\beta_1(t) = \beta_2^*(t), \ \ \beta_3(t) = \beta_3^*(t). \qquad (2.9b)$$

When these are substituted into (2.3), one finds, using (2.8), the set of linear differential equations

$$\begin{pmatrix} \dot{\beta}_1(t) \\ \dot{\beta}_2(t) \\ \dot{\beta}_3(t) \end{pmatrix} = \begin{pmatrix} 2i\alpha_3(t) & 0 & -2i\alpha_1(t) \\ 0 & -2i\alpha_3(t) & 2i\alpha_1(t) \\ 4i\alpha_1(t) & -4i\alpha_1(t) & 0 \end{pmatrix} \begin{pmatrix} \beta_1(t) \\ \beta_2(t) \\ \beta_3(t) \end{pmatrix}. \qquad (2.10)$$

The functions $\alpha_i(t)$ and $\beta_i(t)$ satisfy the relations

$$\alpha_1(t) = \alpha_2(t) = \frac{K(t) - 1}{4}$$

$$\alpha_3(t) = \frac{K(t) + 1}{2}, \qquad (2.11)$$

and

$$\beta_1(t) = \beta_2^*(t) = \frac{-i\alpha(t) + \frac{\gamma(t) - \beta(t)}{2}}{2},$$

$$\beta_3(t) = \frac{\beta(t) + \gamma(t)}{2}. \qquad (2.12)$$

With initial values given for $\beta(t)$ and $d\beta(t)/dt$, the system of equations (2.10) can be integrated. Using (2.11) and (2.12), one can show that (2.10) is equivalent to the system (2.5).

III. THE NON-LINEAR SEXTUPOLE SYSTEM

The method described above can be extended to the case when a non-linear term is added to the Hamiltonian. However, an approximation is made to obtain a finite closed Lie algebra which contains seven elements. As

an example, one considers an Hamiltonian of the form

$$H(t) = \frac{p^2}{2} + K(t)\frac{q^2}{2} + S(t)q^3, \qquad (3.1)$$

where $S(t)$ is the strength of the sextupole term. Defining functions of a and a^* as

$$A_1 = a^2, A_2 = a^{*2}, A_3 = a^*a, A_4 = a^3,$$
$$A_5 = a^{*3}, A_6 = a^2a^*, A_7 = a^{*2}a, \qquad (3.2)$$

one finds, keeping terms of order less than four in a and a^*, the closed Lie algebra

$$\{A_1, A_2\} = 4iA_3, \{A_1, A_3\} = 2iA_1,$$
$$\{A_1, A_4\} = 0,$$
$$\{A_1, A_5\} = 6iA_7, \{A_1, A_6\} = 2iA_4,$$
$$\{A_1, A_7\} = 4iA_6,$$
$$\{A_2, A_3\} = -2iA_2, \{A_2, A_4\} = -6iA_6,$$
$$\{A_2, A_5\} = 0,$$
$$\{A_2, A_6\} = -4iA_7, \{A_2, A_7\} = -2iA_5,$$
$$\{A_3, A_4\} = -3iA_4,$$
$$\{A_3, A_5\} = 3iA_5, \{A_3, A_6\} = -iA_6,$$
$$\{A_3, A_7\} = iA_7,$$
$$\{A_4, A_5\} = \{A_4, A_6\} = \{A_4, A_7\} = 0$$
$$\{A_5, A_6\} = \{A_5, A_7\} = \{A_6, A_7\} = 0. \qquad (3.3)$$

The Hamiltonian (3.1) may be written in the form

$$H(t) = \sum_{i=1}^{7} \alpha(_i(t)A_i, \qquad (3.4)$$

where $\alpha_i(t)$, $i = 1 \rightarrow 3$ are given by (2.11), and

$$\alpha_4(t) = \alpha_5(t) = \frac{\sqrt{(2)}S(t)}{4}$$
$$\alpha_6(t) = \alpha_7(t) = 3\alpha_4(t). \qquad (3.5)$$

One can now seek an approximate time-invariant associated with the Hamiltonian (3.1). This is assumed to be of the form

$$I(t) = \sum_{j=1}^{7} \beta_j(t)A_j, \qquad (3.6a)$$

which contains terms up to third order in a and a^*. Since $I(t)$ must be real, one finds

$$\beta_1(t) = \beta_2^*(t), \quad \beta_3(t) = \beta_3^*(t)$$
$$\beta_4(t) = \beta_5^*(t), \quad \beta_6(t) = \beta_7^*(t). \qquad (3.6b)$$

When this, along with the Hamiltonian (3.1), is substituted into (2.3), one finds, using the algebra (3.3), the system of linear first order differential equations

$$\frac{d\vec{\beta}(t)}{dt} = \mathbf{M}(t)\vec{\beta}(t), \qquad (3.7)$$

where

$$\vec{\beta}(t) = \begin{pmatrix} \beta_1(t) \\ \vdots \\ \beta_7(t) \end{pmatrix},$$

and $\mathbf{M}(t)$ is

$$\begin{pmatrix} 2i\alpha_3 & 0 & -2i\alpha_1 & 0 & 0 & 0 & 0 \\ 0 & -2i\alpha_3 & 2i\alpha_2 & 0 & 0 & 0 & 0 \\ -4i\alpha_2 & 4i\alpha_1 & 0 & 0 & 0 & 0 & 0 \\ 2i\alpha_6 & 0 & -3i\alpha_4 & 3i\alpha_3 & 0 & -2i\alpha_1 & 0 \\ 0 & -2i\alpha_7 & 3i\alpha_5 & 0 & -3i\alpha_3 & 0 & 2i\alpha_2 \\ 4i\alpha_7 & -6i\alpha_4 & -i\alpha_6 & 6i\alpha_2 & 0 & i\alpha_3 & -4i\alpha_1 \\ 6i\alpha_5 & -4i\alpha_6 & i\alpha_7 & 0 & -6i\alpha_1 & 4i\alpha_2 & -i\alpha_3 \end{pmatrix}.$$

In these expressions the $\alpha(t)'^s$ are given in (2.11) and (3.5). The first three $\beta(t)'^s$ are given in (2.12). The remaining $\beta(t)'^s$ are found as solutions to a system of first order differential equations. Using (3.6), the approximate invariant may be written in the form

$$I \sim I_0(t) + 2\Re(\beta_4(t)A_4 + \beta_6(t)A_6). \qquad (3.8)$$

The first term $I_0(t)$ is the function (2.2), which is an invariant for the linear system. The remaining term may be expressed in the form

$$c_1(t)q^3 + c_2(t)q^2p + c_3(t)qp^2 + c_4(t)p^3,$$

with

$$\sqrt{2}c_1(t) = \Re\beta_4(t) + \Re\beta_6(t)$$
$$\sqrt{2}c_2(t) = -(3\Im\beta_4(t) + \Im\beta_6(t)) \qquad (3.9)$$
$$\sqrt{2}c_3(t) = -(3\Re\beta_4(t) - \Re\beta_6(t))$$
$$\sqrt{2}c_4 1(t) = \Im\beta_4(t) - \Im\beta_6(t).$$

The functions $c_i(t)$, $i = 1 \rightarrow 4$, satisfy the following system of first order differential equations:

$$\dot{c}_1(t) = K(t)c_2(t) + 3S(t)\alpha(t)$$
$$\dot{c}_2(t) = -3c_1(t) + 2K(t)c_3(t) + 3S(t)\beta(t)$$
$$\dot{c}_3(t) = -2c_2(t) + 3K(t)c_4(t)$$
$$\dot{c}_4(t) = -c_3(t). \qquad (3.10)$$

IV. RESULTS AND CONCLUSIONS

Numerical results are given which confirm the analytical development in the previous section. The equation of motion for the non-linear time-dependent Hamiltonian (3.1) is found from Hamilton's equations

$$\dot{q} = \frac{\partial H(t)}{\partial p}$$
$$\dot{p} = -\frac{\partial H(t)}{\partial q} \qquad (4.1)$$

to be

$$\ddot{q} + K(t)q + 3S(t)q^2 = 0. \qquad (4.2)$$

The approximate invariant associated with this Hamiltonian is

$$\frac{\epsilon}{2\pi} = I(t) = \frac{\beta(t)p^2 + 2\alpha(t)pq + \gamma(t)q^2}{2}$$
$$+c_1(t)q^3 + c_2(t)q^2p + c_3(t)qp^2 + c_4(t)p^3, \quad (4.3)$$

where ϵ is the emittance. The time-dependent functions $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $\beta_4(t)$, and $\beta_6(t)$ can be found from (2.5), (3.9), and (3.10). This system of equations is equivalent to the system of linear equations (3.7).

The periodic nature of the functions $c_i(t)$ allows the determination of their values at a fixed point in the lattice. The values of the functions q and p are determined from (4.2) using non-linear tracking for the first five circuits of the lattice. At the jth turn, the approximate invariant becomes

$$I(j) = I_0(j) + c_1(j)g(1,j) + c_2(j)g(2,j)$$
$$+c_3(j)g(3,j) + c_4(4,j), \qquad (4.4)$$

with

$$g(1,j) = q(j)^3, \; g(2,j) = q(j)^2p(j),$$
$$g(3,j) = q(j)p(j)^2, \; g(4,j) = p(j)^3. \qquad (4.5)$$

From the requirement that

$$I(k) - I(1) = 0,$$

for $k = 2 \rightarrow 5$, one finds the system of linear equations

$$\Delta(k) = \sum_{i=1}^{4} c_i(j)\Delta g(i,k), \qquad (4.6a)$$

with

$$\Delta(k) = -(I_0(k) - I_0(1))$$
$$\Delta g(i,k) = g(i,k) - g(i,1). \qquad (4.6b)$$

The system (4.6) is solved numerically to find the coefficients $c_i(j)$. The thin lens approximation is used for a lattice made of a single thin sextupole element and identical cells of length L. Each cell consists of a focusing and a defocusing magnet separated by a bending drift magnet, i.e. a FODO lattice. The focal length of the focusing and defocusing magnets is f, and the phase advance per cell μ is found from

$$\sin(\mu/2) = \frac{L}{2f}. \qquad (4.7)$$

The tune ν is obtained from

$$\nu = \mu N_c/2\pi, \qquad (4.8)$$

where N_c is the number of cells. The maximum value of $\beta(t)$ occurs when $\alpha(t) = 0$, and

$\beta(t) = 1/\gamma(t)$, and it is found to be

$$\beta(t)_{max} = 2f\left(\frac{1+\sin(\mu/2)}{1-\sin(\mu/2)}\right). \qquad (4.9)$$

The results for the invariant $I_0(t)$ for the linear system and the approximate invariant $I(t)$ are shown in Figure. 1. For the example considered, the values $N_c = 4$, $\mu = 90^o$, $L/2 = 8875$ cm, and $\nu = 0.33666667 + N_c\mu/360$ have been used. The initial values $q = .1$ cm, $\beta_{max}p = 0$, along with the sextupole strength $3s_e = .1 \times 10^{-5}$ cm^{-2}, have been used. The sextupole function is approximated by $S(t) = (s_e/3)\delta(t - jT_0)$, where T_0 is the orbital period. For the present case, the values of the periodic functions are

$$c_1 = -7.6159501059673D-06 \ \text{cm}^{-2}$$

$$c_2/\beta_{max} = -6.7207474250385D-07 \ \text{cm}^{-2}$$

$$c_3/\beta_{max}^2 = +2.2284434898092D-05 \ \text{cm}^{-2}$$

$$c_4/\beta_{max}^3 = -9.0800984544563D-08 \ \text{cm}^{-2}, \qquad (4.10)$$

with $\beta_{max} = 38389.279$ cm. It is clear from the Figure 1. that the approximate invariant is nearly stable. It remains this way for increasingly larger number of turns, and it oscillates with small amplitude and with a period of 100 turns. The amplitude of the oscillation depends upon the strength of the sextupole non-linearity, and the period results from the nearness of the fractional tune to the resonance value at 1/3. Although, the present approximate invariant, which includes terms in q and p through third order, becomes increasingly unstable for large values of the sextupole strength or large initial values of the amplitude q, it is clear that the method can be extended to include arbitrarily higher order corrections which will improve the stability of the approximate invariant.

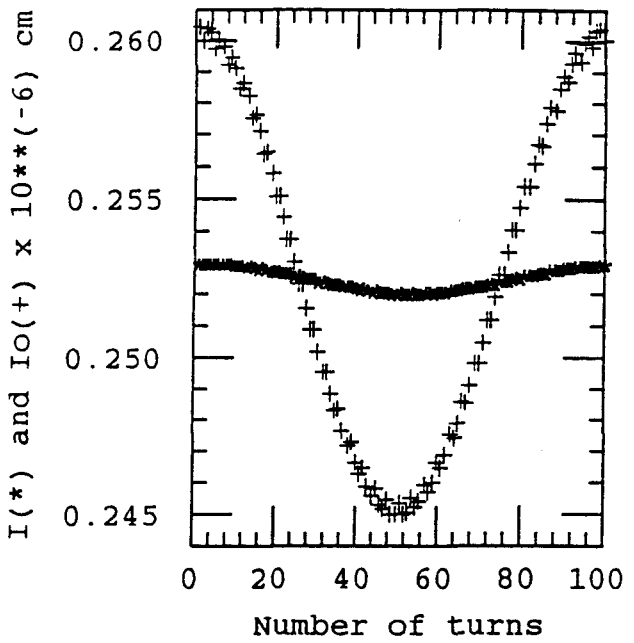

Figure 1. The Courant-Snyder invariant, $\epsilon_0/2\pi = I_0$ cm, and the approximate invariant, $\epsilon/2\pi = I$ cm, for the non-linear sextupole system as a function of turn-number.

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