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Potential Motion for Thomas-Fermi Non-Topological Solitons

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ABSTRACT

In the Thomas-Fermi approximation to theories of coupled fermions and scalars, the equations for spherically-symmetric non-topological solitons have the form of potential motion. This gives a straightforward method for proving the existence of non-topological solitons in a given theory and for finding the constant-density, saturating solutions.

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Solitons in field theory, in the sense usually used in the literature, are finite-energy, non-singular, localized classical solutions to field equations. These solutions can be stable either because of topological reasons, with boundary conditions at infinity preventing decay into lower energy states, or because the solutions themselves are the lowest energy states. We will consider the second kind here, which have been called 'non-topological solitons'.[1,2]

In purely scalar field theories, time-dependent non-topological soliton solutions have been explicitly constructed.[2,3,4] Such solutions correspond in a semi-classical approximation to coherent states of the full quantum theory. In fermion theories, the situation is more complicated: occupation numbers never get larger than one, so classical solutions are never a good approximation. The Dirac equation must be solved to find the true quantum states which are then filled to form a Fermi sea. This has been done for systems with small numbers of fermions, with a boson interaction added as a background field in the Dirac equation, and applied as a model of hadrons.[6] For systems with many fermions, however, it is impossible to solve for all the quantum states. Instead, there is an approximation, due to Thomas and Fermi,[7] that can be used when solutions have fields which are sufficiently slowly-varying functions of position: the fermions may be treated as free particles stacked in a Fermi sea with a position-dependent Fermi momentum. We are interested in theories with fermions coupled to scalars, which provide an attractive force to overcome Fermi repulsion and allow bound states. The Thomas-Fermi approximation in this case incorporates the effects of fermions into the scalar field equation of motion through the assumption of a filled Fermi sea with a Fermi momentum that depends on the value of that field. It was first introduced in the context of non-topological solitons in Ref. 1, followed by many papers applying this idea to specific models, particularly to nuclei (see Ref. 5 for a review).

Here we show that in the Thomas-Fermi approximation, the three dimensional spherically-symmetric scalar field equation has the form of potential motion. This follows simply from energy-momentum conservation, with the potential being the total pressure in the problem, and therefore generalizes easily. As

for scalar field theories, where the mechanical analogy is often used, the potential motion form for fermion field theories is useful for determining the existence of solitons in a given theory and showing whether there are special solutions which can be any size and have constant energy and number densities. Because of the resemblance to what is called the saturation of the interiors of heavy nuclei, we call the special solutions saturating.

Consider for the moment just one scalar field ϕ and one fermion field ψ. Write the Lagrangian as

$$L = L_\phi + L_\psi \tag{1}$$

where L_ϕ contains those terms that involve only ϕ,

$$L_\phi = \tfrac{1}{2}(\partial_\mu \phi)^2 - U(\phi) \ , \tag{2}$$

and L_ψ contains the rest, including interactions between ϕ and ψ:

$$L_\psi = \overline{\psi}\,(i\,\partial\!\!\!/ - m(\phi))\,\psi + \ \ldots \tag{3}$$

The Thomas–Fermi approximation consists of neglecting terms involving spatial derivatives of ϕ in the square of the Dirac equation that follows from (3), based on the assumption that ϕ is slowly varying. Then the fermion states are those of free particles with a dispersion relation which depends on on $\phi(\vec{x})$. Derivative interactions in eqn. (3) contribute at the same order as terms that are being neglected, and therefore have no effect within the approximation.

We can write the conservation of energy–momentum in this theory as

$$\partial_\mu (T^{\mu\nu}_\phi + T^{\mu\nu}_\psi) = 0 \ , \tag{4}$$

with

$$T^{\mu\nu}_\phi = \frac{\partial L_\phi}{\partial(\partial_\mu \phi)}\,\partial^\nu \phi - g^{\mu\nu} L_\phi \tag{5}$$

and the energy–momentum tensor for the fermions given by the perfect fluid of the Thomas-Fermi approximation,

$$T^{\mu\nu}_\psi = (E_\psi + P_\psi)\,U^\mu U^\nu - P_\psi g^{\mu\nu} \ . \tag{6}$$

In the rest frame of the fluid the velocity four-vector $U^\mu = \delta^\mu_0$. Since we are neglecting the dependence of P_ψ on derivatives of ϕ, eqn. (4) becomes

$$\partial_\nu \phi \left(\partial_\mu \frac{\partial L_\phi}{\partial(\partial_\mu \phi)} - \frac{\partial L_\phi}{\partial \phi} - \frac{\partial P_\psi}{\partial \phi} \right) = -\,\delta_{\nu,0} \left(\dot{E}_\psi + \dot{P}_\psi \right) \ . \tag{7}$$

For static fermion densities, this gives

$$\partial_\mu \frac{\partial L_\phi}{\partial(\partial_\mu \phi)} - \frac{\partial L_\phi}{\partial \phi} = \frac{\partial P_\psi}{\partial \phi} \ . \tag{8}$$

The right hand side describes the effect of the fermion fluid on the scalar field equation of motion.

For the Lagrangian (2), static, spherically-symmetric solutions in three space dimensions satisfy

$$\phi'' + \frac{2}{r}\phi' = -\frac{\partial}{\partial \phi}(P_\psi - U) \ , \tag{9}$$

where the prime denotes differentiation with respect to the radial coordinate r. This equation has the form of potential motion for a particle at position ϕ and time r moving in the potential $V = P_\psi - U$, with a friction term whose coefficient is inversely proportional to time.

This form of potential motion can be generalized to more complicated theories. For a theory with n scalars and any number of fermions, $\vec{\phi}$ rolls as a particle in an n-dimensional space moving in a potential $P_\psi - U(\vec{\phi})$, where P_ψ is the sum of all the Fermi pressures. In theories with vector fields, the transverse component will satisfy the same rolling equation, but the potential has the opposite sign, $V = U - P_\psi$, so that the rolling does not apply to theories

containing both scalars and vectors. For a theory with a charged scalar field, scalar field solutions will have a time dependence which adds a quadratic term to the potential U (see Ref. 4). Gravitational effects, which become relevant when the solution becomes heavy enough, can also be included.[8,9] Theories in which the scalar Lagrangian has a more complicated kinetic term, such as chiral perturbation theory, can often be transformed into the form (2), so the equations for the transformed field will have the form of potential motion. For simplicity, we consider here only the one fermion, one scalar theory given by eqns. (1)–(3).

With a given Fermi energy and appropriate boundary conditions on ϕ, $\phi(r \to \infty) = \phi_0$ (vacuum value) and $\phi'(0) = 0$, eqn. (9) can be solved for the field configuration $\phi(r)$, which is then inserted into the dispersion relation derived from eqn. (3) to give the Fermi momentum $k_F(r)$. The number density, energy density, and pressure of the fermions are given by the usual expressions involving $k_F(r)$.[9] The form of eqn. (9) allows the spherically-symmetric saturating solutions to be found without solving any differential equations.

Saturating solutions have constant values for the fields out to large distances, which means that ϕ sits still for a long time. In this case the friction term in eqn. (9) can be ignored, because r is large before ϕ starts moving. Saturating solutions therefore occur when ϕ rolls between two degenerate hills.* One hill occurs at the vacuum value of ϕ, call it ϕ_0, which is the value to which ϕ must roll at large times to insure the energy is finite. The other hill, from which ϕ begins to roll at $r = 0$, is found by solving the algebraic equations

$$\begin{aligned} P_\psi - U &= 0 \\ \frac{\partial}{\partial \phi}(P_\psi - U) &= 0 \end{aligned} \tag{10}$$

for ϵ_F, the Fermi energy, and ϕ_{in}, the value of the scalar field in the interior of the saturating solution. The solution exists, that is, there is a surface connecting

* For a detailed discussion of this in a charged boson theory see, for example, ref. 4. In that theory P_ψ is replaced by $\frac{1}{2}\mu|\phi|^2$, coming from a time dependence in the phase of ϕ. Otherwise, the story about rolling around and saturating solutions is just the same.

the interior and exterior, provided $P_\psi - U < 0$ for $\phi_{in} < \phi < \phi_0$. In practice, these solutions can be found by graphing $U(\phi)$ and $P_\psi(\epsilon_F, \phi)$ then adjusting ϵ_F until the two curves are tangent at a point other than the vacuum.

The properties of the saturating solutions follow as before from knowing ϵ_F and ϕ, where $\phi = \phi_{in}$ is now constant. In particular, the volume energy density is given by $U(\phi_{in}) + E_\psi(\epsilon_F, \phi_{in})$ and can be used to determine whether the solution is energetically bound. Because the surface energy is smaller than the volume energy by a factor $\sim 1/R$, the volume energy is a good approximation to the total energy when the radius R is large. We note that the Thomas-Fermi approximation is extremely good in the interior of these solutions, where the field derivatives are very small, but is less good in the surfaces. Further discussion of these solutions and their applications can be found in previous papers.[9,10]

In this paper we have shown that in theories of coupled fermions and scalars, the spherically-symmetric scalar field equation will have the form of potential motion when the fermions are treated in the Thomas-Fermi approximation. In this case it is straightforward to check whether the field theory has non-topological soliton solutions, either saturating or non-saturating. The most interesting application we know of for these results is in nuclear physics: it is possible that nuclei might be described as solitons in either an effective Lagrangian cooked up for such a purpose, such as the Walecka model,[5,11] or, even more interesting, in effective chiral Lagrangians used to describe hadronic scattering.[12]

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