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Light–Quark, Heavy–Quark Systems

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1. Introduction

1.1. Motivation

There are at least three good reasons for studying in great detail the physics of B- and D- mesons. First, the standard model predicts small but observable signals of CP–violation in decays of B-mesons. Measurement of these asymmetries would give a check on the standard Kobayashi–Maskawa (KM) accounting of CP–violation in the standard model (for recent reviews see [1,2]). It is worth noting that, inasmuch as only one CP violating parameter is known, namely ϵ, almost any model of CP violation can account for it by fixing its free parameters.

Second, the rates for rare decays of heavy mesons are sensitive to departures from the standard model. These rare processes are good probes for new physics since they start at 1–loop order in the standard model. For example, the partial widths for $B \to K^*\gamma$ and $B \to K^*\ell^+\ell^-$ in models with two higgs doublets can easily differ from the standard model's by an order of magnitude. Less (or not at all) rare, but no less interesting, is $B^0 - \bar{B}^0$ mixing. This depends strongly on the top quark mass. The observation[3] of $B^0 - \bar{B}^0$ mixing, with large mixing parameter ($r \sim 0.2$) in 1987, was the first evidence that the top quark was really very heavy.

Last, but not least, precision determination of the elements of the KM matrix is naturally done through study of decays of heavy mesons or baryons[2]. This is important on two counts. The more precisely known these KM elements are, the more strongly constrained models that address the family problem will be. And, also, the standard model predictions for CP asymmetries and rare decay rates, as described above, depend on the KM matrix[1].

[1] Although, it must be said in all fairness, in the case of rare processes, the ratios $\Gamma(\bar{B} \to K^*\gamma)/\Gamma(\bar{B} \to De\bar{\nu})$ and $\Gamma(\bar{B} \to K^*e^+e^-)/\Gamma(\bar{B} \to De\bar{\nu})$ are fairly independent of KM angles.

For these reasons, it is necessary and important to make precise standard model predictions, in terms of standard model parameters, of rates for semileptonic and rare decays, and of CP asymmetries. Discouragingly, these calculations run into the usual difficulties associated with hadronic matrix elements[2]: strong interactions render perturbation theory useless, and we know of no alternative calculational tool[3]. None, that is, until recently, when Isgur and Wise discovered new symmetries of QCD[5].

This paper will describe these recent developments. We will introduce the Heavy Quark Effective Theory (HQET) and then we will find the new symmetries of QCD. We will then put to use these symmetries in a variety of ways. In particular, we will find that the form factors for semileptonic $\bar{B}$ decays to D or D^* mesons can be determined at the point of maximum momentum transfer (that is, when the resulting D or D^* meson is at rest in the $\bar{B}$ rest frame). We will also discuss systematic corrections to these results.

In preparing these notes I have, for the sake of clarity, departed badly from the chronological order in which these developments took place. In so doing, I have undoubtedly offended some, as it may appear I am intent on not giving them due credit. To them, I apologize in advance.

[2] CP asymmetries in decays of $B^0(\bar{B}^0)$ mesons to CP eigenstates are, in some cases, independent of nonperturbative matrix elements. Nevertheless, if the asymmetry is to be predicted, the KM angle must be previously extracted from, say, semileptonic decays, for which understanding of nonperturbative form factors is needed (see section 5.1, below).

[3] Save for numerical simulations of lattice QCD. These, however, convey little physical insight, and are, at present, technologically limited in their ability to produce precise results, *e.g.*, few results are known with dynamical fermions. The methods that are the subject of this paper are valuable on the lattice; for a recent review, see ref. [4]

At least three sets of lectures on this subject have appeared recently, one by Mark Wise[6], one by Howard Georgi[7] and one by myself[8]. The reader interested in the historical developments that led to the HQET can find an account in ref. [7]. This review evolved from reference [8], but it differs from it in two important ways. It is more condensed, so many detailed calculations have been omitted. And it is more complete, for it touches on more recent developments and has more extensive references.

1.2. Physical Intuition

The central idea of the HQET is so simple, it can be described without reference to a single equation. And it should prove useful to refer back to the simple intuitive notion, to be presented below, wherever the formalism and corresponding equations become abstruse.

The HQET is useful when dealing with hadrons composed of one heavy quark and any number of light quarks. More precisely, the quantum numbers of the hadrons are unrestricted as far as isospin and strangeness, but are ± 1 for either B- or C-number. In what follows we shall (imprecisely) refer to these as 'heavy hadrons'.

The successes of the constituent quark model is indicative of the fact that, inside hadrons, strongly bound quarks exchange momentum of magnitude a few hundred MeV. We can think of the typical amount Λ by which the quarks are off-shell in the nucleon as $\Lambda \approx m_p/3 \approx 330$MeV. In a heavy hadron the same intuition can be imported, and again the light quark(s) is(are) very far off-shell, by an amount of order Λ. But, if the mass M_Q of the heavy quark Q is large, $M_Q \gg \Lambda$, then, in fact, this quark is almost on-shell. Moreover, interactions with the light quark(s) typically change the momentum of Q by Λ, but change

the *velocity* of Q by a negligible amount, of the order of $\Lambda/M_Q \ll 1$. It therefore makes sense to think of Q as moving with constant velocity, and this velocity is, of course, the velocity of the heavy hadron.

In the rest frame of the heavy hadron, the heavy quark is practically at rest. The heavy quark effectively acts as a static source of gluons. It is characterized by its flavor and color–$SU(3)$ quantum numbers, but not by its mass. In fact, since spin–flip interactions with Q are of the type of magnetic moment transitions, and these involve an explicit factor of g_s/M_Q, where g_s is the strong interactions coupling constant, the spin quantum number itself decouples in the large M_Q case. Therefore, ***the properties of heavy hadrons are independent of the spin and mass of the heavy source of color.***

The HQET is nothing more than a method for giving these observations a formal basis. It is useful because it gives a procedure for making explicit calculations. But more importantly, it turns the statement 'M_Q is large' into a *systematic* perturbative expansion in powers of Λ/M_Q. Each order in this expansion involves QCD to all orders in the strong coupling, g_s. Also, the statement of mass and spin independence of properties of heavy hadrons appears in the HQET as approximate internal symmetries of the Lagrangian.

Before closing this section, we point out that these statements apply just as well to a very familiar and quite different system: the atom. The rôle of the heavy quark is played by the nucleus, and that of the light degrees of freedom by the electrons (and the electromagnetic field)[4]. That different isotopes have the same chemical properties simply reflects the nuclear mass independence of

[4] An obvious distinction between the atomic and hadronic systems is that in the latter the configuration of the light degrees of freedom is non–computable, due to the difficulties afforded by the non–perturbative nature of strong interactions. The methods that we are describing circumvent the need for a detailed

the atomic wavefunction. Atoms with nuclear spin s are $2s+1$ degenerate; this degeneracy is broken when the finite nuclear mass is accounted for, and the resulting hyperfine splitting is small because the nuclear mass is so much larger than the binding energy (playing the rôle of Λ). It is not surprising that, using M_Q independence, the properties of B and D mesons are related, and using spin independence, those of B and B^* mesons are related, too.

2. The Heavy Quark Effective Theory

2.1. The Effective Lagrangian and its Feynman Rules

We shall focus our attention on the calculation of Green functions in QCD, with a heavy quark line, its external momentum almost on–shell. The external momentum of gluons or light quarks can be far off–shell, but not much larger than the hadronic scale Λ. This region of momentum space is interesting because physical quantities —S–matrix elements— live there. And, as stated in the introduction, we expect to see approximate symmetries of Green functions in that region which are not symmetries away from it. That is, these are approximate symmetries of the S–matrix, but not of the lagrangian.

The effective Lagrangian $\mathcal{L}_{\rm eff}$ is constructed so that it will reproduce these Green functions, to leading order in Λ/M_Q. It is given, for a heavy quark of velocity v_μ ($v^2=1$), by[9],

$$\mathcal{L}_{\rm eff}^{(v)} = \bar{Q}_v iv{\cdot}DQ_v \ , \tag{1}$$

knowledge of the configuration of light degrees of freedom. The price paid is that the range of predictions is restricted. To emphasize the non-computable aspect of the configuration of light degrees of freedom, Nathan Isgur informally referred to it as "brown muck", and the term has somewhat made it into the literature. (Sometimes, in fact, in modified form: J.D. Bjorken has used the term "brown gunk".)

where the covariant derivative is

$$D_\mu = \partial_\mu + ig_s A^a_\mu T^a \ ,$$

and the heavy quark field Q_v is a Dirac spinor that satisfies the constraint

$$\left(\frac{1+\not{v}}{2}\right) Q_v = Q_v \ . \tag{2}$$

In addition, it is understood that the usual Lagrangian $\mathcal{L}_{\text{light}}$ for gluons and light quarks is added to $\mathcal{L}^{(v)}_{\text{eff}}$.

We can see how this arises at tree level, as follows[10]. Consider first the tree level 2-point function for the heavy quark

$$G^{(2)}(p) = \frac{i}{\not{p} - M_Q} \ . \tag{3}$$

We are interested in momentum representing a quark of velocity v_μ slightly off–shell:

$$p_\mu = M_Q v_\mu + k_\mu \ .$$

Here, 'slightly off–shell' means k_μ is of order Λ, and independent of M_Q. Substituting in eq. (3), and expanding in powers of Λ/M_Q, we obtain, to leading order,

$$G^{(2)}(p) = i\left(\frac{1+\not{v}}{2}\right)\frac{1}{v\cdot k} + \mathcal{O}\left(\frac{\Lambda}{M_Q}\right) \ .$$

We recognize the projection operator of eq. (2), and the propagator of the lagrangian in (1).

Similarly, the 3-point function (a heavy quark and a gluon) is given by

$$G^{(2,1)a}_\mu(p,q) = \frac{i}{\not{p} - M_Q}(-ig_s T^a \gamma^\nu)\frac{i}{\not{p} + \not{q} - M_Q}\Delta_{\nu\mu}(q),$$

where $\Delta_{\nu\mu}(q)$ is the gluon propagator. Expanding as above, we have

$$G^{a(2,1)}_\mu(p,q) = \left(\frac{1+\not{v}}{2}\right)\frac{i}{v\cdot k}(-ig_s T^a v^\nu)\frac{i}{v\cdot(k+q)}\Delta_{\mu\nu}(q) + \mathcal{O}\left(\frac{\Lambda}{M_Q}\right),$$

where we have used

$$\left(\frac{1+\not{v}}{2}\right)\gamma_\nu\left(\frac{1+\not{v}}{2}\right) = \left(\frac{1+\not{v}}{2}\right)v_\nu \ .$$

Again, this corresponds to the vertex obtained from the effective Lagrangian in eq. (1). It is straightforward to extend these results to arbitrary tree-level Green functions, provided only one heavy quark is considered and all other (light) particles carry momentum of order Λ.

The effective Lagrangian in (1) is appropriate for the description of a heavy quark, and indeed a heavy hadron, of velocity v_μ. It does, however, break Lorentz covariance. This is not a surprise, since we have expanded the Green functions about one particular velocity: in boosted frames, the expansion in powers of Λ/M_Q becomes invalid, since the boosted momentum k_μ can become arbitrarily large. Lorentz covariance is recovered, however, if we boost the velocity

$$v_\mu \to \Lambda_{\mu\nu} v_\nu$$

along with everything else. It will prove useful to keep this simple observation in mind[5].

[5] In an alternative method, championed by Georgi[11], the effective Lagrangian $\mathcal{L}_{\text{eff}}$ consists of a sum over the different velocity Lagrangians, $\mathcal{L}^{(v)}_{\text{eff}}$, of eq. (1). Lorentz invariance is recovered at the price of "integrating in" the heavy degrees of freedom. This does not lead to overcounting of states, because the sectors of different velocity do not couple to each other, a fact that Georgi refers to as a "velocity superselection rule". See also [12].

2.2. *What is an effective theory?*

In the previous section an effective Lagrangian $\mathcal{L}_{\rm eff}^{(v)}$ was introduced such that Green functions $\widetilde{G}_v(k;q)$ calculated from it agreed, at tree level, with corresponding Green functions $G(p;q)$, in the original field theory (that is, QCD) to leading order in the large mass

$$G(p;q) = \widetilde{G}_v(k;q) + \mathcal{O}(\Lambda/M_Q) \qquad \text{(tree level)} \ . \tag{4}$$

Here, Λ stands for any component of k_μ or of the q's, or for a light quark mass, and $p = M_Q v + k$.

The remarkable thing about eq. (4) is that while the left hand side depends on M_Q, and generally in a complicated way, the first term of the right side is independent of M_Q and is a good approximation to the left side if $M_Q \gg \Lambda$. Albeit remarkable, this fact is useless unless extended beyond tree level.

Does eq. (4) hold beyond tree level? The answer is a resounding 'NO', but the correct version is still close in form to eq. (4), and, more importantly, as we will see, useful:

$$G(p;q;\mu) = \widetilde{C}(M_Q/\mu, g_s)\widetilde{G}_v(k;q;\mu) + \mathcal{O}(\Lambda/M_Q) \qquad \text{(beyond tree level)} \ . \tag{5}$$

This equation will be proved in the following section. The Green functions G and $\widetilde{G}_v$ are renormalized, so they depend on a renormalization point μ. The function $\widetilde{C}$ is independent of momenta or light quark masses: it is independent of the dynamics of the light degrees of freedom. It is there because the left hand side has some terms which grow logarithmically with the heavy mass, $\ln(M_Q/\mu)$. The beauty of eq. (5) is that *all of the logarithmic dependence on the heavy mass factors out.* Better yet, since $\widetilde{C}$ is dimensionless, it is a

function of the ratio M_Q/μ only, and not of M_Q and μ separately[6]. To find the dependence on M_Q it suffices to find the dependence on μ. This in turn is dictated by the renormalization group equation. More on this later.

Equation (5) is useful only to the extent the $\widetilde{G}$ is really independent of M_Q. One should be careful to use M_Q-independent renormalization conditions in the effective theory. This might seem like a trivial point, but in proving (5) we will use an intermediate renormalization which is M_Q-dependent. Also, in general the renormalization scheme and point, μ, need not be the same on both sides of eq. (5). The additional generality translates into practical complications and it is best avoided. One is therefore led to choose a mass-independent subtraction scheme on both sides of eq. (5). In practice, it is convenient to use dimensional regularization with an MS scheme.

It is instructive to note the similarities of the HQET and the more usual kind of effective theory —call it 'normal'— in which a heavy particle is 'integrated out'. Take, for example, the case of weak interactions at low energies, that is, when all the momenta involved are much smaller than the W-boson mass. Everyone knows that we can account for the effects of the W-boson by adding to the Lagrangian terms of the form

$$\Delta\mathcal{L}_{\rm eff} = \frac{1}{M_W^2}\kappa\mathcal{O} \ ,$$

where $\mathcal{O}$ is a 4-fermion operator and κ contains mixing angles and factors of the weak coupling constant. This is simply the statement that a Green function G of the original theory (the standard model including QCD) can be approximated

[6] Actually, additional μ independence is implicit in the definition of the renormalized coupling constant g_s. This reflects itself in the explicit form of $\widetilde{C}$; see section 2.4.

by a Green function $\tilde{G}_{\mathcal{O}}$ of the effective theory (a gauge theory of QCD and electromagnetism) with an insertion of the effective Lagrangian:

$$G = \frac{1}{M_W^2}\kappa\tilde{G}_{\mathcal{O}} + \dots \quad .$$

The ellipses stand for terms suppressed by additional powers of $(M_W)^{-2}$. This equation is very similar to eq. (4). It replaces the task of computing the more complicated left side, which depends on M_W, by the computation in the effective theory which is independent of M_W, and indeed, completely free of the W-boson dynamical degrees of freedom. On the right hand side, the factor of $1/M_W^2$ gives the dependence on the W-boson mass simply and explicitly. And incorrectly! Just as above, the full theory has logarithmic dependence on M_W which has not been made explicit. The correct version is[13]

$$G = \frac{1}{M_W^2}\kappa C(M_W/\mu, g_s)\tilde{G}_{\mathcal{O}} + \dots \quad .$$

The function C is, in this case, also known as the 'short distance QCD effect' first calculated by Altarelli and Parisi[14], and Gaillard and Lee[15].

Summing up, an effective theory (of either the 'normal' or the HQ type) is a method for extracting explicitly the leading large mass dependence of amplitudes. Moreover, the rules of computation of the effective theory are completely independent of the large mass.

2.3. The Effective Theory Beyond Tree Level

In section 2.1 we established the validity of the HQET at tree level, and in section 2.2 we saw that beyond tree level things must get complicated. Here we will describe how the HQET works, and will establish the equivalence between the full and effective theories, as given by eq. (5), to 1-loop. The generalization

to all orders in the loop expansion is straightforward, and not really enlightening (see ref. [10]).

Consider a Green function, both in the full and effective theories, for a heavy quark and $n \geq 2$ gluons. It suffices to prove the equivalence for one-particle irreducible (1PI) functions. In figure 1 the left side is calculated in the full theory and the right side in the HQET. The double line stands for the heavy quark propagator in the HQET.

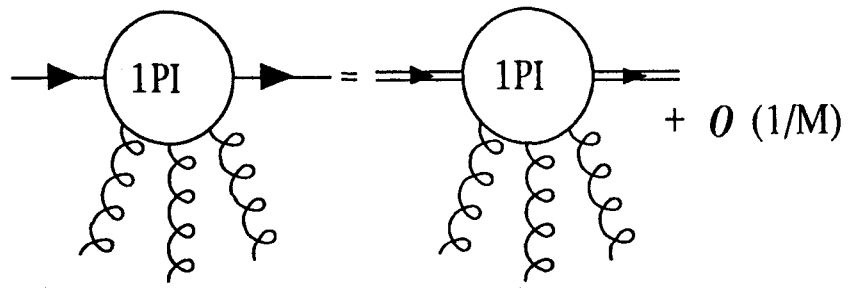

Figure 1

We can prove the validity of the equation represented in figure 1, diagram by diagram (there are several diagrams that contribute to each side of the equation). Consider, for definiteness, the diagrammatic equation in figure 2.

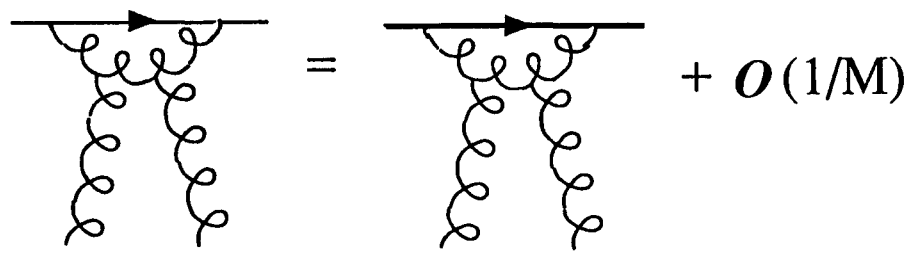

Figure 2

The equation would trivially hold if we could make the propagator replacement

$$\frac{i}{\not{p}+\not{l}-M_Q} \to \left(\frac{1+\not{v}}{2}\right)\frac{i}{v(k+l)}$$

even inside the loop integral. Here $p = M_Q v + k$, and l is the loop momentum. In other words, in the right hand side of figure 2, we take the limit $M_Q \to \infty$ and then integrate, while on the left side we first integrate and then take the limit. Everyone knows that, if both integrals converge, then they agree. And that is the case for figure 2, and, indeed, it is also the case for any 1–loop integral with a heavy quark and $n \geq 2$ external gluons. We have established figure 1 for $n \geq 2$.

We are left with the 2–point ($n = 0$) and 3–point ($n = 1$) functions. These are different from the $n \geq 2$ functions in two ways. First, they receive contributions at tree level. And second, they are divergent at 1–loop. Choose some method of regularization. Dimensional regularization is particularly useful as it preserves gauge invariance (or, more precisely, BRST invariance). The comparison between full and effective theories is simplest if the same gauge and regularization choices are made. For concreteness, consider figure 3.

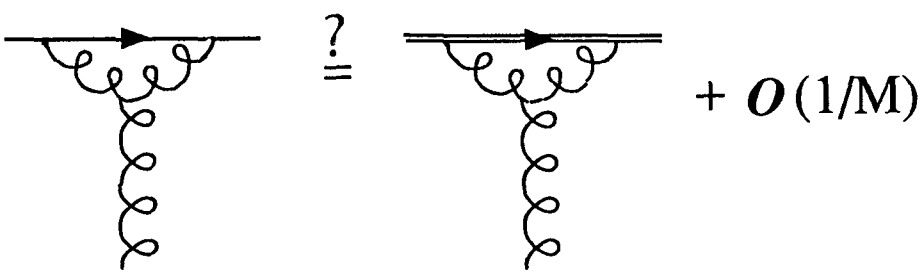

Figure 3

Since both sides are regulated, and therefore finite, we can argue as before. But we run into trouble when we try to remove the regulator. One must renormalize the Green functions by adding counter terms, but there is no guarantee that the counterterms satisfy the same relation as the regulated Green functions of figure 3. To elucidate the relation between counterterms, take a derivative on both sides of figure 3 with respect to either the residual momentum, k_μ, or the gluon external momentum, q_μ. This makes the diagrams finite and the regulator can be removed. Thus, at 1–loop, the relations

$$\frac{\partial}{\partial k_\mu} G^{(2,1)} = \frac{\partial}{\partial k_\mu} \widetilde{G}_v^{(2,1)} + \mathcal{O}\left(\Lambda/M_Q\right) \tag{6}$$

and

$$\frac{\partial}{\partial q_\mu} G^{(2,1)} = \frac{\partial}{\partial q_\mu} \widetilde{G}_v^{(2,1)} + \mathcal{O}(\Lambda/M_Q) \tag{7}$$

hold. The counterterms, or at least the difference between them, are k_μ and q_μ independent. It is a simple algebraic exercise to show, then, that the difference between counterterms is of the form

$$aG^{(2,1)0} + b\widetilde{G}_v^{(2,1)0} \tag{8}$$

where the superscript '0' stands for tree level, and a and b are infinite constants, *i.e.*, independent of k_μ and q_μ. Thus, one can subtract the 1–loop Green functions by standard counterterms, and establish the equality of figure 3.

A similar argument can be constructed for the 2–point function. One must take two derivatives with respect to k_μ, but that is as it should, since the counterterms are linear in momentum.

We have therefore established that, to 1–loop, the renormalized Green functions in the full and effective theories agree. The alert reader must be puzzled

as to the fate of the function $\widetilde{C}(M_Q/\mu, g_s)$ of eq. (5). What has happened is that the constants a and b in the counterterms in eq. (8) are, in general, M_Q dependent. Indeed, if we take derivatives with respect to M_Q, as in eqs. (6) or (7), the degree of divergence is not changed, and one cannot argue that a or b are M_Q independent. The relation between renormalized Green functions that we have derived contains hidden M_Q-dependence in the renormalization prescription for the Green functions in the HQET.

Given two different renormalization schemes, the corresponding renormalized Green functions $\widetilde{G}$ and $\widetilde{G}'$ are related by a finite renormalization

$$\widetilde{G} = z(\mu, g_s)\widetilde{G}' \ .$$

Choosing $\widetilde{G}$ to be the mass-independent subtracted Green function, and $\widetilde{G}'$ the one in our peculiar subtraction scheme, we have that the relation between full and effective theories becomes

$$G^{(2,1)}(p,q;\mu) = \widetilde{C}\,(M_Q/\mu, g_s)\,\widetilde{G}_v^{(2,1)}(k,q;\mu) + \mathcal{O}\,(\Lambda/M_Q) \ ,$$

as advertised in section 2.2. Here, $\widetilde{C}$ is nothing but this finite renormalization $z(\mu, g_s)$. That we can use the same function $\widetilde{C}$ for all Green functions can be established by using the same wave-function renormalization prescription for gluons in the full and effective theories. Otherwise, an additional factor of $z_A^{n/2}$ would have to be included in the relation between $G^{(2,n)}$ and $\widetilde{G}^{(2,n)}$. This completes the argument.

It is worth mentioning that the discussion above *assumes* the renormalizability, preserving BRST invariance, of the effective theory. Although, to my knowledge, this has not been established, there is no obvious reason to doubt that the standard techniques apply in this case.

2.4. *External Currents*

We will often be interested in computing Green functions with an insertion of a current. Consider, the current

$$J_\Gamma = \bar{q}\,\Gamma Q \tag{9}$$

in the full theory, where Γ is some Dirac matrix, and q a light quark. In the effective theory, this is replaced according to

$$J_\Gamma(x) \to e^{-iM_Q vx}\tilde{J}_\Gamma(x) \ , \tag{10}$$

where

$$\tilde{J}_\Gamma = \bar{q}\,\Gamma Q_v \ , \tag{11}$$

and it is understood that in $\tilde{J}_\Gamma$ the heavy quark is that of the HQET, satisfying, in particular, $\not{v}Q_v = Q_v$. The exponential factor in eq. (10) reminds us to take the large momentum out through the current, allowing us to keep the external momentum of light quarks and gluons small. The relation between full and effective theories takes the form of an approximate equation between Green functions —and eventually amplitudes— of insertions of these currents:

$$G_{J_\Gamma}(p,p';q;\mu) = \widetilde{C}(M_Q/\mu, g_s)\widetilde{C}_\Gamma(M_Q/\mu, g_s)\widetilde{G}_{v,\tilde{J}_\Gamma}(k,k';q;\mu) + \mathcal{O}(\Lambda/M_Q) \ , \tag{12}$$

where p and p' are the momenta of the heavy quark and the external current, k and k' the corresponding residual momenta, $p = M_Q v + k$, $p' = M_Q v + k'$, and q stands for the momenta of the light degrees of freedom. The factor $\widetilde{C}\,\widetilde{C}_\Gamma$ accounts for the logarithmic mass dependence, as explained earlier. We see that an additional factor, namely, $\widetilde{C}_\Gamma$, is needed in this case to account for the

different scaling behavior of the currents in the full and effective theories. It is convenient to think of the replacement of currents, not as given by eq. (10), but rather by

$$J_\Gamma(x) \to e^{-iM_Q vx}\widetilde{C}_\Gamma(M_Q/\mu, g_s)\tilde{J}_\Gamma(x) \ . \tag{13}$$

In fact, eq. (12), and therefore the replacement in eq. (13), are not quite correct. To reproduce the matrix elements of the current J_Γ of eq. (9), it is necessary to sum over matrix elements of several different 'currents' in the effective theory. The operator $\tilde{J}_\Gamma$ of eq. (11) is just one of them. In addition, one may have to introduce such operators as $\bar{q}\not{v}\Gamma Q_v$. The correct replacement is therefore

$$J_\Gamma(x) \to e^{-iM_Q vx}\sum_i \widetilde{C}_\Gamma^{(i)}(M_Q/\mu, g_s)\widetilde{\mathcal{O}}^{(i)}(x) \ . \tag{14}$$

Here $\widetilde{\mathcal{O}}^{(i)}(x)$ is the collection of the operators of dimension 3 with appropriate quantum numbers. The first operator in the sum, call it $\widetilde{\mathcal{O}}^{(0)}$, is there even at tree level, and corresponds to the operator $\tilde{J}_\Gamma$ of eq. (11).

Another case of interest is that of the insertion of a current of two heavy quarks

$$J_\Gamma = \bar{Q}'\Gamma Q \ .$$

The replacement now is

$$J_\Gamma(x) \to e^{-iM_Q vx + iM_{Q'}v'x}\sum_i \widehat{C}_\Gamma^{(i)}\left(\frac{M_Q}{\mu}, \frac{M_{Q'}}{M_Q}, vv', g_s\right)\widehat{\mathcal{O}}_\Gamma^{(i)}(x) \ . \tag{15}$$

Again, $\widehat{\mathcal{O}}^{(i)}(x)$ stands for the complete list of operators of dimension 3 in the effective theory with the right quantum numbers. Also, the operator $\widehat{\mathcal{O}}^{(0)} = \bar{Q}'_{v'}\Gamma Q_v$ appears in the sum at tree level.

This deserves some explanation. The Green functions now include two heavy quarks, and to properly establish the validity of eq. (15) we should begin

by considering Green functions with two heavy quarks, no insertion of a current. The function $\widehat{C}$ connecting these full and effective Green functions will now, in general, depend on both M_Q and $M_{Q'}$. Moreover, we can not argue that $\widehat{C}$ is independent of the velocities v and v'. In fact, this was true of the simpler case considered in section 2.3; but there, $\widetilde{C}$ could only depend on v_μ through $v^2 = 1$. In the case at hand there is an additional invariant on which $\widehat{C}$ can depend, namely vv'. These observations apply just as well to the correction factors $\widehat{C}_\Gamma^{(i)}$ in eq. (15).

The explicit functional dependence on M_Q in the functions $\widetilde{C}_\Gamma$ and $\widehat{C}_\Gamma$ can be obtained from a study of their dependence on the renormalization point μ. For clarity of presentation we neglect operator mixing for now. When necessary, this can be incorporated without much difficulty. Taking a derivative $d/d\mu$ on both sides of eqs. (5) and (13), we find

$$\mu\frac{d}{d\mu}\widetilde{C}_\Gamma = (\gamma_\Gamma - \tilde{\gamma}_\Gamma)\widetilde{C}_\Gamma$$

where γ_Γ and $\tilde{\gamma}_\Gamma$ are the anomalous dimensions of the currents J_Γ and $\tilde{J}_\Gamma$ in the full and effective theories, respectively. Of particular interest are the cases $\Gamma = \gamma^\mu$ and $\Gamma = \gamma^\mu\gamma_5$. These correspond, in the full theory, to conserved and partially conserved currents, and therefore the corresponding anomalous dimensions vanish, giving

$$\mu\frac{d\widetilde{C}_\Gamma}{d\mu} = -\tilde{\gamma}_\Gamma\,\widetilde{C}_\Gamma \qquad (\Gamma = \gamma^\mu,\ \gamma^\mu\gamma_5) \ . \tag{16}$$

Before we solve this equation, we recall that

$$\mu\frac{d}{d\mu} = \mu\frac{\partial}{\partial\mu} + \beta(g_s)\frac{\partial}{\partial g_s} \ .$$

Here β is the QCD β–function, with perturbative expansion

$$\frac{\beta(g)}{g} = -b_0 \frac{g^2}{16\pi^2} + b_1 \left(\frac{g^2}{16\pi^2}\right)^2 + \dots \quad ,$$

and

$$b_0 = 11 - \frac{2}{3} n_f \ ,$$

where n_f is the number of quarks in the theory. For our purposes, n_f should *not* include the heavy quark. This is explained in the famous paper by Appelquist and Carrazone[16]; it simply reflects the fact that the logarithmic scaling of g_s is not affected by heavy quark loops, since these are suppressed by powers of M_Q. Now, the solution to (16) is standard:

$$\widetilde{C}_\Gamma(\mu, g_s) = \exp\left(-\int_{\bar{g}_s(\mu_0)}^{g_s} dg' \frac{\tilde{\gamma}_\Gamma(g')}{\beta(g')}\right) \widetilde{C}_\Gamma(\mu_0, \bar{g}_s(\mu_0))$$

where $\bar{g}_s$ is the running coupling constant defined by

$$\mu' \frac{d\bar{g}_s(\mu')}{d\mu'} = \beta(\bar{g}_s(\mu')) \qquad , \qquad \bar{g}_s(\mu) = g_s \ . \tag{17}$$

Choosing $\mu_0 = M_Q$, and restoring the dependence on M_Q, we have then

$$\widetilde{C}_\Gamma(M_Q/\mu, g_s) = \exp\left(-\int_{\bar{g}_s(M_Q)}^{\bar{g}_s(\mu)} dg' \frac{\tilde{\gamma}_\Gamma(g')}{\beta(g')}\right) \widetilde{C}_\Gamma(1, \bar{g}_s(M_Q)) \ .$$

Therefore, the problem of determining $\widetilde{C}_\Gamma(M_Q/\mu, g_s)$ breaks down into two parts. One is the determination of the anomalous dimensions $\tilde{\gamma}_\Gamma$. The other is the calculation of $\widetilde{C}_\Gamma(1, \bar{g}_s(M_Q))$. Both can be done perturbatively, and $\widetilde{C}_\Gamma(M_Q/\mu, g_s)$ can thus be computed, provided μ and M_Q are large enough so that $\bar{g}_s(\mu)$ and $\bar{g}_s(M_Q)$ are small. One finds, for example, that in leading order $\widetilde{C}_\Gamma$ is Γ independent and there is no mixing:

$$\widetilde{C}_\Gamma(M_Q/\mu, g_s) = \left(\frac{\bar{\alpha}_s(M_Q)}{\bar{\alpha}_s(\mu)}\right)^{a_I} \ ,$$

where $\bar{\alpha}_s \equiv \bar{g}_s^2/4\pi$, and[17] $a_I \equiv -c_1/2b_0 = -6/(33 - 2n_f)$.

We now turn to the computation of the coefficient $\widehat{C}_\Gamma$ for the current of two heavy quarks in eq. (15). A new difficulty arises. Because $\widehat{C}_\Gamma$ depends on three dimensionful quantities, namely the masses M_Q and $M_{Q'}$, and the renormalization point μ, its functional dependence is not determined from the renormalization group equation (even if we neglect the implicit dependence of g_s on μ). Two different approximations have been developed to deal with this problem:

I) Treat the ratio $M_{Q'}/M_Q$ as a dimensionless parameter, and study the dependence of $\widehat{C}_\Gamma$ on $M_{Q'}/\mu$ through the renormalization group[18]. This is just like what was done for the heavy-light case, so we can transcribe the result:

$$\widehat{C}_\Gamma\left(\frac{M_{Q'}}{\mu}, \frac{M_{Q'}}{M_Q}, vv', g_s\right) \approx \exp\left(-\int_{\bar{g}_s(M_{Q'})}^{\bar{g}_s(\mu)} dg' \frac{\hat{\gamma}_\Gamma(g')}{\beta(g')}\right) \widehat{C}_\Gamma\left(1, \frac{M_{Q'}}{M_Q}, vv', \bar{g}_s(M_{Q'})\right).$$

Again

$$\widehat{C}_\Gamma\left(1, \frac{M_{Q'}}{M_Q}, vv', \bar{g}_s(M_{Q'})\right) = 1 + \mathcal{O}(\bar{\alpha}_s(M_{Q'})) \ .$$

But, now, the correction of order $\bar{\alpha}_s(M_{Q'})$ is a function of $M_{Q'}/M_Q$. This method has the advantage that the complete functional dependence on $M_{Q'}/M_Q$ is retained, order by order in $\bar{\alpha}_s(M_{Q'})$. Nevertheless, it fails to re–sum the leading–logs between the scales $M_{Q'}$ and M_Q, *i.e.*, it does not include the effects of running of the QCD coupling constant between M_Q and $M_{Q'}$. Therefore, this method is useful when $M_{Q'}/M_Q \sim 1$, or, equivalently, when $(\bar{\alpha}_s(M_{Q'}) - \bar{\alpha}_s(M_Q))/\bar{\alpha}_s(M_Q) \ll 1$.

II) Treat the ratio $M_{Q'}/M_Q$ as small. Expand first in a HQET treating Q as heavy and Q' as light. The corrections are not just of order Λ/M_Q but also $M_{Q'}/M_Q$, but this is assumed to be small (even if much larger than Λ/M_Q).

Then expand from this HQET, in powers of $\Lambda/M_{Q'}$, by constructing a new HQET where both Q and Q' are heavy[19]. The calculation of $\widehat{C}_\Gamma$ then proceeds in two steps. The first gives a factor just like that of the heavy-light current, in (17)

$$\exp\left(-\int_{\bar g_s(M_Q)}^{\bar g_s(\mu)} dg' \frac{\tilde\gamma_\Gamma(g')}{\beta(g')}\right) \widetilde{C}_\Gamma(1,\bar g_s(M_Q)) \ .$$

The second factor is as in method I, above, but neglecting $M_{Q'}/M_Q$. Moreover, the current $\tilde J_\Gamma$ is not conserved, so the anomalous dimension to be used is not $-\hat\gamma_\Gamma$ but $\tilde\gamma_\Gamma - \hat\gamma_\Gamma$. Finally, we must make explicit the fact that in the first and second steps the appropriate β-functions differ in the number of active quarks. We therefore label the one in the second step β' and the corresponding running coupling constant $\bar g_s'$. The second factor is

$$\exp\left(\int_{\bar g_s(M_{Q'})}^{\bar g_s(\mu)} dg' \frac{\tilde\gamma_\Gamma(g')}{\beta(g')} - \int_{\bar g_s'(M_{Q'})}^{\bar g_s'(\mu)} dg'' \frac{\hat\gamma_\Gamma(g'')}{\beta'(g'')}\right) \widehat{C}_\Gamma(1,0,vv',\bar g_s(M_{Q'})) \ .$$

Combining factors gives

$$\widehat{C}_\Gamma\left(\frac{M_{Q'}}{\mu},\frac{M_{Q'}}{M_Q},vv',g_s\right) \approx \exp\left(-\int_{\bar g_s(M_Q)}^{\bar g_s(M_{Q'})} dg' \frac{\tilde\gamma_\Gamma(g')}{\beta(g')} - \int_{\bar g_s'(M_{Q'})}^{\bar g_s'(\mu)} dg'' \frac{\hat\gamma_\Gamma(g'')}{\beta'(g'')}\right) \times \widetilde{C}_\Gamma(1,\bar g_s(M_Q))\widehat{C}_\Gamma(1,0,vv',\bar g_s(M_{Q'})) \ .$$

The advantage of method II over method I is that it does include the effects of running between M_Q and $M_{Q'}$. The disadvantage is that it neglects powers of $M_{Q'}/M_Q$. (Actually, the result can be improved by reincorporating the $M_{Q'}/M_Q$ dependence, as a power series expansion in this ratio).

For example, in method II eqn. (15) becomes, in leading order,[19]

$$\bar c\gamma^\mu b \to \left(\frac{\bar\alpha_s(m_b)}{\bar\alpha_s(m_c)}\right)^{a_I} \left(\frac{\bar\alpha_s'(m_c)}{\bar\alpha_s'(\mu)}\right)^{a_L} \bar c_{v'}[(1+\kappa)\gamma^\mu + (\lambda_b - \lambda_c(vv'))\not v\gamma^\mu]b_v \quad (18)$$

for $\Gamma = \gamma^\mu$, and

$$\bar c\gamma^\mu\gamma_5 b \to \left(\frac{\bar\alpha_s(m_b)}{\bar\alpha_s(m_c)}\right)^{a_I} \left(\frac{\bar\alpha_s'(m_c)}{\bar\alpha_s'(\mu)}\right)^{a_L} \bar c_{v'}[(1+\kappa)\gamma^\mu\gamma_5 - (\lambda_b + \lambda_c(vv'))\not v\gamma^\mu\gamma_5]b_v \quad (19)$$

for $\Gamma = \gamma^\mu\gamma_5$, where

$$\lambda_b = \frac{\alpha_s(m_b)}{3\pi}, \qquad \lambda_c(vv') = \frac{2\alpha_s(m_c)}{3\pi} r(vv') \ ,$$
$$a_L(vv') = \frac{8}{33-2n_f}[vv' r(vv') - 1] \ ,$$
$$r(x) \equiv \frac{1}{\sqrt{x^2-1}} \ln\left(x + \sqrt{x^2-1}\right) \ ,$$

and κ is of order α_s but a subleading–log.

3. Symmetries

3.1. *Flavor – SU(N)*

The Lagrangian for N species of heavy quarks, all with velocity v, is

$$\mathcal{L}_{\rm eff}^{(v)} = \sum_{j=1}^{N} \bar Q_v^{(j)}\, iv{\cdot}D\, Q_v^{(j)} \ .$$

This Lagrangian has a $U(N)$ symmetry[20,5]. The subgroup $U(1)^N$ corresponds to flavor conservation of the strong interactions, and was a good symmetry in the original theory. The novelty in the HQET is then the nonabelian nature of the symmetry group. This leads to relations between properties of heavy hadrons with different quantum numbers. Please note that these will be relations between hadrons of a given velocity, even if of different momentum (since typically $M_{Q_i} \neq M_{Q_j}$ for $i \neq j$). Including the b and c quarks in the HQET, so that $N = 2$, we see that the B and D mesons form a doublet under flavor–$SU(2)$.

This flavor–$SU(2)$ is an approximate symmetry of QCD. It is a good symmetry to the extent that

$$m_c \gg \Lambda \qquad \text{and} \qquad m_b \gg \Lambda \ .$$

These conditions can be met even if $m_b - m_c \gg \Lambda$. This is in contrast to isospin symmetry, which holds because $m_d - m_u \ll \Lambda$.

In the atomic physics analogy of the Introduction, this symmetry implies the equality of chemical properties of different isotopes of an element.

3.2. *Spin* - $SU(2)$

The HQET Lagrangian involves only two components of the spinor Q_v. Recall that

$$\left(\frac{1-\not{v}}{2}\right) Q_v = 0.$$

The two surviving components enter the Lagrangian diagonally, *i.e.*, there are no Dirac matrices in

$$\mathcal{L}_{\text{eff}}^{(v)} = \bar{Q}_v \, iv{\cdot}D \, Q_v .$$

Therefore, there is an $SU(2)$ symmetry of this Lagrangian which rotates the two components of Q_v among themselves[21,5].

Please note that this "spin"–symmetry is actually an *internal* symmetry. That is, for the symmetry to hold no transformation on the coordinates is needed, when a rotation among components of Q_v is made. On the other hand, to recover Lorentz covariance, one does the usual transformation on the light–sector, including a Lorentz transformation of coordinates and in addition a Lorentz transformation on the velocity v_μ. A spin–$SU(2)$ transformation can be added to this procedure, to mimic the original action of Lorentz transformations.

To make it plain that this symmetry has nothing to do with "spin" in the usual sense, consider the large mass limit for a vector particle[22]. Using, again, $p = mv + k$, and expanding the propagator

$$-i\,\frac{g_{\mu\nu} - p_\mu p_\nu/m^2}{p^2 - m^2} = -i\left(\frac{1}{2m}\right)\frac{g_{\mu\nu} - v_\mu v_\nu}{v{\cdot}k} + \mathcal{O}(\frac{1}{m^2}) \ ,$$

we see that the Lagrangian for the HVET (Heavy Vector Effective Theory) is

$$\mathcal{L}_{\text{eff}}^{(v)} = A_{v\mu}^\dagger \, iv{\cdot}D A_{v\mu} \ ,$$

with the constraint

$$(v_\mu v_\nu - g_{\mu\nu}) A_{v\nu} = A_{v\mu} \ .$$

We have rescaled the vector field by $\sqrt{2m}$, so the field has mass dimensions 3/2. The effective Lagrangian is invariant under an $SU(3)$ group of transformations, rotating the three components of the vector field among themselves. Note that the "spin" symmetry is not associated with $SU(2)$ in this case.

The symmetry of the theory is larger than the product of the flavor and spin symmetries. If there are N_S, N_F, and N_V species of heavy scalars, fermions, and vectors, respectively, all transforming the same way under color–$SU(3)$, the symmetry of the effective theory is $SU(N_S + 2N_F + 3N_V)$.

3.3. *Spectrum*

The internal symmetries of the effective Lagrangian are explicitly realized as degeneracies in the spectrum and as relations between transition amplitudes. In this section we will consider the spectrum of the theory[23].

Keep in mind that momenta, and therefore energies and masses, are measured in the HQET relative to $M_Q v_\mu$. Therefore, when we state that in the

HQET the B and D mesons are degenerate, the implication is that the physical mesons differ in their masses by $m_b - m_c$.

For now let us specialize to the rest frame $v = (1, \mathbf{0})$. The total angular momentum operator $\mathbf{J}$, *i.e.*, the generator of rotations, can be written as

$$\mathbf{J} = \mathbf{L} + \mathbf{S} \ ,$$

where $\mathbf{L}$ is the angular momentum operator of the light degrees of freedom, and $\mathbf{S}$, the angular momentum operator for the heavy quark, agrees with the generator of spin-$SU(2)$. Since $\mathbf{J}$ and $\mathbf{S}$ are separately conserved, $\mathbf{L}$ is also separately conserved. Therefore, the states of the theory can be labeled by their $\mathbf{L}$ and $\mathbf{S}$ quantum numbers $(l, m_l; s, m_s)$. Of course, $s = 1/2$, so m_s is $1/2$ or $-1/2$ only.

The simplest state has $l = 0$ and, therefore, $J = 1/2$. We will refer to it as the Λ_Q, by analogy with the nonrelativistic potential constituent quark model of the Λ–baryon, where the strange quark combines with a $l = 0$ combination of the two light quarks.

Next is the state with $l = 1/2$. It leads to $J = 0$ and $J = 1$. We deduce that there is a meson and a vector meson that are degenerate. For the b-quark, the B and B^* fit the bill. They are the lowest lying $B = -1$ states. The lowest lying $C = 1$ states are the D and D^* mesons. These again can very well be assigned to our $J = 0$ and $J = 1$ multiplet. The difference $M_{D^*} - M_D = 145$ MeV is reasonably smaller than the splitting between the D^* and the next state, the D_1, with $M_{D_1} - M_{D^*} = 410$ MeV.

The splittings of B and B^* and of D and D^* result from symmetry breaking effects. These must be corrections of order Λ/M_Q to the HQET predictions.

Therefore, one must have $M_{B^*} - M_B = \Lambda^2/m_b$ and analogously for the D–D^* pair. Therefore

$$\frac{M_{B^*} - M_B}{M_{D^*} - M_D} = \frac{m_c}{m_b} \ .$$

Approximating m_c and m_b by M_D and M_B, respectively, we get $\sim 1/3$ on the right side, in remarkable agreement with the left side. Although these results also follow from potential models of constituent quarks, it is important that they can be derived in this generality, and this simply.

The states with $l = 3/2$ have $J = 1$ and 2. The D_1 and D_2^*, with $M_{D_2^*} - M_{D_1} = 40$ MeV, are remarkably closely spaced (and of course, have the appropriate quantum numbers to form a spin multiplet).

It is convenient to represent the states in a multiplet in a way that makes manifest their properties under the spin symmetry. For fixed l, the spin J of the states related by the spin-$SU(2)$ is $J = l \pm \frac{1}{2}$. The representation we are looking for is a symbol $\chi_{\alpha a}^{(\pm)A}$, where the index α is the heavy quark spin, a and A correspond to the z-components of $\mathbf{L}$ and $\mathbf{J}$ respectively, and the values of l and J are implicitly understood (only the $+/-$ super–index is needed to distinguish between the $J = l + \frac{1}{2}$ and $J = l - \frac{1}{2}$ states). This problem is nothing but the composition of $\mathbf{L}$ and $\mathbf{S}$ into $\mathbf{J}$, and the solution is in Clebsch-Gordan coefficients

$$\chi_{\alpha a}^{(\pm)A} = C\,(l\,a; s\,\alpha | J\,A) = C^{JA}_{l\,a, s\,\alpha} \ ,$$

where $s = \frac{1}{2}$ and $J = l \pm \frac{1}{2}$. (The last equality defines a short version of the symbol.)

3.4. *Strong Transitions*

As an example[23] of the use of the symmetries of the HQET to dynamical processes, consider the amplitudes for the strong decays of any member of the $J = l \pm \frac{1}{2}$ multiplet to the $J = l' \pm \frac{1}{2}$ multiplet, and a light hadron h with orbital angular momentum L_h about the (static) heavy quark, and with total angular momentum J_h, *i.e.*, if the spin of h is $\mathbf{S}_h$ then $\mathbf{J}_h = \mathbf{L}_h + \mathbf{S}_h$.

Using the results of the previous section, we can represent the members of the $J = l \pm \frac{1}{2}$ multiplet by $\chi^{(\pm)A}_{\alpha a}$, and those of the $J = l' \pm \frac{1}{2}$ multiplet by $\chi'^{(\pm)B}_{\beta b}$. The spin-$SU(2)$ symmetry implies that the amplitude must be proportional to

$$\sum_{\alpha} (\chi'^{(\pm)B}_{\alpha b})^* \chi^{(\pm)A}_{\alpha a} \ .$$

In the transition of the "brown muck" with angular momentum l to that with l' and h we must combine the angular momentum of the products to give that of the originating state. To combine l' and h into l we multiply by $C^{la}_{J_h m_h, l' b}$, set $m_h + b = a$ and sum over m_h. Furthermore, to combine the light hadron h and the heavy final state hadron χ' into a state of angular momentum $J = l \pm \frac{1}{2}$, we multiply by $C^{l \pm \frac{1}{2} A}_{J_h m_h, l' \pm \frac{1}{2} B}$, set $m_h + B = A$ and sum over B. Thus we have

$$\begin{aligned} &\mathcal{A}(\chi \to (\chi' h)_{J_h L_h}) = \\ &\qquad \mathcal{A}(l', l, L_h, J_h) \sum (C^{la}_{J_h m_h, l' b})^* C^{l \pm \frac{1}{2} A}_{J_h m_h, l' \pm \frac{1}{2} B} (\chi'^{(\pm)B}_{\alpha b})^* \chi^{(\pm)A}_{\alpha a} \ , \end{aligned} \tag{20}$$

where $\mathcal{A}(l', l, L_h, J_h)$ is the reduced matrix element. The sum is over a and b, with $\alpha = A - a$, $m_h = a - b$ and $B = A - a + b$.

As an application, consider the decays of the states in the $l = 3/2$ multiplet —say the D_1 and D_2^*— to those in the $l = 1/2$ multiplet —the D and D^*— and one pion. From eq. (20) it is easy to check that D_2^* has decay amplitudes in the proportions $\sqrt{2/5} : \sqrt{3/5}$ to the $L_h = 2$ states $D\pi$ and $D^*\pi$, while its multiplet partner decays at the same total rate exclusively to $D^*\pi$. After including kinematic and phase space factors, these predictions work remarkably well[23].

3.5. *Covariant Representation of States*

In the chapters that follow we will be interested in extracting the consequences of the spin and flavor symmetries of the HQET to a variety of processes. It is convenient to develop a formalism that automatically extracts this information for us[19]. I follow the simple presentation of ref. [24].

A prototypical example of an application is the computation of relations between form factors in semileptonic $\bar{B}$ to D and D^* decays. There one needs to study the matrix elements

$$\langle D(v) | \bar{c}_v \Gamma b_{v'} | B(v') \rangle \quad \text{and} \quad \langle D^*(v), \epsilon | \bar{c}_v \Gamma b_{v'} | B(v') \rangle \ . \tag{21}$$

We would like to represent these $l = 1/2$ mesons as the product

$$u_Q \bar{v}_q \ , \tag{22}$$

where u_Q is a spinor representing the heavy quark, $\not{v} u_Q = u_Q$, and v_q is an anti-spinor representing the light stuff with $l = 1/2$, satisfying $\bar{v}_q \not{v} = \bar{v}_q$. The product in (22) is a superposition of states with $J = 0$ and 1. To identify the pseudo-scalar meson P and the vector meson $V(\epsilon)$ with polarization ϵ, $\epsilon v = 0$, we must form appropriate linear combinations of the spin up and down spinors. This is most easily done in the rest frame $v = (1, 0)$; the result will be generalized to arbitrary v by boosting. In the Dirac representation the spin operator is $\mathbf{S} = \gamma^5 \gamma^0 \gamma / 2$ so that the spinor basis $u^{(1)}_\alpha = \delta_{1\alpha}$ and $u^{(2)}_\alpha = \delta_{2\alpha}$ corresponds to spin up and spin down, and the anti-spinor basis $v^{(1)}_\alpha = -\delta_{3\alpha}$ and $v^{(2)}_\alpha = -\delta_{4\alpha}$ corresponds to spin down and spin up. With $\mathbf{S}(u\bar{v}) = (\mathbf{S}u)\bar{v} + u(\mathbf{S}\bar{v})$ it is easy to check that the combination

$$u^{(1)}_Q \bar{v}^{(1)}_q + u^{(2)}_Q \bar{v}^{(2)}_q = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} = \left(\frac{1 + \gamma^0}{2} \right) \gamma^5$$

has zero spin, while

$$u_Q^{(1)}\bar{v}_q^{(2)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \sigma_1 + i\sigma_2 \\ 0 & 0 \end{pmatrix} = \left(\frac{1+\gamma^0}{2}\right)\not{\epsilon}^{(+)}$$

$$u_Q^{(1)}\bar{v}_q^{(1)} - u_Q^{(2)}\bar{v}_q^{(2)} = \begin{pmatrix} 0 & \sigma_3 \\ 0 & 0 \end{pmatrix} = \left(\frac{1+\gamma^0}{2}\right)\not{\epsilon}^{(0)}$$

$$u_Q^{(2)}\bar{v}_q^{(1)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \sigma_1 - i\sigma_2 \\ 0 & 0 \end{pmatrix} = \left(\frac{1+\gamma^0}{2}\right)\not{\epsilon}^{(-)}$$

with $\epsilon^{(\pm)} = (0, 1, \pm i, 0)$ and $\epsilon^{(0)} = (0, 0, 0, 1)$, have total spin 1, with third component 1, 0 and -1, respectively. Thus, for arbitrary velocity v one obtains the representation for pseudoscalar and vector mesons:

$$\widetilde{M}(v) = \left(\frac{1+\not{v}}{2}\right)\gamma^5 \qquad \widetilde{M}^*(v,\epsilon) = \left(\frac{1+\not{v}}{2}\right)\not{\epsilon}\ .$$

By construction, the spin symmetry acts on this representation only on the first index of the matrices $\widetilde{M}(v)$ and $\widetilde{M}^*(v,\epsilon)$.

The power of this machinery can now be displayed. Consider the matrix elements (21). Using the above representation of states and noting that the result should transform under the spin symmetry just as the matrix Γ, we have

$$\langle D(v)|\bar{c}_v\Gamma b_{v'}|\bar{B}(v')\rangle = -\xi(vv')\mathrm{Tr}\,\overline{\widetilde{D}}(v)\Gamma\widetilde{B}(v') \tag{23a}$$

$$\langle D^*(v)\epsilon|\bar{c}_v\Gamma b_{v'}|\bar{B}(v')\rangle = -\xi(vv')\mathrm{Tr}\,\overline{\widetilde{D}}^*(v,\epsilon)\Gamma\widetilde{B}(v')\ , \tag{23b}$$

where $\overline{X} = \gamma^0 X^\dagger \gamma^0$. The common factor $-\xi(vv')$ plays the role of the reduced matrix element in the Wigner–Eckart theorem. We will explore the consequences of eqs. (23) in depth in section 5.2.

An even simpler case is that of the $l = 0$ multiplet. In this case the states must transform as a spinor and are obviously represented by $u^{(s)}$, a Dirac spinor satisfying $\not{v}u = u$. This formalism can be extended[24] to deal with multiplets of arbitrary l.

4. Meson Decay Constants

4.1. Preliminaries

The pseudoscalar decay constant is one of the first physical quantities studied in the context of HQET's. For a heavy-light pseudoscalar meson X of mass M_X, the decay constant f_X, we will see, scales like $1/\sqrt{M_X}$. This was known before the formal development of HQET's, although the arguments relied on models of strong interactions. The HQET will give us a systematic way of obtaining this result. Moreover, it will give us the means of studying corrections to this prediction.

The decay constant f_X is defined through

$$\langle 0|A_\mu^{(0)}|X(p)\rangle = f_X p_\mu\ ,$$

where $A_\mu = \bar{q}\gamma_\mu\gamma_5 Q$ is the heavy-light axial current, and the meson has the standard relativistic normalization

$$\langle X(p')|X(p)\rangle = 2E\delta^{(3)}(\mathbf{p} - \mathbf{p}')\ . \tag{24}$$

Thus, the states have mass-dimension -1. Analogous definitions can be made for other mesons. For example, for the vector meson X^* (the $l = 1/2$ partner of X), has

$$\langle 0|V_\mu(0)|X^*(p,\epsilon)\rangle = f_{X^*}\epsilon_\mu\ . \tag{25}$$

Note that the mass-dimensions of f_X and f_{X^*} are 1 and 2, respectively.

4.2. Pseudoscalar Decay Constant in the HQET

Consider the decay constant of the meson state in the HQET. The effective pseudoscalar decay constant $\tilde{f}_X$ is defined by

$$\langle 0|\widetilde{A}_\mu(0)|\widetilde{X}(v)\rangle = \tilde{f}_X v_\mu \tag{26}$$

The state in the HQET, $|\widetilde{X}\rangle$, is normalized *à la* Bjorken and Drell, to $2E/M_X$ rather than to $2E$:

$$\langle \widetilde{X}(v')|\widetilde{X}(v)\rangle = 2v^0\delta^{(3)}(\mathbf{v}-\mathbf{v}')\ . \tag{27}$$

Actually, defining states in the HQET requires some care, but I will just assume it all works and merely refer the interested reader to the literature[12]. Obviously, since the normalization of states and the dynamics are M_Q independent, so is $\tilde{f}_X$. To relate $\tilde{f}_X$ to the physical f_X simply multiply eq. (26) by $\sqrt{M_X}$, to restore the normalization of states of eq. (24), and write $v_\mu = p_\mu/M_X$. Thus we arrive at

$$f_X = \tilde{f}_X/\sqrt{M_X}\left(\frac{\bar{\alpha}_s(M_Q)}{\bar{\alpha}_s(\mu)}\right)^{a_I},$$

where the last factor comes from the relation between currents in the full and effective theories. The 'constant' $\tilde{f}_X$, is in fact a function of the renormalization point μ; the combination $\tilde{f}_X\bar{\alpha}_s(\mu)^{-a_I}$ is μ-independent to leading-log order.

We have obtained $f_X \sim 1/\sqrt{M_X}$ plus a logarithmic correction. A useful way of quoting the result is, for the physical case of B and D mesons,

$$\frac{f_B}{f_D} = \sqrt{\frac{M_D}{M_B}}\left(\frac{\bar{\alpha}_s(M_B)}{\bar{\alpha}_s(M_D)}\right)^{a_I} \tag{28}$$

4.3. Vector Meson Decay constant

As a simple application of the spin symmetry, consider the pseudoscalar decay constant f_{X^*}. Using the 4×4 notation of section 3.5, the matrix element in eq. (25) that defines the pseudoscalar constant is proportional to

$$\mathrm{Tr}\left(\gamma^\mu\gamma_5\widetilde{M}(v)\right) = \mathrm{Tr}\left(\gamma^\mu\gamma_5\left(\frac{1+\not{v}}{2}\right)\gamma_5\right) = -2v^\mu$$

The matrix element

$$\langle 0|\widetilde{V}^\mu(0)|\widetilde{X}^*(v)\epsilon\rangle = \tilde{f}_{X^*}\epsilon^\mu$$

is proportional to

$$\mathrm{Tr}\left(\gamma^\mu\widetilde{M}^*(v,\epsilon)\right) = \mathrm{Tr}\left(\gamma^\mu\left(\frac{1+\not{v}}{2}\right)\not{\epsilon}\right) = 2\epsilon^\mu$$

with the same constant of proportionality. Therefore

$$\tilde{f}_{X^*} = -\tilde{f}_X$$

The sign is unimportant, since it can be absorbed into a phase redefinition of either state. It is the magnitude that matters. Multiplying by $\sqrt{M_{X^*}} \approx \sqrt{M_X}$ to restore to the standard normalization, we have

$$f_{X^*} = -f_X M_X \tag{29}$$

The predictions eq. (28) and eq. (29) have not been tested experimentally. The difficulty is the small expected branching fraction for the decays $X \to \mu\nu$ or $X^* \to \mu\nu$, for $X = B$ and D. Alternatively, the decay constants f_X and f_{X^*} can be measured in Monte Carlo simulations of lattice QCD. There are indications from such simulations that the $1/M_Q$ corrections to the relation (28) are large[25].

5. Form factors in $\bar{B} \to De\nu$, $\bar{B} \to D^* e\nu$ and $\Lambda_b \to \Lambda_c e\nu$

5.1. *Preliminaries*

The semileptonic decays of a $\bar{B}$-meson to D- or D^*-mesons offer the most direct means of extracting the mixing angle $|V_{cb}|$. In order to extract this angle form experiment, theory must provide the form factors for the $\bar{B} \to D$ and $\bar{B} \to D^*$ transitions. Several means of estimating these form factors can be found in the literature. A popular method consists of estimating the form factor at one value of the momentum transfer $q^2 = q_0^2$, and then introducing the functional dependence on q^2 in some arbitrary, hopefully reasonable, way. The estimate of the form factor at q_0^2 is obtained from some model of strong interactions, like the non-relativistic constituent quark model.

The HQET gives the form factor at the maximum momentum transfer, $q^2 = q^2_{\rm max} = (M_B - M_D)^2$ —the point at which the resulting D or D^* does not recoil in the rest frame of the decaying B-meson. While the functional dependence on q^2 is a non-perturbative problem, it is already progress to have a prediction of the form factor at one point. Moreover, the HQET gives relations between the form factors. One may study these relations experimentally to test the accuracy of the HQET predictions.

The standard definition of form factors in semileptonic $\bar{B}$-meson decays is

$$\langle D(p')|V_\mu|\bar{B}(p)\rangle = f_+(q^2)(p+p')_\mu + f_-(q^2)(p-p')_\mu \tag{30a}$$

$$\langle D^*(p')\epsilon|A_\mu|\bar{B}(p)\rangle = f(q^2)\epsilon^*_\mu + a_+(q^2)\epsilon^* p(p+p')_\mu + a_-(q^2)\epsilon^* p(p-p')_\mu \tag{30b}$$

$$\langle D^*(p')\epsilon|V_\mu|\bar{B}(p)\rangle = ig(q^2)\epsilon_{\mu\nu\lambda\sigma}\epsilon^{*\nu}(p+p')^\lambda(p-p')^\sigma \tag{30c}$$

Here, the states have the standard normalization, eq. (24), and $q^2 \equiv (p - p')^2$. The contribution to the decay rates from the form factors f_- and a_- are suppressed by m_ℓ^2/M_B^2, where m_ℓ is the mass of the charged lepton, and therefore they are usually neglected.

5.2. *Form factors in the HQET*

In the effective theory, we would like to compute the matrix elements of the effective currents $\tilde{V}_\mu$ and $\tilde{A}_\mu$ between states of the $l = \frac{1}{2}$ multiplet. We can take advantage of the flavor and spin symmetries to write these matrix elements in terms of generalized Clebsch-Gordan coefficients and reduced matrix elements, *i.e.*, we use the Wigner-Eckart theorem. We have already introduced the relevant machinery in section 3.5. The matrix elements of the operator $\widetilde{G} = \bar{c}_{v'}\Gamma b_v$ between B and D or D^* states, are given by (*c.f.*, eqs. (23))

$$\langle D(v')|\widetilde{G}|\bar{B}(v)\rangle = -\xi(vv')\mathrm{Tr}\,\overline{\widetilde{D}}(v')\Gamma\widetilde{B}(v) \tag{31a}$$

$$\langle D^*(v')\varepsilon|\widetilde{G}|\bar{B}(v)\rangle = -\xi(vv')\mathrm{Tr}\,\overline{\widetilde{D}}^*(v',\varepsilon)\Gamma\widetilde{B}(v). \tag{31b}$$

Before expanding eqs. (31), we note that the flavor symmetry implies that the B-current form factor between $\bar{B}$-meson states is given by the same reduced matrix element:

$$\langle\bar{B}(v')|\bar{b}_{v'}\Gamma b_v|\bar{B}(v)\rangle = -\xi(vv')\mathrm{Tr}\,\overline{\widetilde{B}}(v')\Gamma\widetilde{B}(v) \tag{32}$$

Using $\Gamma = \gamma^0$, and recalling that B-number is conserved, one finds that ξ is fixed at $v' = v$. With the normalization of states appropriate to the effective theory, eq. (27), and expanding eq. (32) at $v = v'$, one has

$$\xi(1) = 1. \tag{33}$$

The reduced matrix element ξ is the universal function that describes all of the matrix elements of operators $\widetilde{G}$ between $l = \frac{1}{2}$ states. It is known as the Isgur-Wise function after the discoverers of the relations (31) and (32). It

is quite remarkable that the Isgur-Wise function describes both timelike form-factors (as in $\bar{B} \to De\nu$) as well as spacelike form-factors (as in $\bar{B} \to \bar{B}$). The point, of course, is that in both cases it describes transitions between infinitely heavy sources at fixed "velocity-transfer" $(v - v')^2$.

Expanding eq. (31) for $\Gamma = \gamma^\mu$ or $\gamma^\mu\gamma_5$, we have

$$\langle D(v')|\widetilde{V}_\mu|\bar{B}(v)\rangle = \xi(vv')(v_\mu + v'_\mu) \tag{34a}$$

$$\langle D^*(v')\epsilon|\tilde{A}_\mu|\ \bar{B}(v)\rangle = -\xi(vv')[\epsilon^*_\mu(1 + vv') - v'_\mu\epsilon^* v] \tag{34b}$$

$$\langle D^*(v')\epsilon|\widetilde{V}_\mu|\ \bar{B}(v)\rangle = -\xi(vv')[-i\epsilon_{\mu\nu\lambda\sigma}\epsilon^{*\nu}v^\lambda v^\sigma] \tag{34c}$$

It remains to express the physical form factors in terms of the Isgur-Wise functions. We must introduce the coefficient functions $\widehat{C}_\Gamma$ of eq. (15), which in the leading-log approximation are given in (18) and (19). Also, we must multiply by $\sqrt{M_D M_B}$ to restore to the standard normalization of states, and express eqs. (34) in terms of momenta using $v = p/M_B$ and $v' = p'/M_D$. For example, one has,

$$\langle D(p')|V_\nu|B(p)\rangle = \left(\frac{\bar{\alpha}_s(m_b)}{\bar{\alpha}_s(m_c)}\right)^{a_I}\left(\frac{\bar{\alpha}'_s(m_c)}{\bar{\alpha}'_s(\mu)}\right)^{a_L}\xi(vv')\sqrt{M_B M_D}\left(\frac{p_\nu}{M_B} + \frac{p'_\nu}{M_D}\right)$$

It follows that

$$f_\pm(q^2) = \left(\frac{\bar{\alpha}_s(m_b)}{\bar{\alpha}_s(m_c)}\right)^{a_I}\left(\frac{\bar{\alpha}_s(m_c)}{\bar{\alpha}_s(\mu)}\right)^{a_L}\xi(vv')\left(\frac{M_D \pm M_B}{2\sqrt{M_B M_D}}\right) \tag{35}$$

Similarly, f, $a_\pm$ and g can all be written in terms of $\xi(vv')$. Moreover, at $vv' = 1$, one has $q^2 = (M_B v - M_D v)^2 = (M_B - M_D)^2 = q^2_{\max}$ so the normalization eq. (33) gives

$$f_\pm(q^2_{\max}) = \left(\frac{\bar{\alpha}_s(m_b)}{\bar{\alpha}_s(m_c)}\right)^{a_I}\left(\frac{M_D \pm M_B}{2\sqrt{M_B M_D}}\right) \tag{36}$$

We have used $a_L(vv') = 0$ at $vv' = 1$. This is as it should be, for the physical quantity $f_\pm$ is μ-independent. It should be emphasized that there is no μ-dependence of $f_\pm$ in eq. (35): the explicit dependence through $(\bar{\alpha}_s(\mu))^{a_L}$ is cancelled by the implicit dependence on μ of the Isgur-Wise function, $\xi(vv') = \xi(vv', \mu)$.

5.3. *Heavy Baryon Semileptonic Decays*

The same methods can be used to obtain relations among, and normalizations of, the form factors relevant to semileptonic decays of heavy baryons. As an example I will consider here the simplest case, that of transitions between the $l = 0$ states. The case of transitions involving higher l states can be found elsewhere[26][24].

There are three form factors, F_i, for the matrix element of the vector current between Λ_b and Λ_c states, and three more, G_i, for the matrix element of the axial current:

$$\langle\Lambda_c(v', s')|\bar{c}\gamma_\mu b|\Lambda_b(v, s)\rangle = \bar{u}^{(s')}(v')[\gamma_\mu F_1 + v'_\mu F_2 + v_\mu F_3]u^{(s)}(v)\ ,$$

$$\langle\Lambda_c(v', s')|\bar{c}\gamma_\mu\gamma_5 b|\Lambda_b(v, s)\rangle = \bar{u}^{(s')}(v')[\gamma_\mu G_1 + v'_\mu G_2 + v_\mu G_3]\gamma_5 u^{(s)}(v)\ .$$

It is remarkable that all six are given in terms of one universal 'Isgur–Wise' function[26]. This can be derived from arguments similar to those of section 5.2. From the discussion at the end of section 3.5 one has that, in the effective theory, the matrix element of the current is given by

$$\langle\Lambda_c(v', s')|\bar{c}_{v'}\Gamma b_v|\Lambda_b(v, s)\rangle = \zeta(vv')\bar{u}^{(s')}(v')\Gamma u^{(s)}(v)\ , \tag{37}$$

and that ζ is fixed at one point: $\zeta(1) = 1$. Expanding (37) one gets

$$F_1 = G_1 \qquad \text{and} \qquad F_2 = F_3 = G_2 = G_3 = 0.$$

Moreover, $G_1(1) = (\bar{\alpha}_s(m_b)/\bar{\alpha}_s(m_c))^{a_I}$.

5.4. *Form factors in order α_s*

The predicted relations between form factors, and normalizations at $q^2_{\max}$, are only approximate. Indeed, several approximations were made in obtaining

those results. Corrections that arise from subleading order in the $1/M$ expansion will be considered in chapter 6. Here we will discuss corrections of order α_s.

As observed in section 2.4, the vector and axial–vector currents of the full theory, $\bar{c}\Gamma b$, match onto a linear combination of 'currents', *i.e.*, dimension 3 operators, in the effective theory. At one loop, the correspondence between vector and axial currents in the full and effective theories is given by eqs. (18) and (19). The constant λ_b and the function λ_c arise only from 1-loop matching, and are scheme independent. The constant κ receives contributions both from matching at 1-loop, and from 2-loops anomalous dimensions. Leaving out the latter would give a meaningless, scheme dependent, result. Although κ has been computed, it is interesting to note that predictions can be made solely form the 1-loop matching computation.

Indeed, comparing eqs. (34) with eqs. (30), we see that at zeroth order in $\bar{\alpha}_s(m_b)$ or $\bar{\alpha}_s(m_c)$ we have

$$a_+ + a_- = 0.$$

Plugging eq. (19) into eq. (31) we see that, to order $\bar{\alpha}_s(m_c)$ and $\bar{\alpha}_s(m_b)$ there is a computable correction to this combination of form factors, namely

$$\frac{a_+ + a_-}{a_+} = -4\frac{m_c}{m_b}\left[\frac{\bar{\alpha}_s(m_b)}{3\pi} + \frac{2\bar{\alpha}_s(m_c)}{3\pi}r(vv')\right]$$

The constant κ, although difficult to compute, does not change the relations between form factors since it simply rescales the leading order predictions in eq. (5.2.6) by the common factor of $(1+\kappa)$. It does, however, affect the predicted normalization of form factors at $q^2_{\max}$. Since at $v' = v$ the effective vector current is again $\bar{c}_{v'}\gamma_\mu b_v$, but rescaled by $(1+\kappa+\lambda_b-\lambda_c(1))$, the correction to eq. (36) is

$$f_\pm(q^2_{\max}) = (1+\kappa+\lambda_b-\lambda_c(1))\left(\frac{\alpha_s(m_b)}{\alpha_s(m_c)}\right)^{a_I}\left(\frac{M_D \pm M_B}{2\sqrt{M_B M_D}}\right).$$

For a calculation of κ see ref. [27].

6. $1/M_Q$

6.1. *The Correcting Lagrangian*

One of the main virtues of the HQET is that, in contrast to *models* of the strongly bound hadrons, it lets us study systematically the corrections arising from the approximations we have made. To be sure, we've made several approximations already, even within the zeroth order expansion in Λ/M_Q. For example, we have computed the logarithmic dependence on M_Q, *i.e.*, the functions $\widetilde{C}^{(i)}_\Gamma$ and $\widehat{C}^{(i)}_\Gamma$ of eqs. (14) and (15), using perturbation theory. In this section we turn to the corrections of order Λ/M_Q.

The HQET lagrangian was derived, in section 2.1, by putting the heavy quark almost on-shell and expanding in powers of the residual momentum, k_μ, or light quark or gluon momentum, q_μ, over M_Q, which we generally wrote as Λ/M_Q. Let us again derive the effective lagrangian, keeping track, this time, of the terms of order Λ/M_Q.

We will rederive $\mathcal{L}^{(v)}_{\text{eff}}$, including $1/M_Q$ corrections, working directly in configuration space[28]. The heavy quark equation of motion is

$$(i\not{D} - M_Q)Q = 0$$

We can put the quark almost on shell by introducing the redefinition

$$Q = e^{-iM_Q vx}\tilde{Q}_v \tag{38}$$

In terms of $\tilde{Q}_v$, the equation of motion is

$$[i\not{D} + M_Q(\not{v} - 1)]\tilde{Q}_v = 0 \tag{39}$$

If we separate the $(1+\not{v})$ and $(1-\not{v})$ components of $\widetilde{Q}_v$, we see that, as expected, the latter is very heavy and decouples in the infinite mass limit. To project out the components,

$$\widetilde{Q}_v = \widetilde{Q}_v^{(+)} + \widetilde{Q}_v^{(-)}$$

where

$$\widetilde{Q}_v^{(\pm)} = \left(\frac{1 \pm \not{v}}{2}\right) \widetilde{Q}_v \ ,$$

we multiply eq. (6.1.3) by $(\frac{1\pm\not{v}}{2})$. Thus we have the equations

$$iv{\cdot}D\widetilde{Q}_v^{(+)} = -\left(\frac{1+\not{v}}{2}\right) i\not{D}\widetilde{Q}_v^{(-)} \tag{40}$$

and

$$iv{\cdot}D\widetilde{Q}_v^{(-)} + 2M_Q\widetilde{Q}_v^{(-)} = \left(\frac{1-\not{v}}{2}\right) i\not{D}\widetilde{Q}_v^{(+)} \tag{41}$$

These equations can be solved self-consistently by assuming that $\widetilde{Q}_v^{(+)}$ is order $(M_Q)^0$ while $\widetilde{Q}_v^{(-)}$ is order M_Q^{-1}. A recursive solution follows. From eq. (6.1.7)

$$\widetilde{Q}_v^{(-)} = \frac{1}{2M_Q}\left(\frac{1-\not{v}}{2}\right) i\not{D}\widetilde{Q}_v^{(+)} - i\frac{v{\cdot}D}{2M_Q}\widetilde{Q}_v^{(-)}$$

Plugging into eq. (6.1.6) and dropping terms of order $1/M_Q^2$, we have

$$iv{\cdot}D\widetilde{Q}_v^{(+)} = -\left(\frac{1+\not{v}}{2}\right) i\not{D}\frac{1}{2M_Q}\left(\frac{1-\not{v}}{2}\right) i\not{D}\widetilde{Q}_v^{(+)}$$

The right hand side involves

$$\left(\frac{1+\not{v}}{2}\right)\not{D}\left(\frac{1-\not{v}}{2}\right)\not{D}\left(\frac{1+\not{v}}{2}\right)$$
$$= \left(\frac{1+\not{v}}{2}\right)\left[D^2 - (v{\cdot}D)^2 + \tfrac{1}{2}g_s\sigma^{\mu\nu}G_{\mu\nu}\right]\left(\frac{1+\not{v}}{2}\right)$$

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ and $G_{\mu\nu} = \frac{1}{ig_s}[D_\mu, D_\nu]$ is the QCD field strength tensor. This equation of motion is obtained from the lagrangian

$$\mathcal{L}_{\text{eff}}^{(v)} = \bar{Q}_v iv{\cdot}DQ_v + \frac{1}{2M_Q}\bar{Q}_v\left[D^2 - (v{\cdot}D)^2 + \frac{g_s}{2}\sigma^{\mu\nu}G_{\mu\nu}\right]Q_v \tag{42}$$

Here I have reverted to the notation Q_v for $\widetilde{Q}_v^{(+)}$. How to include higher order terms in the $1/M_Q$ expansion into $\mathcal{L}_{\text{eff}}^{(v)}$ should be clear.

The $1/M_Q$ term in $\mathcal{L}_{\text{eff}}^{(v)}$ is treated as small. If it is not, it does not make sense to talk about a HQET in the first place. It is therefore appropriate to use perturbation theory to compute its effects. In this perturbative expansion, the corrections of order $1/M_Q$ to Green functions, and therefore to physical observables, are computed by making a single insertion of the perturbation

$$\Delta\mathcal{L} = \frac{1}{2M_Q}\bar{Q}_v\left[D^2 - (v{\cdot}D)^2 + \frac{g_s}{2}\sigma^{\mu\nu}G_{\mu\nu}\right]Q_v \ .$$

The symmetries of the HQET, discussed at length in chapters 1 and 3, are broken by $\Delta\mathcal{L}$. Under the $SU(N_f)$–flavor symmetry, $\Delta\mathcal{L}$ transforms as a combination of the Adjoint and Singlet representations, while only the chromomagnetic moment operator

$$\bar{Q}_v\sigma^{\mu\nu}G_{\mu\nu}Q_v$$

breaks the spin-$SU(2)$ symmetry: it transforms as a 3 of spin–$SU(2)$.

A single insertion of $\Delta\mathcal{L}$ does include all orders in QCD, and it will often prove difficult to make precise calculations of $1/M_Q$ effects. Since $\Delta\mathcal{L}$ is treated as a simple insertion in Green functions, its treatment in the HQET is entirely analogous to that of current operators of section 2.3. There are coefficient functions that connect the HQET results with the full theory. It is convenient to include them directly into the effective lagrangian as[28,29,30]

$$\Delta\mathcal{L} = \frac{1}{2M_Q}\bar{Q}_v\left[c_1D^2 + c_2(v{\cdot}D)^2 + \tfrac{1}{2}c_3g_s\sigma^{\mu\nu}G_{\mu\nu}\right]Q_v. \tag{43}$$

Here

$$c_i = c_i(M_Q/\mu, g_s)$$

can be determined through the methods discussed extensively in chapter 2. In leading-log, one finds

$$c_1 = -1$$
$$c_2 = 3\left(\frac{\bar{\alpha}_s(\mu)}{\bar{\alpha}_s(M_Q)}\right)^{-8/(33-2n_f)} - 2$$
$$c_3 = -\left(\frac{\bar{\alpha}_s(\mu)}{\bar{\alpha}_s(M_Q)}\right)^{-9/(33-2n_f)} .$$

6.2. *The Corrected Currents*

Just as the lagrangian is corrected in order $1/M_Q$, any other operator is too. In particular, the current operators studied in chapters 2-5, are modified in this order. At tree level, these corrections are given by the change of variables of last section:

$$J_\Gamma = \bar{q}\Gamma Q \to \bar{q}\Gamma e^{-iM_Q v x}\left[Q_v + \frac{1}{2M_Q}\left(\frac{1-\not{v}}{2}\right) i\not{D} Q_v\right].$$

Beyond tree level, this sum of two terms has to be replaced by a more general sum over operators of the right dimensions and quantum numbers. The replacement is

$$J_\Gamma \to e^{-iM_Q v x}\left(\sum_i \widetilde{C}_\Gamma^{(i)} \bar{q}\Gamma_i Q_v + \frac{1}{2M_Q}\sum_j \widetilde{D}_\Gamma^{(j)} \mathcal{O}_j\right) \tag{44}$$

where $\mathcal{O}_j$ are operators of dimension 4 that include, for example, the operators

$$\bar{q}\Gamma i\not{D} Q_v \quad , \quad \bar{q}\Gamma i(v\cdot D) Q_v \quad , \quad \bar{q}\not{v}\Gamma i\not{D} Q_v \ .$$

A complete set of operators, and the corresponding coefficients, $\widetilde{D}_\Gamma^{(i)}$, for the cases $\Gamma = \gamma^\mu$ and $\Gamma = \gamma^\mu\gamma_5$, can be found in refs. [29,31] in the leading-log approximation.

The case of two heavy currents is similar. A straightforward calculation gives

$$J_\Gamma = \bar{Q}'\Gamma Q \to e^{-iM_Q v x + iM_{Q'} v' x}\Big[\bar{Q}'_{v'}\Gamma Q_v + \frac{1}{2M_Q}\bar{Q}'_{v'}\Gamma\left(\frac{1-\not{v}}{2}\right) i\not{D} Q_v + \frac{1}{2M_{Q'}}\bar{Q}'_{v'} i\overleftarrow{\not{D}}\left(\frac{1-\not{v}}{2}\right)\Gamma Q_v\Big] \tag{45}$$

Again, beyond tree level we must replace this expression by a more general sum over operators of dimension four,

$$J_\Gamma \to e^{-iM_Q v x + iM_{Q'} v' x}\left[\sum_i \widehat{C}_\Gamma^{(i)} \bar{Q}'_{v'}\Gamma_i Q_v + \frac{1}{2M_Q}\sum_j \widehat{D}_\Gamma^{(j)} \mathcal{O}_j + \frac{1}{2M_{Q'}}\sum_j \widehat{D}_\Gamma^{\prime(j)} \mathcal{O}_j\right] \tag{46}$$

It is worth pointing out that, in the computation of the coefficient functions $\widetilde{D}_\Gamma^{(j)}$, $\widehat{D}_\Gamma^{(j)}$ and $\widehat{D}_\Gamma^{\prime(j)}$, there is a contribution from the term of order $(1/M_Q)^0$. In computing the coefficient functions to order $1/M_Q$ one must not forget graphs with one insertion of the zeroth order term in the current and one insertion of the first order term in the HQET lagrangian.

6.3. *Corrections of order m_c/m_b*

In the case of semileptonic decays of a beauty hadron to charmed hadron, we introduced earlier an approximation method ("Method II" in section 2.4) in which m_c/m_b was treated as a small parameter. Now, $m_c/m_b \sim 1/3$ and you may justifiably worry that this is not a good expansion parameter. We will see in this section that the corrections are actually of the order of $\alpha_s/\pi(m_c/m_b)$ and therefore small. Moreover, they are explicitly calculable.

The strategy is[29] to look at those corrections of order $1/m_b$ which may be accompanied by a factor of m_c. In the first step of the approximation scheme

we construct a HQET for the b-quark, treating the c-quark as light. We must, of course, keep terms of order $1/m_b$ in this first step. The second step is to go over to a HQET in which the m_c-quark is also heavy. For now, we care only about terms in this HQET that have positive powers of m_c.

In the first step, the hadronic current $\bar{c}\Gamma b$, with $\Gamma = \gamma^\mu$ or $\Gamma = \gamma^\mu\gamma_5$, is replaced according to eq. (44). The question is, which terms in eq. (44) can give factors of m_c when we replace the c-quark by a HQET quark, $c_{v'}$. Recall that, once we complete the second step, all of the m_c dependence is explicit. The answer is that any operators in eq. (44) which have a derivative acting on the c-quark will give a factor of m_c. From eq. (38) we see that a derivative $i\partial_\mu$ acting on the charm quark becomes, in the effective theory, the operation $m_c v'_\mu + i\partial_\mu$. So the prescription is simple: take J_Γ in eq. (44) and replace

$$i\partial_\mu \to m_c v'_\mu$$

in those terms where $i\partial_\mu$ is acting on the charm quark.

For example, if the operator

$$\frac{1}{m_b}\bar{c}\, i\overleftarrow{\not{D}}\,\Gamma b_v$$

is generated at some order in the loop expansion, it gives an operator

$$-\frac{m_c}{m_b}\bar{c}_{v'}\not{v}'\Gamma b_v = -\frac{m_c}{m_b}\bar{c}_{v'}\Gamma b_v \tag{47}$$

after step two is completed.

It is really interesting to note that the resulting correction does not introduce any new unknown form factors. For example, the matrix element of (47) between a $\bar{B}$ and a D is given by eq. (31) only with an additional factor of $-m_c/m_b$ in front.

The calculation described here has been performed in the leading-log approximation in ref. [29]. The correction to the vector current is

$$\Delta V_\mu = \frac{m_c}{m_b}\bar{c}_{v'}(a_1\gamma_\mu + a_2 v_\mu + a_3 v'_\mu)b_v$$

where the coefficients $a_i = a_i(\mu)$, written in terms of

$$z = \frac{\bar{\alpha}_s(m_c)}{\bar{\alpha}_s(m_b)}$$

are

$$\begin{aligned}
a_1 &= \frac{5}{9}(vv'-1) - \frac{1}{18}z^{-6/25} + \frac{2vv'+12}{27}z^{-3/25} - \frac{34vv'-9}{54}z^{6/25} \\
&\qquad - \frac{8}{25}vv'z^{6/25}\ln z \\
a_2 &= \frac{5}{9}(1-2vv') - \frac{13}{9}z^{-6/25} - \frac{44vv'-6}{27}z^{-3/25} - \frac{14vv'-18}{27}z^{6/25} \\
a_3 &= \frac{15}{9} - \frac{2}{3}z^{-3/25} - z^{6/25}
\end{aligned}$$

In particular, this gives a contribution to the form factor, at $v = v'$, of

$$\frac{m_c}{m_b}(a_1 + a_2 + a_3)|_{vv'=1} \simeq .07$$

This is not negligible! It is reassuring that this type of corrections can be extracted explicitly. On the other hand, it should be remembered that both corrections of order $(m_c/m_b)^2$ and of subleading-log order can still be considerable and should be, but have not been, computed.

6.4. Corrections of order $\bar{\Lambda}/m_c$ and $\bar{\Lambda}/m_b$.

Corrections to the form factors for semileptonic decays of B's and Λ_b's that arise from the terms of order $1/m_c$ in the effective lagrangian eq. (42) and the currents eqs. (44) and (46) are, in principle, as large or larger than those

considered in the previous section. It is a welcome surprise that the corrections to the combination of form factors that contribute to the semileptonic decay vanish at the endpoint $vv' = 1$. Thus, the predicted normalization of form factors persists, although, as we will see, not so the relations between form factors.

The decay[32] $\Lambda_b \to \Lambda_c e\nu$ is simpler to analyze than the decays[33] $\bar{B} \to De\nu$ and $\bar{B} \to D^* e\nu$. Moreover, it turns out that for the baryonic decay some relations between form factors survive at this order. For these reasons, we will present here the baryonic case. We will briefly return to the decay of the meson at the end of this section, where we will describe the result.

There are two types of corrections to consider[32], coming from either the modified lagrangian of from the modified current. We start by considering the former. The c_1 and c_2 terms in the effective lagrangian (43) transform trivially under the spin symmetry, contributing to the form factors in the same proportion as the leading term in eq. (37). This effectively renormalizes the function ζ but does not affect relations between form factors.

Moreover, the normalization at the symmetry point $vv' = 1$ is not affected. This is a straightforward application of the Ademollo-Gatto theorem. If j_μ is a symmetry generating current of a hamiltonian H_0, then corrections to the matrix element of the current, at zero momentum, from a symmetry-breaking perturbation to the hamiltonian, ϵH_1, are of order ϵ^2. In the case at hand the Ademollo-Gatto theorem implies that corrections to the normalization of ζ at the symmetry point are of order $(1/m_c)^2$.

The chromomagnetic moment operator in the lagrangian (43) does not give a contribution at all. The spin symmetries imply

$$\langle \Lambda_c(v',s')|\mathrm{T}\int d^4x\,(\bar{c}_{v'}\sigma^{\mu\nu}G_{\mu\nu}c_{v'})(x)(\bar{c}_{v'}\Gamma b_v)(0)|\Lambda_b(v,s)\rangle$$
$$= \zeta_{\mu\nu}(v,v')\bar{u}^{(s')}(v')\sigma^{\mu\nu}\left(\frac{1+\not{v}}{2}\right)\Gamma u^{(s)}(v).$$

The function $\zeta_{\mu\nu}$ must be an antisymmetric tensor and must therefore be proportional to $v'_\mu v_\nu - v'_\nu v_\mu$. But

$$\left(\frac{1+\not{v}'}{2}\right)\sigma^{\mu\nu}\left(\frac{1+\not{v}'}{2}\right)v'_\mu = 0.$$

This, we see, is an enormous simplification. There is no analogous simple reason for the matrix element of the chromomagnetic moment operator to vanish in the case of a meson transition. The chromomagnetic matrix element gives, in that case, uncalculable corrections to the relations between form factors.

We turn next to the contribution from the modification to the current. We need the matrix element of the local operators of order $1/m_c$ in eq. (46). Since the coefficients $\widehat{D}_\Gamma^{(j)}$ in eq. (46) are known only to leading-log order, let us concentrate on the operators that arise from tree level matching, eq. (45). Consider the matrix element

$$\langle \Lambda_c(v',s')|\bar{c}_{v'}i\overleftarrow{D}_\mu\Gamma b_v|\Lambda_b(v,s)\rangle = \bar{u}^{(s')}(v')\Gamma u^{(s)}(v)[Av_\mu + Bv'_\mu],$$

where the form of the right hand side follows again from the spin symmetries. The form factors A and B are not independent. Rather, they are given in terms of ζ. To see this, note that, contracting with v'_μ and using the equations of motion,

$$B = -vv'A. \tag{48}$$

Also, if the mass of the $l = 1/2$ state in the effective theory is $\bar{\Lambda}$, then

$$\langle \Lambda_c(v',s')|i\partial_\mu(\bar{c}_{v'}\Gamma b_v)|\Lambda_b(v,s)\rangle = \bar{\Lambda}(v_\mu - v'_\mu)\zeta(vv')\bar{u}^{(s')}(v')\Gamma u^{(s)}(v).$$

Contracting with v_μ, using the equations of motion and eq. (48) we have

$$A(1-(vv')^2) = \bar{\Lambda}(1-vv')\zeta$$

Therefore, the matrix element of interest is

$$\langle \Lambda_c(v',s')|\bar{c}_{v'} i \overleftarrow{\not{D}} \Gamma b_v|\Lambda_b(r,s)\rangle = \bar{\Lambda}\zeta(vv')\frac{v_\nu - (vv')v'_\nu}{1+vv'}\bar{u}^{(s')}(v')\gamma^\nu \Gamma u^{(s)}(v)$$

where $\Gamma = \gamma^\mu$ or $\gamma^\mu\gamma_5$. Putting it all together, one finds

$$F_1 = G_1\left[1 + \frac{\bar{\Lambda}}{m_c}\left(\frac{1}{1+vv'}\right)\right]$$
$$F_2 = G_2 = -G_1\frac{\bar{\Lambda}}{m_c}\left(\frac{1}{1+vv'}\right)$$
$$F_3 = G_3 = 0$$

Moreover,

$$G_1(1) = \left(\frac{\bar{\alpha}_s(m_b)}{\bar{\alpha}_s(m_c)}\right)^{a_I}$$

as before. Up to an unknown constant, $\bar{\Lambda}$, there are still five relations among six form factors. We can estimate $\bar{\Lambda}$ by writing $\bar{\Lambda} = M_{\Lambda_c} - m_c = (M_{\Lambda_c} - M_D) + (M_D - m_c)$. If the 'constituent' quark mass in the D meson is $\simeq 300MeV$, then $\bar{\Lambda} \simeq 700MeV$. With this, we can estimate the next order corrections to be of the order of $\left(\bar{\Lambda}/2m_c\right)^2 \sim 5\%$. There are, of course, additional computable corrections, of order $\bar{\Lambda}/2m_b$ and $\alpha_s(m_c)/\pi\left(\bar{\Lambda}/2m_c\right)$.

The result of $1/m_c$ corrections to the mesonic transitions is quite different. There both the matrix elements of the correction to the current and of the time order product with the chromo-magnetic moment operator lead to new form factors. The result is that there are incalculable corrections, of order $\bar{\Lambda}/2m_c$, to all the leading order relations between form factors. Even if $\bar{\Lambda}$ is smaller in this case, presumably $\bar{\Lambda} \sim 300$MeV, these corrections may be large, say 10%–20%. Remarkably, at the symmetry point, $v'v = 1$, there are no corrections of order $\bar{\Lambda}/2m_c$ to the leading order predictions. Thus, one may still extract the mixing angle $|V_{cb}|$ with high precision from measurements at the end of the spectrum of the semileptonic decay rates for $B \to De\nu$ and $B \to D^*e\nu$.

7. Further Developments

In this last chapter we try to introduce the reader to some other applications of the HQET. This is not meant to be a complete and thorough compilation of new developments. Space limitations and the fast pace of development of the field prevent us from being thorough. We can only choose a few from a large selection of topics. Moreover, these are not treated in any detail. The interested reader is urged to consult the references for further study.

7.1. Inclusive Semileptonic Decay Rates

It has long been held that the inclusive semileptonic decay rate of a $\bar{B}$ into charmed hadronic states is well approximated by the underlying quark decay width

$$\sum_{X_c}\Gamma\left(\bar{B} \to X_c e\bar{\nu}\right) \approx \Gamma\left(b \to ce\bar{\nu}\right) \tag{49}$$

The HQET provides a derivation of this statement[34]. The result is even finer than the doubly integrated result in Eq. (7.1.1). It can be shown that

$$\left\langle \sum_{X_c}\frac{d^2\Gamma}{dxdy}(\bar{B} \to X_c e\bar{\nu})\right\rangle_f \approx \left\langle \frac{d^2\Gamma}{dxdy}(b \to ce\bar{\nu})\right\rangle_f \tag{50}$$

where $x = q^2/M_B^2 = (p_e + p_{\bar{\nu}})^2/M_B^2$ and $y = p_B p_e/M_B^2$, and the averaging is defined by

$$\langle F(x,y)\rangle_f \equiv \int_{y_{min}}^{y_{max}} dy\, f(y)F(x,y)\ .$$

Here f must be a smooth function, and, in particular, $f = 1$ is a possible choice, leading to eq. (49). Because this is derived systematically, corrections of order Λ/m_b and of order $\bar{\alpha}_s(m_b)/\pi$ can be systematically studied.

It is very important to understand the rôle played by smoothing over y in eq. (50). Although frequently used, the corresponding identity without averaging, *i.e.*, eq. (50) with $f(y) = \delta(y - y_0)$, is not valid. This is most easily seen by considering the region of y close to $y_{\max}$, which is dominated by resonances, *e.g.*, the D and D^*. This means, in particular, that the corresponding formula for $b \to ue\bar{\nu}$, should not be trusted close to the end of the electron energy spectrum. It is not a good idea to extract $|V_{ub}|$ from a study of this kinematic region that uses the free quark model.

7.2. $\bar{B} \to \pi e\bar{\nu}$ and $\bar{B} \to \omega e\bar{\nu}$

The decays $\bar{B} \to De\bar{\nu}$ and $\bar{B} \to D^* e\bar{\nu}$ can be used, as we have seen, to extract the mixing angle $|V_{cb}|$ with high precision. The determination of $|V_{ub}|$ is far more complicated. Experimentally, the fact that $|V_{ub}|/|V_{cb}| \ll 1$, and that charm decays fast to light hadrons makes the process $b \to ue\bar{\nu}$ difficult to observe. While the experimental effort has concentrated mainly on establishing the occurrence of the inclusive process $B \to Xe\nu$ in the kinematic regime inaccessible to the underlying $b \to c$ transition, it is expected that it will shift towards the determination of exclusive modes, such as $\bar{B} \to \omega e\bar{\nu}$. These will give clean, unquestionable evidence of the observation of the underlying $b \to u$ transition. Theoretically, the calculation of either inclusive or exclusive rates is very untrustworthy, the existing results (mainly from phenomenological models) varying wildly and depending sensitively on the choice of parameters.

The HQET suggests a method[35] that may afford higher accuracy in the extraction of $|V_{ub}|$ from exclusive decays. At the very least, since the method follows from the HQET, the corrections to the lowest order predictions can be

studied. This should give us some idea of the uncertainty in the determination of the CKM angle (something that can hardly be said about the existing alternatives).

The idea is simple. As opposed to what was done in the $\bar{B} \to D$ case, the symmetries of the HQET cannot be used to relate the initial and final states anymore: the 'brown muck' of the $\bar{B}$ is not at all the same as that of the ρ (or the π, or any other light quark resonance, for that matter). Nevertheless, the HQET flavor symmetry can be used to relate, for example, the $\bar{B} \to \rho$ and $D \to \rho$ matrix elements. Measure the form factors for the latter and use them in the former. One could even use the light quark flavor $SU(3)$ symmetry to relate the form factors for $D \to \rho$ and the Cabibbo allowed $D \to K^*$.

There are altogether 6 form factors in the $\bar{B} \to \pi$ and $\bar{B} \to \rho$ matrix elements, defined in a way entirely analogous to eqs. (30), and which we label with a B superscript, *e.g.*, $f_\pm^B$. The corresponding D form factors are labeled with the superscript D. The HQET gives

$$\langle \pi(p')|j_\mu|\bar{B}(v)\rangle = \langle \pi(p')|j_\mu|D(v)\rangle$$

$$\langle \rho(p')\epsilon|j_\mu|\bar{B}(v)\rangle = \langle \rho(p')\epsilon|j_\mu|D(v)\rangle,$$

Here, j_μ stands for either V_μ or A_μ. It is straightforward to find the relations between form factors. For example,

$$f_\pm^B(pp') = \frac{1}{2}\left(\sqrt{\frac{M_D}{M_B}} \pm \sqrt{\frac{M_B}{M_D}}\right) f_+^D\left(\frac{M_D}{M_B}pp'\right) + \frac{1}{2}\left(\sqrt{\frac{M_D}{M_B}} \mp \sqrt{\frac{M_B}{M_D}}\right) f_-^D\left(\frac{M_D}{M_B}pp'\right),$$

where we have written $f_\pm^X$ as a function of pp' rather than $q^2 = -2pp' + m_X^2 + m_\pi^2$. Moreover, $f_\pm^X$ scale with the heavy mass as $f_+^X \pm f_-^X \sim M_X^{\mp 1/2}$, and therefore $f_-^X \approx -f_+^X$.

There is a practical difficulty with this proposal. One is that the end of the spectrum for the B decay, $(pp')_{\max} = (m_b^2 + m_\pi^2)/2$ goes well beyond the end of the spectrum for the D decay. Neglecting the pion mass for simplicity, we see that $f_\pm^B(pp')$ cannot be inferred from $f_\pm^D$ for $m_D m_b/2 \leq pp' \leq m_b^2/2$.

To our knowledge, the $1/m_c$ corrections to this process have not been studied.

7.3. *Rare $\bar{B}$ decays*

As mentioned in the introduction, rare $\bar{B}$ decays are believed to be a good probe of new physics. For 'calibration' it is important to obtain precise predictions from the standard model. Unfortunately, this often involves hadronic matrix elements which, needless to say, we can't compute. The HQET gives us a handle on this problem[35]. Again, the trick is to relate the matrix element of interest, say $\bar{B} \to \bar{K}e^+e^-$, to a more easily measured process, like $D \to \bar{K}e\nu$. In fact, it is easy to analyze this example in some detail, for we only need the matrix element of the vector current between the heavy and light pseudoscalar meson states. But this is precisely the same problem as studied in the previous section. Now, though, the semileptonic D decay is not Cabibbo suppressed, so one can do much better!

The processes $\bar{B} \to \bar{K}^*\gamma$ and $\bar{B} \to \bar{K}^*e^+e^-$ receive contributions from a transition magnetic moment operator, *i.e.*, one needs to compute

$$\langle \bar{K}^*(p')\epsilon|\bar{s}_L\sigma^{\mu\nu}b_R|\bar{B}(p)\rangle.$$

It is remarkable that the spin-$SU(2)$ symmetry of the HQET allows us to relate this to the matrix element of a current (which is itself related to semileptonic D decay). To see how this goes, consider, for example, the $\mu = 0$, $\nu = i$ terms,

i.e., the matrices σ^{0i}. These are proportional to $\gamma^0\gamma^i - \gamma^i\gamma^0 = -2\gamma^i\gamma^0$. Now, in the rest frame of the B meson, $v = (1, 0)$, and the projection operator in the HQET is $(1 + \not{v})/2 = (1 + \gamma^0)/2$. So in the HQET, one can substitute

$$\bar{s}_L\sigma^{0i}b_R \to -i\bar{s}_L\gamma^i b_L.$$

Unfortunately, the two-body decay $\bar{B} \to \bar{K}^*\gamma$ has a fixed $\bar{K}^*$ momentum outside the kinematic range of the appropriately rescaled momentum in the corresponding semileptonic D decay.

7.4. $e^+e^- \to B\bar{B}$

In e^+e^- annihilation into a pair of heavy quarks, the HQET can be used to relate cross sections for different exclusive processes[18,36]. For example, the flavour symmetry can be used to relate $\sigma(e^+e^- \to B\bar{B})$, at a center of mass energy of $\sqrt{s} = m_b\sqrt{(v+v')^2}$, to $\sigma(e^+e^- \to D\bar{D})$, at $\sqrt{s} = m_D\sqrt{(v+v')^2}$. Also, the spin symmetry can be used to relate $\sigma(e^+e^- \to B\bar{B})$, $\sigma(e^+e^- \to B^*\bar{B})$ and $\sigma(e^+e^- \to B^*\bar{B}^*)$. They are in the ratios,

$$1 + h \;:\; s/2m_b^2 \;:\; 3(1 + s/3m_b^2 + h)\ ,$$

where

$$h = -\frac{2\alpha_s}{3\pi}\sqrt{1 - 4m_b^2/s}\,\log\left(s/2m_b^2 - 1 + s/2m_b^2\sqrt{1 - 4m_b^2/s}\right).$$

7.5. $\Lambda_b \to \Lambda_c D_s$ *vs.* $\Lambda_b \to \Lambda_c D_s^*$

Here is an example of an application to a purely hadronic weak decay[37]. To exploit the symmetries of the HQET, one constructs an effective hamiltonian

for the underlying process $b \to c\bar{c}s$. It is a sum of four quark operators (à la Fermi), roughly of the form

$$(\bar{s}\gamma^\mu(1-\gamma_5)\underline{c}_{\underline{v}})(\bar{c}_{v'}\gamma_\mu(1-\gamma_5)b_v),$$

where $\underline{c}_{\underline{v}}$ is the charm anti-quark field with velocity $\underline{v}$ in the HQET. From this, one can study the implications of the spin and flavor symmetries on the amplitudes for, for example, $\Lambda_b \to \Lambda_c D_s$.

There are two independent terms in the amplitude for $\Lambda_b \to \Lambda_c D_s$, and two for $\Lambda_b \to \Lambda_c D_s^*$. The symmetries give the latter in terms of the former. The amplitude $\mathcal{A}(\Lambda_b(v) \to \Lambda_c(v')D_s(\bar{v}))$ is given by

$$\bar{u}(v',s')[S+P\gamma_5]u(v,s)\ ,$$

where S and P are respectively amplitudes for the D_s to be in an S-wave and P-wave orbital angular momentum state. The amplitude $\mathcal{A}(\Lambda_b(v) \to \Lambda_c(v')D_s^*(\bar{v},\epsilon))$ is then given by

$$\tfrac{1}{2}\bar{u}(v',s')(1+\gamma_5)\left[(A+2Bv\cdot v')\not{\epsilon}^* - 2B(\epsilon^*\cdot v)\not{v} + B\not{v}\not{\epsilon}^*\right]u(v,s),$$

where

$$A = S - P \qquad \text{and} \qquad B = -\left(\frac{m_b}{m_c}\right)(S-P) - (S+P)\ .$$

7.6. Factorization

Factorization in two body decays of heavy pseudoscalar mesons was resurrected a few years ago as a means of estimating their rate, using existing calculations of semileptonic decay form factors. For the factorization assumption no justification was given. Here by factorization I mean, for example,

$$\langle D\pi|j^\mu J_\mu|\bar{B}\rangle \approx \langle D|J_\mu|\bar{B}\rangle\langle\pi|j^\mu|0\rangle, \tag{51}$$

where J_μ and j_μ are the $V-A$ currents for $b \to c$ and $u \to d$, respectively. Comparing the rate for $\bar{B} \to D\pi$ to the semileptonic one, using this identity naïvely, one finds

$$\frac{\Gamma(\bar{B} \to D\pi)}{\left.\frac{d\Gamma(\bar{B}\to De\bar{\nu})}{dm^2_{e\nu}}\right|_{m^2_{e\nu}=m^2_\pi}} = 6\pi^2 f_\pi^2. \tag{52}$$

The problem with eq. (51) is that it makes no sense. The left and right sides have different dependence on the renormalization point. Such a relation cannot have physical content. The HQET, together with mild extensions of the method, furnish a way[38] of correcting eq. (51). Of course, it is more than that, because the corrected version can actually be shown to hold. In fact, as far as we know, this and the large-N_{color} one are the only *proofs* of factorization in some limit of the underlying theory. The result is a slight modification of eq. (52):

$$\frac{\Gamma(\bar{B} \to D\pi)}{\left.\frac{d\Gamma(\bar{B}\to De\bar{\nu})}{dm^2_{e\nu}}\right|_{m^2_{e\nu}=m^2_\pi}} = A^2 6\pi^2 f_\pi^2,$$

where A is a calculable correction factor. In the leading-log approximation $A \approx 1.05$, in remarkable agreement with experiment[39]. The violations to factorization are expected to be of the order of $\Lambda/(M_B/2) \sim \Lambda/2M_D \sim 10\%$, and of order[40] $\bar{\alpha}_s(m_b)/\pi \sim 10\%$.

Just as interesting is that the same method cannot be used to prove factorization in some other processes, such as $B \to \pi\pi$ and $B \to D\bar{D}$. In fact, it suggests that factorization does not hold for them. A striking confirmation (or refutation) of these ideas could be provided by studying of $B \to D\pi\pi$. The methods of ref. [38] suggest factorization holds only in the limit of colinear pions, and it would be interesting to plot the decay rate as a function of the angle between the pions, in units of the rate computed by assuming factorization.

7.7. *Concluding Remarks*

As this manuscript was being completed, a workshop and a topical conference on "Heavy Quark Symmetry: Theory and Applications" were held at the Institute for Theoretical Physics, Santa Barbara. A considerable number of new results were presented, and many interesting issues were debated. A sampler:

1. An effective lagrangian for heavy mesons and light pseudo-scalars that incorporates both the spin and flavor symmetries of heavy quarks and the light quark chiral symmetries has been written[41]. This allows a study of semileptonic decays $\bar{B} \to DXe\bar{\nu}$ and $\bar{B} \to D^*Xe\bar{\nu}$ where X is a low momentum state of one or more pseudo-Goldstone bosons. The leading contributions, in chiral perturbation theory, to $f_{D_S}/f_D - 1$ and $B_{B_S}/B_B - 1$ have been computed[42].
2. Shifman has proposed a new theoretical limit for factorization[43]. His argument relies on quark–hadron duality. Turning it around, he criticizes the work of ref. [38] as being inconsistent with quark–hadron duality. Discussions during the ITP workshop resulted in the suggestion that duality is saturated in channels for which the assumptions of ref. [38] are not met. Specifically, ref. [38] assumes the final state created by the light quark current has a wavefunction that is not dominated by quarks or gluons with small momentum fraction. Duality is presumably saturated by jet–like final states, which have fragmentation functions that peak at zero momentum. More work is needed to verify this. On the experimental side, there is now strong experimental evidence that purely hadronic two body decays of D and D_S mesons don't factorize[44], as was suggested in ref. [38].

3. Isgur (unpublished, to appear in the proceedings of the ITP Topical Conference) has given a simple model of hadronization that highlights the non-perturbative nature of the high energy end of the spectrum for inclusive semileptonic B–decays. There are many interesting questions in connection to the justification for quark–hadron duality described in section 7.1. How are these non–perturbative effects accounted for? In computing the inclusive rate, is the meson mass, M_B, rather than the quark mass, m_b, the one that fixes the limits of phase space?
4. Can non–perturbative effects ruin the perturbative construction of the HQET in subleading orders, much as non–perturbative effects may ruin the perturbative calculation of Wilson coefficients in the OPE for R[45]? Even if this is not the case, non-perturbative renormalization of the operators that appear in subleading orders in the $1/M$ expansion may be necessary[46].
5. So far there is little that the HQET predicts (or postdicts) that had not been predicted (or postdicted) by hadronic models, like the constituent nonrelativistic quark model. The normalization of form factors for $\bar{B} \to De\bar{\nu}$ and $\bar{B} \to D^*e\bar{\nu}$ of chapter 5, above, has been seen to hold approximately in such models. It is becoming clear that these models are far too predictive and can be incorrect. Experimentally there is evidence that the branching fractions for semileptonic $\bar{B}$ decays to exited charmed–meson states (*e.g.*, the $l = 3/2$ states D_1 and D_2^*) are grossly underestimated by such models[47]. This represents no problem for the HQET. Quark–hadron duality, as in eq. (50), together with the normalization of form factors in $\bar{B} \to De\bar{\nu}$ and $\bar{B} \to D^*e\bar{\nu}$ and an assumption about the smoothness of the form factors leads to an upper bound on the branching fractions into other

states, but this is consistent with experiment (see also [48]). It has been suggested that HQET ideas may be successfully integrated into superior phenomenological models[49].

6. The most burning question of them all: what scale defines the onset of the large mass limit? The answer depends on which quantity one looks at. Lattice computations will be crucial in resolving this issue.

Thus, this is the subject of much current research and is undergoing rapid growth. This review is therefore necessarily incomplete. But it is hopefully timely, and will hopefully help the reader join in the development of the subject at this early stage.

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