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Perturbation Approach and the Constant of Motion for One-Dimensional Dynamical Systems

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Abstract

A perturbation technic is used to find the constant of motion of a one-dimensional autonomous system. The convergence of the method is discussed through some examples. In addition, the approach is extended to one-dimensional non-autonomous systems where some examples are given.

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I. INTRODUCTION

The new Lagrangian formulation of the Mechanics, for one-dimensional autonomous systems [1–6], is based on the constant of motion associated to the system. Once the constant of motion is found, the Lagrangian, Hamiltonian and Generalized Linear Momentum can be found, at least in principle. So, the constant of motion plays the central function in this new formulation.

In general, the constant of motion associated to a dynamical system arises from the solution of a partial differential equation [5] which most of the time is not easy to solve. Therefore, some approximation method is required in order to obtain its expression at any desired order. In this paper, the normal perturbation technic [7] is applied to find in some degree of approximation the constant of motion associated to a one-dimensional system. The approach is used in autonomous system and is extended to non-autonomous systems.

II. CONSTANT OF MOTION FOR AUTONOMOUS SYSTEMS

An autonomous system is characterized by the fact that the force, F, acting on the motion of the particle does not depend explicitly on time, t. Therefore, the motion of the particle is described by Newton's equation of motion,

$$m\, d^2x/dt^2 = F(x, dx/dt)\ , \tag{1}$$

where x represents the coordinate of the system, and m is the mass at rest of the particle. By defining a new variable, $v = dx/dt$, this equation can be written as a one-dimensional autonomous dynamical system,

$$dx/dt = v \tag{2a}$$

and

$$dv/dt = F(x,v)/m \ . \tag{2b}$$

A constant of motion associated to this system, K, is given by the solution of the following partial differential equation [5]

$$v \ \partial K/\partial x + (F(x,v)/m) \ \partial K/\partial v = 0 \ . \tag{3}$$

This equation is, in general, difficult to integrate, but suppose that the force can be separated in two parts,

$$F(x,v) = f_o(x,v) + f_I(x,v) \ . \tag{4}$$

The first part, f_o, is the part of the force for which the Eq. (3) is integrable and has the known solution K_o,

$$v \ \partial K_o/\partial x + (f_o(x,v)/m) \ \partial K_o/\partial v = 0 \ , \tag{5}$$

and the second part, f_I, is the part which is considered a perturbation to the system. Suppose also that the constant of motion can be expressed of the form

$$K(x,v) = K_o(x,v) + \sum_{n=1}^{\infty} K^{(n)}(x,v) \ , \tag{6}$$

where $K^{(n)}$ is of the order $(f_I)^n$ in force strength, that is

$$\mathcal{O}\left(K^{(n)}\right) \sim (f_I)^n \ . \tag{7}$$

Defining $K^{(0)} = K_o$ and substituting (4) and (6) in (3), it follows

$$v\,\frac{\partial K_o}{\partial x}+\frac{f_o(x,v)}{m}\,\frac{\partial K_o}{\partial v}+$$
$$+\sum_{n=1}^{\infty}\left\{v\frac{\partial K^{(n)}}{\partial x}+\frac{f_o(x,v)}{m}\,\frac{\partial K^{(n)}}{\partial v}+\frac{f_I(x,v)}{m}\frac{\partial K^{(n-1)}}{\partial v}\right\}+$$
$$+\sum_{n=1}^{\infty}\left\{\frac{f_I(x,v)}{m}\,\frac{\partial K^{(n)}}{\partial v}\right\}=0\ . \tag{8}$$

Using (5) in (8) and neglecting the terms of the order f_I^{n+1}, the next equation is obtained

$$v\,\frac{\partial K^{(n)}}{\partial x}+\frac{f_o(x,v)}{m}\,\frac{\partial K^{(n)}}{\partial v}=-\frac{f_I(x,v)}{m}\,\frac{\partial K^{(n-1)}}{\partial v}\ ,\ \text{for } n=1,2\ldots \tag{9}$$

This equation can be solved by the characteristics method where the equations for the characteristics are given by

$$\frac{dx}{v}=\frac{m\ dv}{f_o(x,v)}=\frac{m\ dK^{(n)}}{-f_I(x,v)K_v^{(n-1)}}\ , \tag{10}$$

where $K_v^{(n-1)}$ represents the partial differentiation of $K^{(n-1)}$ with respect to v. From the first two terms of (10), the first characteristic curve is obtained which, indeed, can be the known constant

$$K_o=K_o(x,v)\ . \tag{11}$$

This one can be used to express (whenever possible) v as a function of K_o and x,

$$v=v(K_o,x)\ , \tag{12}$$

which, in turns, can be substituted in the first and last terms of (10) to obtain the following general solution

$$K^{(n)} = -\frac{1}{m}\int^{x} \frac{f_I(\xi, v(K_o,\xi))K_v^{(n-1)}}{v(K_o,\xi)}\, d\xi + C(K_o)\ , \tag{13}$$

where $C(K_o)$ is an arbitrary function of the characteristic curve K_o. By selecting this function equal to zero, the solution of (9) is given by

$$K^{(n)} = -\frac{1}{m}\int^{x} \frac{f_I(\xi, v(K_o,\xi))K_v^{(n-1)}}{v(K_o,\xi)}\, d\xi\ , \tag{14}$$

or applying iteratively (14), it follows

$$K^{(n)} = \frac{(-1)^n}{m^n}\int^{x} \frac{dx_{n-1} f_I}{v}\frac{\partial}{\partial v}\int^{x_{n-1}} \frac{dx_{n-2} f_I}{v}\frac{\partial}{\partial v}\cdots\int^{x_1} \frac{dx_o f_I}{v}\frac{\partial K_o}{\partial v} \tag{15a}$$

that can be defined symbolically as

$$K^{(n)} = \left[(-1)^n \prod_{i=1}^{n}\frac{1}{m}\int^{x_i} \frac{dx_{i-1} f_I(x_{i-1})}{v(x_{i-1})}\frac{\partial}{\partial v}\right] K_o\ . \tag{15b}$$

In the next subsections, some examples of application of this approach will be given.

A. *Free Particle Under a Position Depending Force*

The motion of a particle of mass m, under the action of a force which depends only on the position, $f(x)$, can be analyzed by the following dynamical system

$$dx/dt = v \tag{16a}$$

and

$$dv/dt = f(x)/m\ . \tag{16b}$$

Consider $f(x)$ as the perturbation of the free particle motion which has associated the constant of motion

$$K_o = mv^2/2 \ . \tag{17}$$

Then, using (14), the following functions are obtained

$$K^{(1)} = -\int^{x} f(x')dx' \tag{18a}$$

and

$$K^{(j)} = 0 \ , \ \text{for } j \geq 2 \ . \tag{18b}$$

As a result, the constant of motion obtained for the system (16) converges for any strength in the force to the expected expression

$$K(x,v) = mv^2/2 - \int^{x} f(x') \ dx' \ . \tag{19}$$

B. *Free Particle Moving in a Dissipative Medium*

Suppose that a particle is moving in a dissipative medium characterized by a force that is proportional to the square velocity, $f(v) = -\alpha v^2$, where α is the constant of proportionality which depends on the medium itself. Its motion can be studied through the dynamical system

$$dx/dt = v \tag{20a}$$

and

$$dv/dt = -\alpha v^2/m \ . \tag{20b}$$

Considering $f(v)$ as the perturbation to the free motion which has the constant of motion (17), the application of the operator (14) to (17) brings about the functions

$$K^{(n)} = mv^2 \ (2\alpha x/m)^n \ /2n! \ , \ \text{for } n \geq 1 \ . \tag{21}$$

So, from (6) and (21), the solution is given by

$$K(x,v) = (mv^2/2) \ \exp(2\alpha x/m) \tag{22}$$

which is the expected solution [5], and its convergence is independent on the dissipation strength.

C. *Harmonic Oscillator in Dissipative Medium*

The motion of the harmonic oscillation of a particle of mass m, dampened by a frictional force proportional to the particle velocity, $f(v) = -\alpha \ v$, can be studied with the dynamical system

$$dx/dt = v \tag{23a}$$

and

$$dv/dt = -\omega^2 x - \alpha v/m \ , \tag{23b}$$

where ω is the angular frequency of the free oscillations. Assume that the friction term is the perturbation to the harmonic oscillations characterized by the constant of motion

$$K_o = mv^2/2 + m\omega^2 x^2/2 \ . \tag{24}$$

Therefore, from the expression (14), it follows [8]

$$K^{(1)} = \alpha \int^{x} \sqrt{2K_0/m - \omega^2\xi^2}\, d\xi =$$
$$= \frac{\alpha}{2\omega}\left\{\omega x v + (v^2 + \omega^2 x^2)\arcsin\left(\frac{\omega x}{\sqrt{v^2+\omega^2x^2}}\right)\right\} \tag{25}$$

which brings about the solution at first order in perturbation as

$$K \simeq \frac{m}{2}(v^2 + \omega^2 x^2) + \frac{\alpha}{2\omega}\left\{\omega x v + (v^2 + \omega^2 x^2)\arcsin\left(\frac{\omega x}{\sqrt{v^2+\omega^2x^2}}\right)\right\} . \tag{26}$$

This solution can be compared with the exact solution shown in Refs. [4] and [5] (see also Eq. (40a) below). The multivalued function "arcsin" brings about the shrinking behavior of the trajectories in the phase space. It is clear from (16) and (14) that higher order perturbation terms are not so trivial to be calculated.

III. EXTENSION TO NON-AUTONOMOUS SYSTEMS

Suppose now that the force depends on time and is given by

$$F(x,v,t) = f_o(x,v) + f_I(x,v,t) , \tag{27}$$

where f_o is the part of the force which the dynamical system

$$dx/dt = v \tag{28a}$$

and

$$dv/dt = f_o(x,v)/m , \tag{28b}$$

has associated the known constant of motion

$$K_o = K_o(x, v) \tag{29}$$

which satisfies the Eq. (5). By adding a time depending force to this system,

$$dx/dt = v \tag{30a}$$

and

$$dv/dt = f_o(x, v)/m + f_I(x, v, t)/m \ , \tag{30b}$$

the system becomes non-autonomous and the constant of motion associated, K, must satisfy the following equation

$$v\frac{\partial K}{\partial x} + \frac{1}{m}\left[f_o(x, v) + f_I(x, v, t)\right]\frac{\partial K}{\partial v} + \frac{\partial K}{\partial t} = 0 \ . \tag{31}$$

Assume a solution of the form

$$K(x, v, t) = K_o(x, v) + \sum_{n=1}^{\infty} K^{(n)}(x, v, t) \ , \tag{32}$$

where $K^{(n)}$ is of the same order of f_I^n. Substituting (32) in (31), it follows

$$\begin{aligned} &\frac{\partial K_o}{\partial t} + v\frac{\partial K_o}{\partial x} + \frac{f_o(x, v)}{m}\frac{\partial K_o}{\partial v} + \\ &+ \sum_{n=1}^{\infty}\left\{\frac{\partial K^{(n)}}{\partial t} + v\frac{\partial K^{(n)}}{\partial x} + \frac{f_o(x, v)}{m}\frac{\partial K^{(n)}}{\partial v} + \frac{f_I(x, v)}{m}\frac{\partial K^{(n-1)}}{\partial v}\right\} + \\ &+ \sum_{n=1}^{\infty} f_I(x, v, t)\frac{\partial K^{(n)}}{\partial v} = 0 \ . \end{aligned} \tag{33}$$

Using the fact that K_o does not depend explicitly on time and satisfies (5), and dropping the terms of order f_I^{n+1} in (33), the time depending function $K^{(n)}$ is the solution of the equation

$$\frac{\partial K^{(n)}}{\partial t} + v\frac{\partial K^{(n)}}{\partial x} + \frac{f_o(x,v)}{m}\frac{\partial K^{(n)}}{\partial v} = -\frac{f_I(x,v,t)}{m}\frac{\partial K^{(n-1)}}{\partial v} \ . \tag{34}$$

Similarly, this equation can be solved through the characteristics method. The characteristic equations for (34) are given by

$$\frac{dx}{v} = \frac{m\ dv}{f_o(x,v)} = dt = \frac{m\ dK^{(n)}}{-f_I(x,v,t)K_v^{(n-1)}} \ , \tag{35}$$

and its first characteristic curve, C_1, obtained from the first two terms of (35), can be the known function K_o,

$$C_1 = K_o(x,v) \ . \tag{36}$$

From this relation, there are two equivalent ways to obtain the second characteristic curve, depending upon either v is expressed in terms of C_1 and x, or x is expressed in terms of C_1 and v. The result is

$$C_2 = \begin{cases} \displaystyle\int^{x} \frac{dx'}{v(C_1,x')} - t \\ \text{or} \\ \displaystyle\int^{v} \frac{m\ dv'}{f_o(x(C_1,v'),v')} - t \end{cases} \tag{37}$$

From (36) and (37), the variables x and v may be known as a function of time. Then, from the last two terms of (35), the solution of (34) could be given by

$$K^{(n)} = -\frac{1}{m}\int^{t} f_I(x(C_1,C_2,\xi),v(C_1,C_2,\xi),\xi)K_v^{(n-1)}\, d\xi + A(C_1,C_2)\ , \tag{38a}$$

where the function A is arbitrary. So, the solution (32) can be written symbolically as

$$K = K_o + \sum_{n=1}^{\infty}(-1)^n \left[\prod_{i=1}^{n}\frac{1}{m}\int^{x_i} dx_{i-1} f_I(x_{i-1})\frac{\partial}{\partial v}\right] K_o\ . \tag{38b}$$

However, it is necessary to mention that special care must be taken when the Lagrangian for this type of system is looked for (see Ref. [2]). Two application examples follow.

A. *Dissipative Harmonic Oscillation with Driving Term*

This example is the same as the example 2.3 above, yet it is adding an extra periodic function which depends on time, $f(t)$. The associated dynamical system can be written as

$$dx/dt = v \tag{39a}$$

and

$$dv/dt = -\omega^2 x - \alpha v/m + f(t)\ . \tag{39b}$$

Assuming that the driving force, $f(t)$, is the perturbation caused in the dissipative harmonic oscillator which is characterized, for small dissipation, by the constant of motion [5]

$$K_o = \frac{m}{2}(v^2+\omega^2x^2) + \frac{m\omega_\alpha}{\omega}\left[\omega x v - (v^2+\omega^2x^2)\arctan\left(\frac{\omega_\alpha}{\omega}+\frac{v}{x\omega}\right)\right]\ , \tag{40a}$$

where ω_α is defined by

$$\omega_\alpha = \alpha/2m \ . \tag{40b}$$

Considering only one branch of the "arctan" function and for very weak dissipation, (40a) can be approximated by

$$C_1 = \frac{m}{2}(v^2 + \omega^2 x^2) + m\omega_\alpha x v \ , \tag{41}$$

and the second characteristic curve is given, from (37), by

$$C_2 = \int^x \frac{d\xi}{\sqrt{K_o/2m - \Omega^2\xi^2} - \omega_\alpha \xi} - t \ , \tag{42a}$$

where $\Omega > 0$ is defined as

$$\Omega^2 = \omega^2 - \omega_\alpha^2 \ . \tag{42b}$$

From (41) and (42a), the variables x and v can be expressed in terms of the constants C_1, C_2 and the time t,

$$x = x(C_1, C_2, t)$$

and

$$v = v(C_1, C_2, t) \ .$$

With these expressions and (38), the constant of motion at first order approximation is given by

$$K(x, v, t) = K_o(x, v) + K^{(1)}(x, v, t) \ , \tag{43}$$

where K_o is given by (42) and $K^{(1)}$ is given by

$$K^{(1)} = -\int^{t} [v(C_1, C_2, \tau) + \omega_\alpha x(C_1, C_2, \tau)]\, f(\tau)\, d\tau \ . \tag{44}$$

B. *Perturbation of a Charged Particle in an Accelerator Ring*

When the motion of a charged particle along an accelerator ring is displaced because of errors in the magnetic field, its transversal motion [9], called betatron motion, is governed by the dynamical system

$$d\eta/d\phi = \rho \tag{45a}$$

and

$$d\rho/d\phi = -\nu^2\eta + \nu^2\beta^{3/2}(s)F(s) \ , \tag{45b}$$

where s is the length coordinate along the ring; $\beta(s)$ is called the beta function; ν is a constant that measure the number of oscillations per turn made by the particles in the ring which is called the "tune" of the machine; ϕ is a function which is related with the beta function as

$$\phi(s) = \frac{1}{\nu}\int_0^s \frac{d\sigma}{\beta(\sigma)} \ ; \tag{46a}$$

$\eta(s)$ is a function which is related with the displacement, $y(s)$, and the beta function as

$$\eta(s) = \beta^{-1/2}(s)y(s) \ ; \tag{46b}$$

and $F(s)$ is related with the magnetic field errors, ΔB, as

$$F(s) = (\Delta B)(s)/(B\rho) \ , \tag{46c}$$

where $(B\rho)$ is a constant called "Magnetic Rigidity." Note that ϕ plays the role of time in (30), η plays the role of the coordinate x, and ρ plays the role of the velocity v. Assume that these errors are the perturbation of the system (45), which has the constant of motion

$$K_o = \frac{1}{2}(\rho^2 + \nu^2\eta^2) \tag{47a}$$

when $F = 0$. This constant of motion is ν^2 times the so called "emittance" of the particle, and is the first characteristic curve of the associated system (34). The second characteristic curve is given by (37) as

$$C_2 = \frac{1}{\nu}\arcsin\left(\frac{\nu\eta}{\sqrt{2K_o}}\right) - \phi\ . \tag{47b}$$

From (47a) and (47b), it follows

$$\eta(K_o, C_2, \phi) = \frac{\sqrt{2K_o}}{\nu}\sin(\nu C_2 + \nu\phi) \tag{48a}$$

and

$$\rho(K_o, C_2, \phi) = \sqrt{2K_o}\cos(\nu C_2 + \nu\phi)\ . \tag{48b}$$

Thus, at first order approximation in (38a), it follows

$$K^{(1)} = \nu^2\sqrt{2K_o}\int^{\phi}\cos(\nu C_2 + \nu\phi')\beta^{3/2}(s(\phi'))F(s(\phi'))\ d\phi'\ , \tag{49}$$

or using again (47a) and (47b), the constant of motion at first order approximation is given by

$$K(\eta, \rho, \phi) = \frac{1}{2}(\rho^2 + \nu^2\eta^2) + \nu^2\sqrt{\rho^2 + \nu^2\eta^2}\ \times$$

$$\times \int^{\phi} \cos\left[\arcsin\left(\frac{\nu\eta}{\sqrt{\rho^2+\nu^2\eta^2}}\right) - \nu\phi + \nu\phi'\right] \beta^{3/2}(s(\phi'))F(s(\phi'))\, d\phi' \,. \tag{50}$$

It is pointed out that the relation (46a) establishes a one-to-one relationship between ϕ and s since the beta function is a positive defined function for any s value.

IV. CONCLUSION

The constant of motion associated to a one-dimensional autonomous system is found using the standard perturbation method. The examples given show that the approach can converge very rapidly and independently on the force strength, suggesting a rather far freedom of applications and showing that may be a useful approach in some cases where the constant of motion is not so obvious to find. The application of the approach to non-autonomous systems is straightforwardly extended although the complication to obtain higher order approximation is manifested.

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