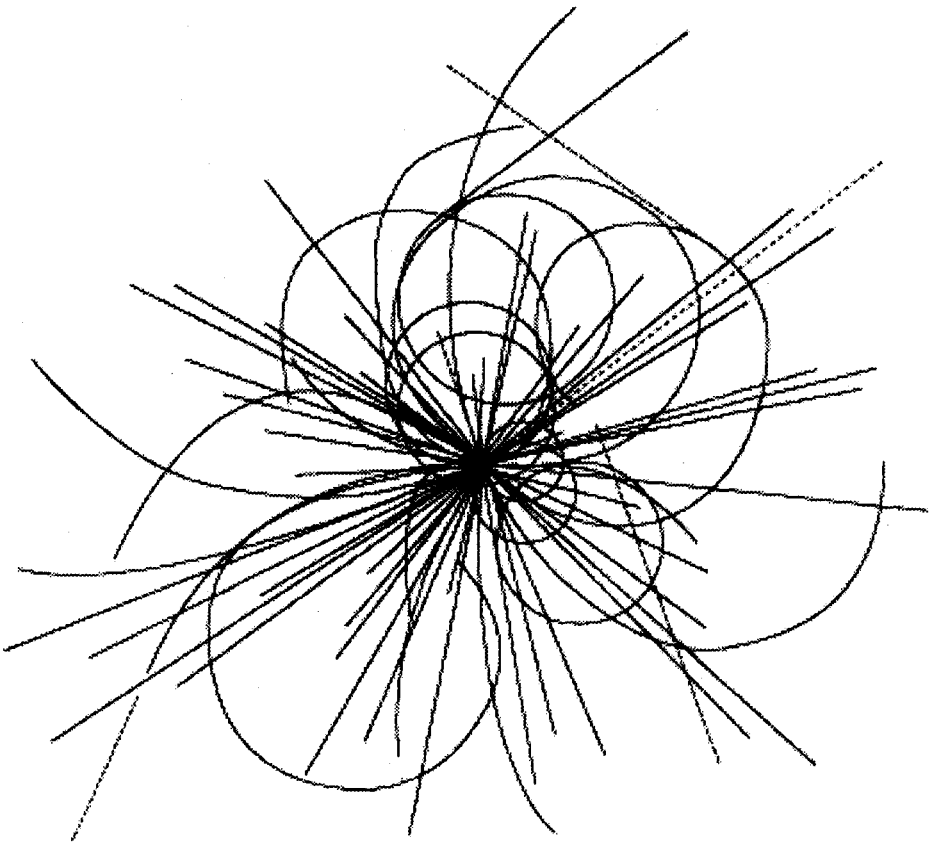
SSCL-625

SSCL-625
May 1993
Distribution Category: 414

G. López

Analytical Studies of the Emittance Growth Due to the Feedback System



Superconducting Super Collider Laboratory

SSCL-625

Analytical Studies of the Emittance Growth Due to the Feedback System

G. López

Superconducting Super Collider Laboratory*
2550 Beckleymeade Avenue
Dallas, TX 75237

May 1993

* Operated by the Universities Research Association, Inc., for the U.S. Department of Energy under Contract No. DE-AC35-89ER40486.

Analytical Studies of the Emittance Growth Due to the Feedback System

G. López

Abstract

An analytical study is done of the change in the Courant-Snyder Invariant (CSI) of particles due to the feedback system. Given the tune of the machine, ν_o, the tune spread in the bunch, $\sigma_\nu = \sqrt{<(\Delta\nu)^2>}$, and the gain of the feedback system, g, the known relation, $4\pi\sigma_\nu/g << 1$, is confirmed in the general case, implying $g \geq 0.05$ for the transverse feedback system during collision. In addition, results show that the feedback noise transmitted to the particles, σ_x, must be $\sigma_x \leq 0.25$ μm for the feedback system not to cause a significant increase of beam emittance during 24 h of Collider operation.

1.0 INTRODUCTION

Recent studies on emittance growth[1,2] indicate that it can be suppressed by using a low-noise feedback system and by damping the oscillation faster than the decoherent time, τ_d, of the beam. This decoherent time is about 0.34 s for an rms tune spread of $\Delta\nu \simeq 10^{-4}$, which is caused by the nonlinear magnetic field at injection (ignoring chromaticity). However, the approach followed by the author was not general, so one may think that the result is not true for the general case. The general case will be presented in this paper, confirming the validity of this result in general, although the noise contribution expression differs by a factor of 8.

2.0 SINGLE PARTICLE MOTION

The transverse motion of a particle traveling in a synchrotron linear machine perturbed by a pure time-dependent force can be described by the equation[3]

$$\frac{d^2x}{ds^2} + K_x(s)x = F(s) \ , \tag{1}$$

where the longitudinal coordinate, s, has been used instead of time; x is the transverse displacement of the particle; $K_x(s)$ is defined by the linear lattice; and $F(s)$ describes the perturbation in the particles around the machine. Using the transformation

$$\eta(s) = x(s)/\sqrt{\beta(s)} \tag{2a}$$

$$\rho(s) = \nu[\alpha(s)x(s)/\sqrt{\beta(s)} + \sqrt{\beta(s)}\acute{x}(s)] \tag{2b}$$

and

$$\phi(s) = \frac{1}{\nu}\int_0^s d\sigma/\beta(\sigma) \ , \tag{2c}$$

Eq. (1) can be written as the dynamical system

$$\frac{d\eta}{d\phi} = \rho \tag{3a}$$

$$\frac{d\rho}{d\phi} = -\nu^2\eta + f(\phi) \ , \tag{3b}$$

where f is the function $f(\phi) = \nu^2\beta^{3/2}F\Big(s(\phi)\Big)$, the identity $2\beta\ddot{\beta} - (\acute{\beta})^2 + 4\beta^2K = 4$ and the notation $d/ds ='$ have been used, and α and β are the Courant-Snyder parameters. The Courant-Snyder Function (CSF), ϵ, of the particle is given by

$$\epsilon = \eta^2 + (\rho/\nu)^2 = \left[x^2 + (\alpha x + \beta\acute{x})^2\right]/\beta \ . \tag{4}$$

Actually, it is incorrect to call this expression "invariant," since the interaction $f(\phi)$ breaks this property.

3.0 GENERAL EXPRESSION FOR EMITTANCE GROWTH

For perturbations depending only on the variable ϕ, as shown in Eqs. (3), it is possible to give a general expression for the emittance growth. Assume the motion of a particle in the bunch, relative to the centroid $q(\phi) = \eta(\phi) - \eta_c(\phi)$ and $p(\phi) = \rho(\phi) - \rho_c(\phi)$, is described by the equation

$$\frac{d^2q}{d\phi^2} + \nu^2 q = H(\phi) \ , \tag{5}$$

where $H(\phi)$ is an arbitrary function. The solution of this equation satisfying the initial conditions $q(0) = q_o$ and $dq(0)/dt = p(0) = p_o$ is well known[4] and is given by (including $p = dq/dt$) :

$$q(\phi) = q_h(\phi) + \frac{1}{\nu}\int_0^\phi H(\hat{\phi}) \sin\nu(\phi - \hat{\phi}) d\hat{\phi} \ , \tag{6a}$$

$$\frac{p(\phi)}{\nu} = \frac{p_h(\phi)}{\nu} + \frac{1}{\nu}\int_0^\phi H(\hat{\phi}) \cos\nu(\phi - \hat{\phi}) d\hat{\phi} \ , \tag{6b}$$

where $q_h(\phi)$ is the solution of the homogeneous equation. From these equations, a general expression for the CSF, $\epsilon = q^2 + (p/\nu)^2$, can be given as

$$\begin{aligned}\epsilon = \epsilon_h + \frac{2}{\nu}\int_0^\phi H(\acute{\phi})[q_h(\phi)\sin\nu(\phi - \acute{\phi}) + p_h(\phi)\cos\nu(\phi - \acute{\phi})]d\acute{\phi} \\ + \frac{1}{\nu^2}\int_0^\phi\int_0^\phi H(\acute{\phi})H(\acute{\acute{\phi}})\cos\nu(\acute{\acute{\phi}} - \acute{\phi})d\acute{\phi}d\acute{\acute{\phi}} \ .\end{aligned} \tag{7a}$$

Making averages over the random variables or the tune spread, the second term on the right hand side normally vanishes $\Big(< H(\acute{\phi})q_h(\phi) >= 0$ and $< H(\acute{\phi})p_h(\phi) >= 0\Big)$. Therefore, the following general expression can be used for the change in the CSF:

$$\Delta\epsilon = \frac{1}{\nu^2}\int_0^\phi\int_0^\phi H(\acute{\phi})H(\acute{\acute{\phi}})\cos\nu(\acute{\acute{\phi}} - \acute{\phi})d\acute{\phi}d\acute{\acute{\phi}} \ . \tag{7b}$$

Thus, by knowing the right hand side of Eq. (5), the emittance growth estimation can be reduced to the calculation of the integration (7b).

4.0 FEEDBACK EFFECT ON THE EMITTANCE GROWTH

Assume a Beam Position Monitor (BPM) located upstream of a Kicker (K) separated by a phase advance of $\pi/2$, as shown in Figure 1. The BPM measures the centroid of a bunch, and K makes a correction to the bunch trajectory providing a kick,

$$\Delta\acute{x} = \frac{g}{\sqrt{\beta_1\beta_2}}x_1 \ , \tag{8}$$

where g is the gain of the feedback system, and β_1 and β_2 are the beta functions at the location of the BPM and K. Since the initial state $(x_1, \acute{x}_1)$ is connected to the final state $(x_2, \acute{x}_2)$ through a Courant-Snyder map,

$$\begin{pmatrix} x_2 \\ \acute{x}_2 \end{pmatrix} = \begin{pmatrix} \alpha_1\sqrt{\frac{\beta_2}{\beta_1}} & \sqrt{\beta_1\beta_2} \\ -\frac{1+\alpha_1\alpha_2}{\sqrt{\beta_1\beta_2}} & -\alpha_2\sqrt{\frac{\beta_1}{\beta_2}} \end{pmatrix} \begin{pmatrix} x_1 \\ \acute{x}_1 \end{pmatrix} \quad , \tag{9}$$

the coordinate x_1 can be written in terms of x_2 and $\acute{x}_2$ as $x_1 = -\alpha_2\sqrt{\beta_1/\beta_2}x_2 - \sqrt{\beta_1\beta_2}\acute{x}_1$, and the kick, Eq. (8), can be written as

$$\Delta\acute{x} = -(g/\nu\sqrt{\beta_2})\nu[(\alpha_2 x_2/\sqrt{\beta_2}) + \sqrt{\beta_2}\acute{x}_2] \ . \tag{10a}$$

Using Eqs. (2b) and (10a), it follows that

$$\Delta\rho = -g\rho_2 \ . \tag{10b}$$

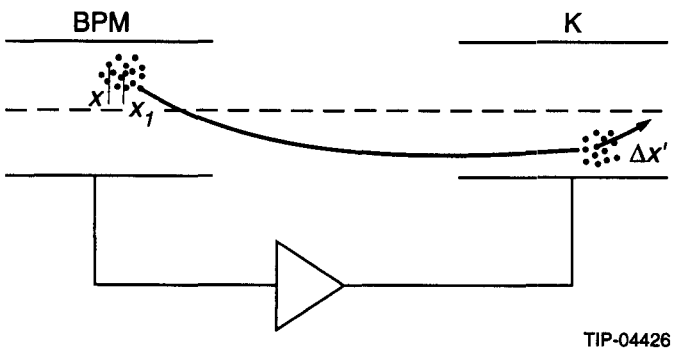


Figure 1. Bunch Behavior Model.

This implies that the behavior of the centroid and any particle in the bunch can be described by the two-dynamical system:

$$\frac{d\eta_c}{d\phi} = \rho_c \tag{11a}$$

$$\frac{d\rho_c}{d\phi} = -\nu_o^2\eta_c - g[\rho_c + (\delta\rho)_c]\delta_{2\pi}(\phi) \tag{11b}$$

and

$$\frac{d\eta}{d\phi} = \rho \tag{12a}$$

$$\frac{d\rho}{d\phi} = -\nu^2\eta - g[\rho_c + (\delta\rho)_c]\delta_{2\pi}(\phi) \ , \tag{12b}$$

where (η_c, ρ_c) and (η, ρ) represent the state of the centroid and the particle in the bunch; $(\delta\rho)_c$ represents the noise on the variable ρ_c in the feedback system; $\delta_{2\pi}(\phi)$ is the delta function of period 2π:

$$\delta_{2\pi}(\phi) = \sum_{n=0}^{J} \delta(\phi - 2\pi n) \ ; \tag{13}$$

and ν_o and ν ($\nu = \nu_o + \Delta\nu$) are the tune ($\int_o^C d\sigma/\beta(\sigma)2\pi$) of the centroid and the particle. J is a fixed number of turns. Note that because of the characteristic of the feedback system, any particle in the bunch is affected by the same centroid kick, Eq. (8). Taking only the time-independent average component of $\delta_{2\pi}(\phi)$,

$$1/\mu = 1/2\pi =< \delta_{2\pi} >= \frac{1}{2\pi J} \int_0^{\phi} \sum_{n=0}^{J} \delta(\acute{\phi} - 2\pi n) d\acute{\phi} \ , \tag{14}$$

on the term proportional to ρ_c, the motion of the centroid is governed by the equation

$$\frac{d^2\eta_c}{d\phi^2} + \nu_o^2 \eta_c + \frac{g}{\mu}\frac{d\eta_c}{d\phi} = -g(\delta\rho)_c \delta_{2\pi}(\phi) \ . \tag{15}$$

The solution of this equation satisfying initial conditions $\eta_c(0) = \eta_{co}$ and $(d\eta_c/d\phi)_{\phi=0} = \rho_{co}$ is well known and is given by

$$\eta_c(\phi) = \eta_{ch}(\phi) - \frac{g\exp(-g\phi/2\mu)}{\Omega} \int_0^{\phi} (\delta\rho)_c(\hat{\phi}) \delta_{2\pi}(\hat{\phi}) \exp(g\hat{\phi}/2\mu) \sin\Omega(\phi - \hat{\phi}) d\hat{\phi} \ , \tag{16}$$

where $\eta_{ch}(\phi)$ is the solution of the homogeneous equation,

$$\eta_{ch}(\phi) = \left[\eta_{co}\cos\Omega\phi + \left(\frac{\rho_{co}}{\Omega} + \frac{g\eta_{co}}{2\mu\Omega}\right)\sin\Omega\phi\right]\exp(-g\phi/2\mu) \tag{17a}$$

$$\rho_{ch}(\phi) = \left[\rho_{co}\cos\Omega\phi - \left(\frac{\eta_{co}\nu_o^2}{\Omega} + \frac{g\rho_{co}}{2\mu\Omega}\right)\sin\Omega\phi\right]\exp(-g\phi/2\mu) \ , \tag{17b}$$

(including its velocity ρ_c), where Ω is defined as

$$\Omega = \sqrt{\nu_o^2 - (g/2\mu)^2} \ . \tag{18}$$

To simplify the analysis, consider the long time behavior ($g\phi/2\mu >> 1$) of the second term on the right hand side of Eq. (16):

$$\begin{aligned} &\exp(-g\phi/2\mu)\left|\int_0^{\phi} (\delta\rho)_c \delta_{2\pi}(\hat{\phi}) \exp(g\hat{\phi}/2\mu) \sin\Omega(\phi - \hat{\phi}) d\hat{\phi}\right| \\ &\leq \exp(-g\phi/2\mu) \int_0^{\phi} |(\delta\rho)_c| \delta_{2\pi}(\hat{\phi}) \exp(g\hat{\phi}/2\mu) d\hat{\phi} \\ &\rightarrow (2\mu/g)|(\delta\rho)_c|\delta_{2\pi}(\phi) \ . \end{aligned} \tag{19}$$

Therefore, Eq. (16) can be written in this limit in the form

$$\eta_c(\phi) = \eta_{ch}(\phi) - \frac{2\mu}{\Omega}(\delta\rho)_c\delta_{2\pi}(\phi) \ . \tag{20}$$

Using Eqs. (11) and (12), the motion of the particles relative to the bunch centroid, $q(\phi) = \eta(\phi) - \eta_c(\phi)$ and $p(\phi) = \rho(\phi) - \rho_c(\phi)$, can be described by the equation

$$\frac{d^2q}{d\phi^2} + \nu^2 q = -2\nu_o(\Delta\nu)\eta_c(\phi) \ , \tag{21}$$

where the second-order term in $\Delta\nu$ has been neglected. Defining the right hand side of Eq. (15) as $H(\phi) = -(2\nu_o\Delta\nu)\eta_c(\phi)$, the relation (7b) can be used to determine the change in the CSF of the particle,

$$\begin{aligned}\Delta\epsilon =& \Big(\frac{2\nu_o\Delta\nu}{\nu}\Big)^2 \int_0^\phi \int_0^\phi \eta_{ch}(\acute{\phi})\eta_{ch}(\acute{\acute{\phi}}) \cos\nu(\acute{\phi} - \acute{\acute{\phi}}) d\acute{\phi} d\acute{\acute{\phi}} \\ &+ \Big(\frac{2\nu_o\mu\Delta\nu}{\nu\Omega}\Big)^2 \int_0^\phi \int_0^\phi K(\acute{\phi} - \acute{\acute{\phi}})\delta_{2\pi}(\acute{\phi})\delta_{2\pi}(\acute{\acute{\phi}}) \cos\nu(\acute{\phi} - \acute{\acute{\phi}}) d\acute{\phi} d\acute{\acute{\phi}} \ ,\end{aligned} \tag{22}$$

where the average over the random variables has been used, and the correlation function, $K(\acute{\phi} - \acute{\acute{\phi}}) =< (\delta\rho)_c(\acute{\phi})(\delta\rho)_c(\acute{\acute{\phi}}) >$, has been defined. The first term on the right hand side of Eq. (22) represents the coherent contribution to the emittance growth, and the second term gives the feedback noise contribution to the emittance.

For the collider, $\nu_o >> g/4\pi$. Therefore, the following relations,

$$\frac{(\Omega + \nu)^2 + (g/2\mu)^2}{(\Omega - \nu)^2 + (g/2\mu)^2} >> 1 \ , \tag{23a}$$

$$\Omega - \nu \simeq \Delta\nu \ , \tag{23b}$$

and

$$(\Omega - \nu)^2 + (g/2\mu)^2 \simeq (g/2\mu)^2 + (\Delta\nu)^2 \ , \tag{23c}$$

and long time behavior (ϕ large), $g\phi/2\mu >> 1$, can be used to simplify the first term on the right hand side of Eq. (22). In addition, using Eq. (13) and assuming white noise, $K(2\pi n - 2\pi m) = \sigma_\rho\delta_{n,m}$, Eq. (22) can be written as

$$\Delta\epsilon = \Big(\frac{\nu_o\Delta\nu}{\nu}\Big)^2 \Big[\frac{\epsilon_{co}}{(g/2\mu)^2 + (\Delta\nu)^2} + \frac{4(\Delta\nu)(g/2\mu)\rho_{co}\eta_{co}/\nu}{[(g/2\mu)^2 + (\Delta\nu)^2]^2}\Big] + \Big(\frac{8\pi\sigma_\rho\Delta\nu}{\nu_o}\Big)^2 J \ , \tag{24}$$

where $\epsilon_{co} = \eta_{co}^2 + (\rho_{co}/\nu_o)^2$ is the CSF of the centroid. From Eq. (24), it is seen that there is a large increment in the CSF of the beam if $\Delta\nu \geq g/2\mu$. Therefore, the case of real interest for minimizing the effect of the feedback kick on the CSF is $\Delta\nu << g/2\mu$. In this

case and averaging over the tune of the particles, the second term inside the bracket in Eq. (24) is canceled, and the emittance growth is given by

$$< \Delta\epsilon >= \frac{16\pi^2\sigma_\nu^2}{g^2}\epsilon_{co} + \left(\frac{8\pi\sigma_\nu\sigma_\rho}{\nu_o}\right)^2 J \ , \tag{25}$$

where σ_ν is the tune spread in the bunch. It is important to observe that the restriction $4\pi\sigma_\nu << g$ avoids having gain very close to the zero value. For gain equal to zero, it is clear from Eqs. (15) and (16) that there will not be any emittance growth. Using the relations (2b) and (10), the noise $\delta\rho$ can be associated with the noise δx. To do this, observe that at the kicker location the following relation follows:

$$\delta\rho = \nu_o\left[\frac{\alpha_2}{\sqrt{\beta_2}}(\delta x_2) + \sqrt{\beta_2}(\delta\acute{x}_2)\right] \ . \tag{26}$$

But from (9), the relation $\acute{x}_2 = -x_1/\sqrt{\beta_1\beta_2} - \alpha_2 x_2/\beta_2$ is obtained. Thus, the noisy part, $\delta\acute{x}_2$, can be written as

$$\delta\acute{x}_2 = -\frac{1}{\sqrt{\beta_1\beta_2}}(\delta x_1) - \frac{\alpha_2}{\beta_2}(\delta x_2) \ . \tag{27}$$

Substituting this in relation (26), the noise relation is given by

$$\delta\rho = -\frac{\nu_o}{\sqrt{\beta_1}}(\delta x_1) \ , \tag{28}$$

and the rms value is given by

$$\sigma_\rho = \frac{\nu_o}{\sqrt{\beta_1}}\sigma_x \ , \tag{29}$$

which can be used in Eq. (25) to get

$$< \Delta\epsilon >= \left(\frac{4\pi\sigma_\nu}{g}\right)^2\epsilon_{co} + \left(\frac{8\pi\sigma_\nu\sigma_x}{\sqrt{\beta_1}}\right)^2 J \ , \tag{30}$$

where β_1 is the beta function at the BPM location. The first term on the right hand side of Eq. (30) agrees with that of Reference 1, but the second term differs by a factor of 8. The linear increase of the emittance with respect to the number of turns described by the second term on the right hand side of Eq. (30) was expected.[5] Thus, given the tune spread of the bunch, the first term on the right hand side of Eq. (30) determines the gain, g, and the second term the BPM resolution, σ_x. In terms of the damping time, $\tau_g = 2T_o/g$ (T_o is the revolution time), and the decoherent time, $\tau_\nu = T_o/\sigma_\nu$, the emittance growth rate, $d < \epsilon > /dt = f_o < \Delta\epsilon > /J$, can be written as

$$\frac{d < \epsilon >}{dt} = \left(\frac{4\pi\tau_g}{\tau_\nu}\right)^2\frac{d\epsilon_{co}}{dt} + \frac{4f_o}{\beta_1}(4\pi\sigma_\nu\sigma_x)^2 \ . \tag{31}$$

The first and second terms on the right hand side of this expression are really independent since the centroid motion (therefore, $d\epsilon_{co}/dt$) is driven by the instabilities of the beam (mainly by the dipole mode multibunch instability), and this instability can be driven by the feedback noise only if this noise is very high.[6] So, each term can be studied separately.

To avoid emittance growth due to decoherence of the bunch, the condition $g/2\mu >> \sigma_\nu$ can be expressed as

$$\tau_g << \tau_\nu/4\pi \ . \tag{32}$$

For a tune spread of approximately 10^{-4} due to nonlinear magnetic field at injection[7] (1.8×10^{-3} for the beam-beam interaction[8]), the decoherent time is approximately $\tau_\nu \simeq$ 2.9 s (0.16 s), and the damping time resulting from Eq. (27) is $\tau_g << 0.23$ s ($\tau_g << 0.013$ s). The gain is then $g > 0.0025$ ($g > 0.04$), *i.e.*, the damping must be carried out in less than $2/g < 800$ turns (50 turns) to avoid emittance growth due to decoherence.

The doubling time due to the feedback noise, $T_2 = \epsilon_o/(d\epsilon/dt)_{\text{noise}}$, is written as

$$T_2 = \frac{\beta_1 \epsilon_o}{4 f_o (4\pi\sigma_\nu\sigma_x)^2} \ , \tag{33}$$

which brings about the following BPM resolution expression:

$$\sigma_x = \frac{1}{4\pi\sigma_\nu}\sqrt{\frac{\beta_1\epsilon_o}{4 f_o T_2}} \ . \tag{34}$$

Figure 2 shows the BPM resolution as a function of the tune spread in the bunch for doubling times of 24 h, *i.e.*, during collision operation ($\beta_1 = 400$ m and $\epsilon_o = 5 \times 10^{-10}$). The tune spread at injection may be approximately 10^{-4}, so the BPM resolution could be approximately 2.5 μm. However, if the total tune spread during collision due to beam-beam interaction is approximately 1.8×10^{-3}, then the BPM resolution must be approximately 0.25 μm to avoid degradation in the luminosity due to emittance growth.

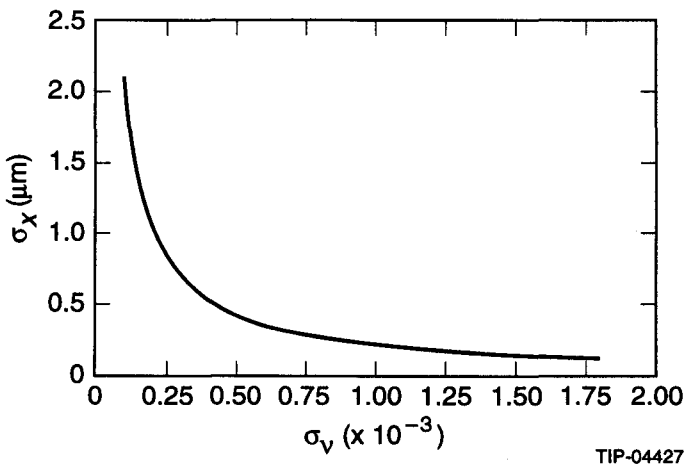


Figure 2. BPM Resolution as a Function of Tune Spread.

5.0 CONCLUSIONS

Using the Courant-Snyder function of particles, an analytical study was made of the emittance growth due to the feedback system. The relation $4\pi\sigma_\nu/g << 1$ was confirmed within this general approach. It was assumed that the gain and the BPM resolution can be studied separately since the feedback noise is too low to cause coherent perturbation. Relation (32) determines the gain of the feedback system, which can vary as long as this relation is maintained. For the collider, a gain of $g \geq 0.05$ can be established independent of the tune spread. On the other hand, the BPM resolution depends much more strongly on the tune spread and is dominated by the tune spread during collision (1.8×10^{-3}), which determines a tune BPM resolution of $\sigma_x = 0.25$ μm. Selection of these values guarantees control of the dipole mode multibunch instabilities and emittance growth due to the feedback system during collider operation. In addition, these limit values should be established independently on the frequency bandwidth of the feedback, *i.e.*, they must be applied to the resistive wall and rf transverse multibunch instabilities feedback system. Other external perturbations on the beam—such as ground vibration, quadrupole random errors, vibration due to fluid or gas motion inside the magnets, *etc.*—are important when considering the total emittance growth of the beam. Although they were not considered here, the reader can find the treatment of these problems in References 1, 2, and 9.

ACKNOWLEDGEMENTS

I wish to thank Dr. V. Lebedev, Dr. V. V. Parkhomchuk, Dr. G. Stupakov, and Dr. V. D. Shiltsev for their valuable discussions, and Dr. R. Meinke and Dr. R. Schwitters for their support at the SSC Laboratory.

REFERENCES

1. V. Lebedev *et al.*, SSCL-Preprint-188, March 1993.
2. V. Lebedev *et al.*, Preprint 91-120, Novosibirk, 1991.
3. E. D. Courant and H. S. Snyder, *Ann. of Phys.*, **3**, 1–48 (1958).
4. E. A. Coddington, *An Introduction to Ordinary Differential Equations*, Prentice-Hall, Inc., 1961, p. 89.
5. L. Michelotti and F. Mills, *Proceedings IEEE Part. Accel. Conf.*, Chicago, v. **2**, March 1989, p. 1394.
6. S. Chen and G. López, SSCL-614, 1993.
7. SSC Conceptual Design, SSC-SR-2020, 1986, p. 183.
8. G. López, SSCL-442, 1991.
9. V. V. Parkhomchuk and V. D. Shiltsev, SSCL-622, 1993.