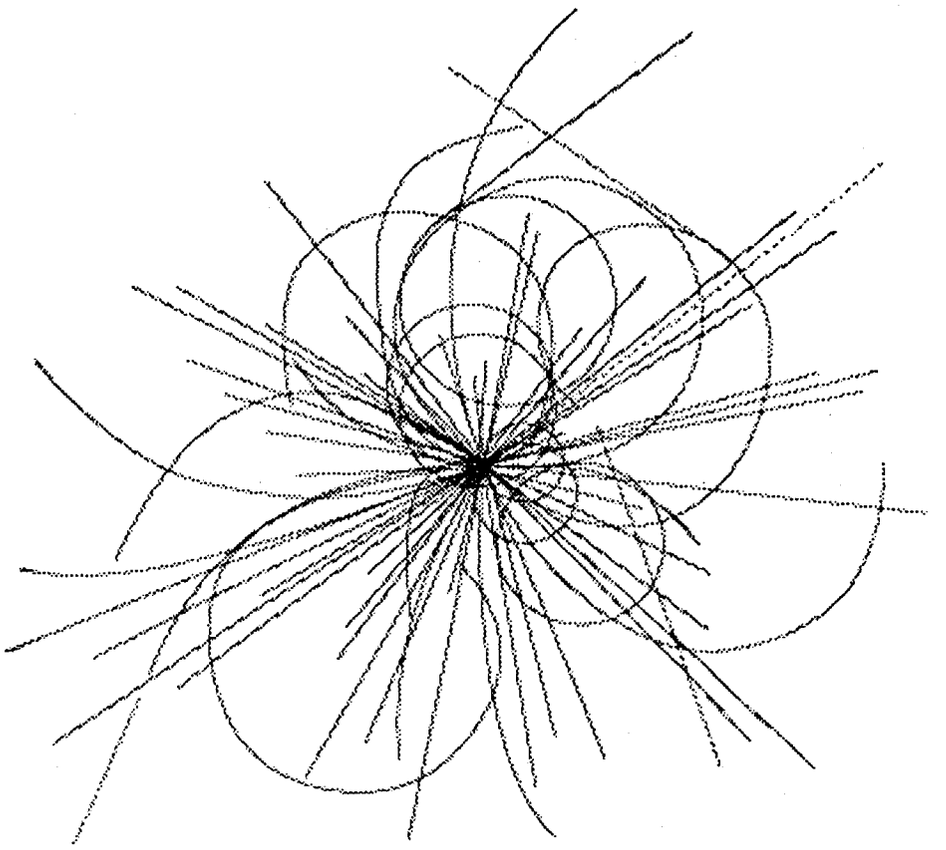
SSCL-604

SSCL-604
October 1992
Distribution Category: 400

M. Syphers
T. Sen

Notes on Amplitude Function Mismatch



Superconducting Super Collider Laboratory

SSCL-604

Notes on Amplitude Function Mismatch

M.J. Syphers and T. Sen

Superconducting Super Collider Laboratory*
2550 Beckleymeade Avenue
Dallas, Texas 75237

October 1992

* Operated by the Universities Research Association, Inc., for the U.S. Department of Energy under Contract No. DE-AC35-89ER40486.

Notes on Amplitude Function Mismatch

M.J. Syphers and T. Sen

Abstract

The design process of modern high-energy synchrotrons involves the development of the accelerator lattice in pieces, typically an arc made up of repetitive *cells* interrupted by occasional *matched insertions* for injection, extraction, acceleration, and various other systems required by the facility. The focusing elements of an insertion must be such that the periodic amplitude functions at the ends of the insertion match those of the cells on either side of the insertion. How well this match has to be and its sensitivity to the global betatron tunes of the accelerator as well as the particle momentum are the underlying themes of this report. Many of the relationships also are of use to the designers of beamlines which are used to transport and inject beams into a synchrotron.

Most of the content of this paper is not new to the accelerator physics community, but we thought it would be useful to place this important, basic information all in one place. Besides the classic work of Courant and Snyder, our sources include other papers, internal reports, and numerous discussions with our colleagues.

1.0 BASICS

The general solution for linear betatron oscillations in one transverse degree of freedom can be written as[1]

$$x(s) = A\sqrt{\beta(s)}\cos[\psi(s) + \delta] \tag{1}$$

where A and δ are constants given by the particle's initial conditions. The phase advance $\psi(s)$ and the amplitude function $\beta(s)$ satisfy the differential equations

$$\psi' = \frac{1}{\beta}, \tag{2}$$

$$2\beta\beta'' - \beta'^2 + 4\beta^2 K = 4, \tag{3}$$

where $K = e(\partial B_y/\partial x)/p$, with e = charge, p = momentum, $\partial B_y/\partial x$ = magnetic field gradient, and $\beta' = d\beta/ds$, *etc.* When one considers the periodic solution of the amplitude function, the motion through a single repeat period can be written in terms of the Courant-Snyder parameters β,

$$\alpha(s) \equiv -\frac{1}{2}\frac{d\beta(s)}{ds}, \quad \text{and} \tag{4}$$

$$\gamma \equiv \frac{1+\alpha^2}{\beta}, \tag{5}$$

in matrix form:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0+C} = \begin{pmatrix} \cos\Delta\psi_C + \alpha\sin\Delta\psi_C & \beta\sin\Delta\psi_C \\ -\gamma\sin\Delta\psi_C & \cos\Delta\psi_C - \alpha\sin\Delta\psi_C \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}. \tag{6}$$

Here, the phase advance is

$$\Delta\psi_C = \int_{s_0}^{s_0+C} \frac{ds}{\beta(s)} \tag{7}$$

where C is the repeat distance of the hardware, which may be the circumference of the accelerator. The *tune* of a synchrotron is

$$\nu \equiv \frac{1}{2\pi}\oint \frac{ds}{\beta(s)}. \tag{8}$$

The matrix of Eq. (6) is often written in a compact way as

$$M = I\cos\Delta\psi_C + J\sin\Delta\psi_C \tag{9}$$

where

$$J \equiv \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}. \tag{10}$$

Noting that $J^2 = -I$, where I is the identity matrix, one may also write Eq. (9) in even more compact form:

$$M = e^{J\Delta\psi c}. \tag{11}$$

The latter form often permits simplification of algebraic manipulations.

The matrix $M(s_1 \rightarrow s_2)$ which propagates motion between two points in the lattice may be written as

$$\begin{pmatrix} \left(\frac{\beta_2}{\beta_1}\right)^{\frac{1}{2}} (\cos\Delta\psi + \alpha_1 \sin\Delta\psi) & (\beta_1\beta_2)^{\frac{1}{2}} \sin\Delta\psi \\ -\frac{1+\alpha_1\alpha_2}{(\beta_1\beta_2)^{\frac{1}{2}}} \sin\Delta\psi - \frac{(\alpha_2-\alpha_1)}{(\beta_1\beta_2)^{\frac{1}{2}}} \cos\Delta\psi & \left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{2}} (\cos\Delta\psi - \alpha_2 \sin\Delta\psi) \end{pmatrix}. \tag{12}$$

In this case, $\Delta\psi$ is the phase advance from s_1 to s_2:

$$\Delta\psi = \int_{s_1}^{s_2} \frac{ds}{\beta(s)}. \tag{13}$$

It is often easier to remember the differential Eq. (3) when written in terms of the Courant-Snyder parameters:

$$K\beta = \gamma + \alpha'. \tag{14}$$

2.0 PROPAGATION OF THE AMPLITUDE FUNCTION

One can see how the amplitude function and its slope propagate through an accelerator section by referring to Figure 1 and noting that

$$M_2 = M(s_1 \rightarrow s_2) M_1 M(s_1 \rightarrow s_2)^{-1} \tag{15}$$

where M_1 and M_2 are the one-turn matrices corresponding to points 1 and 2. Then, using Eq. (9), one sees that

$$J_2 = M(s_1 \rightarrow s_2) J_1 M(s_1 \rightarrow s_2)^{-1}. \tag{16}$$

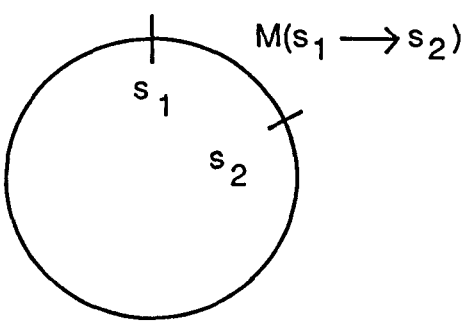


Figure 1. Definition of Matrix $M(s_1 \rightarrow s_2)$.

As an example, consider a thin quadrupole of length ℓ whose matrix is

$$M_q = \begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix} \tag{17}$$

where q is the strength $q = B'\ell/(B\rho)$. Then,

$$\begin{pmatrix} \alpha_2 & \beta_2 \\ -\gamma_2 & -\alpha_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 & \beta_1 \\ -\gamma_1 & -\alpha_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} \tag{18}$$

$$= \begin{pmatrix} \alpha_1 + \beta_1 q & \beta_1 \\ -\gamma_1 - 2q\alpha_1 - \beta_1 q^2 & -(\alpha_1 + \beta_1 q) \end{pmatrix} \tag{19}$$

from which we see that

$$\Delta\alpha = \beta q \tag{20}$$

$$\Delta\beta = 0 \tag{21}$$

$$\Delta\gamma = 2\alpha q + \beta q^2 \tag{22}$$

when passing through a thin quadrupole.

To see how the betatron phase advances through a region, let the transfer matrix for that region be given by

$$M(s_1 \to s_2) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{23}$$

Then, equating this with Eq. (12), we see that

$$\Delta\psi = \tan^{-1}\left(\frac{b}{a\beta_1 - b\alpha_1}\right) + 2\pi n, \quad n = 0, 1, 2, \ldots \tag{24}$$

The value of n cannot be determined from the above relation of matrix elements. The total phase advance must be obtained using Eq. (13) or by dividing the problem into small enough pieces. The phase advance depends upon the initial values of α and β unless the matrix element b is zero (*i.e.*, if the phase advance through the region is a multiple of 180°).

3.0 PROPAGATION OF A THIN GRADIENT ERROR

Let's see how the amplitude function downstream of a thin gradient error is altered. If $J_0(s_0)$ is the matrix of unperturbed Courant-Snyder parameters at the location of the error and $J_0(s)$ contains the unperturbed parameters at a point downstream, then, using Eq. (16),

$$\Delta J(s) = M(s_0 \to s)\Delta J(s_0) M(s_0 \to s)^{-1}, \tag{25}$$

where

$$\Delta J(s) = J(s) - J_0(s) = \begin{pmatrix} \alpha - \alpha_0 & \beta - \beta_0 \\ -(\gamma - \gamma_0) & -(\alpha - \alpha_0) \end{pmatrix}; \tag{26}$$

β is the new value of the amplitude function at s, β_0 is the unperturbed value, *etc.* Using our results for $\Delta\alpha$, $\Delta\beta$, and $\Delta\gamma$ through a thin quad and our expression for $M(s_0 \to s)$ (Eq. (12)), we find that

$$\frac{\Delta\beta(s)}{\beta_0(s)} = -(\beta_i q)\sin 2\psi_0(s-s_0) + \frac{1}{2}(\beta_i q)^2\left[1-\cos 2\psi_0(s-s_0)\right] \tag{27}$$

where $\psi_0(s-s_0)$ is the unperturbed phase advance between points s_0 and s and $\beta_i \equiv \beta_0(s_0)$. The amplitude function perturbation oscillates at twice the betatron frequency and for $(\beta_i q)$ sufficiently small, the perturbation describes simple harmonic motion. The change in α also propagates at twice the betatron frequency, it being given by

$$\begin{aligned}\Delta\alpha(s) = {} & \beta_i q\left[\cos 2\psi_0(s-s_0) - \alpha_0(s)\sin 2\psi_0(s-s_0)\right] \\ & - \frac{1}{2}(\beta_i q)^2\Big[\sin 2\psi_0(s-s_0) - \alpha_0(s)\Big(1-\cos 2\psi_0(s-s_0)\Big)\Big] \quad .\end{aligned} \tag{28}$$

Introducing this quad error also changes the phase advance across the lattice. The new transfer matrix $\bar{M}(s_0 \to s)$ from point s_0 to s is

$$\bar{M}(s_0 \to s) = M(s_0 \to s)\begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix}. \tag{29}$$

The new phase advance $\psi(s-s_0)$ across this section may be calculated using

$$\sin\psi(s-s_0) = \frac{\bar{M}(s_0 \to s)_{12}}{\sqrt{\beta_i\beta(s)}} \tag{30}$$

where $\bar{M}(s_0 \to s)_{12}$ is the (1,2) element of this matrix and $\beta(s)$ is the new amplitude function at s. Using Eq. (27), we obtain

$$\sin\psi(s-s_0) = \left[1-\beta_i q\sin 2\psi_0(s-s_0) + (\beta_i q)^2\sin^2\psi_0(s-s_0)\right]^{-1/2}\sin\psi_0(s-s_0) \quad . \tag{31}$$

An explicit result for the change in the phase advance may be obtained perturbatively in orders of the quad error q from the above exact expression. To second order in q, we find that the change $\Delta\psi \equiv \psi(s-s_0) - \psi_0(s-s_0)$ is

$$\Delta\psi = \beta_i q\sin^2\psi_0(s-s_0) - (\beta_i q)^2\sin 2\psi_0(s-s_0)\sin^2\psi_0(s-s_0) + O(q^3) \quad . \tag{32}$$

To first order in q, at a point $\pi/2$ away from the location of the error, there is no change in the β function while the change in phase advance is at its maximum value of $\beta_i q$.

The above results may be especially useful in a beamline where they may be used to calculate the changes in lattice functions at any point due to quad errors upstream of this point.

4.0 EQUATION OF MOTION OF β-WAVE

The equation of motion for an amplitude function mismatch is nonlinear when s is taken as the independent variable. A more congenial equation can be developed by using the reduced phase $\phi \equiv \psi/\nu$ as the independent variable. For betatron oscillations the Floquet transformation, where the other variable is $\zeta = x/\sqrt{\beta}$, produces the equation of motion

$$\frac{d^2\zeta}{d\phi^2} + \nu^2\zeta = 0 \tag{33}$$

which is pure simple harmonic motion with frequency (tune) ν. For the amplitude function mismatch, we need to define the reduced phase in terms of the unperturbed functions. That is, let $\phi \equiv \psi_0/\nu_0$, where $d\psi_0/ds = 1/\beta_0$, and ν_0 is the unperturbed tune. We then may derive an equation of motion for $[\beta(\phi) - \beta_0(\phi)]/\beta_0(\phi) \equiv \Delta\beta/\beta_0$ in the absence of gradient errors:

$$\frac{d^2}{d\phi^2}\frac{\Delta\beta}{\beta_0}(\phi) + (2\nu_0)^2\frac{\Delta\beta}{\beta_0}(\phi) = -2\nu_0^2\,\mathrm{det}\Delta J \tag{34}$$

$$= 2\nu_0^2[\Delta\alpha^2 - \Delta\beta\Delta\gamma] \tag{35}$$

where $\Delta\alpha = \alpha(\phi) - \alpha_0(\phi)$, *etc.* The quantity $\mathrm{det}\Delta J$ is an invariant in portions of the lattice without gradient perturbations as can be seen with the aid of Eq. (16).

So, the free amplitude function distortion oscillates with twice the betatron tune and with a constant offset given by the determinant of the ΔJ matrix at any point. This offset must be there since $\beta > 0$ and hence $\Delta\beta/\beta_0$ must always be greater than -1.

Rewritten in terms of the Courant-Snyder parameters,

$$\mathrm{det}\Delta J = -\frac{\left(\frac{\Delta\beta}{\beta_0}\right)^2 + \left(\Delta\alpha - \alpha_0\frac{\Delta\beta}{\beta_0}\right)^2}{1 + \Delta\beta/\beta_0} < 0. \tag{36}$$

Thus, $|\mathrm{det}\Delta J|^{1/2}$ can be interpreted as the relative amplitude of the β mismatch for small perturbations.

One can also express $\mathrm{det}\Delta J$ in terms of the unperturbed and the perturbed Courant-Snyder parameters:

$$\mathrm{det}\Delta J = -2(F - 1), \tag{37}$$

where

$$F \equiv \frac{1}{2}\left[\beta_0\gamma + \gamma_0\beta - 2\alpha_0\alpha\right]. \tag{38}$$

We will see this quantity F again later when we discuss injection mismatches.

The solution to Eq. (34) is just simple harmonic motion with a constant term added:

$$\frac{\Delta\beta}{\beta_0}(\phi) = A\cos 2\nu_0\phi + B\sin 2\nu_0\phi + \frac{1}{2}|\mathrm{det}\Delta J|. \tag{39}$$

The constants A and B are found from the initial conditions:

$$A = \frac{\Delta\beta}{\beta_0}(0) - \frac{1}{2}|\det\Delta J|, \tag{40}$$

$$B = \alpha_0 \frac{\Delta\beta}{\beta_0}(0) - \Delta\alpha(0). \tag{41}$$

Thus, the maximum value of $\Delta\beta/\beta_0$ downstream of our starting point $\phi = 0$ is given by

$$\left(\frac{\Delta\beta}{\beta_0}\right)_{max} = \sqrt{A^2 + B^2} + \frac{1}{2}|\det\Delta J| \tag{42}$$

$$= \frac{|\det\Delta J|}{2} + \sqrt{|\det\Delta J| + \left(\frac{|\det\Delta J|}{2}\right)^2} \tag{43}$$

where use has been made of Eq. (36). The maxima occur at phases where

$$\tan 2\nu_0\phi = B/A \tag{44}$$

$$= \left(\frac{\alpha_0 \frac{\Delta\beta}{\beta_0} - \Delta\alpha}{\frac{\Delta\beta}{\beta_0} - |\det\Delta J|/2}\right)_0 . \tag{45}$$

The usefulness of the above result is, of course, that once one calculates the mismatch of the amplitude function and its slope at one point in the lattice (at the end of a *nearly* matched insertion, for example), then the maximum mismatch downstream can be computed immediately.

If we look once again at the perturbation downstream of a thin quadrupole error, we see that just after the quad,

$$\det\Delta J = \begin{vmatrix} q\beta_i & 0 \\ -\Delta\gamma & -q\beta_i \end{vmatrix} = -(q\beta_i)^2 \tag{46}$$

where $\beta_i = \beta_0$ at the location of the quadrupole. Then,

$$\frac{d^2}{d\phi^2}\frac{\Delta\beta}{\beta_0}(\phi) + (2\nu_0)^2\frac{\Delta\beta}{\beta_0}(\phi) = 2\nu_0^2(q\beta_i)^2 = \text{constant} \ . \tag{47}$$

The initial conditions are $\Delta\beta/\beta(0) = 0$ and $\Delta\alpha(0) = q\beta_i$. Solving for the constants A and B with these initial conditions we arrive once again at the solution Eq. (27). The maximum error in β downstream of the perturbation would be

$$\left(\frac{\Delta\beta}{\beta_0}\right)_{max} = q\beta_i\sqrt{1 + (q\beta_i)^2/4} + \frac{1}{2}(q\beta_i)^2 \tag{48}$$

$$\approx q\beta_i = \sqrt{|\det\Delta J|} \tag{49}$$

where the last line is valid for small perturbations.

To include the driving terms due to gradient errors in the equation of motion for $\Delta\beta/\beta_0$, we let β_0 satisfy the differential equation $K\beta_0 = \gamma_0 + \alpha_0'$, and let β satisfy $(K+k)\beta = \gamma + \alpha'$, where $\beta = \beta_0 + \Delta\beta$, *etc.* Then, the relative β error satisfies

$$\frac{d^2}{d\phi^2}\frac{\Delta\beta}{\beta_0}(\phi) + (2\nu_0)^2\frac{\Delta\beta}{\beta_0}(\phi) = -2\nu_0^2\left[\beta_0^2(\phi)\,k(\phi)\left(1+\frac{\Delta\beta}{\beta_0}(\phi)\right) + \det\Delta J(\phi)\right]. \quad (50)$$

Here, in general, $\det\Delta J(\phi)$ is not invariant as it is altered by gradient perturbations. Noting that

$$\Delta\alpha - \alpha_0\frac{\Delta\beta}{\beta_0}(\phi) = -\frac{1}{2\nu_0}\frac{d}{d\phi}\frac{\Delta\beta}{\beta_0}(\phi)\,, \quad (51)$$

one can easily exhibit the equation of motion entirely in terms of $\Delta\beta/\beta_0$ and its derivatives with respect to ϕ.

For small perturbations we can drop terms which are second order in the small quantities, *e.g.*, $k\Delta\beta$. This reduces the above equation to

$$\frac{d^2}{d\phi^2}\frac{\Delta\beta}{\beta_0}(\phi) + (2\nu_0)^2\frac{\Delta\beta}{\beta_0}(\phi) = -2\nu_0^2\,\beta_0^2\,k(\phi) \quad (52)$$

as appears in Courant and Snyder.

5.0 GENERAL INSERTION ERROR

Up to this point we have not imposed any periodic boundary conditions in our examples of amplitude function mismatch. In this section, we will focus on the effect of an insertion error propagating around the ring. Accordingly we consider the synchrotron to be composed of an insertion described by a transfer matrix P and the rest of the ring described by a matrix U. Errors which are localized within the insertion perturb the transfer matrix P but leave U unchanged. Consider Figure 2. At point 0 the periodic ring matrix is

$$M_0 = P\cdot U\,. \quad (53)$$

Due to the error in the insertion,

$$P \rightarrow \bar{P} = P + \Delta P$$

and this changes the ring matrix at point 0 to

$$\bar{M}_0 = \bar{P}U = \bar{P}\cdot P^{-1}\cdot M_0\ . \quad (54)$$

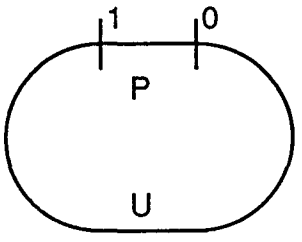


Figure 2. Definition of Insertion.

In terms of the original phase advance μ_0 around the ring and the beam parameters at point 0, the unperturbed ring matrix can be written as

$$M_0 = I\cos\mu_0 + J_0\sin\mu_0 \tag{55}$$

while the changed ring matrix is

$$\bar{M}_0 = I\cos\mu + \bar{J}_0\sin\mu \tag{56}$$

with μ the new phase advance around the ring. This new phase advance may be extracted if we know the matrix of the insertion P and the perturbation ΔP as,

$$\cos\mu = \frac{1}{2}Tr\ \bar{M}_0 = \frac{1}{2}Tr[(I + \Delta P\cdot P^{-1})M_0] \tag{57}$$

$$= \cos\mu_0 + \frac{1}{2}Tr(\Delta P\cdot P^{-1}\cdot M_0)\ . \tag{58}$$

Writing the new matrix $\bar{J}_0$ at point 0 as,

$$\bar{J}_0 = J_0 + \Delta J_0$$

and substituting in Eq. (56), we obtain

$$\Delta J_0 = \frac{1}{\sin\mu}[(I + \Delta P\cdot P^{-1})M_0 - (I\cos\mu + J_0\sin\mu)] \tag{59}$$

$$= \frac{Tr(\Delta P\cdot P^{-1}\cdot M_0)\ I + 2\Delta P\cdot P^{-1}\cdot M_0}{2\sin\mu} + J_0\left(\frac{\sin\mu_0}{\sin\mu} - 1\right)\ . \tag{60}$$

The sign of $\sin\mu$ is the same as that of the $(1,2)$ element of the new ring matrix $\bar{M}_0$. From the above expression we can calculate the change in β and α at the exit of the insertion. The changes at any other point s in the ring can be calculated by application of

$$\Delta J(s) = M(0\to s)\cdot\Delta J_0\cdot M(0\to s)^{-1}$$

where $M(0\to s)$ is the transfer matrix from point 0 to s. At the entrance to the insertion (point 1 in Figure 2), the change to the J-matrix is

$$\Delta J_1 = \frac{1}{\sin\mu}[M_1(I + P^{-1}\cdot\Delta P) - (I\cos\mu + J_1\sin\mu)] \tag{61}$$

$$= \frac{Tr(M_1\cdot P^{-1}\cdot\Delta P)\ I + 2M_1\cdot P^{-1}\cdot\Delta P}{2\sin\mu} + J_1\left(\frac{\sin\mu_0}{\sin\mu} - 1\right) \tag{62}$$

where M_1 is the original ring matrix at point 1 and J_1 contains the unperturbed Courant-Snyder parameters corresponding to point 1.

5.1 Thin Quad Error in a Synchrotron

We specialize the above general case to that of a thin quad error in a synchrotron. In the absence of the error, the matrix P is simply the identity. The presence of the thin quad at point 0 changes this to $\bar{P}$ where

$$\bar{P} = \begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix} , \tag{63}$$

q being the strength of the quad. The new ring matrix at point 0 is

$$\bar{M}_0 = \begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix} \begin{pmatrix} \cos\mu_0 + \alpha_0 \sin\mu_0 & \beta_0 \sin\mu_0 \\ -\gamma_0 \sin\mu_0 & \cos\mu_0 - \alpha_0 \sin\mu_0 \end{pmatrix} . \tag{64}$$

Using Eq. (58), the phase change due to the quad error is

$$\Delta\mu \equiv \mu - \mu_0 = \arccos[\cos\mu_0 - \frac{1}{2} q\beta_0 \sin\mu_0] - \mu_0 . \tag{65}$$

Similarly the change in the amplitude functions at the location of the quad error may be found using

$$\Delta J_0 = \frac{1}{\sin\mu}[\bar{M}_0 - (I\cos\mu + J_0 \sin\mu)] . \tag{66}$$

We obtain

$$\begin{aligned} \Delta\beta &= \beta_0 \left(\frac{\sin\mu_0}{\sin\mu} - 1 \right) \\ \Delta\alpha &= (\alpha_0 + \frac{1}{2} q\beta_0) \frac{\sin\mu_0}{\sin\mu} - \alpha_0 \\ \Delta\gamma &= \frac{1}{\sin\mu}[q\cos\mu_0 + (q\alpha_0 + \gamma_0)\sin\mu_0] - \gamma_0 . \end{aligned} \tag{67}$$

Note that the above are exact results and not limited to being first order in q. To make contact with the usual first order results, we take the limit of small q and neglect terms of $O(q^2)$. In this limit,

$$\cos\mu_0 - \frac{1}{2} q\beta_0 \sin\mu_0 \simeq \cos\left(\mu_0 + \frac{1}{2} q\beta_0\right)$$

and the phase shift is

$$\Delta\mu = \frac{1}{2} q\beta_0 . \tag{68}$$

In the same limit

$$\begin{aligned} \sin\mu &= [1 - (\cos\mu_0 - \frac{1}{2} q\beta_0 \sin\mu_0)^2]^{1/2} \\ &\simeq \sin\mu_0 [1 + \frac{1}{2} q\beta_0 \cot\mu_0] . \end{aligned}$$

Correspondingly, the changes in the amplitude functions at the location of the quad error, to first order in q, are

$$\begin{aligned}
\Delta\beta &= -\frac{1}{2}q\beta_0^2\cot\mu_0 \\
\Delta\alpha &= \frac{1}{2}q\beta_0(1-\alpha_0\cot\mu_0) \\
\Delta\gamma &= q[\alpha_0+\frac{1}{2}(1-\alpha_0^2)\cot\mu_0] \ .
\end{aligned} \tag{69}$$

Downstream of the quadrupole error, the amplitude function mismatch will have maxima

$$\left(\frac{\Delta\beta}{\beta_0}\right)_{max} \approx \sqrt{|\det\Delta J|} = \frac{q\beta_0}{2\sin 2\pi\nu_0} \tag{70}$$

and we see that the mismatch is enhanced when the tune is near an integer or half–integer. This result implies that the higher order tune shifts are also amplified near integer or half–integer tunes, as we shall see below.

5.2 Second Order Tune Shift

Generalizing Eq. (68) we obtain the well known result that the phase shift due to gradient errors $k(s)$ in a ring of circumference C is, to first order,

$$\Delta\mu = \frac{1}{2}\int_0^C \beta_0(s)k(s)ds \ . \tag{71}$$

However the phase shift to second order in the gradient errors is directly related to the beta wave to first order, resulting from the gradient errors. We postpone a detailed derivation of this to another note and merely sketch the details here.

Consider Figure 3 with two quad errors located at points 1 and 2. The new periodic matrix at point 1 in the presence of these two errors is

$$\bar{M}_1 = M(s_2 \to s_1)\cdot P_2 \cdot M(s_1 \to s_2)\cdot P_1 \equiv M_1 + \Delta M_1 \tag{72}$$

where $M(s_1 \to s_2)$ is the transfer matrix from point 1 to point 2, *etc.* and P_1, P_2 are the matrices

$$\begin{aligned}
P_1 &= \begin{pmatrix} 1 & 0 \\ -k_1\Delta s_1 & 1 \end{pmatrix} , \\
P_2 &= \begin{pmatrix} 1 & 0 \\ -k_2\Delta s_2 & 1 \end{pmatrix} .
\end{aligned} \tag{73}$$

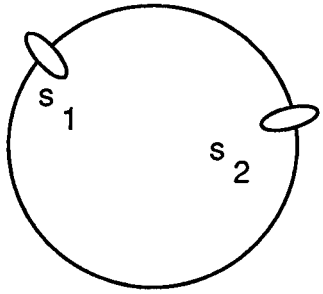


Figure 3. Interaction of Two Quad Errors.

The new phase advance μ can be found from

$$\cos\mu = \cos\mu_0 + \frac{1}{2} Tr \Delta M_1 \ . \tag{74}$$

We may also write

$$\cos\mu = \cos(\mu_0 + \Delta\mu) = \cos\mu_0 \cos\Delta\mu - \sin\mu_0 \sin\Delta\mu \ . \tag{75}$$

Expanding the phase shift in powers of a small quantity ϵ as

$$\Delta\mu = \epsilon\Delta\mu_1 + \epsilon^2\Delta\mu_2 + \dots \ , \tag{76}$$

scaling the gradient errors as

$$k_1 \to \epsilon k_1 \ \ ; k_2 \to \epsilon k_2 \ ,$$

and substituting into Eqs. (74) and (75), we can equate like powers of ϵ, take the limit of infinitesimally-thin quads distributed around the ring and integrate over the circumference. We obtain

$$\begin{aligned} \Delta\mu_1 &= \frac{1}{2}\int_0^C k(s)\beta_0(s)ds \\ \Delta\mu_2 &= \frac{1}{4}\int_0^C k(s)\beta_0(s)\frac{\Delta\beta_1(s)}{\beta_0(s)}ds \end{aligned} \tag{77}$$

where

$$\frac{\Delta\beta_1(s)}{\beta_0(s)} = -\frac{1}{2\sin\mu_0}\int_s^{s+C} k(s')\beta_0(s')\cos\Big[2\Big(\psi_0(s') - \psi_0(s)\Big) - \mu_0\Big]ds' \tag{78}$$

and $\psi_0(s)$ is the unperturbed phase advance to point s. This last expression is the change in β to first order in the gradient errors. This relation between the second order phase shift $\Delta\mu_2$ and the first order beta wave tells us that if we can localize the beta wave due to a set of gradient errors (or adjustments), then the higher order phase shift will be reduced. As a consequence, higher order chromatic effects are smaller if the first order beta wave is

reduced. Second order changes to the amplitude functions and to the chromaticity will be discussed in more detail in a forthcoming note.

6.0 INJECTION MISMATCH

Just as it is important to study the effects of amplitude function mismatches when designing insertions for circular accelerators, it is also of interest to look at the effects of mismatches of amplitude functions upon entrance to these accelerators. As one would expect, a beam which is described by Courant-Snyder parameters which are not the periodic parameters of the accelerator into which it is injected will tend to filament due to nonlinearities and hence have its emittance increased. We wish to look at the severity of such mismatches.

To begin with, we must differentiate between the amplitude function which is being delivered by the beamline from the periodic amplitude function of the synchrotron. Suppose β and α are the Courant-Snyder parameters as delivered by the beamline to a particular point in an accelerator, and β_o, α_o are the periodic lattice functions of the ring at that point. A particle with trajectory (x, x') can be viewed in the $(x, \beta x' + \alpha x) \equiv (x, \eta)$ phase space corresponding to the beamline functions, or in the $(x, \beta_0 x' + \alpha_0 x) \equiv (x, \eta_0)$ phase space corresponding to the lattice functions of the ring. If the phase space motion lies on a circle in the beamline view, then the phase space motion will lie on an ellipse in the ring view.

The equation of the ellipse can be obtained by noting that

$$x' = (\eta - \alpha x)/\beta = (\eta_0 - \alpha_0 x)/\beta_0 \tag{79}$$

or,

$$\eta = \frac{\beta}{\beta_0}\eta_0 + (\alpha - \frac{\beta}{\beta_0}\alpha_0)x \tag{80}$$

$$\equiv \beta_r \eta_0 + \Delta\alpha_r x. \tag{81}$$

If the equation of the circle in the beamline view is

$$x^2 + \eta^2 = \beta A^2 = \beta_r \beta_0 A^2, \tag{82}$$

where A^2 is the Courant-Snyder invariant (the same A as in Eq. (1)), then the equation of the ellipse in the "ring" system will be

$$\frac{(1 + \Delta\alpha_r^2)}{\beta_r} x^2 + 2\Delta\alpha_r \, x\eta_0 + \beta_r \eta_0^2 = \beta_0 A^2. \tag{83}$$

It would be useful to rotate the coordinate axes so that they correspond to the major and minor axes of the ellipse. This amounts to rotating through an angle Θ so that the cross-term in the equation of the ellipse is eliminated, namely

$$\tan 2\Theta = \frac{2\beta_r \Delta\alpha_r}{1 + \Delta\alpha_r^2 - \beta_r^2} \,. \tag{84}$$

The resulting equation in the rotated coordinates $x_e \equiv x\cos\Theta + \eta_0 \sin\Theta$, $\eta_{oe} \equiv -x\sin\Theta + \eta_0\cos\Theta$ will be, after some manipulation,

$$\frac{1}{b_r} x_e^2 + b_r \eta_{oe}^2 = \beta_0 A^2 \tag{85}$$

where

$$b_r \equiv F + \sqrt{F^2 - 1} \tag{86}$$

and F is given by Eq. (38). Note that if $\Delta\alpha_r = 0$, then $b_r = \beta_r$.

There is a physical significance to the quantity b_r; it is the ratio of the areas of two circumscribed ellipses (as shown in Figure 4) which have shapes and orientations given by the two sets of Courant-Snyder parameters found in the matrices J and J_0. (This can be seen most easily by transforming from x-x' phase space to x-$(\alpha_0 x + \beta_0 x')$ phase space and using Eq. (85).)

This might suggest that a beam contained within the smaller ellipse upon injection into the synchrotron (whose periodic functions give ellipses similar to the larger one) will have its emittance increased by a factor b_r. However, this would be an over-estimate of the increase of the average of the emittances of all the particles.

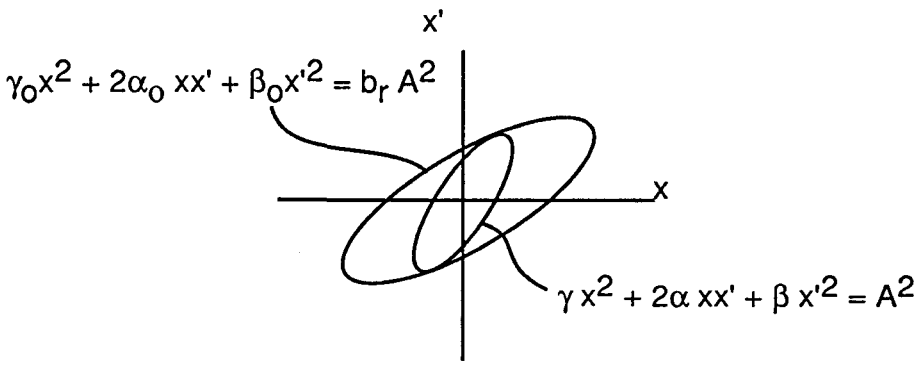


Figure 4. The Ratio of the Areas of the Two Circumscribed Ellipses is Given by b_r.

We now compute the variance of the resulting particle distribution, assuming that the particles undergo betatron oscillations of constant amplitude once in the synchrotron, but that the tune may depend upon the amplitude. For simplicity, we consider the case where the slope of the amplitude function (*i.e.*, α) is matched and equal to zero at our

observation point in the ring but the function β itself is not matched. The final result does not depend upon this assumption. Then we can compare phase space trajectories by looking at Eqs. (82) and (85). We see that in the beamline view, the radius of a particle's trajectory in phase space would be increased by a factor of $\sqrt{b_r}$ ($= \sqrt{\beta_r}$, here), as depicted in Figure 5, due to a mismatch. If in the beamline view the new phase space trajectory is

$$x^2 + \eta^2 = b_r R^2, \tag{87}$$

then in the synchrotron view, the equation of the ellipse would be

$$\frac{x^2}{b_r R^2} + \frac{\eta_0^2}{R^2/b_r} = 1\,. \tag{88}$$

A particle with initial phase space coordinates x_i and η_{oi} will commence describing a circular trajectory in phase space upon subsequent passages through the synchrotron, where the radius of the phase space trajectory is given by $a = \sqrt{x_i^2 + \eta_{oi}^2}$.

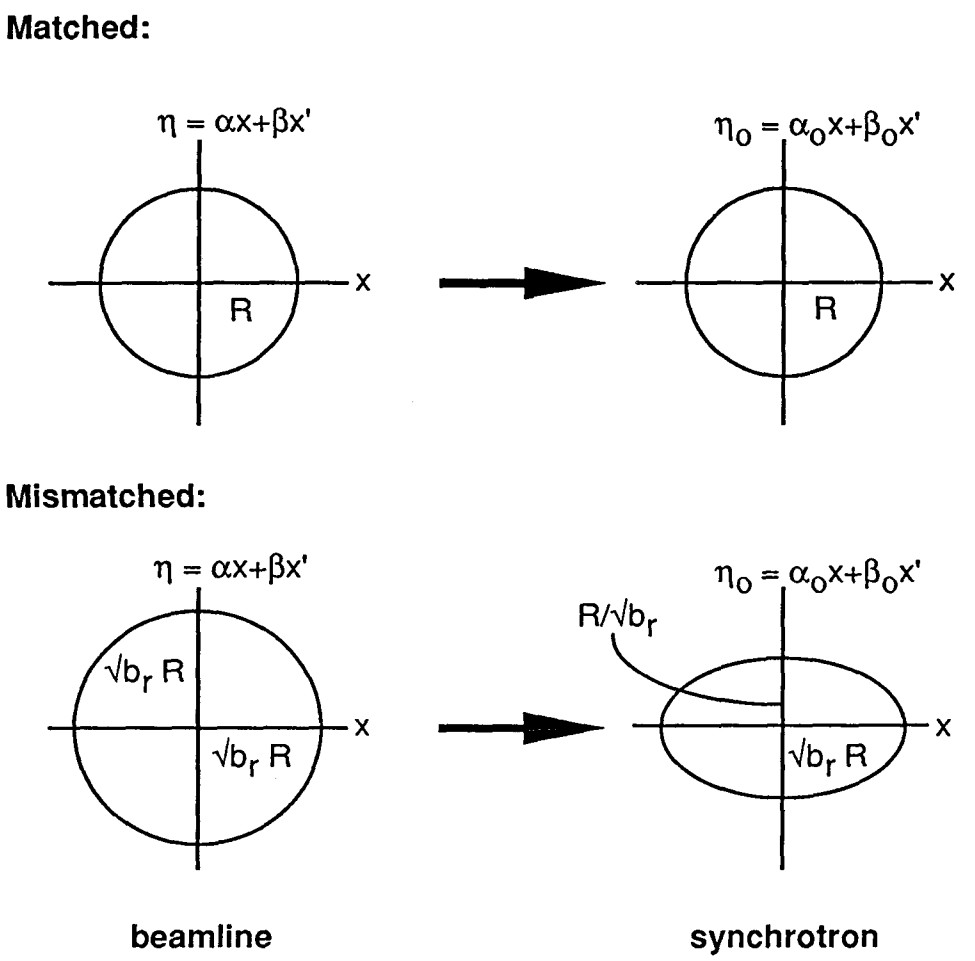


Figure 5. For an "Unmatched" Condition, the Beam Shape Would be Elliptical in the Phase Space Respecting the Synchrotron's Lattice Functions with $\langle x_i^2 \rangle = b_r \sigma_0^2$, and $\langle \eta_{oi}^2 \rangle = \sigma_0^2 / b_r$.

Finally, imagine a beam distribution which has $\langle x_i^2 \rangle = \langle \eta_{oi}^2 \rangle = \sigma_0^2$ for an incoming amplitude function which is perfectly matched. Then a mismatch would result in an initial distribution at the injection point which has

$$\langle x_i^2 \rangle = b_r \sigma_0^2, \tag{89}$$

$$\langle \eta_{oi}^2 \rangle = \frac{1}{b_r} \sigma_0^2 \tag{90}$$

and hence

$$\langle a^2 \rangle = \langle x_i^2 \rangle + \langle \eta_{oi}^2 \rangle \tag{91}$$

$$= b_r \sigma_0^2 + \frac{1}{b_r} \sigma_0^2. \tag{92}$$

It follows that the resulting distribution after filamentation will have variance in the x coordinate

$$\sigma^2 = \langle x^2 \rangle = \frac{\langle a^2 \rangle}{2} = \frac{b_r^2 + 1}{2 b_r} \sigma_0^2 \tag{93}$$

$$= F\, \sigma_0^2. \tag{94}$$

This expression can be rewritten in terms of $\det \Delta J$ which we found in Section 4.0:

$$\frac{\sigma^2}{\sigma_0^2} = 1 + \frac{1}{2} \left| \det(\Delta J) \right| \ . \tag{95}$$

For the case where the slope of the amplitude function is matched and equal to zero, we have

$$\frac{\sigma^2}{\sigma_0^2} = 1 + \frac{1}{2} \left(\frac{\Delta\beta/\beta_0}{\sqrt{1 + \Delta\beta/\beta_0}} \right)^2 . \tag{96}$$

This says that a 20% β mismatch at injection, for example, would cause only a 2% increase in emittance.

It is interesting to note that the change in emittance (*i.e.*, σ^2) generated by an amplitude function mismatch is proportional to the incoming emittance, in contrast to the effect of a steering error where the emittance increase is independent of the initial emittance. For a steering error at injection the resulting variance would be

$$\sigma^2 = \sigma_0^2 + \frac{1}{2} \left[\Delta x^2 + (\beta_0 \Delta x' + \alpha_0 \Delta x)^2 \right]. \tag{97}$$

In principle, the steering error can be corrected by the implementation of a feedback system designed to reduce transverse beam oscillations. The system must be able to damp the oscillations in a time shorter than the time it takes the particle motion to decohere. The

number of turns to decohere would be on the order of $1/\Delta\nu$, where $\Delta\nu$ is the tune spread of the beam due to nonlinearities. While dampers will not help curtail amplitude function mismatches at injection, the results of the preceding paragraph point out that an injection β mismatch is not as critical as injection steering errors, especially at higher energies (and hence, smaller beam sizes).

Perhaps a worse offender would be a mismatch of the dispersion function into a synchrotron, since this would be a trajectory error of off-momentum particles which cannot be corrected with dampers. For this case, the resulting variance of the particle distribution after filamentation at a point in the lattice would be

$$\sigma^2 = \sigma_0^2 + \frac{1}{2}\left[\Delta D^2 + (\beta_0 \Delta D' + \alpha_0 \Delta D)^2\right] \left(\frac{\sigma_p}{p}\right)^2 \tag{98}$$

where $(\Delta D, \Delta D')$ is the error in the dispersion function delivered to that point of the synchrotron, and σ_p/p is the rms relative momentum deviation. In this case, the severity of the mismatch condition depends strongly on the relative momentum spread.

ACKNOWLEDGEMENTS

The authors would like to thank Karl Brown for his suggestions to the document and for our many useful discussions with him on the above topics. Special thanks also go to Don Edwards for reading and checking parts of the work and for initially pointing out the significance of det ΔJ.

REFERENCES

1. E.D. Courant and H.S. Snyder, "Theory of the Alternating-Gradient Synchrotron," *Annals of Physics*, Vol. **3**, No. 1, p. 26 (1958).