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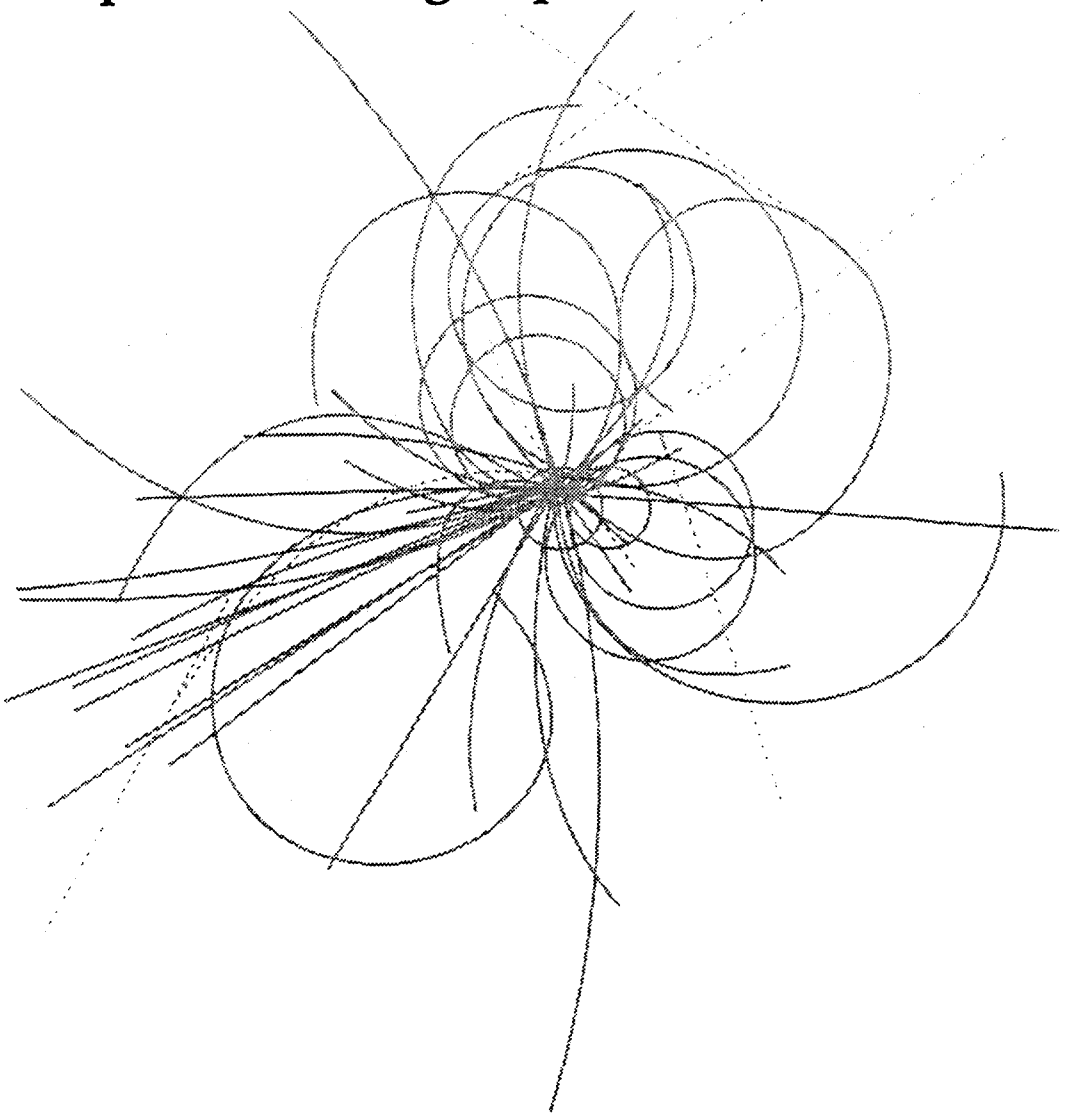

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Abstract

Sextupole vibration due to ground motions in the SSC may resonate with the beam causing exponential emittance growth. The growth rate is proportional to the spectral density of vibrations at the double sideband betatron frequency. The estimates for the SSC show that this effect is much smaller than the quadrupole vibrations with the same amplitude.

1.0 INTRODUCTION

The magnet vibrations in the SSC and their influence on the beam emittance have been studied in a number of works.[1–4] The largest attention has been given the problem of quadrupole vibrations in the transverse direction that perturbs the dipole magnetic field on the orbit. This paper addresses a similar problem related with sextupole vibrations. A displaced sextupole perturbs the quadrupole component of the magnetic field and changes the lattice of the ring. The sextupoles in the collider are used to correct chromaticity of the machine.

Preliminary estimations of the effect have previously been done by A. Chao and D. Douglas.[5]

In this paper we study in detail random vibrations of a single sextupole in the transverse direction and shows that it causes an exponential growth of the emittance of the beam with the growth rate being proportional to the spectral density of vibrations at the double betatron frequency. The estimate for the collider parameters shows that, in terms of the amplitude of the vibrations, the emittance growth caused by vibrations of the sextupoles is much smaller than that of the quadrupoles.

2.0 SEXTUPOLE VIBRATIONS

A displacement of a sextupole in the transverse direction perturbs the quadrupole component of the magnetic field of the closed orbit. The equation that describes how particle transverse position evolves with time in the presence of such a perturbation has the following form

$$y'' + K(s)y = -\delta K(s,t)y, \tag{1}$$

where y is the position offset of a particle, $K(s) = eB'/Pc$ and $\delta K(s)$ is caused by the displacement of a sextupole, $\delta K(s,t) = h(s)e\,B''\Delta(t)/Pc$, where $\Delta(t)$ is the sextupole displacement, B'' is the second derivative of the magnetic field in the sextupole and $h(s)$ denotes the function that is equal to unity inside the sextupole and takes zero values outside of it. The function $h(s)$ indicates that the gradient of the magnetic field on the orbit is perturbed only inside the sextupole.

Since the length of the sextupole l is much smaller than the betatron period, the function $h(s)$ can be replaced by a delta-function (thin lens approximation) according to

$$\frac{1}{l}h(s) \to \delta(s - s_0), \tag{2}$$

where s_0 designates the sextupole position. Putting Eq. (2) into Eq. (1) yields

$$y'' + K(s)y = -\frac{\Delta(t)B''le}{Pc}\,\delta(s - s_0)y. \tag{3}$$

It is convenient to transform from y and s to new variables η and ψ where $\psi(s)$ is the betatron phase and $\eta = \beta^{-1/2}y$. From Eq. (3), using the relation $d\psi/ds = 1/\beta$, one finds

$$\frac{d^2\eta}{d\psi^2} + \eta = -\eta \frac{\Delta(t)B''l\beta_0 e}{Pc} \sum_{m=-\infty}^{\infty} \delta\left(\psi - m\mu - \psi_0\right), \tag{4}$$

where β_0 and ψ_0 refer to the position of the sextupole and the sum on the right-hand side takes into account that due to periodicity of the problem the values of ψ which differ by μ correspond to the same position on the orbit. Each delta-function in Eq. (4) represents a kick that the particle experiences during successive passes through the displaced sextupole.

In Eq. (4), the dimensionless parameter

$$\varepsilon = \frac{\Delta(t)B''l\beta_0 e}{Pc} \tag{5}$$

gives a measure of the effect under consideration. In an analytical approach, the simplest way to find the motion governed by Eq. (4) with a random function $\Delta(t)$, is to calculate the averaged time derivative

$$\gamma \equiv d\Big\langle \ln\left(\eta^2 + \dot{\eta}^2\right)\Big\rangle / dN, \tag{6}$$

where N is the number of turns and the angular brackets stand for the averaging. Notice that the quantity $\eta^2 + \dot{\eta}^2$ is proportional to Courant-Snyder parameter and is conserved in the absence of the perturbation. The calculations in Appendix A show that so defined γ does not depend on time and is given by

$$\gamma = \frac{1}{8}\Omega \left(\frac{B''l\beta_0 e}{Pc}\right)^2 \sum_{m=-\infty}^{\infty} S_\Delta[(2\nu - n)\Omega], \tag{7}$$

where S_Δ is the spectral density of the function $\Delta(t)$ and Ω is the revolution frequency.

So defined γ, however, differs from a standard definition of the emittance growth rate according to which γ should be given by $d\ln\left\langle \eta^2 + \dot{\eta}^2\right\rangle / dN$ (the averaging is performed before taking the logarithm). Numerical simulations showed that the quantity $d\ln\left\langle \eta^2 + \dot{\eta}^2\right\rangle / dN$ is approximately two times larger than that given by Eq. (7). To perform numerical estimates of the effect, we will use for γ Eq. (7) multiplied by a factor of 2.

For the SSC, according to collider specifications, the product $B''l$ for the sextupoles located near the focussing quadrupoles is equal to $2.4 \cdot 10^3$ T/m. These sextupoles make a dominant contribution to Eq. (7), because the beta-function at their positions reaches its local maximum, $\beta_0 = 305$ m. Taking the nominal collider parameter $Pc = 20$ TeV and

remembering that there are about 400 sextupoles* near the focussing quadrupoles in the ring we will require that after 10 hours of operation the emittance must not increase by more than 10%. This gives the following condition for the spectral density of vibrations

$$\sum_{m=-\infty}^{\infty} S_{\Delta}[(2\nu - n)\Omega] < 3 \cdot 10^{-6} \frac{\text{micron}^2}{\text{Hz}}. \tag{8}$$

Typically the vibration spectrum rapidly falls with the frequency. That means that the dominant term in the sum of Eq. (8) is the first one. It corresponds to the vibration frequency equal to the sideband of the double betatron frequency (for a fractional part of tune 0.28 this frequency is 1.9 kHz). The condition of Eq. (8) is much less stringent than that of the analogous requirement for the quadrupole vibrations.[3,4]

* In the range of 1 kHz each of the sextupoles will vibrate independently so that their contributions have to be summed.

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I would like to thank A. Chao who attracted my attention to the problem of sextupole vibrations.

APPENDIX A
PARAMETRIC RESONANCE OF A RANDOMLY DRIVEN OSCILLATOR

Consider first an oscillator whose frequency randomly fluctuates around a given value. We assume that the amplitude of the fluctuations is relatively small. The behavior of such an oscillator is governed by the following equation.

$$\frac{d^2\eta}{d\psi^2} + \Big(1 + \varepsilon(\psi)\Big)\eta = 0, \tag{A1}$$

where the unperturbed frequency of the oscillator is chosen as unity and $\varepsilon(\psi)$ is assumed to be a stationary random function with given statistical properties, $|\varepsilon(\psi)| \ll 1$.

It is convenient to transform from η to new complex variable z,

$$z = \eta - i\dot{\eta}, \tag{A2}$$

where the dot stands for the differentiation with respect to ψ. The equation for z takes the form

$$\dot{z} - iz = -\frac{1}{2}i\varepsilon(z + z^*) \tag{A3}$$

where z^* is the complex conjugate of z. Now, we define the modulus r and phase ϕ of z, $z = r\exp(i\phi)$. The equations for r and ϕ follow from Eq. (A3),

$$\dot{r} = -\frac{1}{2}\varepsilon r \sin 2\phi, \tag{A4}$$

$$\dot{\phi} = 1 - \frac{1}{2}\varepsilon(1 + \cos 2\phi). \tag{A5}$$

An important observation is that Eq. (A5) does not contain the variable r. That allows us to approximately integrate it using the smallness of ε. In the lowest approximation we neglect ε-term to obtain $\phi = t + \phi_0$ where ϕ_0 is the initial phase. Putting this into the argument of cosine in Eq. (A5) one finds

$$\dot{\phi} = 1 - \frac{1}{2}\varepsilon\left(1 + \cos 2(\psi + \phi_0)\right). \tag{A6}$$

Integrating Eq. (A6), we obtain an expression for ϕ that is valid through the first order,

$$\phi = \psi + \phi_0 - \frac{1}{2}\int_0^{\psi} \varepsilon(\psi')\left(1 + \cos 2(\psi' + \phi_0)\right) d\psi'. \tag{A7}$$

Now, putting Eq. (A7) into Eq. (A4) and expanding the sine in small parameter ε one finds

$$\frac{d\ln r}{d\psi} = -\frac{1}{2}\varepsilon(\psi)\sin 2(\psi + \phi_0) + \frac{1}{2}\varepsilon(\psi)\cos 2(\psi + \phi_0)\int_0^{\psi} \varepsilon(\psi')\left(1 + \cos 2(\psi' + \phi_0)\right) d\psi'. \tag{A8}$$

Averaging Eq. (A8) with the use of $\langle\varepsilon\rangle = 0$, and keeping only terms that do not oscillate with time yields

$$\left\langle \frac{d\ln r}{d\psi} \right\rangle = \frac{1}{2}\cos 2(\psi+\phi_0)\int_0^{\psi} K_\varepsilon(\psi-\psi')\left(1+\cos 2(\psi'+\phi_0)\right)d\psi', \tag{A9}$$

where K_ε is the correlation function,

$$K_\varepsilon(\tau) = \Big\langle \varepsilon(\psi)\varepsilon(\psi-\tau)\Big\rangle. \tag{A10}$$

Introducing the new integration variable $\tau = \psi - \psi'$ and noting that for large ψ the integration in Eq. (A9) over τ can be extended up to infinity one finds

$$\left\langle \frac{d\ln r}{d\psi} \right\rangle = \frac{1}{2}\cos 2(\psi+\phi_0)\int_0^{\infty} K_\varepsilon(\tau)\Big(1+\cos 2(\psi-\tau+\phi_0)\Big)d\tau. \tag{A11}$$

Now, keeping only the terms in Eq. (A11) that do not oscillate with time we have

$$\left\langle \frac{d\ln r}{d\psi} \right\rangle = \frac{1}{4}\int_0^{\infty} K_\varepsilon(\tau)\cos 2\tau d\tau = \frac{\pi}{8}S_\varepsilon(2), \tag{A12}$$

where S_ε is the spectral density of $\varepsilon(\psi)$ related to the correlation function K_ε by the following equations

$$S_\varepsilon(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} K_\varepsilon(\tau)e^{i\omega\tau}d\tau, \qquad K_\varepsilon(\tau) = \int_0^{\infty} S_\varepsilon(\omega)\cos\omega\tau d\omega. \tag{A13}$$

Argument 2 of the spectral density S_ε in Eq. (A12) appears because the frequency of the oscillator in our units is equal to 1. In dimensional units, S_ε has to be computed at the double frequency of the oscillator. This is a simple manifestation of the fact that the growth of amplitude of the oscillator (1) occurs as a result of parametric resonance.

Using the same technique as above, a kinetic equation can be also derived for the evolution of the distribution function $\rho(r,t)$ over the amplitudes r. It has the following form

$$\frac{\partial\rho}{\partial\psi} = \frac{\pi}{16}S_\varepsilon(2)\frac{1}{r}\frac{\partial}{\partial r}r^3\frac{\partial\rho}{\partial r}. \tag{A14}$$

We do not present here the derivation of Eq. (A14) because it can be obtained as a particular limit of a more general theory of synchrotron oscillations in the presence of rf noise.[6] It is worth noting that a simple transformation of variable, $r \to \xi \equiv 2t + \ln r$, converts Eq. (A14) to the diffusion equation with a constant diffusion coefficient,

$$\frac{\partial\rho}{\partial\psi} = \frac{\pi}{16}S_\varepsilon(2)\frac{\partial^2\rho}{\partial\xi^2}, \tag{A15}$$

that can be analytically solved for any initial distribution function.

To proceed to "accelerator" case given by Eq. (A4), note that it differs from Eq. (A1) only in that instead of the random function $\varepsilon(\psi)$ we now have the product $\varepsilon(\psi) \sum_{m=-\infty}^{\infty} \delta(\psi - \psi_0 - m\mu)$ where $\varepsilon(\psi)$ is again assumed to be a random function with a given spectrum. Performing calculations in the same fashion as above one can reproduce Eqs. (A12), (A14) and (A15) in which $S_\varepsilon(2)$ should be replaced by a sum,

$$S_\varepsilon(2) \to \mu^{-2} \sum_{m=-\infty}^{\infty} S_\varepsilon(2 - m/\nu). \tag{A16}$$

In dimensional units, the result takes the form

$$\frac{d\left\langle \ln\left(\eta^2 + \dot{\eta}^2\right)\right\rangle}{dN} = \frac{1}{8}\Omega \left(\frac{B'' l \beta_0 e}{Pc}\right)^2 \sum_{m=-\infty}^{\infty} S_\Delta[(2\nu - n)\Omega], \tag{A17}$$

where $N = \psi/\mu$ is the number of turns, Ω is the revolution frequency and S_Δ is the spectral density of the function $\Delta(t)$.

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