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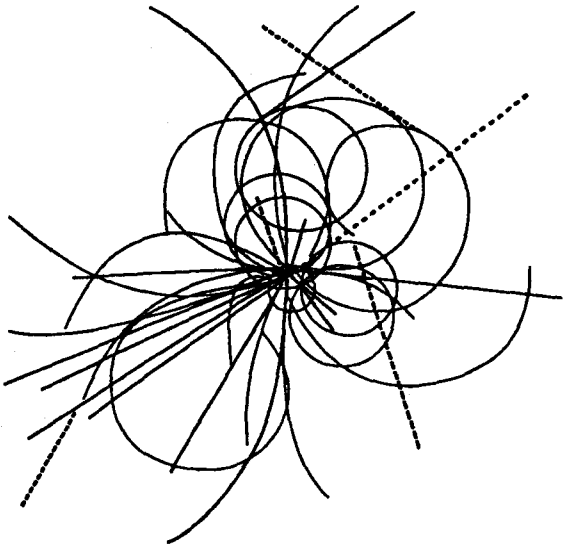

On the Feasibility of Tracking with Differential-Algebra Maps in Long-Term Stability Studies for Large Hadron Colliders

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On the Feasibility of Tracking with Differential-Algebra Maps in Long-Term Stability Studies for Large Hadron Colliders

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Abstract

A time saving alternative to conventional element by element tracking in long-term stability studies consists in the use of truncated Taylor maps. In this report it is studied, how the nonsymplecticity of a moderately high-order truncated Taylor map affects its reliability when being used for tracking over several thousand turns. Various different machines and two different map-constructing programs are compared. It is found, that the discrepancies between the Taylor map results and those obtained by direct tracking grow with amplitude, so that such maps are not guaranteed to be sufficient for long-term tracking over millions of turns without suitable symplectification.

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Introduction

Large proton storage rings like HERA, LHC or SSC suffer from significant and unavoidable magnet nonlinearities. Extensive computer simulations are necessary to investigate the problem of long-term stability in these colliders. In the conventional approach trajectories of many particles are followed over millions of turns element by element through the accelerator structure, using tracking codes, which take into account only the effect of external forces. In view of the restricted computer resources a more time saving method has become highly desirable. This report deals with an apparently promising alternative. It tries to answer the question, whether the complete storage ring consisting of thousands of magnets can be replaced by a moderately high order polynomial map in the simulation, without loosing information about the dynamic aperture.

In the last years it has repeatedly been argued that the Taylor maps obtained by analyzing standard tracking codes like SIXTRACK [1] or TEAPOT [2] with differential-algebra (DA) techniques [3] might be used to perform actual tracking studies. In [4] a 7th-order map of the SSC from after the interaction region to before the interaction region was suggested to be sufficient for the SSC beam-beam studies; in [5] a study of beam survival as a function of initial amplitude and turn number N was performed for the SSC using an 11th-order map; good agreement was reported between DA and direct tracking (DI) results for $N \leq 10^6$.

However such survival results give only a rough global indication of the beam behavior: in some cases, one would also like to be able to make more detailed long-term studies, such as the divergence of nearby trajectories (and the onset of chaotic motion).

In particular, due to the truncation at finite order, DA (Taylor) maps are inherently *not* symplectic and it would be extremely surprising if nowhere any trace of this nonsymplecticity was in evidence. Of course, by increasing the order of the map the nonsymplecticity may be made arbitrarily small, but as the map size grows exponentially, DA tracking decelerates quickly with increasing order and a 14th-order map is (for the LHC) expected to be not faster than direct tracking using Sixtrack. Alternatively, one could imagine restoring symplecticity by modification of the Taylor map. However, this requires further investigation.

For phase space trajectories inside the dynamic aperture the amplitudes should be constant when averaged over a suitably large number of turns (1000, for instance). This is borne out by direct tracking simulations using SIXTRACK which treats the dynamics in a symplectic[1] manner: for all cases that we examined, the DI results show perfectly constant average amplitudes. Deviations from symplecticity in the DA approach may therefore be expected to show up as a time (turn number) dependence of the average amplitude.

It should be stressed that we have examined several different maps: one produced for the SSC using TEAPOT [2], one for the LHC also produced using TEAPOT, and one set of maps for the LHC produced with SIXTRACK [1]. We have studied the amplitude evolution using both DI and DA, for four different cases, denoted as LHC(N), LHC(R), LHC(T) and SSC (see below).

[1] Apart from the effects of rounding errors.

The models under consideration

We have constructed 11th-order Taylor maps for an LHC with the following characteristics: we have used the nominal LHC lattice with interaction zones and nonlinear chromaticity and amplitude-dependent tune shift corrections with sextupoles, octupoles, and decapoles à la Neuffer. In one version of the map (N) there were systematic dipole errors but no random dipole errors, whilst in another version (R) also the random dipole errors were included. In no case have we considered quadrupole and skew errors or closed-orbit corrections.

The production of the 11th-order map using SIXTRACK [1] and the available DA packages [3, 6] takes about 3 CPU hours on the IBM (about 1 CPU hour on the CRAY): the map-tracking procedure is about 10 times as fast as the analogous direct tracking program. We also used a map produced from the equivalent program TEAPOT [2] which, due to splitting each element into several thin elements and due to complex frame transfer between thin elements for better dispersion calculation, takes about 20 CRAY hours for the same LHC lattice: this version is denoted by LHC(T), and contains no random errors (since random errors are implemented in a different way in SIXTRACK and TEAPOT this would make comparison difficult). In addition we have investigated an SSC map which was obtained by analyzing the program Teapot and took about 10 CPU hour on the CRAY. We regret that we did not consider an SSC map from SSCTRK which would take about 0.3 CPU hours in CRAY for an 11th-order SSC map.

We have also constructed a hybrid map in which the cavity is treated exactly, as in DI, and the rest of the machine with DA. Since the DA map only has to deal with 5 variables the resulting code is about twice as fast as full DA tracking: no appreciable differences are evident in the results.

The phase space coordinates of the particles are denoted in the transverse part by x, x', y, and y', and in the longitudinal part by z, z' where the primes denote the momenta. These are understood to be Courant-Snyder transformed so as to make them independent of the β functions. In addition to the Euclidean phase space distance between two points, we have also studied various amplitudes: $A_x = \sqrt{x^2 + x'^2}$, $A_y = \sqrt{y^2 + y'^2}$, $A_z = \sqrt{z^2 + z'^2}$, $A_{xy} = \sqrt{A_x^2 + A_y^2}$, and $A_{xyz} = \sqrt{A_{xy}^2 + A_z^2}$, which we then average over a number of turns in order to separate the global, long-term evolution from short-term fluctuations.

Our reason for disregarding the phase information is that although the correlation in phase between DI and DA tracking results is expected to deteriorate rapidly for even small N this is not a worry in practice as long as the *absolute* amplitudes remain well-behaved. The quantity we are concerned with in this paper is the time evolution of the average amplitudes A, as a function of the initial amplitude.

Results

A first point of interest is how well DA reproduces DI on the local level of one turn. In fig. 1 we present the 'one-turn accuracy' as defined in [5], that is, the phase space distance between a point obtained from DA and one from DI after one turn, starting at the same initial point. We compare, for the LHC(N) case, different orders of the map and various initial amplitudes: we have also added analogous results for the LHC(T) and the SSC. For both map-producing

programs the one-turn accuracy is seen to be essentially the same. As expected the agreement increases with the order of the map and decreases with increasing initial amplitude. Note, however, the saturation of the accuracy for the SSC at about 10^{-8} requires further investigation. The very good agreement between the two different maps LHC(N) and LHC(T) for the LHC, and the essential agreement between the two TEAPOT-produced maps for LHC and SSC, give us confidence in our ability to construct maps correctly and as accurately as DA will allow.

The differences between DI and DA that enter in fig. 1 are of limited practical relevance since a part of the phase space distance is a difference in phase, which we are quite willing to accept as long as the absolute amplitudes are well-behaved. Therefore we have made the following exercise: we follow the DI evolution, predicting at each turn the result of the next turn using DA, and then comparing that result with the DI one. In this case we have disregarded the phase information and only compare the values of A_{xy} (which is 0.5835 for DI in this case). The differences are plotted in fig. 2, for LHC(N). The main feature of this plot is an oscillation over 5 orders of magnitude in the predictive power of DA at twice the synchrotron frequency. We ascribe this to the fact that the dependence of the transverse degrees of freedom on the relative momentum deviation is more important than that on the path length difference: if the momentum deviation happens to be small (twice per synchrotron oscillation), the error is also reduced.

The real object of interest is of course not how well DA predicts the evolution on the time scale of one turn, but rather over many turns. In fig. 3 we have therefore plotted, for the same machine and initial amplitudes as before, the accumulated difference in A_{xy} between the DI and DA tracking results. In analogy with fig. 2 a modulating effect of the synchrotron oscillations can be seen: whenever the momentum deviation becomes large, the difference between DI and DA increases, resulting in a stepwise increase in the accumulated error. For this initial amplitude, the overall picture is that of a linear increase in amplitude.

It is also useful to get an impression of the overall behavior of the deviations between DI and DA. To this end, we computed for N turns the average values A_i^{DI} and A_i^{DA} for DI and DA, respectively, with $i = x, y, z, xy, xyz$. The *average amplitude deviation per turn* δA_i is then defined as $\delta A_i = (A_i^{DA} - A_i^{DI})/(N A_i^{DI})$. In Table 1, we give the results for our four cases LHC(N), LHC(R) LHC(T), and SSC(R). We took $N = 20,000$ for the LHC versions and $N = 10,000$ for the SSC. The different models LHC(N), LHC(R), and LHC(T) are seen to be in general agreement with each other, while the results for the SSC are in agreement with those for the LHC where the initial amplitudes overlap. As expected, δA_{xy} increases with increasing initial amplitude. In particular, for the LHC(N) and LHC(R), some particles with large initial amplitudes (indicated in the Table) were lost after a smaller number of turns. These losses are purely artifacts of the DA approach since for DI tracking all these amplitudes are stable. on the basis of these results one might already conclude that an 11th-order DA tracking are not reliable for the LHC since some particles near (but still inside) the dynamic aperture have already gotten lost within 12,000 turns. As for the SSC, the smaller amplitude deviation $\delta_{xy} = 5.3 \times 10^{-9}$ at the dynamic aperture $A_{xy} = 0.42$ (3 to 4 order of magnitudes smaller compared with those of the LHC at the dynamic aperture) is probably

LHC(N)		LHC(R)		LHC(T)		SSC	
A_{xy}	δA_{xy}	A_{xy}	δA_{xy}	A_{xy}	δA_{xy}	A_{xy}	δA_{xy}
						0.0595	5.76E-11
0.0971	1.36E-11	0.0972	1.80E-11	0.0982	1.83E-11		
						0.1436	4.25E-12
0.1942	6.85E-11	0.1941	9.91E-11	0.1950	7.62E-11		
						0.2341	-1.14E-10
0.2914	8.70E-10	0.2907	1.04E-9	0.2918	8.90E-10		
						0.3262*	-1.29E-9
0.3887	8.32E-9	0.3870	9.09E-9	0.3887	8.32E-9		
						0.4178	5.31E-9
0.4861	5.58E-8	0.4830*	5.69E-8	0.4861	5.51E-8		
						0.5056*	9.26E-6
0.5835	2.92E-7	0.5781	2.74E-7	0.5830	2.79E-7		
0.6813	1.38E-6	0.6745	1.21E-6	0.6803	-6.05E-5		
0.7791(1)	8.27E-6	0.7716(2)	9.27E-6	0.7779	-1.00E-5		
0.8283(3)	1.55E-5	0.8119(4)	1.05E-5				
0.8779*	-9.70E-6	0.8628(5)	1.95E-5				
0.9242(6)	5.26E-5	0.9104(7)	5.71E-5				

Table 1: Amplitudes and average deviation increase per turn for the total transverse amplitude A_{xy} for the LHC(N), LHC(R), LHC(T), and SSC, for various initial amplitudes. Where indicated by superscripts the particle was lost in DA after (1):12,000, (2):10,000 (3):6000, (4):8000, (5):3000, (6):1400, and (7):1800 turns, respectively. Entries with an asterisk are discussed in greater detail in the text. Note that the dynamic aperture for the SSC is $A_{xy} = 0.42$. The dynamic aperture for the LHC with random errors is $A_{xy} = 0.77$, and without random errors is $A_{xy} = 0.88$.

the reason that the one-million-turn survival plots from the 11th-order map tracking and from the element-by-element tracking agree globally [5]. However, it should be addressed that there is no a priori way of judging whether such an 11th-order SSC map can be used for even longer ($> 10^6$) turn tracking without checking the results afterwards.

Fig. 4 is a graphical representation of table 1. It shows the absolute value of the relative amplitude error per turn as a function of amplitude. For all amplitudes up to the dynamic aperture the error is approximately the same in the four cases. We conclude that the causes for these errors are solely related to the order of the map and not to the specific model under investigation. Fig. 4 can therefore be used, with some care, to evaluate the error that is to be expected for other models and to estimate a maximum number of turns for which DA can be usefully applied, depending on one's taste for what is an acceptable accumulated error. For comparison, the corresponding relative averaged amplitude decrease per turn for SIXTRACK due to computer roundoff errors is estimated to be about 2×10^{-13} [7] at an amplitude of $A_x = 3.0$ which is three times the maximum amplitude depicted in fig. 4 ! It should be noted that even in the case of LHC with no random errors the particles with initial amplitude of

LHC(N)		LHC(R)		description
DI	DA	DI	DA	
	.681		.675	false onset of chaos in DA
0.878		0.772		onset of chaos
	.779		.772	false particle loss in DA
1.021		1.021		particle loss before 20,000 turns

Table 2: Stability behavior for direct and DA tracking, for the LHC with and without random errors. For each case the initial amplitude at which qualitatively new behavior occurs is entered.

0.779 are already lost after 12,000 turns, while DI tracking shows that the *actual* dynamic aperture in that case is a good deal larger (see table 2).

A more important observation to be made is that the difference between DI and DA tracking does not appear to behave in an easily analyzable way. To illustrate this point we present in figs. 5 to 8 different examples of amplitude evolution.

The first example, fig. 5, is A_{xy} for the LHC with random errors and an initial amplitude of 0.483. An increase is evident which on our time scale of 20,000 turns appears linear. Other, quite similar, initial amplitudes for the SSC give a linear *decrease*. This is the (undesirable but regular) behavior one expects from a map with a small symplecticity violation.

In fig. 6 we give an example for the SSC with an initial amplitude of 0.326, where on the average the amplitude is decreasing. On top of that, however, we find that the amplitude oscillates more and more rapidly.

A really weird case is presented in fig. 7, for the LHC with no random errors at an initial amplitude of 0.878. At first the amplitude increases drastically, and the particle would appear to be lost after some 3500 turns. However the trajectory miraculously recovers and the transverse amplitude A_{xy} starts decreasing steadily. It should be noted that this behavior falls in the middle of the region which under DA is unstable: for amplitudes 0.779 and 0.924 the particle is simply lost after 6000 and 1400 turns, respectively.

Finally, in fig. 8 we present the results for the SSC with an initial amplitude of 0.506. Note that this amplitude (0.506) is outside of the dynamic aperture (0.42). Here we find that the DA prediction, after faithfully following the DI results for some 2000 turns, suddenly exhibits a rapid growth of amplitude.

One of the most interesting parameters in the long-term stability studies is the Lyapunov exponent which describes the transition from the regular to the chaotic regime [8]. This can be studied by tracking two particles with a small difference in initial coordinates. We calculated this phase space distance for DI and DA tracking, and determined the regularity or chaoticity of the time evolution.

In table 2 we present the different predictions of stability in DA and DI, both for LHC(N) and LHC(R). In both cases the amplitude at which chaotic motion appears is largely under-

estimated by some 15-20% in DA tracking, and the same holds for its estimates of amplitudes where particle loss starts to occur before the $20,000^{th}$ turn.

Conclusions

To summarize: we have compared direct and the 11th-order DA tracking for typical LHC and SSC lay-outs. It is observed that, due to the lack of enough symplecticity, DA trackings leads to deviations from the expected behavior of the trajectories in ways which are essentially unpredictable: we have observed both increases and decreases in amplitude, increasingly rapid oscillations and cusp-like behavior. Moreover, one and the same map may display any of those behaviors, depending on the starting conditions. These deviations are not very large for small initial amplitudes, but still large enough so that local behavior of the motion may be dramatically changed when going from DI to DA. In addition, analysis of the Lyapunov exponent indicates (for the LHC) that the regular-motion region is considerably reduced, even if this does not lead directly to particle loss on our time scales. Finally, the amplitude at which an actual particle loss does take place is reduced by a similar amount, independent of whether random dipole errors are included or not.

We therefore conclude that the differences between direct tracking and DA predictions are in all cases of practical interest sufficiently large that, in general, Taylor maps of moderate order (11th-order) are not guaranteed to be reliable for long-term tracking studies unless such maps be suitably symplectified. This requires further studies.

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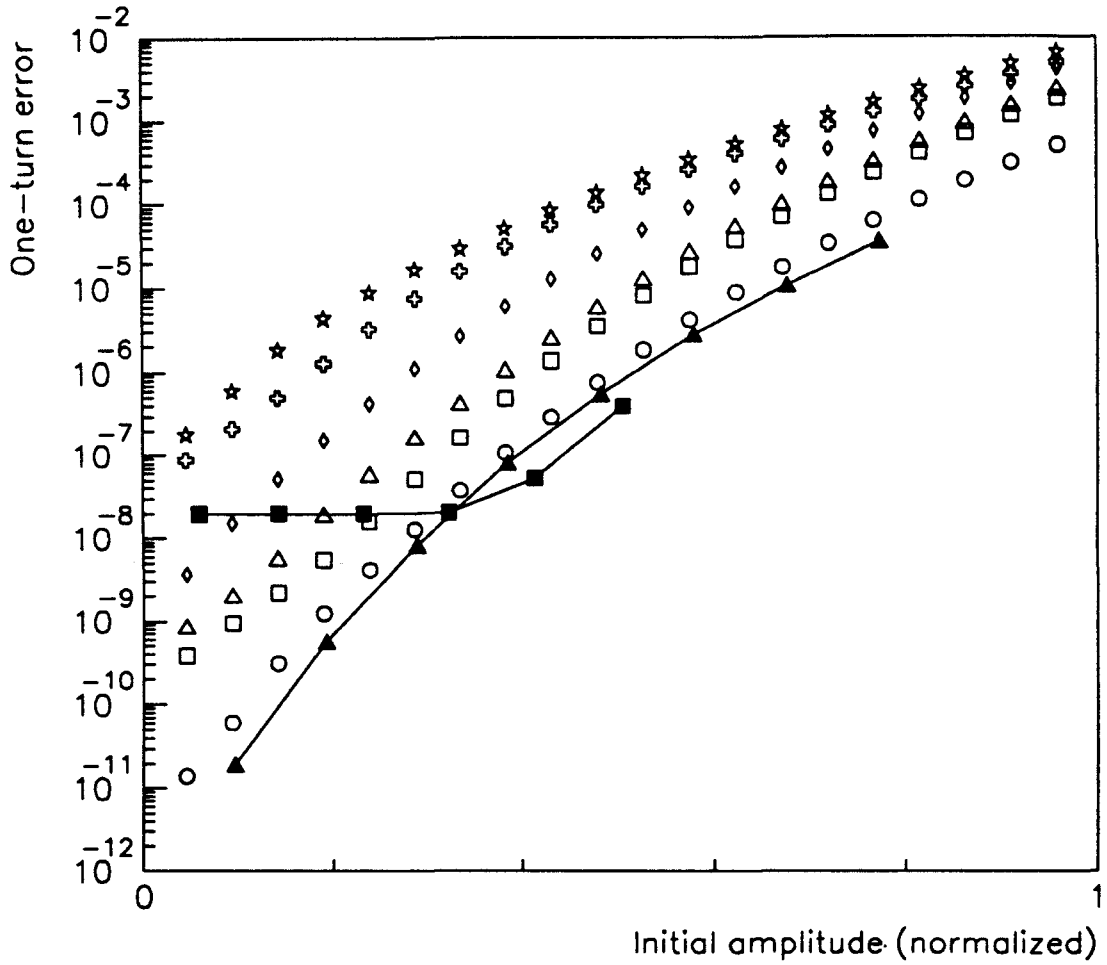

Figure 1: One-turn accuracies for various initial amplitudes and orders of the Taylor map.

Open symbols refer to LHC(N) with map order from 11 (circles) down to 6 (stars). Closed symbols refer to LHC(T) (triangles) and SSC (squares), both for 11th-order maps. At the dynamic aperture, the 11th-order one-turn errors are 2×10^{-4} for the LHC(N) (initial amplitude = 0.88), 3×10^{-5} for the LHC(T) (initial amplitude = 0.77), and 3×10^{-8} for the SSC (initial amplitude = 0.42).

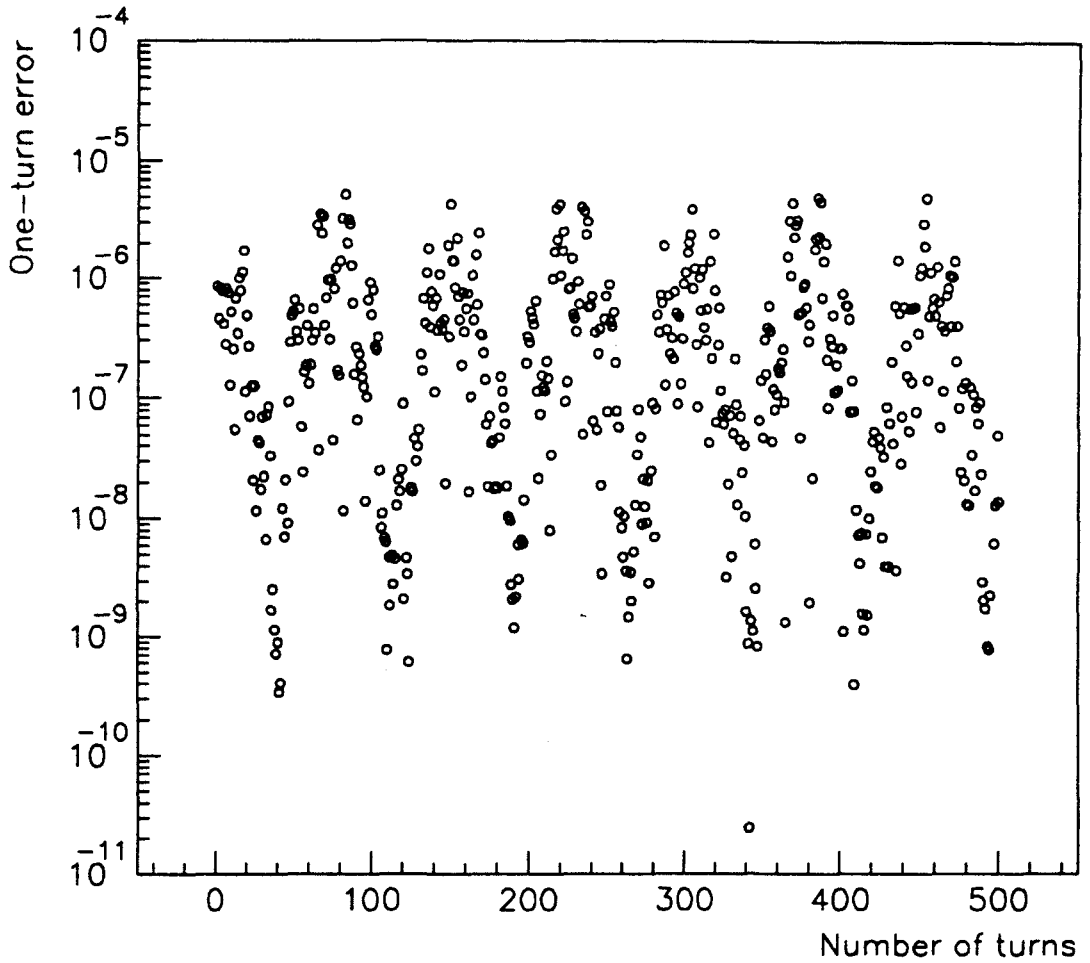

Figure 2: One-turn error as function of turn number.

The variations in the error over 5 orders of magnitude has twice the frequency of the synchrotron oscillation.

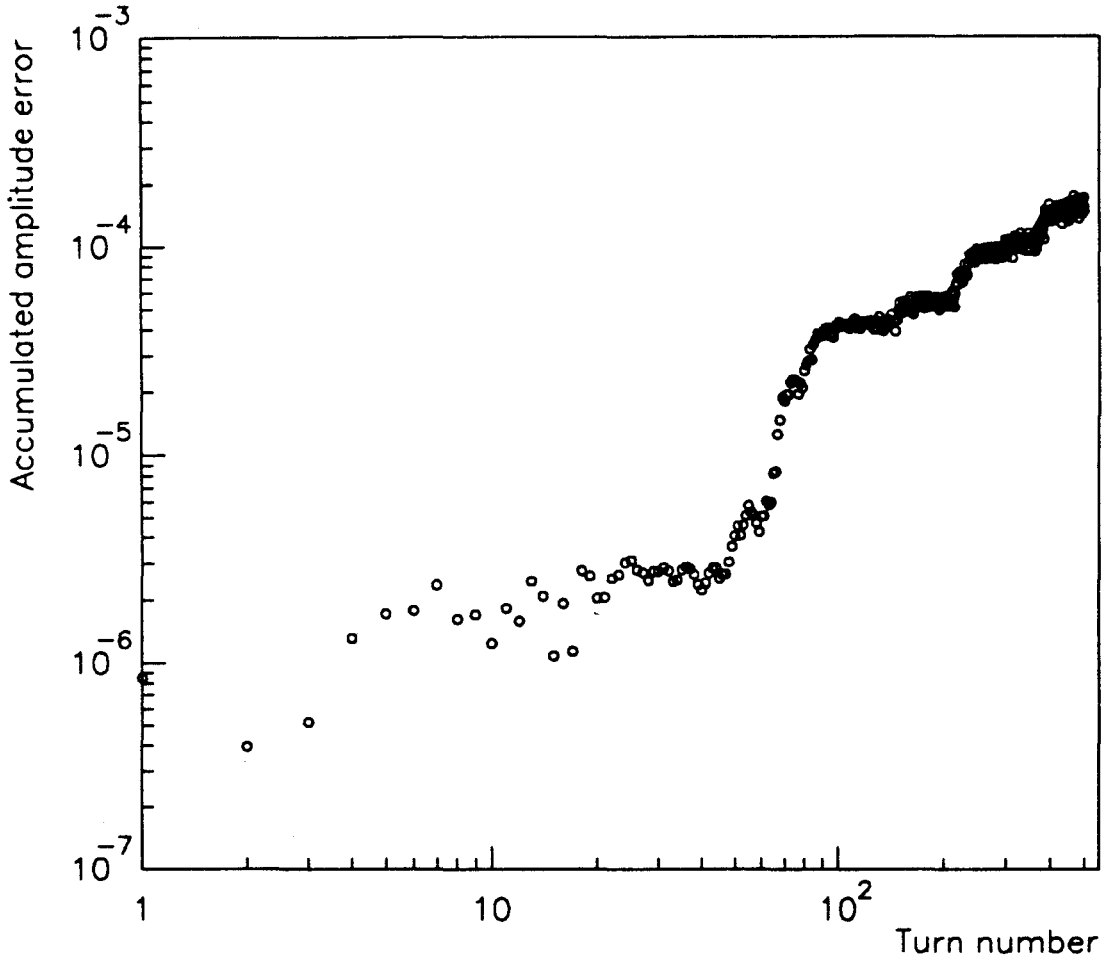

Figure 3: Accumulated amplitude error as a function of turn number.

The amplitude error grows approximately linearly with time, and the step-like increases are a reflection of the oscillations in the one-turn error evident in figure 2.

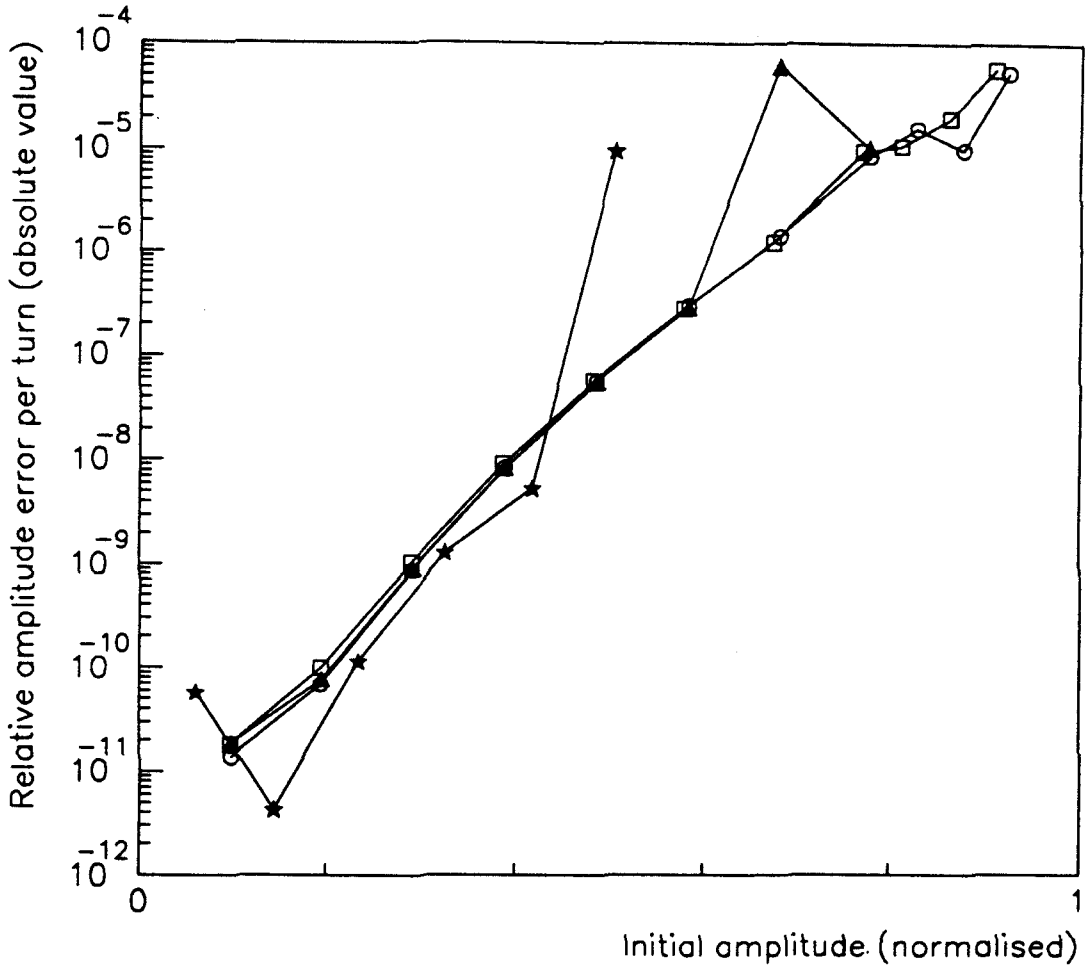

Figure 4: Absolute value of the relative amplitude error per turn as a function of amplitude.

The circles, squares, triangles and stars represent the results for LHC(N), LHC(R), LHC(T) and SSC respectively. The fact that the amplitude errors are in all four cases approximately the same leads us to the conclusion that they are due to the order of the map only. The graph can therefore be used to estimate the errors for other tracking models.

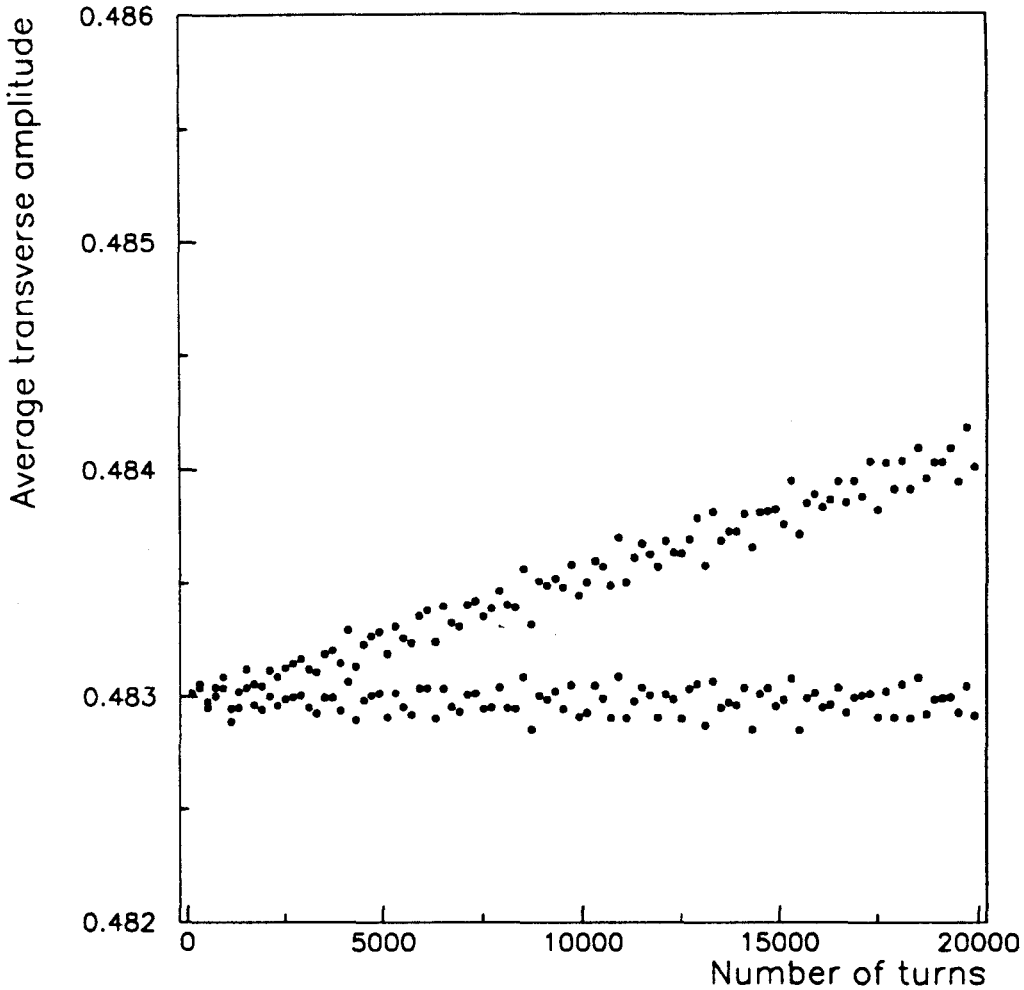

Figure 5: Amplitude evolution as a function of turn number for the LHC.

In direct tracking the amplitude is perfectly constant, while with map tracking the amplitude grows approximately linearly with time. This behavior is what we typically find for not too large initial amplitudes and our 11th-order map.

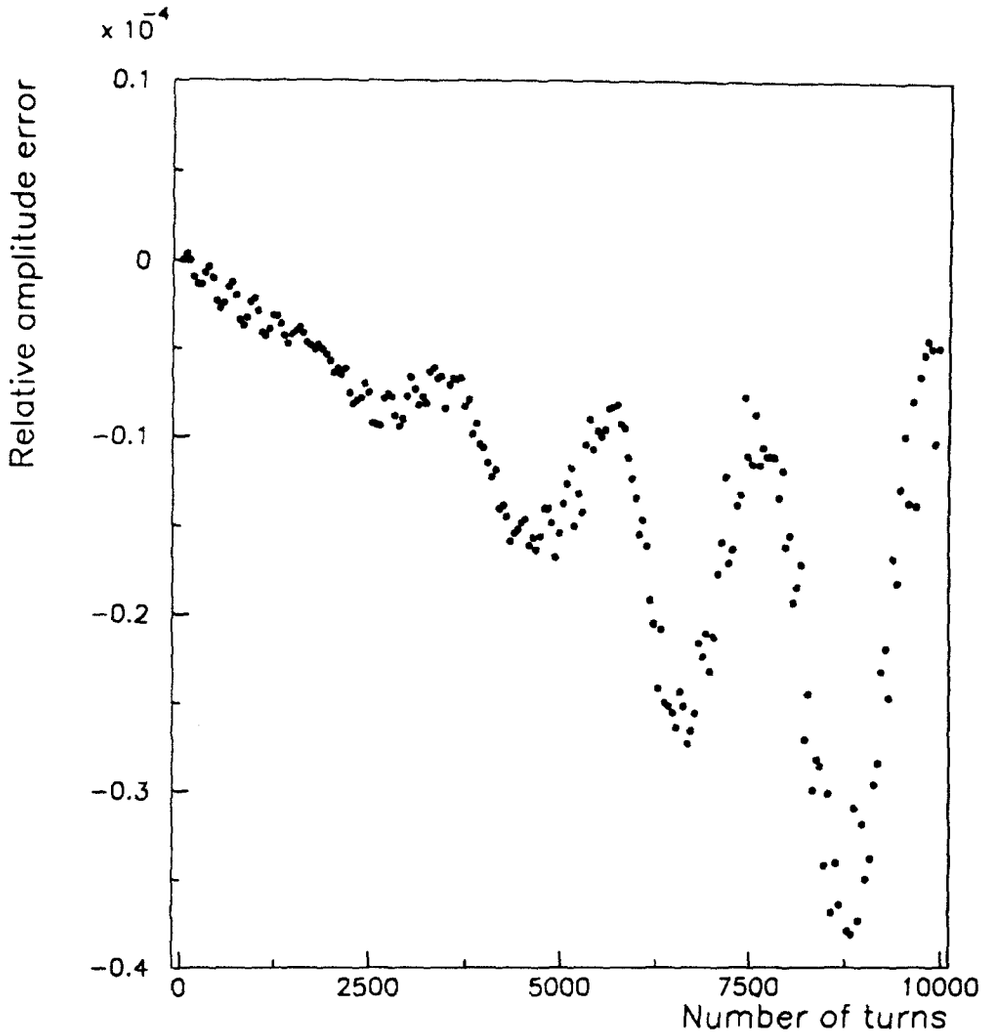

Figure 6: Relative amplitude error as a function of turn number for the SSC.

In the case of the SSC we typically find a decrease of the DA amplitude compared to direct tracking. Due to large oscillations in the direct-tracking amplitude we present here the relative difference of the amplitudes A_{xy}^{DA} and A_{xy}^{DI} each averaged over 50 turns.

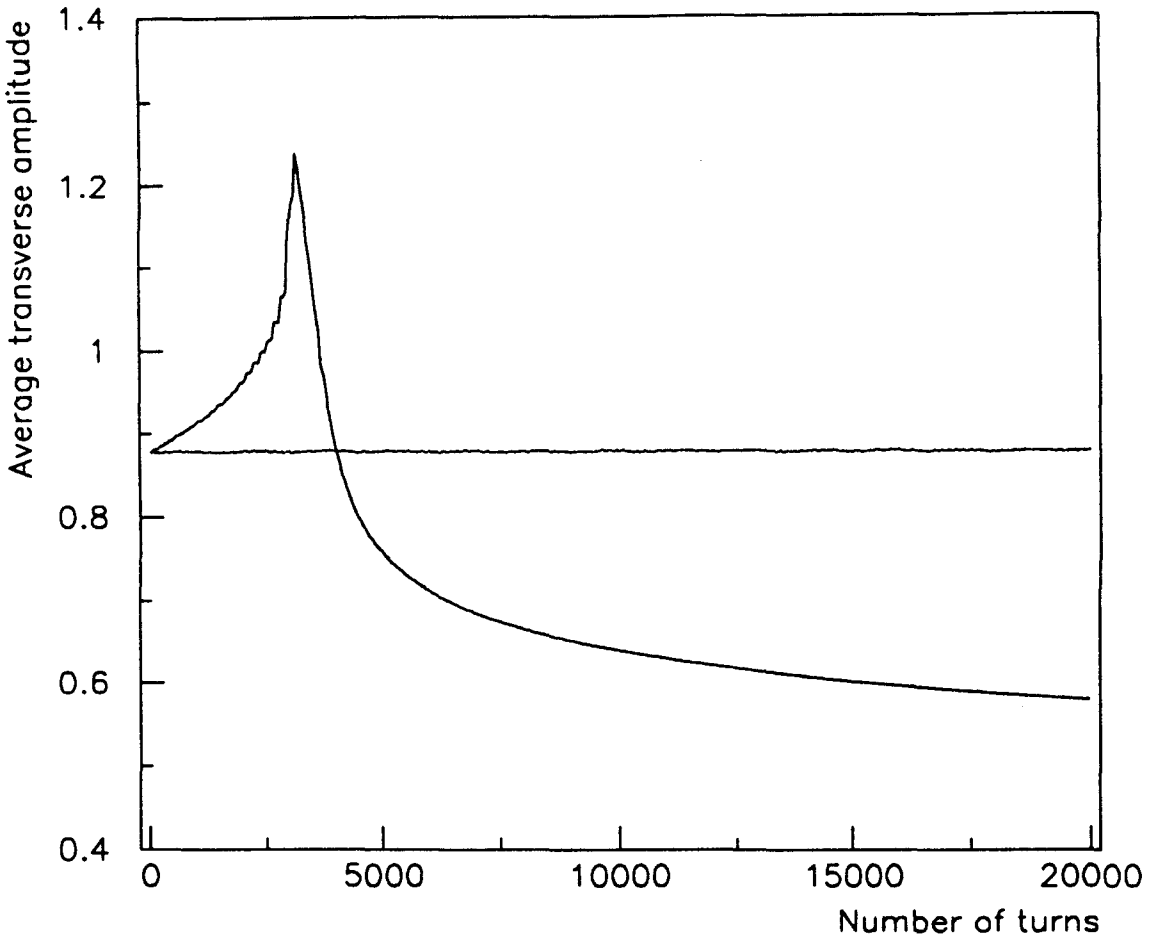

Figure 7: Amplitude evolution as a function of turn number for the LHC.

For large amplitudes (in the chaotic regime) the direct-tracking amplitude is still a constant over this time range. The DA amplitude, after an initial rapid growth, recovers miraculously and starts to settle down to zero. Note that this amplitude falls in the middle of a range where in DA tracking the particle is artificially lost after a small number of turns.

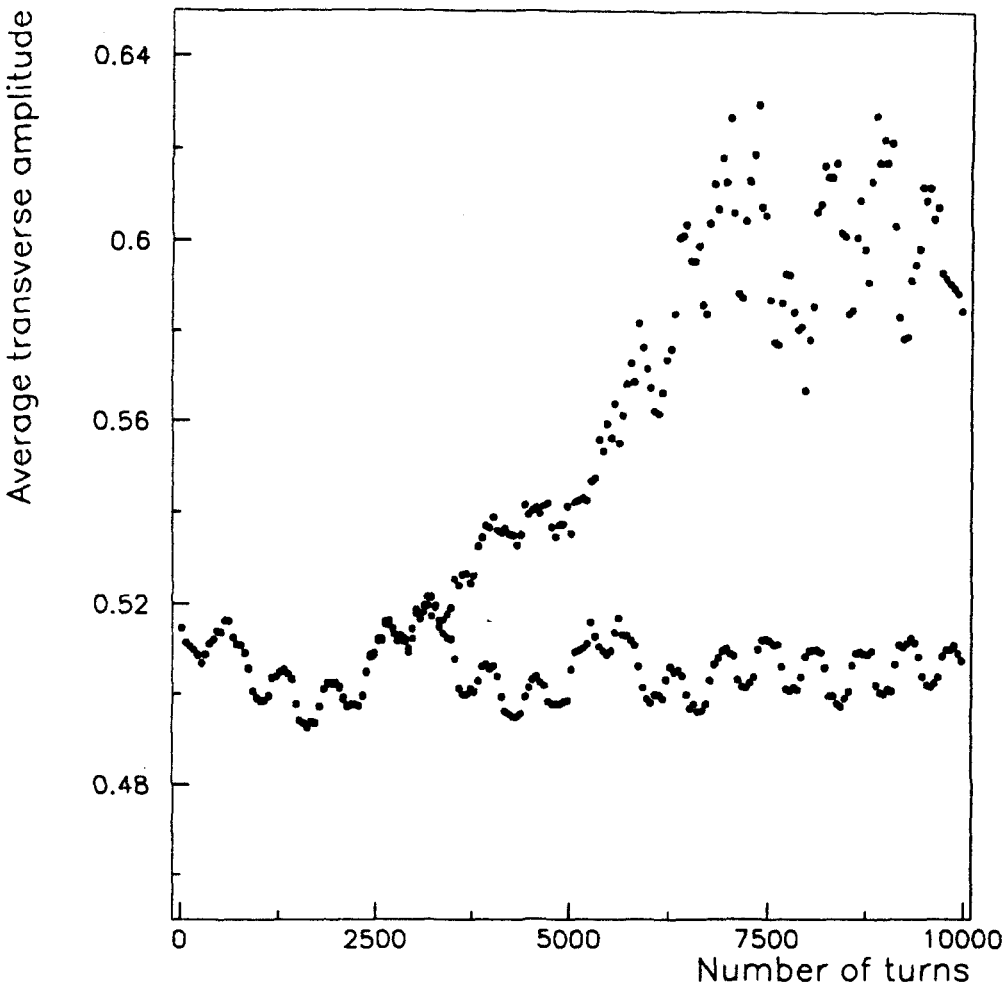

Figure 8: Amplitude evolution as a function of turn number for the SSC.

For the large amplitude 0.5056 (about 18% larger than the dynamic aperture at 0.42) the DA results, after following the direct-tracking ones for some time, suddenly diverge appreciably.