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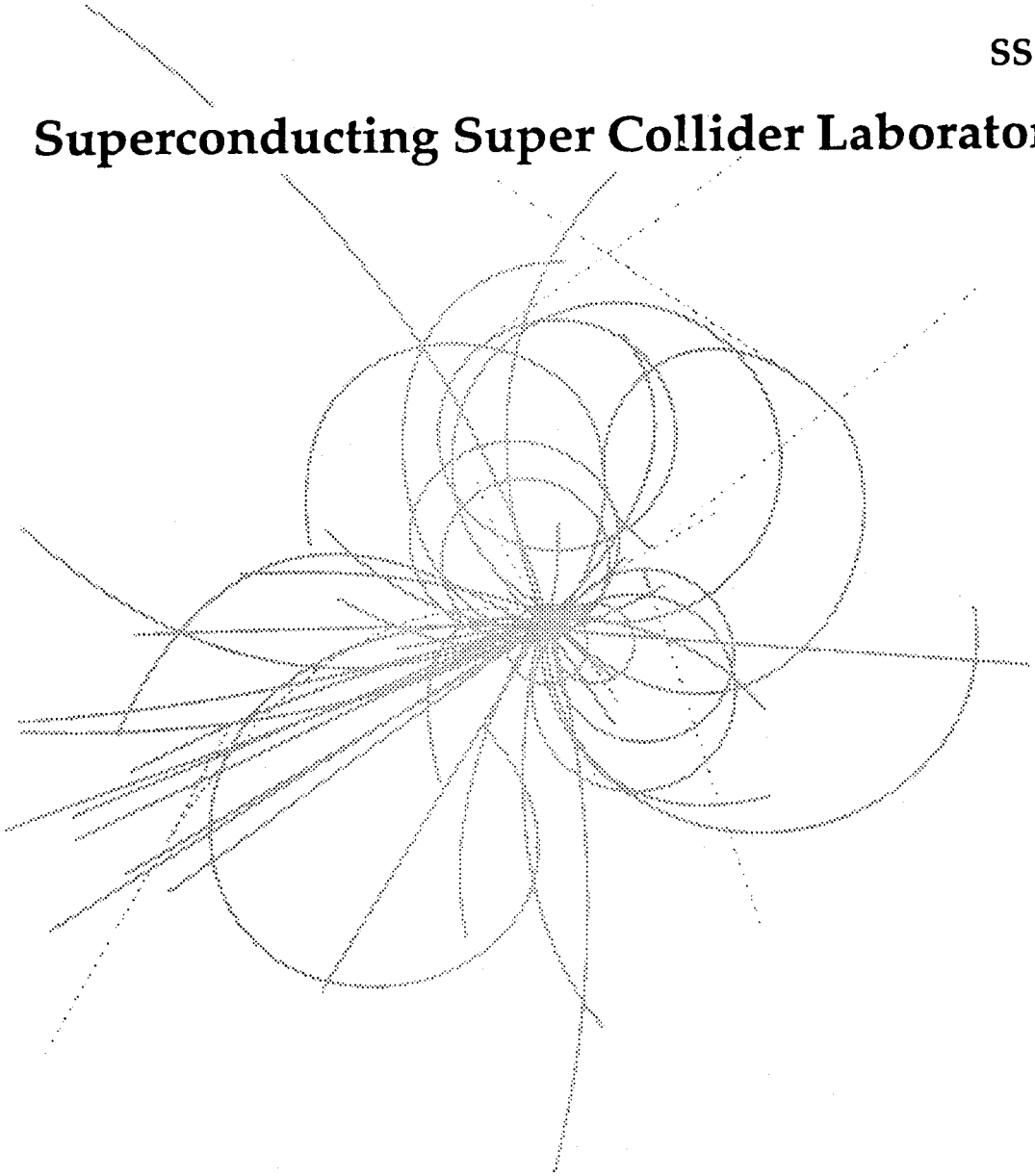

Luminosity Dilution Due to Random Offset Beam-Beam Interaction

G. Stupakov

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G. Stupakov*

Accelerator Design and Operations Division
Superconducting Super Collider Laboratory†
2550 Beckleymeade Avenue
Dallas, TX 75237

November 1991

* Guest scientist visiting from the Institute of Nuclear Physics, Novosibirsk, 630090, USSR.

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Abstract

We consider beam-beam interaction in a collider in the case when the beams randomly displace around the equilibrium orbit at the interaction point. Due to the random part of the interaction, particles diffuse over the betatron amplitude causing an emittance growth of the beam. A Fokker-Planck equation is derived in which a diffusion coefficient is related with the spectral density of the noise. Estimations for the Superconducting Super Collider parameters give a tolerable level of the high-frequency beam offset at the interaction point.

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1.0 INTRODUCTION

The required confinement time of protons in the Superconducting Super Collider (SSC) (10–20 hours)[1] makes it necessary to carefully study even small effects that can gradually dilute the luminosity of the collider over a long time. A possible physical mechanism which can influence the beam performance is related to different types of noise in the machine. These include ground motion causing random displacements of magnets, sound vibrations, and rf noise. These noises could excite betatron and synchrotron oscillations in the beams over an allowable level and result in the growth of the emittance of the beams.

A particular mechanism of beam degradation that we study in this paper is related to random beam displacements at the interaction points caused by noisy betatron oscillations of the bunches around the equilibrium orbits. We do not specify here the physical mechanism which drives these oscillations, but we do make some natural assumptions about statistical properties of the amplitudes of the oscillations. Due to nonlinearity of the beam-beam interaction, the random setoffs of the interacting beams result in a diffusion process that increases the radial extension of the beams and, hence, decreases the luminosity. The main purpose of this paper is to find constraints on the noise level (in terms of the beam offset at the interaction point) at which the effect of the luminosity degradation is not important for the SSC.

2.0 GENERAL CONSIDERATION

We consider a simple model in which two bunches are collided periodically at one interaction point. For the sake of simplicity we choose a one-dimensional model, assuming that the width of the bunch in the z-direction is much larger than its dimension in the x-direction, that is $\sigma_x \ll \sigma_z$. We will also neglect the influence of one beam upon the other (the weak-strong beam model), assuming that one of the colliding beams (a strong one) has a fixed distribution of particles over the betatron amplitudes and phases, and that this distribution function is not changing in time. However, the beam-beam interactions evolve the distribution function of the other beam, and the problem consists in describing its evolution.

Throughout this paper we will use dimensionless units normalizing the length on σ_x, momentum on $P\sigma_x/\beta^*$ (P is the particles' momentum, β^* is the value of the β function at the interaction point), and action on $P\sigma_x^2/\beta^*$.

The initial density profile $n_0(x)$ of the bunch in the x-direction is assumed to be a Gaussian:

$$n_0(x) = \frac{N_B}{\sqrt{2\pi}\ l} e^{-x^2/2}, \tag{1}$$

where l is the bunch length and N_B is the number of particles in the bunch per unit length in the z-direction. In Eq. (1) we assume a step-like density distribution in the bunch along its orbit.

To describe betatron oscillations, we use canonically conjugate action J and phase ϕ variables. The phase ϕ is chosen so that it is constant for the unperturbed (that is, neglecting the beam-beam interaction) betatron motion, $\phi = \text{const}$, so that in our dimensionless units,

$$x = \sqrt{2J\beta/\beta^*}\,\sin(\nu\theta + \varphi(\theta) + \phi), \tag{2}$$

where ν is the betatron tune, and $\varphi(\theta)$ is the periodic part of the betatron phase, $\varphi(\theta) = \varphi(\theta + 2\pi)$. For what follows, we will need to evaluate x at the interaction point only where $\beta = \beta^*$ and φ can be chosen to be zero. At this point, Eq. (2) takes the form

$$x = \sqrt{2J}\sin(\nu\theta + \phi). \tag{3}$$

The initial distribution of particles of the bunch in the action variable is given by a function $F_0(J)$. One can easily check that the distribution function which corresponds to the given density profile, Eq. (1), has the following form:

$$F_0(J) = \frac{1}{2\pi} e^{-J}. \tag{4}$$

It is normalized so that $\int_0^\infty dJ \int_0^{2\pi} d\phi F_0 = 1$.

In the next sections we will see that due to the beam-beam interaction of the beams with a random setoff, the distribution function F gets a perturbation around its equilibrium value, $F=F_0(J) + \Delta F(J)$. (We will show later that an average ΔF does not depend on ϕ.) The density perturbation $\Delta n(x)$ associated with ΔF is given by

$$\Delta n(x) = \frac{N_B}{l} \int_0^\infty dJ \int_0^{2\pi} d\phi \Delta F(J) \delta[x-\sqrt{2J}\sin(\nu\theta + \phi)] =$$

$$= \frac{2N_B}{l} \int_{x^2/2}^\infty dJ \frac{\Delta F(J)}{\sqrt{2J-x^2}}. \tag{5}$$

Because the density of the bunch differs from the initial one, the luminosity of the machine also changes. Its relative change, $\Delta L/L_0$, is

$$\frac{\Delta L}{L_0} = \frac{\int n_0 \Delta n(x) dx}{\int n_0^2 dx} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} x^2 \frac{\Delta n}{n_0} dx. \tag{6}$$

Putting Eqs. (1) and (5) into Eq. (6) and performing the integration over x, one finds

$$\frac{\Delta L}{L_0} = 2^{3/2}\pi \int_0^\infty dJ \Delta F(J)\, e^{-J/2} I_0(J/2) dJ, \tag{7}$$

where I_0 is the modified Bessel function of the zeroth order.

Now, the problem is to find the perturbation of the distribution function ΔF and to evaluate the integral (7). In the next three sections we compute ΔF, assuming that it is small compared with F_0, $\Delta F \ll F_0$. With this assumption, we will be able to consider an initial stage of the degradation of the luminosity when ΔL (being negative) linearly grows in time. Though not valid on a long time scale, this approach allows us to predict the strength of the effect and to qualitatively estimate its danger for a real machine.

3.0 AMPLITUDE AND PHASE CHANGE DUE TO MULTIPLE INTERACTIONS OF THE BUNCHES

A dimensionless potential energy V_m governing the interaction of the two colliding bunches with their centers displaced by a distance ζ_m in the x-direction is given by (see, for example, Reference 2):

$$V_m = 4\pi\xi h(x-\zeta_m)\delta(\theta - 2\pi(m-1)), \tag{8}$$

where ξ is the conventional interaction parameter:

$$\xi = \frac{N_B r_p \beta^*}{\sqrt{2\pi}\,\sigma\gamma} \tag{9}$$

(r_p is the classical proton radius) and the function $h(x)$ is

$$h(x) = \int_0^x dy \int_0^y dz e^{-z^2}. \tag{10}$$

The subscript m denotes the collision number, $m = 1,2,...,$ and we assume that the first interaction occurs at $\theta = 0$.

The effects under consideration are related to small, $\zeta_m \ll 1$, random setoffs of the beams with respect to each other. For small ζ_m, the function $h(x-\zeta_m)$ can be expanded in a power series. Keeping only the first two terms in this series, we have

$$h(x-\zeta_m) = h(x) - \zeta_m h'(x), \tag{11}$$

where the prime denotes derivative with respect to x. The first term on the right side of Eq. (11) refers to the head-on interaction of the beams. For small ξ, the main effect of this interaction is a non-linear tune shift, $\Delta\nu$, that can be included in ν. For our purposes, the crucial one is the second term in Eq. (11), with the corresponding contribution $\tilde{V}_m$ to the potential energy given by

$$\tilde{V}_m = \tilde{v}_m \delta[\theta - 2\pi(m-1)], \tag{12}$$

where

$$\tilde{v}_m = 4\pi\xi\zeta_m h'(x). \tag{13}$$

The term $\tilde{V}_m$ in the Hamiltonian produces perturbations δJ_m and $\delta\phi_m$ of action and phase in the mth interaction of the bunches. These can be found from the Hamiltonian equations of motion. A simple integration yields

$$\delta J_m = -\frac{\partial \tilde{v}_m}{\partial \phi} = -4\pi\xi\zeta_m h''\{\sqrt{2J_{m-1}}\sin[2\pi\nu(m-1)+\phi_{m-1}]\} \times$$

$$\times\sqrt{2J_{m-1}}\cos[2\pi\nu(m-1)+\phi_{m-1}], \tag{14}$$

$$\delta\phi_m = \frac{\partial \tilde{v}_m}{\partial J} = 4\pi\xi\zeta_m h''\{(\sqrt{2J_{m-1}}\sin[2\pi\nu(m-1)+\phi_{m-1}]\} \times$$

$$\times\frac{1}{\sqrt{2J_{m-1}}}\sin[2\pi\nu(m-1)+\phi_{m-1}], \tag{15}$$

where J_{m-1} and ϕ_{m-1} denote the value of J and ϕ before the mth interaction. Total change of J and ϕ resulting from M interactions, ΔJ_M and $\Delta\phi_M$, can be found by summing Eqs. (14) and (15) for $m = 1,2,...M$:

$$\Delta J_M = -4\pi\xi \sum_{m=0}^{m=M-1} \zeta_m \sqrt{2J_m} h''[\sqrt{2J_m}\ \sin(2\pi\nu m+\phi_m)]\cos(2\pi\nu m+\phi_m), \tag{16}$$

$$\Delta\phi_M = -4\pi\xi \sum_{m=0}^{m=M-1} \zeta_m \frac{1}{\sqrt{2J_m}} h''[\sqrt{2J_m}\ \sin(2\pi\nu m+\phi_m)]\cos(2\pi\nu m+\phi_m). \tag{17}$$

Using the explicit expression $h''(x) = \exp(-x^2/2)$ (see Eq. (10)), one can expand the exponential into a Fourier series:

$$e^{-x^2} = e^{-J\sin^2(2\pi\nu m+\phi)} = e^{-J/2}[I_0(\frac{J}{2})+2\sum_{k=1}^{\infty} I_k(\frac{J}{2})\cos 2k(2\pi\nu m+\phi)], \tag{18}$$

where I_k denotes the modified Bessel function of kth order. Putting Eq. (18) into Eqs. (16) and (17), we find

$$\Delta J_M = -4\pi\xi \sum_{m=0}^{m=M-1} \zeta_m \sqrt{2J_m} e^{-J_m/2} \sum_{k=0}^{\infty}[I_k(\frac{J_m}{2})+I_{k+1}(\frac{J_m}{2})]\cos[(2k+1)(2\pi\nu m+\phi_m)], \tag{19}$$

$$\Delta\phi_M = 4\pi\xi \sum_{m=0}^{m=M-1} \zeta_m e^{-J_m/2} \frac{1}{\sqrt{2J_m}} \sum_{k=0}^{\infty}[I_k(\frac{J_m}{2})+I_{k+1}(\frac{J_m}{2})]\sin[(2k+1)(2\pi\nu m+\phi_m)]. \tag{20}$$

Now let us take into account the smallness of the parameter $\xi\zeta$ in Eqs. (19) and (20) and consider a case when $M \gg 1$ but $\Delta J_m, \Delta\phi_m \ll 1$. In other words, the changes of the action and phase (and hence the distribution function and the luminosity) are assumed to be relatively small on the time interval under consideration. In linear approximation in the parameter $\xi\zeta$ one can substitute J_m and ϕ_m by J_0 and ϕ_0 on the right side of Eqs. (19) and (20):

$$\Delta J_M = -4\pi\xi \sum_{m=0}^{m=M-1} \zeta_m \sqrt{2J_0} e^{-J_0/2} \sum_{k=0}^{\infty}[\ I_k(\frac{J_0}{2}) + I_{k+1}(\frac{J_0}{2})]\cos[(2k+1)(2\pi\nu m+\phi_0)]. \tag{21}$$

A similar expression can be obtained for $\Delta\phi_M$.

4.0 AVERAGING

To calculate how luminosity changes with time due to random off-set beam-beam interactions, we will need first to average the change ΔJ_M and the quadratic quantities $(\Delta J_M)^2$ and $\Delta J_M \Delta\phi_M$. In what follows, we denote the average quantities by the angle brackets, $\langle\Delta J_M\rangle$, $\langle(\Delta J_M)^2\rangle$ and $\langle\Delta J_M \Delta\phi_M\rangle$. Performing the averaging, we will assume that the mean value of the random variable ζ_m vanishes:

$$\langle\zeta_m\rangle = 0. \tag{22}$$

We also define the auto-correlation function $K(n)$:

$$K(n) = \langle \zeta_m \zeta_{m+n} \rangle. \tag{23}$$

This is an even function, $K(n) = K(-n)$, and the setoff variance is simply $\langle \zeta_m^2 \rangle = K(0)$.

Consider first $\langle (\Delta J_M)^2 \rangle$. The calculations performed in Appendix A give:

$$\langle (\Delta J_M)^2 \rangle = 16\pi^2 \xi^2 J_0 e^{-J_0 M} \sum_{n=-\infty}^{\infty} K(n) \sum_{k=0}^{\infty} (I_k + I_{k+1})^2 \cos 2\pi\nu n(2k+1). \tag{24}$$

The argument of the modified Bessel functions in Eq. (24) is equal to $J_0/2$. The derivation of Eq. (24) is based on the assumption that $M \gg 1$ and ν is not close to a rational number.

Now we rewrite Eq. (24) in terms of the noise spectral density $S(\omega)$. To define $S(\omega)$, we consider ζ as a continuous random function of time, $\zeta = \zeta(t)$, with a given auto-correlation function $K(\tau)$:

$$K(\tau) = \langle \zeta(t) \zeta(t-\tau) \rangle. \tag{25}$$

According to the spectral theory of random functions, the Fourier transformation of the auto-correlation function gives the spectral density $S(\omega)$:

$$K(\tau) = \int_0^{\infty} d\omega S(\omega) \cos \omega\tau. \tag{26}$$

One can conceive that the function $\zeta(t)$ of the continuous time t gives the relative displacement of the bunches about the closed orbit in the course of revolutions. What we need from this function are its values $\zeta_m = \zeta(mT)$ taken at the moments when interactions occur. (Here T denotes the period of revolution, $T = 2\pi/\Omega$, where Ω is the revolution frequency.) Using the identity

$$\sum_{n=-\infty}^{\infty} \cos n\alpha = 2\pi \sum_{m=-\infty}^{\infty} \delta(\alpha - 2\pi m) \tag{27}$$

and changing the order of summations in Eq. (24), one finds

$$\langle (\Delta J_M)^2 \rangle = 16\pi^2 \xi^2 J_0 e^{-J_0 M} \sum_{k=0}^{\infty} (I_k + I_{k+1})^2 R_k, \tag{28}$$

where the factors R_k are given by

$$R_k \equiv \sum_{n=-\infty}^{\infty} K(n) \cos 2\pi\nu n(2k+1) =$$

$$\pi \int_0^{\infty} d\omega S(\omega) \left\{ \sum_{m=-\infty}^{\infty} \delta[\omega T - 2\pi m - 2\pi\nu n(k+1)] + \sum_{m=-\infty}^{\infty} \delta[\omega T - 2\pi m + 2\pi\nu n(k+1)] \right\} =$$

$$= \frac{\pi}{T} \sum_{m=o}^{\infty} [S(\omega_{km}^{(1)}) + S(\omega_{km}^{(2)})]. \tag{29}$$

The two sets of frequencies $\omega_{km}^{(1)}$ and $\omega_{km}^{(2)}$ are determined by the following equations:

$$\omega_{km}^{(1)} = \Omega\{[\nu(2k+1)] + m\}, \quad \omega_{km}^{(2)} = \Omega\{[1 - \nu(2k+1)] + m\}, \tag{30}$$

where the brackets denote the fractional part of a number.

The physical meaning of Eq. (28) is that it explicitly expresses the averaged square of ΔJ_M in terms of the spectral density of the noise. Note that only a discrete set of frequencies enters Eq. (28); these are equal to Ω multiplied by the distance between ν (or its odd harmonic $\nu(2k+1)$) and the nearest integer. In the next section we will find the contributions of these harmonics to the luminosity dilution.

To finish this section, we calculate $\langle \Delta J_M \rangle$. Averaging Eq. (21) with the use of Eq. (22) immediately yields

$$\langle \Delta J_M \rangle = 0. \tag{31}$$

We should emphasize here that this result is valid only in a linear approximation in the parameter ζ. Taking into account second-order terms, generally speaking, gives a non-vanishing contribution to $\langle \Delta J_M \rangle$. However, it is worth noting that in the next section, where we find a perturbation of the distribution function, only the first-order averaged value of ΔJ_M enters the result.

Finally, using the same approach as for calculation of $\langle (\Delta J_M)^2 \rangle$ the averaging of $\langle \Delta J_M \Delta\phi_M \rangle$ is performed in Appendix A. It turns out that $\langle \Delta J_M \Delta\phi_M \rangle = 0$.

5.0 DISTRIBUTION FUNCTION OF THE BUNCH

In this section, we will find the perturbation of the distribution function in the bunch caused by multiple offset beam-beam interactions.

Let us denote by J_M and ϕ_M the values of action and phase after M interactions with an oncoming bunch of a particle which initially had J_0 and ϕ_0 :

$$J_M = J_0 + \Delta J_M(J_0, \phi_0), \quad \phi_M = \phi_0 + \Delta\phi_M(J_0, \phi_0), \tag{32}$$

where ΔJ_M is given by Eq. (21). In what follows, we will not need an explicit expression for $\Delta\phi_M$. Since we assume that ΔJ_M and $\Delta\phi_M$ are small, the Eqs. (32) can be iteratively solved for the initial variables J_0 and ϕ_0, giving J_0 and ϕ_0 as functions of the variables J_M and ϕ_M. Performing the calculations, we keep terms up to the second order:

$$J_0 = J_M - \Delta J_M(J_M - \Delta J_M, \phi_M - \Delta\phi_M) \approx J_M - \Delta J_M + \frac{\partial \Delta J_M}{\partial J_M} \Delta J_M + \frac{\partial \Delta J_M}{\partial \phi_M} \Delta\phi_M. \tag{33}$$

In Eq. (33) it is understood that the arguments J_0 and ϕ_0 in ΔJ_M and $\Delta\phi_M$ are replaced by J_M and ϕ_M.

Using the property of phase space conservation in Hamiltonian motion, the distribution function after M beam-beam interactions, $F(J_M, \phi_M)$, can be simply obtained by putting the expression for J_0 given by Eq. (33) into the initial distribution function $F_0(J_0)$. Omitting for convenience the subscripts of variables J_M and ϕ_M and using a power series expansion up to the second order, one finds:

$$F(J, \phi) = F_0(J_0) - \left(\Delta J_M - \frac{\partial \Delta J_M}{\partial J_M} \Delta J_M - \frac{\partial \Delta J_M}{\partial \phi_M} \Delta\phi_M \right) \frac{\partial F_0}{\partial J} + \frac{1}{2} (\Delta J_M)^2 \frac{\partial^2 F_0}{\partial J^2}. \tag{34}$$

Since we are now keeping second-order terms in our power expansions, ΔJ_M in Eq. (24) is also to be calculated with this accuracy. However, in Sec. 3.0, we have found ΔJ_M (Eq. (21)) in the first order only. Slightly changing for a moment our notation, let us denote the right side of Eq. (21) by $\Delta J_M^{(1)}$ so that

$$\Delta J_M = \Delta J_M^{(1)} + \Delta J_M^{(2)}, \tag{35}$$

where $\Delta J_M^{(2)}$ is the second-order contribution to ΔJ_M. As shown in Appendix B, for a Hamiltonian motion, the second-order term can be explicitly expressed through the first-order one:

$$\Delta J_M^{(2)} = \frac{1}{2}\frac{\partial \Delta J_M^{(1)}}{\partial J}\Delta J_M^{(1)} + \frac{1}{2}\frac{\partial \Delta J_M^{(1)}}{\partial \phi}\Delta\phi_M^{(1)}. \tag{36}$$

Putting this in Eq. (35), we can find $F(J, \phi)$ given by Eq. (34) through the second order. Now, using the equality

$$\frac{\partial \Delta J_M^{(1)}}{\partial J} + \frac{\partial \Delta\phi_M^{(1)}}{\partial \phi} = 0, \tag{37}$$

which is a simple manifestation of the phase volume conservation in the first order, and averaging Eq. (34) with account of $\langle \Delta J_M \Delta\phi_M \rangle = 0$ and $\langle \Delta J_M \rangle = 0$, one finds a Fokker-Planck type equation for the averaged perturbation $\Delta F = F - F_0$ of the distribution function

$$\langle \Delta F \rangle = \frac{1}{2}\frac{\partial}{\partial J}\langle (\Delta J_M)^2 \rangle \frac{\partial F_0}{\partial J}. \tag{38}$$

In Eq. (38) we returned to original notation, omitting the superscript$^{(1)}$ in $\Delta J_M^{(1)}$. Note that $\langle \Delta F \rangle$ depends only on the variable J.

6.0 LUMINOSITY DILUTION

Substituting Eq. (38) into Eq. (6) and integrating by parts gives a relative change of the luminosity due to offset beam-beam interactions:

$$\frac{\Delta L}{L_0} = -2^{3/2}\int_0^\infty dJ \langle (\Delta J_M)^2 \rangle e^{-3J/2}[I_0(J/2) - I_1(J/2)]. \tag{39}$$

Putting Eq. (28) into Eq. (39) and performing numerical integration yields

$$\frac{\Delta L}{L_0} = -\xi^2 M(a_0 R_0 + a_1 R_1 + a_2 R_2 + a_3 R_3 + \ldots), \tag{40}$$

where the coefficients a_k are

$$a_0 = 12.45,\ a_1 = 0.73,\ a_2 = 0.038,\ a_3 = 0.002. \tag{41}$$

As we see, the largest coefficient in the series (40) is a_0; for large k, a_k rapidly fall down. The factor R_0 corresponding to a_0 contains different harmonics of the noise spectrum. In a typical case, the noise

spectrum is a decreasing function of ω, and the main contribution to R_0 comes from the first term in Eq. (29):

$$R_0 \simeq \frac{\pi}{T} S(\Omega \Delta \nu), \tag{42}$$

where $\Delta\nu$ is the distance from ν to the nearest integer. Depending on the working tunes, the next term in Eq. (40) proportional to $a_1 R_1$ may also be important because R_1 contains different harmonics of the noise.

7.0 DISCUSSION

To understand the role of the effects under study in the SSC, let us estimate a tolerable setoff of the beams at the interaction point.

Note that according to Eqs. (25) and (26) the spectral density S has a dimension of the length squared over the frequency. Hence, the product $\Omega\Delta\nu S(\Omega\Delta\nu)$, which has a dimension of length squared, can be associated with the beam displacement squared, δx^2, at the frequency $\Omega\Delta\nu$. In terms of δx, Eq. (40) takes the form

$$\frac{\Delta L}{L_0} = -6.2\xi^2 M \frac{\delta x^2}{\Delta \nu}. \tag{43}$$

Now, putting in Eq. (43) $\Delta\nu = 0.26$ and $\xi = 0.001$ (for one interaction point) and requiring $\Delta L/L_0$ to be less than 0.5 after 10 hours of operation ($M = 10^8$), we obtain:

$$\delta x < 1.1 \times 10^{-2}\, \sigma_x. \tag{44}$$

In Eq. (44) we returned to dimensional variables, putting σ_x on the right side. For $\sigma_x = 5$ μm at the low beta interaction point, this gives a serious constraint $\delta x < 5.5 \times 10^{-2}$ μm on a possible beam offset.

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APPENDIX A

Let us first calculate $\langle(\Delta J_M)^2\rangle$. Averaging $(\Delta J_M)^2$ and using Eq. (23), one finds

$$(\Delta J_M)^2 = 32\pi^2\xi^2 J_0 e^{-J_0} M \sum_{l,m=0}^{M-1} K(m-l) \times$$

$$\times \sum_{k=0}^{\infty} [\mathrm{I}_k + \mathrm{I}_{k+1}] \cos[(2k+1)(2\pi\nu m + \phi_0)] \times$$

$$\times \sum_{r=0}^{\infty} [\mathrm{I}_r + \mathrm{I}_{r+1}] \cos[(2r+1)(2\pi\nu l + \phi_0)], \tag{A1}$$

where the argument of the modified Bessel functions is equal to $J_0/2$. Now define a new summation index, $n = m-l$, and change the order of summation using the identity

$$\sum_{m=0}^{M-1} \sum_{n=m-M+1}^{m} = \sum_{n=1}^{M-1} \sum_{m=n}^{M-1} + \sum_{m=0}^{M-1} \{n=0\} + \sum_{n=-M+1}^{-1} \sum_{m=0}^{n+M-1}. \tag{A2}$$

After this, one can sum up over m because this index enters cosine arguments only. Then, using elementary trigonometric identities for the product of two cosines, we obtain a formula that contains the following expressions:

$$\sum_{\mathrm{m}} \cos\{2\pi\nu m[(2r+1)\pm(2k+1)] - 2\pi\nu n(2r+1) + [(2r+1)\pm(2k+1)]\phi_0\}, \tag{A3}$$

where the number of additives in (A3) is equal to $M-n$. Assume that ν is not a rational number. In this case, for $r \neq k$ the sum (A3) is of the order of unity even in the limit $M \gg 1$, and only for $r=k$ is this sum equal to

$$(M-n)\cos 2\pi\nu n(2k+1) \approx M \cos 2\pi\nu n(2k+1), \tag{A4}$$

that is, it grows linearly with M. In (A4) we took into account that $M \gg n$, because n is, in fact, restricted from above by $K(n) \to 0$ for large n.

Keeping only the largest terms (proportional to M) in Eq. (A1), one finds

$$\langle(\Delta J_M)^2\rangle = 16\pi^2\xi^2 J_0 e^{-J_0} M \sum_{n=-M+1}^{M-1} K(n) \sum_{k=0}^{\infty} (\mathrm{I}_k + \mathrm{I}_{k+1})^2 \cos 2\pi\nu n(2k+1). \tag{A5}$$

Since the sum over n in (A5) converges when $|n| \to \infty$, one can extend summation from $-\infty$ to ∞. Using the evenness of the function $K(n)$, we arrive at Eq. (24).

Using the same approach as above, one can calculate the averaged value of $\langle\Delta J_M \Delta\phi_M\rangle$. The result is

$$\langle\Delta J_M \Delta\phi_M\rangle = 8\pi^2\xi^2 J_0 e^{-J_0} M \sum_{n=-M+1}^{M-1} K(n) \sum_{k=0}^{\infty} (\mathrm{I}_k + \mathrm{I}_{k+1})^2 \sin 2\pi\nu n(2k+1). \tag{A6}$$

However this is equal to zero because the product $K(n)\sin 2\pi\nu n(2k+1)$ is an odd function of n.

APPENDIX B

Let us consider a time-independent Hamiltonian $H(p,q)$ with the corresponding Hamiltonian equation:

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}, \qquad \frac{dq}{dt} = \frac{\partial H}{\partial p}. \tag{B1}$$

Let p_0 and q_0 denote initial values of the phase variables. Let us consider a motion in which p and q change by small values, Δp and Δq. In the first approximation, they can be found by integrating Eq. (B1) over time and putting in the Hamiltonian p_0 and q_0 instead of p and q :

$$\Delta p^{(1)} = -\int_{t_0}^{t} \frac{\partial H(p_0,q_0)}{\partial q_0} dt' = -(t-t_0)H_q(p_0,q_0), \tag{B2}$$

$$\Delta q^{(1)} = \int_{t_0}^{t} \frac{\partial H(p_0,q_0)}{\partial p_0} dt' = (t-t_0)H_p(p_0,q_0), \tag{B3}$$

where the indices p and q denote partial derivatives with respect to the corresponding variables.

In the next approximation, while integrating (B1) one should take into account that p and q differ from p_0 and q_0 in the first order:

$$\frac{dp}{dt} = -H_q(p_0,q_0) - H_{qq}(p_0,q_0)\Delta q^{(1)} - H_{qp}(p_0,q_0)\Delta p^{(1)}. \tag{B4}$$

Integration of Eq. (B4), taking into account Eqs. (B2) and (B3), now yields

$$\Delta p = \Delta p^{(1)} + \frac{1}{2}\frac{\partial \Delta p^{(1)}}{\partial q}\Delta q^{(1)} + \frac{1}{2}\frac{\partial \Delta p^{(1)}}{\partial p}\Delta p^{(1)}. \tag{B5}$$

One can easily check that the result (B5) is also valid for a Hamiltonian H depending versus time through a factor $f(t)$,

$$H(p,q,t) = f(t)H_0(p,q). \tag{B6}$$

This is just the case relevant to the problem of beam-beam interaction, for which the Hamiltonian is given by Eq. (12) with the time variable equal to θ. Rewriting Eq. (B5) in terms of action and phase, we arrive at Eqs. (36).