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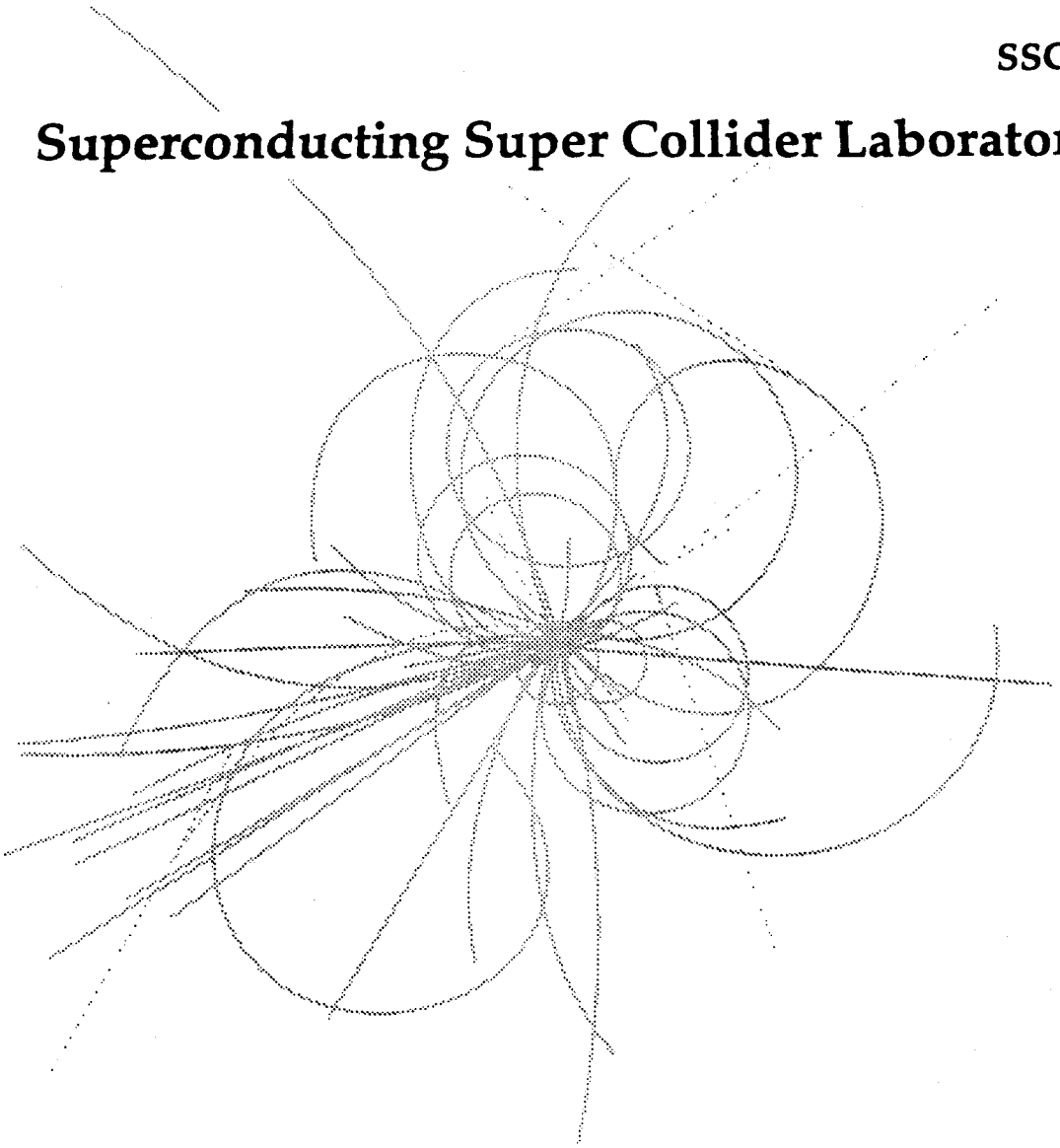

Tune Shift Effect Due to the Multipole Longitudinal Periodic Structure in the Superconducting Dipole Magnets

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Abstract

Neglecting the curvature terms, the magnetic field and the vector potential which generate the multipole longitudinal periodic structure in a superconducting dipole magnet are found. Using this field and the standard Hamiltonian perturbation theory, the tune shifts due to this periodic pattern in the superconducting dipole magnets are estimated for the Superconducting Super Collider (SSC) machine. The results suggest that this tune shift is very small for most of the multipoles and could be ignored for the SSC. However, for the quadrupole longitudinal oscillation pattern, the tune shift relative to the amplitude of this oscillation could be of the order of 10^{-5} and may not be ignored.

FIGURES

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1.0 INTRODUCTION

The discovery of the sextupole, dipole, and quadrupole longitudinal periodic structure due to the persistent current field in the HERA superconducting magnets at Deutsches Elektronen-Synchrotron[1] (DESY) raised many questions about the possible effects of this pattern in the dynamic of the beam. Experiments suggest that this periodic pattern is due to the strand pitch in the superconducting (sc) cable, and measurements indicate that its wavelength is approximately equal to this strand pitch (9.1 ± 0.5 cm for the outer coil of the SSC sc dipoles). The sextupole pattern has already been confirmed in a short 50-mm R&D dipole magnet;[2] the effect on the dynamic of the SSC beam must be confidently estimated, even if it is known already that the effect must be small. (At Fermi National Accelerator Laboratory, the Tevatron was designed—and works—without any knowledge of this matter,[3] although small variations have been observed.[4]) Obtaining detailed dynamic effects would require a 3D-Tracking Code. Unfortunately, there is no such code at this time, nor is one likely in the near future. However, it is possible to see this effect by calculating the tune shift through the Hamiltonian formalism.

To calculate the tune shift, the superconvergent Hamiltonian perturbation method[5] is used. First, the vector potential is calculated through the expression of the magnetic field, which is obtained from the scalar potential approach (the curvature is neglected). Then, using the lowest order in the vector potential and the nonrelativistic Hamiltonian, the tune shift is calculated by applying the standard canonical transformations and averaging.[6]

2.0 MAGNETIC FIELD AND VECTOR POTENTIAL

From Maxwell equations, the relation between the vector potential, $\mathbf{A}$, and the magnetic field, $\mathbf{B}$, and the relation between the magnetic field and the scalar potential, Φ, are given[7] by

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{1}$$

and

$$\mathbf{B} = -\nabla\Phi \ . \tag{2}$$

Because the magnetic field will be calculated in the vacuum beam pipe, the following differential equations are satisfied:

$$\nabla \times \mathbf{B} = \mathbf{0} \tag{3a}$$

and

$$\nabla \cdot \mathbf{B} = 0 \ . \tag{3b}$$

Therefore, the scalar potential must satisfy the Laplace equation:

$$\nabla^2\Phi = 0 \ . \tag{4}$$

2.1 The Scalar Potential

Neglecting the curvature effect, the Laplacian in the coordinate system based on a closed planar reference curve is similar to a cylindrical coordinate and can be written as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial s^2} = 0 \ , \tag{5}$$

where s is the longitudinal coordinate, and $r = \sqrt{x^2+y^2}$ measures the radial displacement with respect to the close planar reference curve. Suppose there is a multipole of order m which is oscillating along the s-axis with a periodic structure given by $v(s)$. Furthermore, assume the next anzat for the scalar potential:

$$\Phi = -\sin m\theta \left[v(s)r^m + \sum_{n=1}^{\infty} v_n(s) r^n \right] \ , \tag{6}$$

where the functions $v_m(s)$ are arbitrary at the moment. The reason to propose this anzat is clear: The m-multipole symmetry must be preserved. Substituting Eq. (6) in Eq. (5) and rearranging terms, the following expression is obtained:

$$\begin{aligned}
&(-1+m^2)v_1 r^{-1} + (-2^2+m^2)v_2 + [(-3^2+m^2)v_3 - \ddot{v}_1]r + \\
&\qquad + \sum_{n<m} \left\{ -[(n+2)^2+m^2]v_{n+2} - \ddot{v}_n \right\} r^n + \\
&\qquad + \left\{ [-(m+2)^2+m^2]v_{m+2} - \ddot{v}_m - \ddot{v} \right\} r^m + \\
&\qquad + \sum_{n>m} \left\{ [-(n+2)^2+m^2]v_{n+2} - \ddot{v}_n \right\} r^n = 0 \ ,
\end{aligned} \tag{7}$$

where $\ddot{\xi}$ is the second derivation of the variable ξ with respect to s. It is always possible to make the following selection:

$$v_j = 0 \ , \ \text{for } j < m \tag{8a}$$

and

$$v_m(s) = v(s) \ , \tag{8b}$$

which leads to

$$v_{m+2} = \frac{2\ddot{v}}{[-(m+2)^2 + m^2]} \tag{9a}$$

and

$$v_{n+2} = \frac{\ddot{v}_n}{[-(n+2)^2 + m^2]} \ , \ \text{for } n > m \ . \tag{9b}$$

From Eqs. (9a) and (9b), it follows that

$$v_{m+2k} = \frac{2^{2k} v^{(2k)}}{\prod_{j=1}^{k}[-(m+2j)^2 + m^2]} \ , \ k \geq 1, \tag{10}$$

where $v^{(2k)}$ represents the $2k$-derivative of v with respect to s, and $\prod$ represents the product of the elements. In this way, the coefficients of Eq. (6) are determined, and the solution is given by

$$\Phi(r, \theta, s) = -r^m \sin m\theta \left[v(s) + \sum_{k=1}^{\infty} \frac{r^{2k} v^{(2k)} 2^{2k}}{\prod_{j=1}^{k}[-(m+2j)^2 + m^2]} \right] \ . \tag{11}$$

2.2 Magnetic Field and Relation with Experiments

If the scalar potential is known, the magnetic field can be calculated using Eq. (2). The components of the magnetic field are given by

$$B_r = r^{m-1} \sin m\theta \left\{ v(s) + \sum_{k=1}^{\infty} \frac{(m+2k) r^{2k} v^{(2k)} 2^{2k}}{\prod_{j=1}^{k}[-(m+2j)^2 + m^2]} \right\} \ , \tag{12a}$$

$$B_\theta = m r^{m-1} \cos m\theta \left\{ v(s) + \sum_{k=1}^{\infty} \frac{r^{2k} v^{(2k)} 2^{2k}}{\prod_{j=1}^{k}[-(m+2j)^2 + m^2]} \right\} \ , \tag{12b}$$

and

$$B_s = r^m \sin m\theta \left\{ v^{(1)} + \sum_{k=1}^{\infty} \frac{r^{2k} v^{2k+1} 2^{2k}}{\prod_{j=1}^{k}[-(m+2j)^2 + m^2]} \right\} \ . \tag{12c}$$

To know the function $v(s)$, assume that at the radius $r = r_o$ the B_r component of the m-multipole magnetic field is measured, yielding the expression

$$B_r^{exp} = m^{-1} r_o^{m-1} \sin m\theta [b_m + \alpha_m \sin \kappa s] \ , \tag{13}$$

where b_m is the systematic component of the m-multipole, κ is the wave number of the longitudinal periodic pattern, and α_m is its amplitude of oscillation (in Gauss/cm^{m-1}). If the function $v(s)$ is assumed to be of the form

$$v(s) = m^{-1}[b_m + c_m \sin \kappa s] \ , \tag{14}$$

Eq. (12a) is written as

$$B_r(r = r_o) = m^{-1} r_o^{m-1} \sin m\theta \Bigg\{ b_m + c_m \sin \kappa s + \\ + c_m \sin \kappa s \sum_{k=1}^{\infty} \frac{(-1)^k (2k+m)(2r_o\kappa)^{2k}}{\prod_{j=1}^{k}[-(m+2j)^2 + m^2]} \Bigg\} \ ; \tag{15}$$

combining this expression with Eq. (13), the coefficients c_m are defined by the experimental values:

$$c_m(\kappa, r_o) = \frac{\alpha_m}{1 + \sum_{k=1}^{\infty} \frac{(-1)^k (2k+m)(2\kappa r_o)^{2k}}{\prod_{j=1}^{k}[-(m+2j)^2 + m^2]}} \ . \tag{16}$$

It is interesting to note that the actual experimental amplitude values, α_m, are related with an infinity series. Eq. (16) brings about the complete solution for the magnetic field.

2.3 The Vector Potential

In the above coordinate system, Eq. (1) is written in terms of its components as

$$\frac{1}{r}\frac{\partial A_s}{\partial \theta} - \frac{\partial A_\theta}{\partial s} = B_r \ , \tag{17a}$$

$$\frac{\partial A_r}{\partial s} - \frac{\partial A_s}{\partial r} = B_\theta \ , \tag{17b}$$

and

$$\frac{1}{r}\frac{\partial (r A_\theta)}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \theta} = B_s \ . \tag{17c}$$

Assume the following anzat for the vector potential:

$$A_r = \sum_n a_n(s) r^{n+m} \cos m\theta + \sum_n b_n(s) r^{n+m} \sin m\theta \ , \tag{18a}$$

$$A_\theta = \sum_n g_n(s) r^{n+m} \cos m\theta + \sum_n h_n(s) r^{n+m} \sin m\theta \ , \tag{18b}$$

and

$$A_s = \sum_n q_n(s) r^{n+m} \cos m\theta + \sum_n p_n(s) r^{n+m} \sin m\theta \ . \tag{18c}$$

Substituting these expressions in Eqs. (17), using Eqs. (12) in Eqs. (17), and rearranging terms, the following algebraic relations are obtained:

$$m p_o + \sum_{n=1}^{\infty} (m p_n - \acute{g}_{n-1}) r^n = 0 \ , \tag{19a}$$

$$-m q_o - \sum_{n=1}^{\infty} (m q_n + \acute{h}_{n-1}) r^n = m v + \sum_{k=1}^{\infty} (m + 2k) v_{m+2k} r^{2k} \ , \tag{19b}$$

$$m p_o + \sum_{n=1}^{\infty} [\acute{b}_{n-1} - (n+m) p_n] r^n = 0 \ , \tag{20a}$$

$$-m q_o + \sum_{n=1}^{\infty} [\acute{a}_{n-1} - (n+m) q_n] r^n = m v + \sum_{k=1}^{\infty} v_{m+2k} r^{2k} \ , \tag{20b}$$

$$(n + m + 1) g_n - m b_n = 0 \ , \tag{21a}$$

and

$$\sum_{n=1}^{\infty} [(n + m + 1) h_n + m a_n] r^n = r \acute{v} + \sum_{k=1}^{\infty} \acute{v}_{m+2k} r^{2k+1} \ , \tag{21b}$$

where v_{m+2k} has been defined in Eq. (10). From Eqs. (19a), (20a), and (21a) it is not difficult to see that it is always possible to choose

$$b_n = g_n = p_n = 0, \text{ for } n \geq 0 \ , \tag{22}$$

and from Eqs. (19b), (20b), and (21b), the following relations are obtained:

$$q_o = -v \ , \tag{23}$$

$$(m+1)h_o + ma_o = 0 \ , \tag{24a}$$

$$(m+2)h_1 + ma_1 = \acute{v} \ , \tag{24b}$$

$$\acute{a}_{2k-1} - (2k+m)q_{2k} = mv_{m+2k}, \ k \geq 1 \ , \tag{25a}$$

$$(2k+m+2)h_{2k+1} + ma_{2k+1} = \acute{v}_{m+2k}, \ k \geq 1 \ , \tag{25b}$$

and

$$mq_{2k} + \acute{h}_{2k-1} = -(m+2k)v_{m+2k}, \ k \geq 1 \ . \tag{25c}$$

The above system is self-consistent (see Appendix A), and it is possible to choose the gauge of the vector potential such that

$$a_n = 0 \ , \text{ for } n \geq 1 \ , \tag{26}$$

obtaining the following coefficients:

$$h_o = 0 \ , \tag{27a}$$

$$h_{2l+1} = \frac{\acute{v}_{m+2l}}{2l+m+2} \ , \ l \geq 0 \ , \tag{27b}$$

and

$$q_{2l+2} = -\frac{mv_{m+2l+2}}{2l+m+2} \ , \ l \geq 0 \ , \tag{27c}$$

where a new parameter, $l = k-1$, has been chosen. Therefore, the components of the vector potential in these coordinates are given by

$$A_r = 0 \ , \tag{28a}$$

$$A_\theta = r^m \sin m\theta \sum_{k=0}^{\infty} \frac{\acute{v}_{m+2k} r^{2k+1}}{2k+m+2} \ , \tag{28b}$$

and

$$A_s = -r^m \cos m\theta \left\{ v + \sum_{k=0}^{\infty} \frac{m v_{m+2k+2} r^{2k+2}}{2k+m+2} \right\} . \tag{28c}$$

These relations can be expressed in the coordinates x, y, s, where x and y are related to r and θ by

$$x = r \cos\theta$$

and

$$y = r \sin\theta \ ,$$

through a rotation of the vector (A_r, A_θ) by an angle θ,

$$A_x = -\sin\theta \ A_\theta \tag{29a}$$

and

$$A_y = \cos\theta \ A_\theta \ . \tag{29b}$$

In addition, using the identities

$$r^m \sin m\theta = Im \sum_{\rho=0}^{m} i^\rho \binom{m}{\rho} x^{m-\rho} y^\rho \tag{30a}$$

and

$$r^m \cos m\theta = Re \sum_{\rho=0}^{m} i^\rho \binom{m}{\rho} x^{m-\rho} y^\rho \ , \tag{30b}$$

where $\binom{m}{\rho}$ represents the combinatory coefficient:

$$\binom{m}{\rho} = \frac{m!}{\rho!(m-\rho)!} \ . \tag{30c}$$

The components of the vector potential in these coordinates are given by

$$A_x = -Im \sum_{\rho=0}^{m} i^\rho \binom{m}{\rho} x^{m-\rho} y^{\rho+1} \sum_{k=0}^{\infty} \frac{\acute{v}_{m+2k} (x^2+y^2)^k}{2k+m+2} \ , \tag{31a}$$

$$A_y = +Im \sum_{\rho=0}^{m} i^\rho \binom{m}{\rho} x^{m-\rho+1} y^\rho \sum_{k=0}^{\infty} \frac{\acute{v}_{m+2k}(x^2+y^2)^k}{2k+m+2} \ , \tag{31b}$$

and

$$A_s = -Re \sum_{\rho=0}^{m} i^\rho \binom{m}{\rho} x^{m-\rho} y^\rho \left\{ v + \sum_{k=0}^{\infty} \frac{m v_{m+2k+2}(x^2+y^2)^{k+1}}{2k+m+2} \right\} \ . \tag{31c}$$

In particular, the expressions for the dipole ($m = 1$), quadrupole ($m = 2$), and sextupole ($m = 3$) at lowest order are given by

$$A_x^{(1)} = -y^2 \acute{v}/3 \ , \tag{32a}$$

$$A_y^{(1)} = +xy\acute{v}/3 \ , \tag{32b}$$

$$A_s^{(1)} = -xv \ , \tag{32c}$$

$$A_x^{(2)} = -xy^2 \acute{v}/2 \ , \tag{33a}$$

$$A_y^{(2)} = +x^2 y\acute{v}/2 \ , \tag{33b}$$

$$A_s^{(2)} = -(x^2 - y^2)v \ , \tag{33c}$$

$$A_x^{(3)} = -(3x^2y^2 - y^4)\acute{v}/5 \ , \tag{34a}$$

$$A_y^{(3)} = +(3x^3y - xy^3)\acute{v}/5 \ , \tag{34b}$$

and

$$A_s^{(3)} = -(x^3 - 3xy^2)v \ . \tag{34c}$$

In order to simplify the analysis, the systematic multipole component will not be considered in the following discussion, and the amplitude of the longitudinal oscillation pattern, c_m, will be denoted by a.

3.0 HAMILTONIAN FORMALISM AND TUNE SHIFTS

The nonrelativistic Hamiltonian for the transverse motion of a synchronous charged particle traveling around an accelerator ring can be written as[8]

$$H = \frac{1}{2}\left(P_x - \frac{e}{cp}A_x\right)^2 + \frac{1}{2}\left(P_y - \frac{e}{cp}A_y\right)^2 - \frac{e}{cp}A_s + \\ + \frac{1}{2}K_1(s)y^2 - \frac{1}{2}\left(K_1(s) - \frac{1}{\rho^2}\right)x^2 \ , \tag{35}$$

where p is the longitudinal momentum of the particle, $P_x = p_x/p$ and $P_y = p_y/p$ are its normalized transversal momenta, e is the charge of the particle, c is the speed of light, $\rho(s)$ is the curvature of the accelerator ring, $K_1(s)$ describes the linear lattice of the machine (without longitudinal oscillation pattern), and the vector potential components, A, represent the additional electromagnetic field due to other sources (longitudinal multipole periodic pattern along the sc dipoles, for example). This Hamiltonian can be written as

$$H(x,y,s) = H_o(x,y,s) + V(s,y,s) + U(x,y,s) \ , \tag{36}$$

where H_o, $V = V^{(1)} + V^{(2)}$, and U are defined by

$$H_o(x,y,s) = \frac{1}{2}\left(P_x + K_x(s)x^2\right) + \frac{1}{2}\left(P_y + K_y(s)y^2\right) \ , \tag{37a}$$

$$V^{(1)}(x,y,s) = -\left(\frac{e}{cp}\right)A_s \ , \tag{37b}$$

$$V^{(2)}(x,y,s) = -\left(\frac{e}{cp}\right)\left(P_xA_x + P_yA_y\right) \ , \tag{37c}$$

and

$$U(x,y,s) = \frac{1}{2}\left(\frac{e}{cp}\right)^2\left(A_x^2 + A_y^2\right) \ . \tag{37d}$$

For tune shift calculations, it is more convenient to express the Hamiltonian in the canonical variable (J,ϕ), where J and ϕ are the vectors

$$J = (J_1, J_2)$$

and

$$\phi = (\phi_1, \phi_2) \ .$$

This can be accomplished through the generating function

$$F(s, x, y, \phi) = -\sum_{i=1}^{2} \frac{x_i}{\beta_i(s)} (\tan \phi_i - \acute{\beta}_i/2) \ , \tag{38}$$

where $\beta_i(s)$ is the beta function associated with the motion of the particle in the i (x for $i = 1$, y for $i = 2$) direction, $\acute{\beta}_i$ is its derivative with respect to s, and $\phi_i(s)$ is the betatron phase related to the beta function through

$$\phi_i(s) = \phi_i(0) + \int_0^s \frac{d\sigma}{\beta_i(\sigma)} \ . \tag{39}$$

The action, J_i, the coordinates, and the canonical momenta are given by

$$J_i = -\frac{\partial F}{\partial \phi_i} = \frac{1}{2\beta_i} \left[x_i^2 + (\beta_i \acute{x}_i - \acute{\beta}_i x_i/2)^2 \right] \ , \tag{40}$$

$$x_i = \sqrt{2 J_i \beta_i} \cos \phi_i \ , \tag{41}$$

and

$$P_i = -\sqrt{\frac{2J_i}{\beta_i}} \left(\sin \phi_i - \frac{1}{2} \acute{\beta}_i \cos \phi_i \right) \ , \tag{42}$$

for $i = 1, 2$, i.e., $i = x, y$. Furthermore, Eq. (37a) becomes

$$\mathcal{H}_o = \sum_{i=1}^{2} \frac{J_i}{\beta_i(s)} \ , \tag{43}$$

and the other expressions also become functions of the action-angle variables. Their explicit expression depends upon their explicit (P_x, P_y, x, y) dependency. To calculate the tune

shift, the average of the Hamiltonian along the whole machine, $\mathcal{C}$, and all over the betatron phases must be obtained:

$$< \mathcal{H} >= \frac{1}{2\pi} \int_0^{\mathcal{C}} ds \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} d\phi_1 d\phi_2 \; \mathcal{H}(s, \phi, J) \; . \tag{44}$$

Hence, the partial derivation of this quantity with respect to the action brings about the tune of the machine:

$$\nu_i = \frac{\partial < \mathcal{H} >}{\partial J_i} \; , \tag{45}$$

which is mainly given by the average of Eq. (43). The other terms in Eqs. (37b) to (37d) give the tune shift of this value.

4.0 FIRST ORDER IN PERTURBATION STRENGTH

At the first order in perturbation strength, the lowest order in the action variable is taken into consideration in Eq. (36). It is clear from Eqs. (37) that $U(x, y, s)$ corresponds to a second order. From Eqs. (32), (33), (34), (37), (41), and (42), and from the relation:

$$\frac{1}{2\pi} \int_0^{2\pi} d\phi \cos^n(\phi) = \frac{1}{2^n} \frac{n!}{\left(\frac{n}{2}!\right)^2} \delta_{n,2\mathcal{N}} \; , \tag{46}$$

where $\mathcal{N}$ represents any natural number (including zero), the only non-zero term which contributes at this order is the quadrupole term (Eq. (33c)). The action-angle variable is written as

$$\mathcal{V}^{(1)}_{m=2} = \left(\frac{e}{cp}\right) 2v \left[J_1 \beta_1 \cos^2 \phi_1 - J_2 \beta_2 \cos^2 \phi_2\right] \; . \tag{47}$$

With the help of Eq. (46), the average of this expression is given by

$$< \mathcal{V}^{(1)}_{m=2} > = \left(\frac{e}{cp}\right) a(J_1 g_1 - J_2 g_2) \; , \tag{48}$$

where a is the amplitude of the longitudinal quadrupole oscillation pattern, and the function numbers g_1 and g_2 are defined as

$$g_1 = \frac{1}{4\pi} \int_0^{\mathcal{C}} \sin \kappa s \; \beta_1(s) ds \tag{49a}$$

and

$$g_2 = \frac{1}{4\pi} \int_0^C \sin \kappa s \; \beta_2(s) ds \; . \tag{49b}$$

The tune shift is then given by

$$\Delta\nu_1 = \frac{\partial < \mathcal{V}^{(1)} >}{\partial J_1} = \frac{a g_1}{\gamma} \sqrt{\frac{r_p}{mc^2}} \tag{50a}$$

and

$$\Delta\nu_2 = \frac{\partial < \mathcal{V}^{(1)} >}{\partial J_2} = -\frac{a g_2}{\gamma} \sqrt{\frac{r_p}{mc^2}} \; , \tag{50b}$$

where the following identity has been used:

$$\left(\frac{e}{cp}\right)^2 = \frac{r_p}{mc^2 \gamma^2} \; , \tag{51}$$

where $r_p = e^2/mc^2 = 1.5348 \times 10^{-16}$ cm is the classical radius of the proton, $mc^2 = 1.6 \times 10^{-3}$ ergs is its energy at rest, and $\gamma = \sqrt{1 - v^2/c^2}$ is its relativistic factor. To calculate the numbers g_1 and g_2 along the Collider and High Energy Booster (HEB) of the SSC machine, the computer program MAD[12] was used to print out the beta function in three different locations of each magnet (at the beginning, at the center, and at the end). The beta function was fit with a second-order polynomial per magnet:

$$\beta(s) = x_1 + x_2 s + x_3 s^2 \; ,$$

where the coefficients are calculated from the three known beta values. The wavelength of the longitudinal oscillation pattern,

$$\lambda = \frac{2\pi}{\kappa} \; ,$$

was chosen as $\lambda = 9$ cm. A computer program was developed to make these calculations and the integration of Eqs. (49). The results are shown in Table 1; calculations for the

Tevatron lattice design are shown for reference.

TABLE 1. NUMERICAL INTEGRATION OF g_1 AND g_2.

	SSC-Collider	SSC-HEB	Tevatron
g_1	-0.650×10^6	-2.050×10^5	2607.5
g_2	-0.475×10^6	-1.975×10^5	1815.7

At injection ($\gamma_{Collider} = 2000$, $\gamma_{HEB} = 200$, and $\gamma_{Tevatron} = 150$), the calculated tune shifts are shown in Table 2.

TABLE 2. TUNE SHIFTS* AT INJECTION.

	SSC-Collider	SSC-HEB	Tevatron
$\Delta\nu_x/a$	-1.007×10^{-4}	-3.175×10^{-4}	$+5.3 \times 10^{-6}$
$\Delta\nu_y/a$	$+0.735 \times 10^{-4}$	$+3.050 \times 10^{-4}$	-3.7×10^{-6}

* Amplitude of Oscillations, a, in Gauss/cm.

However, the values calculated in Table 1 are strongly dependent on the wavelength of the longitudinal oscillation pattern, as can be seen in Figures 1–3. These figures show that the sign and the order of the coefficients (Eqs. (49)) may not be well defined. In addition, the SSC-HEB and the Tevatron may actually have the same order of magnitude in the relative tune shift (10^{-4}). Moreover, since the amplitude of the quadrupole oscillation pattern is expected to be random from magnet to magnet, this relative tune shift should be divided by the square root of the total number of magnets. So, a tune shift of the order of 10^{-5} is likely for the SSC-HEB and the Tevatron, and probably of one order of magnitude lower for the SSC-Collider.

5.0 SECOND ORDER IN PERTURBATION STRENGTH

At first sight, it seems unnecessary to study higher orders of perturbation strength since, in principle, these would have smaller values than the first order has. However, there are three reasons why is not worthless to go to the second order of perturbation strength. First, it is useful to check how small they are. Second, if they are not too small to ignore, they would be nonlinear, in general, producing tune shift spread in the machine. Finally, going to a second order, there is a "legal" way[9] to cancel the terms that originally were cut out in the averaging process made at the first order of perturbation.

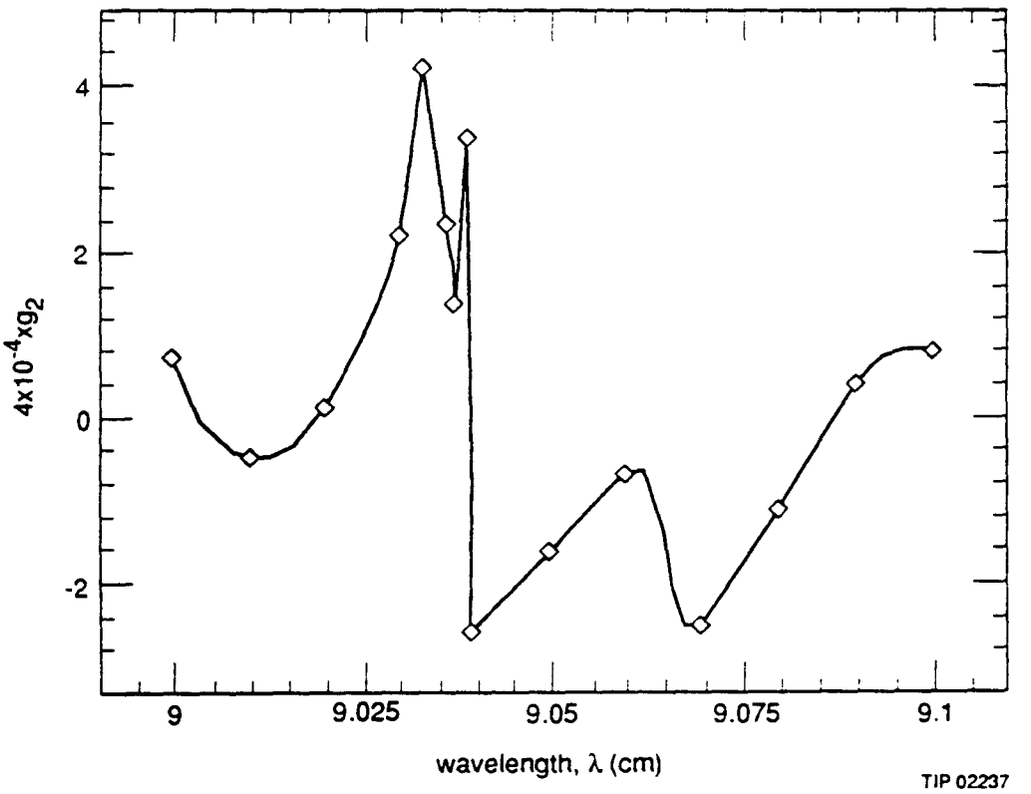

Figure 1. Variations of the g_2 Value for the Tevatron Lattice With Respect to the Longitudinal Oscillation Pattern Wavelength.

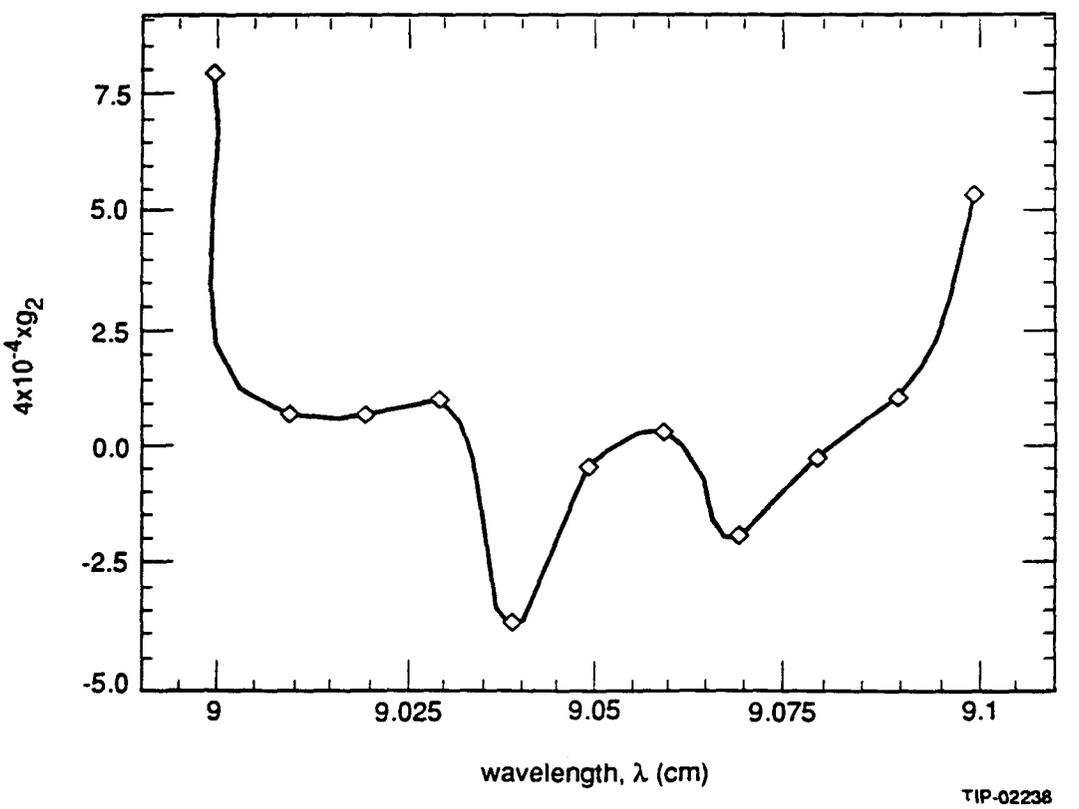

Figure 2. Variation of the g_2 Value for the SSC-HEB Lattice With Respect to the Longitudinal Oscillation Pattern Wavelength.

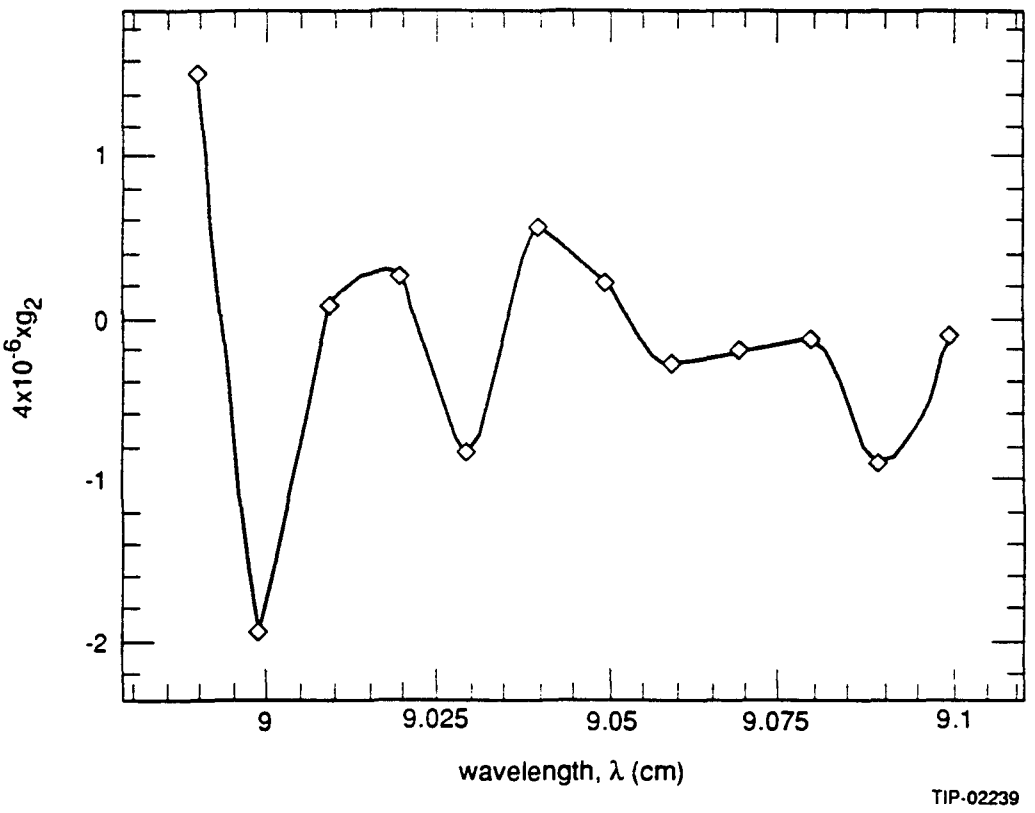

Figure 3. Variation of the g_2 Value for the SSC-Collider Lattice With Respect to the Longitudinal Oscillation Pattern Wavelength.

To go to a second-order perturbation theory, a new canonical transformation must be made (Φ, K). This canonical transformation is close to the identity (the original action-angle transformation of Eq. (38)) and is characterized by the generating function

$$F_{new}(s, \phi, K) = \sum_{i=1}^{2} K_i \phi_i + G(s, \phi, K) \ , \tag{52}$$

where G is a function to be determined. The relation between the new variables (Φ, K) and the old ones (ϕ, J) is given by the expressions

$$J_i = \frac{\partial F_{new}}{\partial \phi} = K_i + G_{\phi_i} \ , \tag{53a}$$

and

$$\Phi_i = \frac{\partial F_{new}}{\partial K_i} = \phi_i + G_{K_i} \ , \tag{53b}$$

where the subindex means partial differentiation. In addition, the new Hamiltonian, $\widehat{\mathcal{H}}(s,\Phi,K)$, is given by

$$\widehat{\mathcal{H}} = \sum_{i=1}^{2} \frac{K_i}{\beta_i(s)} + \sum_{i=1}^{2} \frac{1}{\beta_i(s)} \frac{\partial G}{\partial \phi_i} + \frac{\partial G}{\partial s} + \\ + \mathcal{V}(s, \Phi - G_K, K + G_\phi) + U(s, \Phi - G_K, K + G_\phi) \ . \tag{54}$$

Doing a Taylor expansion of the two last terms on the right-hand side of Eq. (54), it follows that

$$\widehat{\mathcal{H}} = \sum_{i=1}^{2} \frac{K_i}{\beta_i(s)} + \mathcal{V}^{(1)}_{m=2}(s,\Phi,K) + \\ + \sum_{i=1}^{2} \frac{1}{\beta_i(s)} \frac{\partial G}{\partial \phi_i} + \frac{\partial G}{\partial s} + \mathcal{V} + \\ + \sum_{i=1}^{2} \left[\mathcal{V}_{K_i} G_{\phi_i} - \mathcal{V}_{\phi_i} G_{K_i} \right] + \mathcal{U}(s,\Phi,K) + \\ + \dots \ , \tag{55}$$

where the quadrupole term of Eq. (47) has been extracted from $\mathcal{V}$ to put it together with the first-order, zero-average terms (first line in Eq. (55)), and the term $\mathcal{U}$ has been put together with the second-order terms (third line in Eq. (55)). In this expression, it is possible to make

$$\sum_{i=1}^{2} \frac{1}{\beta_i(s)} \frac{\partial G}{\partial \phi_i} + \frac{\partial G}{\partial s} + \mathcal{V} = 0 \ , \tag{56}$$

"legally" deleting the term $\mathcal{V}$ from the Hamiltonian. The solution of this partial differential equation (see Appendix B) brings about the following expression for G:

$$G(s,\phi,K) = -\int_0^s \mathcal{V}\big(\xi, \phi - \psi(s) + \psi(\xi), K\big) \ d\xi \ , \tag{57}$$

where the components of the function ψ are defined by

$$\psi_i(s) = \int_0^s \frac{d\sigma}{\beta_i(\sigma)} \ . \tag{58}$$

Using this expression in Eq. (56), the full second-order approximation can be solved, neglecting higher-order terms. Consequently, the second order in perturbation Hamiltonian

can be written as

$$\widehat{\mathcal{H}} = \sum_{i=1}^{2} \frac{K_i}{\beta_i(s)} + \mathcal{V}^{(1)}_{m=2}(s,\phi,K) + \mathcal{H}_{11} + \mathcal{H}_{12} + \mathcal{H}_{22} \ , \tag{59}$$

where $\mathcal{H}_{11}$, $\mathcal{H}_{12}$, and $\mathcal{H}_{22}$ are given by

$$\mathcal{H}_{11} = \sum_{i=1}^{2} \left[\mathcal{V}^{(1)}_{K_i} G^{(1)}_{\phi_i} - \mathcal{V}^{(1)}_{\phi_i} G^{(1)}_{K_i} \right] \ , \tag{60a}$$

$$\mathcal{H}_{12} = \sum_{i=1}^{2} \left[\mathcal{V}^{(1)}_{K_i} G^{(2)}_{\phi_i} - \mathcal{V}^{(1)}_{\phi_i} G^{(2)}_{K_i} \right] + \sum_{i=1}^{2} \left[\mathcal{V}^{(2)}_{K_i} G^{(1)}_{\phi_i} - \mathcal{V}^{(2)}_{\phi_i} G^{(1)}_{K_i} \right] \ , \tag{60b}$$

and

$$\mathcal{H}_{22} = \sum_{i=1}^{2} \left[\mathcal{V}^{(2)}_{K_i} G^{(2)}_{\phi_i} - \mathcal{V}^{(2)}_{\phi_i} G^{(2)}_{K_i} \right] + \mathcal{U} \ , \tag{60c}$$

where, using Eqs. (37b) and (37c), the discomposition

$$\mathcal{V} = \mathcal{V}^{(1)} + \mathcal{V}^{(2)} \tag{61a}$$

has been made, and $G^{(i)}$ for $i = 1, 2$ has been defined as

$$G^{(i)} = - \int_0^s \mathcal{V}^{(i)}(\xi, \phi - \psi(s) + \psi(\xi), K) \ d\xi \ . \tag{61b}$$

For a given m-multipole, it is not difficult to see from Eqs. (31) (with $k = 0$), (37), (41), and (42) the following order of dependence in the action for the second-order terms of the Hamiltonian:

$$\mathcal{O}(\mathcal{H}_{11}) \sim K^{m-1} \ , \tag{62a}$$

$$\mathcal{O}(\mathcal{H}_{12}) \sim K^{m} \ , \tag{62b}$$

and

$$\mathcal{O}(\mathcal{H}_{22}) \sim K^{m+1} \ . \tag{62c}$$

So, at first sight, the most important contribution would come from term (62a). The terms (62c) and (62b) are expected to be very small, and they will be studied in the next section.

5.1 Tune Shift Arising from $\mathcal{H}_{22}$

The easiest term to calculate is $\mathcal{U}$ because its average value is different from zero. Using Eqs. (31a) and (31b) with $k = 0$, substituting Eq. (41) in these expressions, and rearranging terms in the expression resulting from Eq. (37d), it follows that

$$\mathcal{U} = \left(\frac{e}{cp}\right)^2 \frac{2^m(\acute{v}_m)^2}{(m+2)^2} \sum_{\rho=0}^{2\rho+1\leq m} \sum_{\sigma=0}^{2\sigma+1\leq m} (-1)^{\rho+\sigma} \binom{m}{2\rho+1}\binom{m}{2\sigma+1} K_1^{m-\rho-\sigma}$$
$$K_2^{\rho+\sigma+1}\Big\{K_2K_1^{-1}\beta_1^{m-\rho-\sigma-1}\beta_2^{\rho+\sigma+2}(\cos\phi_1)^{2m-2\rho-2\sigma}(\cos\phi_2)^{2\rho+2\sigma+4}+$$
$$+\beta_1^{m-\rho-\sigma}\beta_2^{\rho+\sigma+1}(\cos\phi_1)^{2m-2\rho-2\sigma}(\cos\phi_2)^{2\rho+2\sigma+2}\Big\}\,. \tag{63}$$

With the help of Eqs. (46) and (14), the following average results:

$$<\mathcal{U}>= \left(\frac{e}{cp}\right)^2 \frac{(a\kappa)^2}{2^m 4m^2(m+2)^2} \sum_{\rho=0}^{2\rho+1\leq m} \sum_{\sigma=0}^{2\sigma+1\leq m} (-1)^{\rho+\sigma} \binom{m}{2\rho+1}\binom{m}{2\sigma+1}$$
$$K_1^{m-\rho-\sigma}K_2^{\rho+\sigma+1}\Big\{K_2K_1^{-1}\frac{(2m-2\rho-2\sigma-2)!(2\rho+2\sigma+4)!}{[(m-\rho-\sigma-1)!(\rho+\sigma+2)!]^2}g_{\rho\sigma}^m+$$
$$+\frac{(2m-2\rho-2\sigma)!(2\rho+2\sigma+2)!}{[(m-\rho-\sigma)!(\rho+\sigma+1)!]^2}h_{\rho\sigma}^m\Big\}\,, \tag{64}$$

where the numbers $g_{\rho\sigma}^m$ and $h_{\rho\sigma}^m$ are given by

$$g_{\rho\sigma}^m = \frac{1}{2\pi}\int_0^C \beta_1^{m-\rho-\sigma-1}(s)\beta_2^{\rho+\sigma+2}(s)\cos^2\kappa s\; ds \tag{65a}$$

and

$$h_{\rho\sigma}^m = \frac{1}{2\pi}\int_0^C \beta_1^{m-\rho-\sigma}(s)\beta_2^{\rho+\sigma+1}(s)\cos^2\kappa s\; ds\,. \tag{65b}$$

To estimate these numbers, assume that the accelerator ring contains FODO cells where the maximum and minimum values of the beta functions are given by β_M and β_m,

respectively. Assume also that variation of the beta function in each half-cell is given by

$$\beta_1(s) = \beta_M - \alpha s^2 \tag{66a}$$

and

$$\beta_2(s) = \beta_m + \alpha s^2 \ , \tag{66b}$$

where α is determined in terms of β_M, β_m, and the half-cell length, L, as

$$\alpha = \frac{\beta_M - \beta_m}{L^2} \ . \tag{66c}$$

If N_c is the total number of cells in the ring, then $\mathcal{C} = 2N_cL$, the maximum values of the integrals (Eqs. (65)) are

$$g^m_{\rho\sigma-max} = (-1)^{m-\rho-\sigma-1}\frac{\mathcal{C}(\beta_M - \beta_m)^{m+1}}{4\pi(2m+3)} \tag{67a}$$

and

$$h^m_{\rho\sigma-max} = (-1)^{m-\rho-\sigma}\frac{\mathcal{C}(\beta_M - \beta_m)^{m+1}}{4\pi(2m+3)} \ , \tag{67b}$$

and the average value (Eq. (64)) is given in this approximation as

$$\begin{aligned} <\mathcal{U}> = (-1)^m \Upsilon_m \sum_{\rho=0}^{2\rho+1\leq m} \sum_{\sigma=0}^{2\sigma+1\leq m} \binom{m}{2\rho+1}\binom{m}{2\sigma+1} K_1^{m-\rho-\sigma} K_2^{\rho+\sigma+1} \\ \left\{ -K_2K_1^{-1}\frac{(2m-2\rho-2\sigma-2)!(2\rho+2\sigma+4)!}{[(m-\rho-\sigma-1)!(\rho+\sigma+2)!]^2} + \right. \\ \left. + \frac{(2m-2\rho-2\sigma)!(2\rho+2\sigma+2)!}{[(m-\rho-\sigma)!(\rho+\sigma+1)!]^2} \right\} \ , \end{aligned} \tag{68}$$

where the parameter Υ_m is defined, using Eq. (51), as

$$\Upsilon_m = \frac{r_p}{mc^2\gamma^2}\frac{(a\kappa)^2\mathcal{C}(\beta_m - \beta_m)^{m+1}}{16\pi 2^m m^2(m+2)^2(2m+3)} \ . \tag{69}$$

The action variable is essentially the emittance, ϵ, of a beam of particles circulating in the accelerator, which is related to the normalized emittance, ϵ_N, by a γ factor. So, it is

possible to make the following identification:

$$K_i = \frac{\epsilon_N}{2\gamma} \,. \tag{70}$$

The tune shift is then given by

$$|\Delta\nu_1^m| \leq \left|\frac{\partial \mathcal{U}}{\partial K_1}\right|_{K_i=\epsilon_N/2\gamma} = \frac{\Upsilon_m \epsilon_N Q_{m1}}{(2\gamma)^m} \tag{71a}$$

and

$$|\Delta\nu_2^m| \leq \left|\frac{\partial \mathcal{U}}{\partial K_2}\right|_{K_i=\epsilon_N/2\gamma} = \frac{\Upsilon_m \epsilon_N Q_{m2}}{(2\gamma)^m} \,, \tag{71b}$$

where the factors Q_{m1} and Q_{m2} are defined by

$$\begin{aligned} Q_{m1} = & \sum_{\rho=0}^{2\rho+1\leq m} \sum_{\sigma=0}^{2\sigma+1\leq m} (-1)^{\rho+\sigma} \binom{m}{2\rho+1} \binom{m}{2\sigma+1} \times \\ & \times \left\{ \frac{(\rho+\sigma+1)(2m-2\rho-2\sigma)!(2\rho+2\sigma+2)!}{[(m-\rho-\sigma)!(\rho+\sigma+1)!]^2} + \right. \\ & \left. - \frac{(\rho+\sigma+2)(2m-2\rho-2\sigma-2)!(2\rho+2\sigma+4)!}{[(m-\rho-\sigma-1)!(\rho+\sigma+2)!]^2} \right\} \end{aligned} \tag{72a}$$

and

$$\begin{aligned} Q_{m2} = & \sum_{\rho=0}^{2\rho+1\leq m} \sum_{\sigma=0}^{2\sigma+1\leq m} (-1)^{\rho+\sigma} \binom{m}{2\rho+1} \binom{m}{2\sigma+1} \times \\ & \times \left\{ \frac{(m-\rho-\sigma)(2m-2\rho-2\sigma)!(2\rho+2\sigma+2)!}{[(m-\rho-\sigma)!(\rho+\sigma+1)!]^2} + \right. \\ & \left. - \frac{(m-\rho-\sigma-1)(2m-2\rho-2\sigma-2)!(2\rho+2\sigma+4)!}{[(m-\rho-\sigma-1)!(\rho+\sigma+2)!]^2} \right\} \,. \end{aligned} \tag{72b}$$

In order to compare the relative effect of different multipoles, the quantity $\Delta\nu_1^m/\Delta\nu_1^n$ must be estimated. From Eq. (71a), it follows that

$$\left|\frac{\Delta\nu_1^m}{\Delta\nu_1^n}\right| = \left[\frac{(\beta_M - \beta_m)\epsilon_N}{2\gamma}\right]^{m-n} \left(\frac{n^2+2n}{m^2+2m}\right)^2 \left(\frac{2n+3}{2m+3}\right) \left|\frac{Q_{m1}}{Q_{n1}}\right| \,, \tag{73}$$

which shows the expected behavior. For higher-order multipoles ($m - n > 0$), the tune shift is smaller (due mainly to the γ factor). As a result, the dipole longitudinal oscillation

pattern would cause the highest value in Eqs. (71). For the nominal values of the Collider ($\epsilon_N = 10^{-4}$ cm, $\gamma_{injection} = 2000$, $\mathcal{C} = 8712000$ cm, $L = 9000$ cm, $\beta_M - \beta_m = 25000$ cm), the tune shift due to the dipole oscillation is

$$\left|\Delta\nu_1^1\right|/a^2 \simeq \left|\Delta\nu_2^1\right|/a^2 \leq 10^{-13}\,\text{Gauss}^{-2}\ . \tag{74}$$

Because the other term contained in $\mathcal{H}_{22}$ will have the same order value in the tune shift, it will not be analyzed. Thus, the term $\mathcal{H}_{22}$ contributes negligibly to the tune shift.

5.2 Tune Shift Arising from $\mathcal{H}_{12}$

Detailed calculations of the tune shift are presented in Appendix C. Here, the results will be presented, and an approximation will be done to estimate its value. The quadrupole part does not appear here because it was extracted from $\mathcal{V}^{(1)}$ for the calculation at first order. The dipole longitudinal oscillation pattern calculations in Appendix C yield the following expression for the averaged Hamiltonian:

$$< \mathcal{H}_{12}^{m=1} >= \left(\frac{e}{cp}\right)^2 \frac{K_2\Upsilon}{3}\ , \tag{75}$$

where Υ is given in terms of Eqs. (C6) of Appendix C, as

$$\Upsilon = W_1^{0\acute{0}} + W_1^{0\acute{2}} - W_2^{1\acute{0}} - W_2^{1\acute{2}} + W_4^{1\acute{0}} + W_{4;1}^{1\acute{2}} + \Gamma_{0;1} + \Gamma_{1;2} - \Gamma_{1;4}\ . \tag{76}$$

The tune shift is then given by

$$\Delta\nu_1 = 0 \tag{77a}$$

and

$$\Delta\nu_2 = \frac{r_p\Upsilon}{mc^2\gamma^2 3}\ . \tag{77b}$$

If the same approximation in Section 5.1 is used here, the following relations result:

$$|\Gamma_{0;1}| = \left|W_1^{0\acute{0}}\right| = \left|W_1^{0\acute{2}}\right| = \frac{a^2(\beta_M - \beta_m)\mathcal{C}L}{8\lambda}\ ; \tag{78}$$

if the same values for the Collider in Section 4.0 are used, it follows that

$$|\Delta\nu_2|/a^2 < \frac{r_p}{mc^2\gamma^2}\frac{(\beta_M - \beta_m)\mathcal{C}L}{8\lambda} \approx 2.6\times 10^{-8}\,\text{Gauss}^{-2}\ . \tag{79}$$

Because the sextupole contribution is expected to be even smaller than this value, it will not be calculated. However, the functions $\mathcal{V}^{(i)}$ and $G^{(i)}$ necessary to perform these calculations are given in Appendix C.

As a result, the term $\mathcal{H}_{12}$ also has a negligible contribution to the tune shift.

5.3 Tune Shift Arising from $\mathcal{H}_{11}$

This section will be a little more developed than the previous one because an uncertainty in the tune shift is found when the same approximation of Section 5.1 is used in the results (see below). First, as seen in Appendix D, the dipole longitudinal oscillation pattern does not contribute to the tune shift since its associated average Hamiltonian does not depend on the action variables. The quadrupole part of $\mathcal{V}^{(1)}$ was extracted and considered in Section 4.0. Therefore, there is no quadrupole contribution. However, the sextupole longitudinal oscillation pattern has a quite small contribution which will be discussed below.

From Eqs. (34c), (37b), (41), and (61b), it follows that

$$\mathcal{V}^{(1)}_{m=3} = \left(\frac{e}{pc}\right) 2^{3/2} v \left[(K_1\beta_1)^{3/2}\cos^3\phi_1 - 3(K_1\beta_1)^{1/2}K_2\beta_2\cos\phi_1\cos^2\phi_2\right] \tag{80a}$$

and

$$G^{(1)} = -\left(\frac{e}{cp}\right) 2^{3/2}\Bigg\{K_1^{3/2}\sum_{\rho=0}^{3}\binom{3}{\rho}\cos^{3-\rho}\phi_1\sin^{\rho}\phi_1\; g^{\rho}_{11}(s) + \\ - 3K_1^{1/2}K_2\sum_{\rho=0}^{1}\sum_{\acute{\rho}=0}^{2}\binom{2}{\acute{\rho}}\cos^{1-\rho}\phi_1\sin^{\rho}\phi_1\cos^{2-\acute{\rho}}\phi_2\sin^{\acute{\rho}}\phi_2\; g^{\rho\acute{\rho}}_{12}(s)\Bigg\}\,, \tag{80b}$$

where the functions g^{ρ}_{11} and $g^{\rho\acute{\rho}}_{12}$ are defined by

$$g^{\rho}_{11}(s) = \int_0^s v(\xi)\beta_1^{3/2}(\xi)\cos^{3-\rho}\delta_1\sin^{\rho}\delta_1\; d\xi \tag{81a}$$

and

$$g^{\rho\acute{\rho}}_{12}(s) = \int_0^s v(\xi)\beta_1^{1/2}(\xi)\beta_2(\xi)\cos^{1-\rho}\delta_1\sin^{\rho}\delta_1\cos^{2-\acute{\rho}}\delta_2\sin^{\acute{\rho}}\delta_2\; d\xi\,, \tag{81b}$$

and the following identity has been used:

$$\cos^n(\phi_i - \delta_i) = \sum_{\rho=0}^{n}\binom{n}{\rho}\cos^{n-\rho}\phi_i\sin^{\rho}\phi_i\cos^{n-\rho}\delta_i\sin^{\rho}\delta_i\,, \tag{82}$$

where δ_i is defined as

$$\delta_i(s,\xi) = \psi_i(s) - \psi_i(\xi) \ , \ i = 1,2 \ . \tag{83}$$

Doing the derivation of Eqs. (80), the following average values are obtained after making some rearrangements:

$$\begin{aligned} < \mathcal{V}^{(1)}_{\phi_1} G^{(1)}_{K_1} > = & - \left(\frac{e}{cp}\right)^2 2^3 \Bigg\{ -\frac{9}{2} K_1^2 \sum_{\rho=0}^{3} \binom{3}{\rho} (5-\rho|\rho+1) \ q^{\rho}_{11;1} + \\ & + \frac{9}{2} K_1 K_2 \sum_{\rho=0}^{1} \sum_{\acute{\rho}=0}^{3} \binom{2}{\acute{\rho}} (3-\rho|\rho+1)(2-\acute{\rho}|\acute{\rho}) \ Q^{\rho\acute{\rho}}_{12;1} + \\ & + \frac{9}{2} K_1 K_2 \sum_{\rho=0}^{3} \binom{3}{\rho} (3-\rho|\rho+1)(2|0) \ q^{\rho}_{11;2} + \\ & - \frac{9}{2} K_2^2 \sum_{\rho=0}^{1} \sum_{\acute{\rho}=0}^{2} \binom{2}{\acute{\rho}} (1-\rho|\rho+1)(4-\acute{\rho}|\acute{\rho}) \ Q^{\rho\acute{\rho}}_{12;2} \Bigg\} \ , \end{aligned} \tag{84a}$$

$$\begin{aligned} < \mathcal{V}^{(1)}_{\phi_2} G^{(1)}_{K_2} > = & - \left(\frac{e}{cp}\right)^2 2^3 \Bigg\{ 18 K_1 K_2 \sum_{\rho=0}^{1} \sum_{\acute{\rho}=0}^{2} \binom{2}{\acute{\rho}} (2-\rho|\rho) \times \\ & \times (3-\acute{\rho}|\acute{\rho}+1) \ Q^{\rho\acute{\rho}}_{12;2} \Bigg\} \ , \end{aligned} \tag{84b}$$

$$\begin{aligned} < \mathcal{V}^{(1)}_{K_1} G^{(1)}_{\phi_1} > = & - \left(\frac{e}{cp}\right)^2 2^3 \times \\ & \times \Bigg\{ \frac{3}{2} K_1^2 \sum_{\rho=0}^{3} \binom{3}{\rho} [-(3-\rho)(5-\rho|\rho+1) + \rho(7-\rho|\rho-1)] \ q^{\rho}_{11;1} + \\ & - \frac{9}{2} K_1 K_2 \sum_{\rho=0}^{1} \sum_{\acute{\rho}=0}^{2} \binom{2}{\acute{\rho}} [-(1-\rho)(3-\rho|\rho+1) + \rho(5-\rho|\rho-1)](2-\acute{\rho}|\acute{\rho}) \ Q^{\rho\acute{\rho}}_{12;1} + \\ & - \frac{3}{2} K_1 K_2 \sum_{\rho=0}^{3} \binom{3}{\rho} [-(3-\rho)(3-\rho|\rho+1) + \rho(5-\rho|\rho-1)](2|0) \ q^{\rho}_{11;2} + \\ & + \frac{9}{2} K_2^2 \sum_{\rho=0}^{1} \sum_{\acute{\rho}=0}^{2} \binom{2}{\acute{\rho}} [-(1-\rho)(1-\rho|\rho+1) + \rho(3-\rho|\rho-1)](4-\acute{\rho}|\acute{\rho}) \ Q^{\rho\acute{\rho}}_{12;2} \Bigg\} \ , \end{aligned} \tag{84c}$$

and

$$< \mathcal{V}_{K_2}^{(1)} G_{\phi_2}^{(1)} > = -\left(\frac{e}{cp}\right)^2 2^3 \Big\{ 9K_1K_2 \sum_{\rho=0}^{1} \sum_{\acute{\rho}}^{2} \binom{2}{\acute{\rho}} (2-\rho|\rho) \times$$

$$\times \left[-(2-\acute{\rho})(3-\acute{\rho}|\acute{\rho}+1) + \acute{\rho}(5-\acute{\rho}|\acute{\rho}-1)\right] Q_{12;2}^{\rho\acute{\rho}} \Big\} , \qquad (84d)$$

where the following definitions have been made:

$$q_{11;1}^{\rho} = \frac{1}{2\pi} \int_0^C v(s)\beta_1^{3/2}(s) g_{11}^{\rho}(s)\, ds , \qquad (85a)$$

$$q_{11;2}^{\rho} = \frac{1}{2\pi} \int_0^C v(s)\beta_1^{1/2}(s)\beta_2(s) g_{11}^{\rho}(s)\, ds , \qquad (85b)$$

$$Q_{12;1}^{\rho\acute{\rho}} = \frac{1}{2\pi} \int_0^C v(s)\beta_1^{3/2}(s) g_{12}^{\rho\acute{\rho}}(s)\, ds , \qquad (85c)$$

$$Q_{12;2}^{\rho\acute{\rho}} = \frac{1}{2\pi} \int_0^C v(s)\beta_1^{1/2}(s)\beta_2(s) g_{12}^{\rho\acute{\rho}}(s)\, ds , \qquad (85d)$$

and

$$(n|m) = \frac{1}{2\pi} \int_0^{2\pi} \cos^n\theta \sin^m\theta \, d\theta . \qquad (85e)$$

The last definition clearly has the following properties:

$$(n|0) = \frac{1}{2^n} \frac{n!}{\left(\frac{n}{2}!\right)^2} \delta_{n,2\mathcal{N}} \qquad (86a)$$

and

$$(n|m) = 0, \text{ wherever } n \text{ or } m \text{ is odd} . \qquad (86b)$$

Using Eqs. (84), the average value of Eq. (60a) is given by

$$< \mathcal{H}_{11}^{m=3} > = -\left(\frac{e}{cp}\right)^2 \Big\{ K_1^2 \left[27 q_{11;1}^{1} - 9 q_{11:1}^{3}\right] +$$

$$+K_1K_2\left[-9Q^{10}_{12;1}+18Q^{12}_{12;1}-9q^{1}_{11;2}-9q^{3}_{11;2}+108Q^{01}_{12;2}\right]+$$

$$\left.+K_2^2\left[27Q^{10}_{12;2}-9Q^{12}_{12;2}\right]\right\}. \tag{87}$$

Thus, the tune shift is given by the partial derivation of this expression:

$$\left(\Delta\nu_1^{m=3}\right)_{K_i=\epsilon_N/2\gamma}=\frac{r_p\epsilon_N\lambda_1}{mc^2\gamma^3} \tag{88a}$$

and

$$\left(\Delta\nu_2^{m=3}\right)_{K_i=\epsilon_N/2\gamma}=\frac{r_p\epsilon_N\lambda_2}{mc^2\gamma^3}\ , \tag{88b}$$

where the identity in Eq. (51) has been used, and λ_1, λ_2 are defined as

$$\lambda_1=-27q^{1}_{11;1}+9q^{3}_{11;1}+\frac{9}{2}Q^{10}_{12;1}-9Q^{12}_{12;1}+\frac{9}{2}q^{1}_{11;2}+\frac{9}{2}q^{3}_{11;2}-54Q^{01}_{12;2} \tag{89a}$$

$$\lambda_2=9Q^{10}_{12;1}-18Q^{12}_{12;1}+9q^{1}_{11;2}+9q^{3}_{11;2}-108Q^{01}_{12;2}-27Q^{10}_{12;2}+9Q^{12}_{12;2}\ . \tag{89b}$$

If the same approximation made in Section 5.0 were done here, the tune shift would have quite a large value:

$$2|\Delta\nu_1|/a^2\approx|\Delta\nu_2|/a^2<0.0012\,\mathrm{Gauss}^{-2}\mathrm{cm}^4\ .$$

Hence, this approximation does not produce a confident low value for this case, and numerical integration of Eqs. (85) must be carried out to obtain a confident value. Using the same procedure as in Section 4.0, Table 3 shows the results of these integrations. (The same approximation was made for the betatron phases.)

As can be seen from these numerical values, the term $\mathcal{H}_{11}$ is not important to the tune shift of the machine.

5.4 Symmetry Considerations

Suppose that the beta function is an even function with respect to the initial reference point "0" in an accelerator ring of length $\mathcal{C}$; that is,

$$\beta_i(-s)=\beta_i(s)\ . \tag{90}$$

In a FODO cell structure, there are in fact three points where this happens, corresponding to the location of the three quadrupoles which make up the FODO structure. In this case,

TABLE 3. NUMERICAL INTEGRATION OF q, Q, AND λ, AND CORRESPONDING TUNE SHIFTS.

Numerical Integration		
25×	SSC-Collider	SSC-HEB
$q^1_{11;1}$	-1.546×10^{15}	-5.555×10^{12}
$q^1_{11;2}$	-8.926×10^{14}	-1.734×10^{12}
$q^3_{11;1}$	-8.500×10^{14}	$+11.477 \times 10^{15}$
$q^3_{11;2}$	-3.363×10^{14}	$+8.616 \times 10^{15}$
$Q^{10}_{12;1}$	-9.072×10^{14}	-1.719×10^{12}
$Q^{12}_{12;1}$	-2.057×10^{11}	$+2.141 \times 10^{10}$
$Q^{10}_{12;2}$	-1.546×10^{15}	-5.479×10^{12}
$Q^{12}_{12;2}$	-7.227×10^{10}	$+1.158 \times 10^{10}$
λ_1	$+5.660 \times 10^{17}$	$+1.422 \times 10^{17}$
λ_2	$+1.895 \times 10^{17}$	$+7.805 \times 10^{16}$
Tune Shifts*		
$\Delta\nu_1/a^2$	2.7×10^{-11}	6.82×10^{-9}
$\Delta\nu_2/a^2$	9.1×10^{-12}	3.74×10^{-9}

* Amplitude of Oscillations, a, in Gauss/cm^2.

from Eq. (58) it follows that

$$\psi_i(-s) = \int_0^{-s} \frac{d\sigma}{\beta_i(\sigma)} = -\int_0^{s} \frac{d\sigma}{\beta_i(-\sigma)} = -\int_0^{s} \frac{d\sigma}{\beta_i(\sigma)} = -\psi_i(s) \; ; \tag{91}$$

that is, ψ_i is an odd function. Furthermore, from Eqs. (83) and (91), the function δ_i is also odd in the following sense:

$$\delta_i(-s,-\xi) = \psi_i(-s) - \psi_i(-\xi) = -[\psi_i(s) - \psi_i(\xi)] = -\delta_i(s,\xi) \; . \tag{92}$$

Therefore, the following symmetry is obtained for Eqs. (81):

$$g^{\rho}_{11}(-s) = (-1)^{\rho} \; g^{\rho}_{11}$$

and

$$g^{\rho\dot{\rho}}_{12}(-s) = (-1)^{\rho+\dot{\rho}} g^{\rho\dot{\rho}}_{12} \; .$$

Using these symmetries in Eqs. (85), it follows that

$$q^{\rho}_{11;i} = (-1)^{\rho} q^{\rho}_{11;i}$$

and

$$Q^{\rho\acute{\rho}}_{12;i} = (-1)^{\rho+\acute{\rho}} Q^{\rho\acute{\rho}}_{12;i} \ .$$

Therefore, if the symmetry in Eq. (90) were established in the accelerator ring, the tune shift of Eqs. (88) would be exactly zero. But, note that the Collider and the HEB do not have this symmetry, even though the contribution of the term $\mathcal{H}_{11}$ can be ignored.

6.0 DISTORTION OF THE INVARIANT CURVES AND ACTION BLOW-UP

The main contribution to the tune shift comes from the quadrupole oscillation pattern; this will also cause a distortion in the invariant curves, K_i, given by Eq. (53a). In order to compute this distortion, the generating function, G, must be known. With the help of Eqs. (47) and (57), the generating function is given by

$$G = -\left(\frac{e}{cp}\right) a \sum_{\rho=0}^{2} \binom{2}{\rho} \left\{K_1 \cos^{2-\rho}\phi_1 \sin^{\rho}\phi_1 \ g_{1\rho}(s) - K_2 \cos^{2-\rho}\phi_2 \sin^{\rho}\phi_2 \ g_{2\rho}(s)\right\} \ , \quad (93)$$

where the functions $g_{1\rho}$ and $g_{2\rho}$ are defined with the help of Eq. (83) as

$$g_{1\rho}(s) = \int_0^s \beta_1(\xi) \sin\kappa\xi \cos^{2-\rho}\delta_1 \sin^{\rho}\delta_1 \ d\xi \qquad (94a)$$

and

$$g_{2\rho}(s) = \int_0^s \beta_2(\xi) \sin\kappa\xi \cos^{2-\rho}\delta_2 \sin^{\rho}\delta_2 \ d\xi \ . \qquad (94b)$$

The partial differentiation of Eq. (93) with respect to the betatron phases is

$$G_{\phi_i} = -\left(\frac{e}{cp}\right) 2aK_i \left\{(\Delta g)_i(s) \sin 2\phi_i + g_{1i}(s) \cos 2\phi_i\right\} \ , \text{ for } i = 1, 2 \ , \qquad (95)$$

where $(\Delta g)_i$ is the difference $g_{12}(s) - g_{10}(s)$. Using Eq. (51), the variation in the invariant curve can be written as

$$\Delta J_i(s) = -\frac{2aK_i}{\gamma}\sqrt{\frac{r_p}{mc^2}} \left\{(\Delta g)_i(s) \sin 2\phi_i + g_{1i}(s) \cos 2\phi_i\right\} \ . \qquad (96)$$

The functions (Eqs. (94)) have the same period of oscillation as the longitudinal oscillation pattern. This fast oscillations are modulated by the betatron oscillation. So, taking the

mean value of the beta function, $\bar{\beta}$, the following approximated behavior is obtained:

$$(\Delta g_i)(s) \approx \frac{\bar{\beta}\lambda}{\pi} \cos \frac{2\pi s}{\lambda} [\sin \psi_i(s) - \cos \psi_i(s) - 1] \tag{97a}$$

and

$$g_{1i}(s) \approx \frac{\bar{\beta}\lambda}{4\pi} [\cos 2\psi_i(s) - \sin 2\psi_i(s)] \ , \tag{97b}$$

which have the maximum values

$$(\Delta g_i)_{max} = \frac{2\bar{\beta}\lambda}{\pi}$$

and

$$(g_{1i})_{max} = \frac{\bar{\beta}\lambda}{4\pi} \ .$$

Assuming that this maximum occurs at $s = s_*$, the deformation of the invariant curve at this point, for the Collider at injection and using Eq. (70), is given by

$$(\Delta J_i)_* = -a\bar{\xi} \left\{ 2 \sin 2\phi_i + \frac{1}{4} \cos 2\phi_i \right\} \ , \tag{98}$$

where the amplitude of the longitudinal quadrupole oscillations, a, is given in Gauss/cm, and the parameter $\bar{\xi}$ is

$$\bar{\xi} = \frac{2\epsilon_N \bar{\beta}\lambda}{\pi\gamma^2} \sqrt{\frac{r_p}{mc^2}} \approx 2.79 \times 10^{-15} \text{ Gauss}^{-1}\text{cm}^2 \ .$$

This number shows that this deformation can be completely ignored. The distortion given by Eq. (98) is shown in Figure 4.

6.1 Resonant Structure

The resonant structure of the Hamiltonian is given in perturbation theory by the behavior of the generating function (Eq. (57)). If there is a value in the parameter for which the function G_{ϕ_i} is unbounded, the action of the particle (Eq. (53a)) will be unbounded, too, and the particle will eventually leave the physical region in the phase space. For the

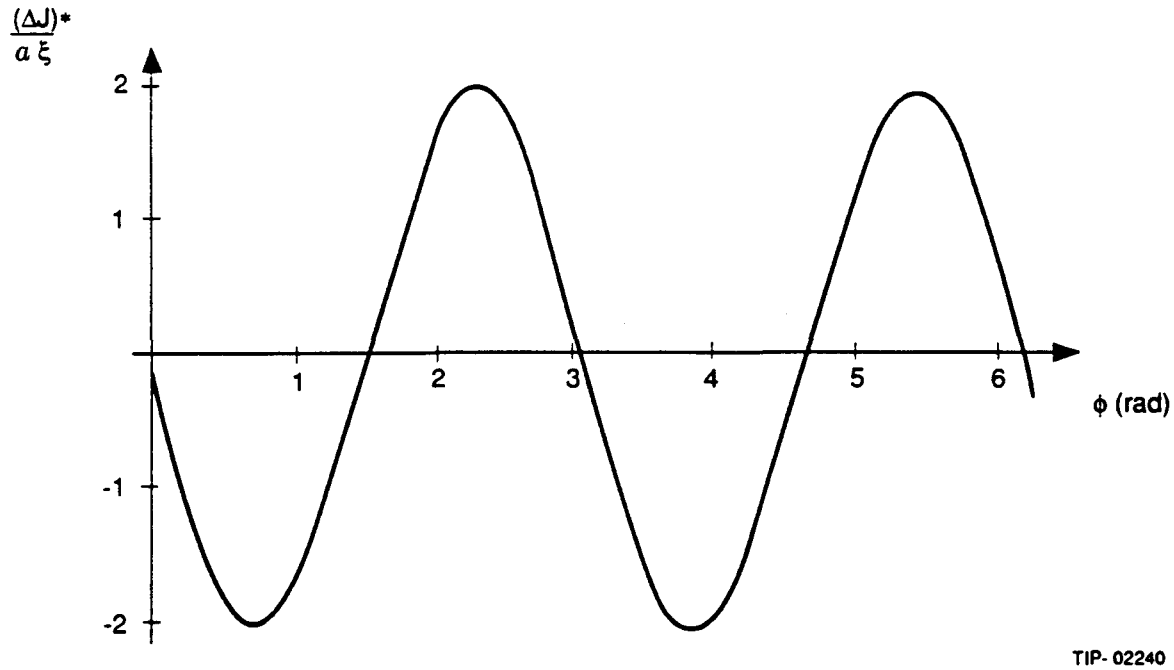

Figure 4. Distortion Caused by the Quadrupole Oscillation Pattern.

quadrupole oscillation pattern, the expressions of interest are Eqs. (93) and (95). From Eqs. (94), it is not difficult to obtain the following expressions:

$$(\Delta g)_1(s) = 2Re \sum_n \beta_{1n} \left\{ \int_0^s e^{i\frac{2\pi n}{C}\xi} \sin\kappa\xi \; d\xi + \right.$$

$$\sin\psi_1(s) \int_0^s e^{i\frac{2\pi n}{C}\xi} \sin\kappa\xi \cos\psi_1(\xi) \; d\xi +$$

$$\left. + \cos\psi_1(s) \int_0^s e^{i\frac{2\pi n}{C}\xi} \sin\kappa\xi \sin\psi_1(\xi) \; d\xi \right\} \tag{99a}$$

and

$$g_{11}(s) = \frac{1}{2} Re \sum_n \beta_{1n} \left\{ \sin 2\psi_1(s) \int_0^s e^{i\frac{2\pi n}{C}\xi} \sin\kappa\xi \cos 2\psi_1(\xi) + \right.$$

$$\left. - \cos 2\psi_1(\xi) \int_0^s e^{i\frac{2\pi n}{C}\xi} \sin\kappa\xi \sin 2\psi_1(\xi) d\xi \right\} , \tag{99b}$$

where $\mathcal{C}$ is the circumference of the ring, β_{1n} are complex numbers, and the Fourier expansion of the beta function has been used:

$$\beta_1(\xi) = Re \sum_n \beta_{1n} e^{i\frac{2\pi n}{\mathcal{C}}\xi} \ . \tag{100}$$

Making the change of variable

$$\theta = \xi/\mathcal{C} \ ,$$

considering $\theta\mathcal{C} \ll 1$, which allows the approximation

$$\psi_1(\theta\mathcal{C}) = 2\pi\nu_1\theta \ ,$$

and integrating, the following expression results:

$$\begin{aligned}(\Delta g)_1(s) = \frac{\mathcal{C}}{4\pi} Re \sum_n \beta_{1n} \Bigg\{ & 4\frac{e^{i2\pi(n+\mathcal{C}/\lambda)s/\mathcal{C}} - 1}{n+\mathcal{C}/\lambda} - 4\frac{e^{i2\pi(n-\mathcal{C}/\lambda)s/\mathcal{C}} - 1}{n-\mathcal{C}/\lambda} + \\ & + \sin\psi_1(s) \Bigg[\frac{e^{i2\pi(n-\mathcal{C}/\lambda+\nu_1)s/\mathcal{C}} - 1}{n-\mathcal{C}/\lambda+\nu_1} + \frac{e^{i2\pi(n-\mathcal{C}/\lambda-\nu_1)s/\mathcal{C}} - 1}{n-\mathcal{C}/\lambda-\nu_1} + \\ & - \frac{e^{i2\pi(n+\mathcal{C}/\lambda+\nu_1)s/\mathcal{C}} - 1}{n+\mathcal{C}/\lambda+\nu_1} - \frac{e^{i2\pi(n+\mathcal{C}/\lambda-\nu_1)s/\mathcal{C}} - 1}{n+\mathcal{C}/\lambda-\nu_1} \Bigg] + \\ & + \cos\psi_1(s) \Bigg[\frac{e^{i2\pi(n-\mathcal{C}/\lambda+\nu_1)s/\mathcal{C}} - 1}{n-\mathcal{C}/\lambda+\nu_1} - \frac{e^{i2\pi(n-\mathcal{C}/\lambda-\nu_1)s/\mathcal{C}} - 1}{n-\mathcal{C}/\lambda-\nu_1} + \\ & - \frac{e^{i2\pi(n+\mathcal{C}/\lambda+\nu_1)s/\mathcal{C}} - 1}{n+\mathcal{C}/\lambda+\nu_1} + \frac{e^{i2\pi(n+\mathcal{C}/\lambda-\nu_1)s/\mathcal{C}} - 1}{n+\mathcal{C}/\lambda-\nu_1} \Bigg] \Bigg\}\end{aligned} \tag{101a}$$

and

$$\begin{aligned} g_{11}(s) = \frac{\mathcal{C}}{16\pi} Re \sum_n \beta_{1n} \Bigg\{ & \sin 2\psi_1(s) \Bigg[\frac{e^{i2\pi(n-\mathcal{C}/\lambda+2\nu_1)s/\mathcal{C}} - 1}{n-\mathcal{C}/\lambda+2\nu_1} + \frac{e^{i2\pi(n-\mathcal{C}/\lambda-2\nu_1)s/\mathcal{C}} - 1}{n-\mathcal{C}/\lambda-2\nu_1} + \\ & - \frac{e^{i2\pi(n+\mathcal{C}/\lambda+2\nu_1)s/\mathcal{C}} - 1}{n+\mathcal{C}/\lambda+2\nu_1} - \frac{e^{i2\pi(n+\mathcal{C}/\lambda-2\nu_1)s/\mathcal{C}} - 1}{n+\mathcal{C}/\lambda-2\nu_1} \Bigg] + \end{aligned}$$

$$- \cos 2\psi_1(s) \left[\frac{e^{i2\pi(n-\mathcal{C}/\lambda+2\nu_1)s/\mathcal{C}} - 1}{n - \mathcal{C}/\lambda + 2\nu_1} - \frac{e^{i2\pi(n-\mathcal{C}/\lambda-2\nu_1)s/\mathcal{C}} - 1}{n - \mathcal{C}/\lambda - 2\nu_1} + \right.$$

$$\left. - \frac{e^{i2\pi(n+\mathcal{C}/\lambda+2\nu_1)s/\mathcal{C}} - 1}{n - \mathcal{C}/\lambda + 2\nu_1} + \frac{e^{i2\pi(n+\mathcal{C}/\lambda-2\nu_1)s/\mathcal{C}} - 1}{n + \mathcal{C}/\lambda - 2\nu_1} \right] \Bigg\} . \tag{101b}$$

Since, in general, $\mathcal{C}/\lambda$ is a very large number (about 10^6 for the Collider), and the tune of the machine is relatively small (about 123 for the Collider), the denominators in Eqs. (101) do not cause any problem to the dynamic of the beam. Notice that $n + \mathcal{C}/\lambda - 2\nu_1 \to 0$ does not mean that there is a singularity in these functions, but that the action associated with the motion of the particle will start to blow up linearly. To see this fact in more detail, assume that the relation $n + \mathcal{C}/\lambda - 2\nu_1 \approx 0$ is satisfied. Thus, extracting the most important term of Eqs. (101), the expression in Eq. (95) would be given by

$$G_{\phi_1} = \left(\frac{e}{cp}\right) \frac{aK_1\mathcal{C}}{8\pi} \cos 2\phi_1 \left[\sin 2\psi_1(s) + \cos 2\psi_1(s)\right] \times$$

$$\times \, Re \sum_n \beta_{1n} \frac{e^{i2\pi(n+\mathcal{C}/\lambda-2\nu_1)s/\mathcal{C}} - 1}{n + \mathcal{C}/\lambda - 2\nu_1} . \tag{102}$$

Taken to the limit when $n + \mathcal{C}/\lambda - 2\nu_1 \to 0$,

$$\lim_{n+\mathcal{C}/\lambda-2\nu_1 \to 0} \frac{e^{i2\pi(n+\mathcal{C}/\lambda-2\nu_1)s/\mathcal{C}} - 1}{n + \mathcal{C}/\lambda - 2\nu_1} = i2\pi s/\mathcal{C} ,$$

the following expression arises:

$$G_{\phi_1} = \left(\frac{e}{cp}\right) \frac{aK_1\tilde{\beta}_1}{4} \cos 2\phi_1 \; \left[\sin 2\psi_1(s) + \cos 2\psi_1(s)\right] s , \tag{103a}$$

where $\tilde{\beta}_1$ is given by

$$\tilde{\beta}_1 = Re \sum_n i\beta_{1n} . \tag{103b}$$

The function in Eq. (103a) diverges linearly, which is consistent with the condition in Eq. (B6) of Appendix B. So, the phase space of the particle will blow up. This is a general feature, within the above approximation, that is not difficult to demonstrate for any resonant value.

7.0 CONCLUSION AND COMMENTS

The magnetic field which generates the longitudinal multipole oscillation pattern was calculated, as was the vector potential. In these calculations, the curvature was neglected to make the estimations much easier. It is not difficult to see that introduction of the curvature does not make a significant difference in the results. However, the calculations are much more difficult.

The tune shift analysis was made at a first and a second order in perturbation strength. The only longitudinal multipole oscillation pattern which cannot be neglected in the SSC machine is the quadrupole. The result of Section 4.0 suggests that the tune shift induced in the HEB and Collider machines by the quadrupole oscillation pattern could be at least of the order 10^{-5} and 10^{-6}, respectively, (assuming random oscillation amplitudes from magnet to magnet), at injection energy for a synchronous particle.

The sextupole oscillation pattern does not affect a synchronous particle, but it has a first-order, systematic contribution to an off-momentum particle, in the region where the "dispersion" is non-zero. The contribution is about a factor of $(\delta p/p)D_{max}$ the previously mentioned quadrupole contribution, where D_{max} is the maximum dispersion in the lattice (about 280 cm for the Collider). So, even for a very large off-momentum particle, $\delta p/p \sim 10^{-3}$, the induced tune shift is one order of magnitude lower.

The value obtained in Section 4.0 depends on the quadrupole amplitude of oscillation, which is not yet known for the SSC dipole magnets, and on its wavelength. The amplitude of the quadrupole oscillations seems to be of the same order of the tolerances. In addition, experiments with the HERA dipole magnets indicate that this parameter also depends on time. This parameter may be important for a final estimation, and experiments on the SSC dipole magnets are required to determine completely the behavior of this parameter.

Finally, if the long-term effect of the longitudinal periodic pattern in the dynamics of the particle were required (dynamics aperture), a 3-D tracking code would be needed. This would be important mainly at injection energy in the Collider.

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APPENDIX A. Consistency of the System (Eqs. (25))

Defining Eqs. (25) $2l+1 = 2k-1$, this system can be written as

$$\acute{a}_{2l+1} - (1l+m+2)q_{2l+2} = mv_{m+2l+2} \ , \tag{A1a}$$

$$(2l+m+4)h_{2l+3} + ma_{2l+3} = \acute{v}_{m+2l+2} \ , \tag{A1b}$$

and

$$mq_{2l+2} + \acute{h}_{2l+1} = -(m+2l+2)v_{m+2l+2} \ . \tag{A1c}$$

Taking the derivative of Eq. (A1b) with respect to s, it follows that

$$(2l+m+4)\acute{h}_{2l+3} + ma_{2l+3} = \ddot{v}_{m+2l+2} \ . \tag{A2}$$

Now, using Eqs. (A1a) and (A1c) in Eq. (A2), the next expression is obtained after making some rearrangements:

$$[m^2 - (m+2l+4)^2]v_{m+2l+4} = \ddot{v}_{m+2l+2} \ . \tag{A3}$$

But, from Eq. (22), we obtain

$$\ddot{v}_{m+2l+2} = \frac{2^{2l+4}v^{2l+4}}{\prod_{j=1}^{l+1}[-(m+2j)^2+m^2]} =$$

$$= \frac{[m^2-(m+2l+4)^2]v^{2l+4}2^{2l+4}}{\prod_{j=1}^{l+2}[-(m+2j)^2+m^2]} = [m^2-(m+2l+4)^2]v_{m+2l+4} \ .$$

That is, Eq. (A3) is an identity. Hence, the system (Eqs. (A1)) is self-consistent.

APPENDIX B. Solution of Equation (56)

The partial differential equation, Eq. (56):

$$\sum_{i=1}^{2} \frac{1}{\beta_i(s)} \frac{\partial G}{\partial \phi_i} + \frac{\partial G}{\partial s} = -\mathcal{V}(s, \phi_1, \phi_2) \ , \tag{B1}$$

can be solved by the characteristic method,[10] by which the equations for the characteristics are

$$\beta_1(s)\ d\phi_1 = \beta_2(s)\ d\phi_2 = ds = -\frac{dG}{\mathcal{V}(s, \phi_1, \phi_2)} \ . \tag{B2}$$

From the first three terms, the following characteristic curves are obtained:

$$C_1 = \phi_1 - \psi_1(s) \tag{B3a}$$

and

$$C_2 = \phi_2 - \psi_2(s) \ , \tag{B3b}$$

where the functions $\psi_i,\ i = 1, 2$, are defined as

$$\psi_i(s) = \int_0^s \frac{d\sigma}{\beta_i(\sigma)} \ . \tag{B4}$$

Substituting Eqs. (B3) in the last term of Eq. (B2), and integrating with respect to s, it follows that

$$G(s, \phi) = -\int_0^s \mathcal{V}\big(\xi, C_1 + \psi_1(\xi), C_2 + \psi_2(\xi)\big)\ d\xi + A(C_1, C_2) \ , \tag{B5}$$

where the function A is arbitrary. By choosing the condition

$$G(0, \phi) = 0 \ , \tag{B6}$$

and using Eqs. (B3) again, the following unique solution is obtained:

$$G(s, \phi) = -\int_0^s \mathcal{V}\big(\xi, \phi_1 - \psi_1(s) + \psi(\xi), \phi_2 - \psi_2(s) + \psi_2(\xi)\big)\ d\xi \ . \tag{B7}$$

It is not difficult to see that the known expression[11] is equivalent to this one. However, there is no real physical reason why the condition of Eq. (B6) must always be required. So,

in general, the gauge of the generating function, A, will depend on ϕ, s, and the ignorable action-variable, K:

$$A(s, \phi, K) = A\big(\phi - \psi(s), K\big) \ .$$

APPENDIX C. $\mathcal{H}_{12}$ Average Value Calculations

C.1 Dipole Average Value

The $\mathcal{V}^{(1)}_{m=1}$ and $\mathcal{V}^{(2)}_{m=1}$ components for the dipole oscillation pattern are given from Eqs. (32), (37), (41), and (42) by

$$\mathcal{V}^{(1)}_{m=1} = \left(\frac{e}{cp}\right) v\sqrt{2K_1\beta_1(s)}\cos\phi_1 \tag{C1a}$$

and

$$\mathcal{V}^{(2)}_{m=1} = -\left(\frac{e}{cp}\right)\frac{2^{3/2}\acute{v}K_1^{1/2}K_2}{3}\left\{\beta_2\beta_1^{-1/2}\left[\cos^2\phi_2\sin\phi_1 - \frac{\acute{\beta}_1}{2}\cos^2\phi_2\cos\phi_1\right] - \right.$$
$$\left. -\beta_1^{1/2}\left[\cos\phi_2\sin\phi_2\cos\phi_1 - \frac{\acute{\beta}_2}{2}\cos^2\phi_2\cos\phi_1\right]\right\}. \tag{C1b}$$

The functions $G^{(1)}$ and $G^{(2)}$, Eq. (61b), are then given by

$$G^{(1)}_{m=1} = -\left(\frac{e}{cp}\right)2^{1/2}K_1^{1/2}\sum_{\rho=0}^{1}\cos^{1-\rho}\phi_1\sin^{\rho}\phi_1 g^{\rho}(s) \tag{C2a}$$

and

$$G^{(2)}_{m=1} = \left(\frac{e}{cp}\right)\frac{2^{3/2}}{3}K_1^{1/2}K_2\times$$
$$\times\left\{\sum_{\rho=0}^{1}(-1)^{\rho}\sum_{\acute{\rho}=0}^{2}\binom{2}{\acute{\rho}}\sin^{1-\rho}\phi_1\cos^{\rho}\phi_1\cos^{2-\acute{\rho}}\phi_2\sin^{\acute{\rho}}\phi_2\ h_1^{\rho\acute{\rho}}(s)+\right.$$
$$-\sum_{\rho=0}^{1}\sum_{\acute{\rho}=0}^{2}\binom{2}{\acute{\rho}}\cos^{1-\rho}\phi_1\sin\rho\phi_1\cos^{2-\acute{\rho}}\phi_2\sin^{\acute{\rho}}\phi_2\ h_2^{\rho\acute{\rho}}(s)+$$
$$+\sum_{\rho=0}^{1}\sum_{\acute{\rho}=0}^{2}\binom{2}{\acute{\rho}}\cos^{1-\rho}\phi_1\sin^{\rho}\phi_1\sin^{2-\acute{\rho}}2\phi_2\cos^{\acute{\rho}}2\phi_2\ h_3^{\rho\acute{\rho}}(s)+$$
$$\left.+\sum_{\rho=0}^{1}\sum_{\acute{\rho}=0}^{2}\binom{2}{\acute{\rho}}\cos^{1-\rho}\phi_1\sin^{\rho}\phi_1\cos^{2-\acute{\rho}}\phi_2\sin^{\acute{\rho}}\phi_2\ h_4^{\rho\acute{\rho}}(s)\right\}, \tag{C2b}$$

where the following functions have been defined:

$$g^{\rho}(s) = \int_0^s v\beta_1^{1/2} \cos^{1-\rho}\delta_1 \sin^{\rho}\delta_1 \; d\xi \; , \tag{C3}$$

$$h_1^{\rho\acute{\rho}}(s) = \int_0^s \acute{v}\beta_2\beta_1^{-1/2} \cos^{1-\rho}\delta_1 \sin^{\rho}\delta_1 \cos^{2-\acute{\rho}}\delta_2 \acute{\rho}\delta_2 \; d\xi \; , \tag{C4a}$$

$$h_2^{\rho\acute{\rho}}(s) = \int_0^s \acute{v}\frac{\acute{\beta}_1}{2}\beta_2\beta_1^{-1/2} \cos^{1-\rho}\delta_1 \sin^{\rho}\delta_1 \cos^{2-\acute{\rho}}\delta_2 \sin^{\acute{\rho}}\delta_2 \; d\xi \; , \tag{C4b}$$

$$h_3^{\rho\acute{\rho}}(s) = \frac{1}{2}\int_0^s \acute{v}\beta_1^{1/2} \cos^{1-\rho}\delta_1 \sin^{\rho}\delta_1 \cos^{1-\acute{\rho}} 2\delta_2 \sin^{\acute{\rho}} 2\delta_2 \; d\xi \; , \tag{C4c}$$

and

$$h_4^{\rho\acute{\rho}}(s) = \int_0^s \acute{v}\frac{\acute{\beta}_2}{2}\beta_1^{1/2} \cos^{1-\rho}\delta_1 \sin^{\rho}\delta_1 \cos^{2-\acute{\rho}}\delta_2 \sin^{\rho}\delta_2 \; d\xi \; . \tag{C4d}$$

Taking the derivatives of Eqs. (C1) and (C2), obtaining their products, and taking into account Eq. (46), the following average values are obtained:

$$< \mathcal{V}_{\phi_1}^{(1)} G_{K_1}^{(2)} >= -\left(\frac{e}{cp}\right)^2 \frac{2K_2}{3}\frac{1}{4}\left\{W_1^{0\acute{0}} + W_1^{0\acute{2}} - W_2^{1\acute{0}} - W_2^{1\acute{2}} + W_4^{1\acute{0}} + W_4^{1\acute{2}}\right\} \; , \tag{C5a}$$

$$< \mathcal{V}_{K_1}^{(1)} G_{\phi_1}^{(2)} >= -\left(\frac{e}{cp}\right)^2 \frac{2K_2}{3}\frac{1}{4}\left\{W_1^{0\acute{0}} + W_1^{0\acute{2}} - W_2^{1\acute{0}} - W_2^{1\acute{2}} + W_4^{1\acute{0}} + W_4^{1\acute{2}}\right\} \; , \tag{C5b}$$

$$< \mathcal{V}_{\phi_1}^{(2)} G_{K_1}^{(1)} >= \left(\frac{e}{cp}\right)^2 \frac{2K_2}{3}\frac{1}{4}\left\{\Gamma_{0;1} + \Gamma_{1;2} - \Gamma_{1;4}\right\} \; , \tag{C5c}$$

and

$$< \mathcal{V}_{K_1}^{(2)} G_{\phi_1}^{(1)} >= \left(\frac{e}{cp}\right)^2 \frac{2K_2}{3}\frac{1}{4}\left\{-\Gamma_{0;1}/2 - \Gamma_{1;2} + \Gamma_{1;4}\right\} \; , \tag{C5d}$$

where the numbers W and Γ have been defined as

$$W_i^{\rho\acute{\rho}} = \frac{1}{2\pi}\int_0^C v(s)\beta_1^{1/2}(s)h_i^{\rho\acute{\rho}}(s)\ ds\ , \qquad (C6a)$$

$$\Gamma_{\rho;1} = \frac{1}{2\pi}\int_0^C \acute{v}(s)\beta_2(s)\beta_1^{-1/2}(s)g^{\rho}(s)\ ds\ , \qquad (C6b)$$

$$\Gamma_{\rho;2} = \frac{1}{2\pi}\int_0^C \acute{v}(s)\frac{\acute{\beta}_1(s)}{2}\beta_2(s)\beta_1^{-1/2}(s)g^{\rho}(s)\ ds\ , \qquad (C6c)$$

and

$$\Gamma_{\rho;4} = \frac{1}{2\pi}\int_0^C \acute{v}(s)\frac{\acute{\beta}_2(s)}{2}\beta_1^{1/2}(s)g^{\rho}(s)\ ds\ . \qquad (C6d)$$

Thus, the average of the Hamiltonian Eq. (60b) is written as

$$< \mathcal{H}_{12}^{m=1} >= \left(\frac{e}{cp}\right)^2 \frac{K_2}{3}\Upsilon\ , \qquad (C7a)$$

where Υ is given by

$$\Upsilon = W_1^{0\acute{0}} + W_1^{0\acute{2}} - W_2^{1\acute{0}} - W_2^{1\acute{2}} + W_4^{1\acute{0}} + W_{4;1}^{1\acute{2}} + +\Gamma_{0;1} + \Gamma_{1;2} - \Gamma_{1;4}\ . \qquad (C7b)$$

C.2 Sextupole Average Value

The $\mathcal{V}^{(1)}$ and $\mathcal{V}^{(2)}$ components for the sextupole oscillation pattern are taken from Eqs. (34), (37), (41), and (42):

$$\mathcal{V}_{m=3}^{(1)} = \left(\frac{e}{cp}\right) 2^{3/2} v\left[(K_1\beta_1)^{3/2}\cos^3\cos\phi_1 - 3\sqrt{K_1\beta_1}(k_2\beta_2)\cos\phi_1\cos^2\phi_2\right]\ , \qquad (C8a)$$

$$\mathcal{V}_{m=3}^{(2)} = -\left(\frac{e}{cp}\right)\frac{2^{5/2}\acute{v}}{5}\times$$

$$\times\left\{\sqrt{\frac{K_1}{\beta_1}}\left(\sin\phi_1 - \frac{\acute{\beta}_1}{2}\cos\phi_1\right)\left(3K_1k_2\beta_1\beta_2\cos^2\phi_1\cos^2\phi_2 - K_2^2\beta_2^2\cos^4\phi_2\right) +\right.$$

$$-\sqrt{\frac{K_2}{\beta_2}}\left(\sin\phi_2-\frac{\acute{\beta}_2}{2}\cos\phi_2\right)\Big(3(K_1\beta_1)^{3/2}(K_2\beta_2)^{1/2}\cos^3\phi_1\cos\phi_2+$$

$$-(K_1\beta_1)^{1/2}(K_2\beta_2)^{3/2}\cos\phi_1\cos^3\phi_2\Big)\Big\}. \quad (C8b)$$

The functions $G^{(1)}$ and $G^{(2)}$, Eq. (61b), are then given by

$$G^{(1)}_{m=3}=-\left(\frac{e}{cp}\right)2^{3/2}\Big[K_1^{3/2}\sum_{\rho=0}^{3}\binom{3}{\rho}\cos^{3-\rho}\phi_1\sin^{\rho}\phi_1 g^{\rho}_{11}(s)+$$

$$-3K_1^{1/2}K_2\sum_{\rho=0}^{1}\sum_{\acute{\rho}=0}^{2}\binom{2}{\acute{\rho}}\cos^{1-\rho}\phi_1\sin^{\rho}\phi_1\cos^{2-\acute{\rho}}\phi_2\sin^{\acute{\rho}}\phi_2 g^{\rho\acute{\rho}}_{12}(s)\Big] \quad (C9a)$$

and

$$G^{(2)}_{m=3}=\left(\frac{e}{cp}\right)\frac{2^{5/2}}{5}\times$$

$$\times\Big\{3K_1^{3/2}K_2\sum_{\rho=0}^{2}\sum_{\acute{\rho}}^{2}\sum_{\tilde{\rho}}^{1}(-1)^{\tilde{\rho}}\cos^{2-\rho+\tilde{\rho}}\phi_1\sin^{1+\rho-\tilde{\rho}}\phi_1\cos^{2-\acute{\rho}}\phi_2\sin^{\acute{\rho}}\phi_2\ g^{\rho\acute{\rho}\tilde{\rho}}_{21}(s)+$$

$$-K_1^{5/2}\sum_{\rho=0}^{1}\sum_{\acute{\rho}=0}^{4}(-1)^{\rho}\binom{4}{\acute{\rho}}\sin^{1-\rho}\phi_1\cos^{\rho}\cos^{4-\acute{\rho}}\phi_2\sin^{\acute{\rho}}\phi_2\ g^{\rho\acute{\rho}}_{22}(s)+$$

$$-3K_1^{3/2}K_2\sum_{\rho=0}^{3}\sum_{\acute{\rho}=0}^{2}\binom{3}{\rho}2\acute{\rho}\cos^{3-\rho}\phi_1\sin^{\rho}\phi_1\cos^{2-\acute{\rho}}\phi_2\sin^{\acute{\rho}}\phi_2\ g^{\rho\acute{\rho}}_{23}(s)+$$

$$+K^{5/2}\sum_{\rho=0}^{1}\sum_{\acute{\rho}=0}^{4}\binom{4}{\acute{\rho}}\cos^{1-\rho}\phi_1\sin^{\rho}\phi_1\cos^{4-\acute{\rho}}\phi_2\sin^{\acute{\rho}}\phi_2\ g^{\rho\acute{\rho}}_{24}(s)+$$

$$-3K_1^{3/2}K_2\sum_{\rho=0}^{3}\sum_{\acute{\rho}=0}^{1}\sum_{\tilde{\rho}=0}^{1}(-1)^{\tilde{\rho}}\binom{3}{\rho}\cos^{3-\rho}\phi_1\sin^{\rho}\phi_1\cos^{1-\acute{\rho}+\tilde{\rho}}\phi_2\sin^{1+\acute{\rho}-\tilde{\rho}}\phi_2 g^{\rho\acute{\rho}\tilde{\rho}}_{25}(s)+$$

$$+K_1^{1/2}K_2^2\sum_{\rho=0}^{1}\sum_{\acute{\rho}=0}^{3}\sum_{\tilde{\rho}=0}^{1}(-1)^{\tilde{\rho}}\binom{3}{\acute{\rho}}\cos^{1-\rho}\phi_1\sin^{\rho}\phi_1\cos^{3-\acute{\rho}+\tilde{\rho}}\phi_2\sin^{1+\acute{\rho}-\tilde{\rho}}\phi_2 g^{\rho\acute{\rho}\tilde{\rho}}_{26}(s)+$$

$$-3K_1^{3/2}K_2\sum_{\rho=0}^{3}\sum_{\acute{\rho}=0}^{2}\binom{3}{\rho}\binom{2}{\acute{\rho}}\cos^{3-\rho}\phi_1\sin^{\rho}\phi_1\cos^{2-\acute{\rho}}\phi_2\sin^{\acute{\rho}}\phi_2\ g^{\rho\acute{\rho}}_{27}(s)+$$

$$+K_1^{1/2}K_2^2\sum_{\rho=0}^{1}\sum_{\acute{\rho}=0}^{4}\binom{4}{\acute{\rho}}\cos^{1-\rho}\phi_1\sin^{\rho}\phi_1\cos^{4-\acute{\rho}}\phi_2\sin^{\acute{\rho}}\phi_2\ g^{\rho\acute{\rho}}_{28}(s)\Big\}\ ,$$

$$(C9b)$$

where the g functions are defined as

$$g_{11}^{\rho}(s) = \int_0^s v(\xi)\beta_1^{3/2}(\xi)\cos^{3-\rho}\delta_1 \sin^{\rho}\delta_1 \; d\xi \; , \qquad (C10a)$$

$$g_{12}^{\rho\acute{\rho}\ddot{\rho}}(s) = \int_0^s v\beta_1^{1/2}\beta_2 \cos^{1-\rho}\delta_1 \sin^{\rho}\delta_1 \cos^{2-\acute{\rho}}\delta_2 \sin^{\acute{\rho}}\delta_2 \; d\xi \; , \qquad (C10b)$$

$$g_{21}^{\rho\acute{\rho}\ddot{\rho}}(s) = \int_0^s \acute{v}\beta_1^{1/2}\beta_2 \cos^{2-\rho+\ddot{\rho}}\delta_1 \sin^{1+\rho-\ddot{\rho}}\delta_1 \cos^{2-\acute{\rho}}\delta_2 \sin^{\acute{\rho}}\delta_2 \; d\xi \; , \qquad (C10c)$$

$$g_{22}^{\rho\acute{\rho}}(s) = \int_0^s \acute{v}\beta_1^{-1/2}\beta_2^2 \cos^{2-\rho}\delta_1 \sin^{\rho}\delta_1 \cos^{4-\acute{\rho}}\delta_2 \sin^{\acute{\rho}}\delta_2 \; d\xi \; , \qquad (C10d)$$

$$g_{23}^{\rho\acute{\rho}}(s) = \int_0^s \acute{v}\beta_1^{1/2}\beta_2 \frac{\acute{\beta}_1}{2} \cos^{3-\rho}\delta_1 \sin^{\rho}\delta_1 \cos^{2-\acute{\rho}}\delta_2 \sin^{\acute{\rho}}\delta_2 \; d\xi \; , \qquad (C10e)$$

$$g_{24}^{\rho\acute{\rho}}(s) = \int_0^s \acute{v}\beta_1^{-1/2}\beta_2^2 \frac{\acute{\beta}_1}{2} \cos^{1-\rho}\delta_1 \sin^{\rho}\delta_1 \cos^{4-\acute{\rho}}\delta_2 \sin^{\acute{\rho}}\delta_2 \; d\xi \; , \qquad (C10f)$$

$$g_{25}^{\rho\acute{\rho}\ddot{\rho}}(s) = \int_0^s \acute{v}\beta_1^{3/2} \cos^{3-\rho}\delta_1 \sin^{\rho}\delta_1 \cos^{1-\acute{\rho}+\ddot{\rho}}\delta_2 \sin^{1+\acute{\rho}-\ddot{\rho}}\delta_2 \; d\xi \; , \qquad (C10g)$$

$$g_{26}^{\rho\acute{\rho}\ddot{\rho}}(s) = \int_0^s \acute{v}\beta_1^{1/2}\beta_2 \cos^{1-\rho}\delta_1 \sin^{\rho}\delta_1 \cos^{3-\acute{\rho}+\ddot{\rho}}\delta_2 \sin^{1+\acute{\rho}-\ddot{\rho}}\delta_2 \; d\xi \; , \qquad (C10h)$$

$$g_{27}^{\rho\acute{\rho}}(s) = \int_0^s \acute{v}\beta_1^{3/2} \frac{\acute{\beta}_2}{2} \cos^{3-\rho}\delta_1 \sin^{\rho}\delta_1 \cos^{2-\acute{\rho}}\delta_2 \sin^{\acute{\rho}}\delta_2 \; d\xi \; , \qquad (C10i)$$

$$g_{28}^{\rho\acute{\rho}}(s) = \int_0^s \acute{v}\beta_1^{1/2}\beta_2 \frac{\acute{\beta}_2}{2} \cos^{1-\rho}\delta_1 \sin^{\rho}\delta_1 \cos^{4-\acute{\rho}}\delta_2 \sin^{\acute{\rho}}\delta_2 \; d\xi \; . \qquad (C10j)$$

APPENDIX D. Dipole $\mathcal{H}_{11}$ Average Value Calculations

From Eqs. (32c), (37b), (41) and (61b), the potential and the generating function are given by

$$\mathcal{V}^{(1)}_{m=1} = \left(\frac{e}{cp}\right) v\sqrt{2K_1\beta_1}\cos\phi_1 \tag{D1}$$

and

$$G^{(1)} = -\left(\frac{e}{cp}\right) 2^{1/2}K_1^{1/2}\sum_{\rho=0}^{1}\cos^{1-\rho}\phi_1\sin^{\rho}\phi_1\; g_\rho(s)\ , \tag{D2}$$

where the function g_ρ is given by

$$g_\rho(s) = \int_0^s v(\xi)\beta_1^{1/2}(\xi)\cos^{1-\rho}\delta_1\sin^{\rho}\delta_1\ . \tag{D3}$$

The function δ_1 is defined by Eq. (83). Taking the derivatives of Eqs. (D1) and (D2), the following average values are obtained:

$$< \mathcal{V}^{(1)}_{K_1}G^{(1)}_{\phi_1} >= -\left(\frac{e}{cp}\right)^2\sum_{\rho=0}^{1}\left[-(1-\rho)(1-\rho|\rho+1)+\rho(3-\rho|\rho-1)\right]Q_\rho \tag{D4a}$$

and

$$< \mathcal{V}^{(1)}_{\phi_1}G^{(1)}_{K_1} >= \left(\frac{e}{cp}\right)^2\sum_{\rho=0}^{1}Q_\rho\ , \tag{D4b}$$

where the number Q_ρ is given by

$$Q_\rho = \frac{1}{2\pi} = \int_0^C v(s)\beta_1^{1/2}(s)g_\rho(s)\ ds\ . \tag{D5}$$

It is not necessary to do more calculations. Because Eqs. (D4) do not depend on the action variables, the tune shift contribution of this term is zero.