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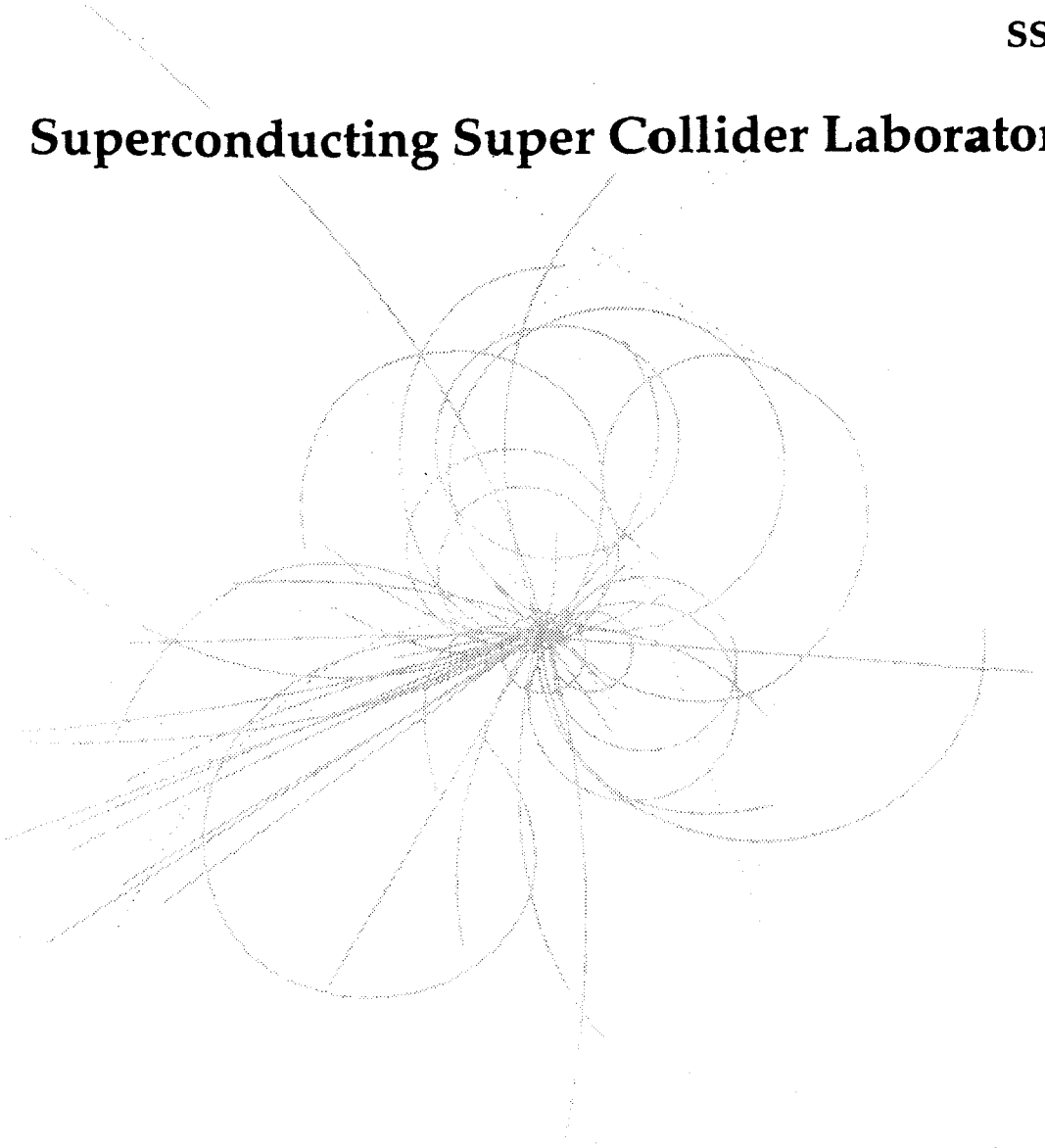

Heat Equation and Time Dependence of the Temperature Rise due to Energy Deposition

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Heat Equation and Time Dependence of the Temperature Rise due to Energy Deposition

G. López

Abstract

Using the Heat Conduction Equation in cartesian coordinates, some studies are made about the temperature rise due to time dependent energy deposited in materials. Thermal conductivity and specific heat are taken as constants. An exact analytical solution for an arbitrary distribution of energy can be given which justifies the use of instantaneous temperature rise at room temperature for high-frequency energy deposition events. Several energy deposition distributions are considered.

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1.0 INTRODUCTION

Temperature rise normally influences accelerator component design. This temperature rise is a principal concern in determining a safe operating margin of the components (scrapers, lambertson magnets, dump abort system, superconducting magnets, etc.). This temperature may be limited by the melting or cracking point of the materials or some other failure limitations (quench limits in superconducting magnets, cryogenic capacity to remove the heat generated, or radiation damage). Preliminary estimations are very useful and important guides, and may be done through analytical solutions of the heat equation. However, a computer simulation final analysis program is advised to determine the peak and distribution of the temperature rise.

The temperature time-dependent solution is of interest here. Some of the most important mechanisms which involve energy deposition, like hadronic and electromagnetic showers developed in the materials by the interaction of high energy charged or neutral particles, are time dependent. Consequently, a time-dependent solution of the heat equation is required for an estimation of the temperature rise.

This report restricts itself to time-dependent solutions of the heat equation at room temperature without boundary conditions (specific heat and thermal conductivity are constants and the materials are mathematically infinite in extension). Moreover, this restriction is not essential for the fundamental result to address. The heat equation is given by

$$(\rho c_p)\,\frac{\partial\theta}{\partial t} = k\,\nabla^2\theta + \sum_{i=0}^{n}\frac{\epsilon_i(\vec{r})}{\Delta t_i}\delta(t-t_i)\ , \tag{1a}$$

where ∇^2 is given by

$$\nabla^2 = \sum_{j=1}^{3}\frac{\partial^2}{\partial x_j^2}\ , \tag{1b}$$

c_p is the specific heat, ρ is the density, k is the thermal conductivity of the materials. $\epsilon_i(\vec{r})$ represents the density of energy deposited in the materials at the time t_i, $i = 1, \ldots, N$. $\Delta t_i = t_i - t_{i-1}$ is the interval of time of each energy deposition event. Finally, $\theta = T - T_o$ represents the difference between the new temperature at the position $\vec{r}$ and time t, $T(\vec{r}, t)$, and the initial uniform temperature, T_o.

2.0 SOLUTION WITH ZERO THERMAL CONDUCTIVITY

Assuming that thermal conductivity is zero, Eq. (1) can be solved, even for a specific heat which depends on the temperature, through direct integration. However, a constant c_p will be assumed to introduce the basic idea for the more general case. Assume also that the

energy density is constant and the same for every i-event which occurs at the same interval of time, Δt. Therefore, Eq. (1) can be written as

$$\frac{d\theta}{dt} = \frac{\epsilon f_o}{(\rho c_p)} \sum_{i=0}^{n} \delta(t - t_i) \ , \tag{2a}$$

where f_o is defined as

$$f_o = 1/(\Delta t) \ . \tag{2b}$$

Taking the Fourier transformation of Eq. (2a), defined by Eq. (A1) in Appendix A, it follows

$$\theta_\alpha = \frac{\epsilon f_o}{(\rho c_p)} \sum_{i=0}^{n} \frac{e^{-i\alpha t_i}}{-i\alpha} \ . \tag{3}$$

Using the inverse Fourier transformation, defined by Eq. (A2) in Appendix A, the following solution results

$$\theta(t) = \frac{\epsilon f_o}{(\rho c_p)} \sum_{i=0}^{n} H(t - t_i) \ , \tag{4}$$

where H is the Heaviside function,

$$H(t - t_i) = \begin{cases} 1 & \text{if } t - t_i \geq 0 \\ 0 & \text{if } t - t_i < 0 \end{cases} \ . \tag{5}$$

Assuming that the event time separation is infinitesimally close (frequency of energy deposition, f_o, is very high), the summation in Eq. (4) can be approximated by an integration,

$$\sum_{i=0}^{n} H(t - t_i) = \int_0^{\tau_n} H(t - \tau) \, d\tau \ . \tag{6}$$

But according to Eq. (5), this integration will be zero for $t > \tau$. So, the result of the integration of Eq. (6) is

$$\int_0^{\tau_n} H(t - \tau) \, d\tau = \begin{cases} t & \text{if } t \leq \tau \\ \tau_n & \text{if } t > \tau_n \end{cases} \ . \tag{7}$$

The solution to Eq. (6) can be given as

$$\theta(t) = \begin{cases} \frac{\epsilon f_o}{(\rho c_p)} \, t & \text{if } t \leq \tau \\ \frac{\epsilon f_o}{(\rho c_p)} \, \tau_n & \text{if } t > \tau_n \end{cases} \ . \tag{8}$$

Figure 1 shows both solutions, Eqs. (5) and (8). For small τ_n and very high frequency f_o such that $f_o\tau_n \rightarrow 1$, the term $\epsilon/(\delta c_p)$ corresponds to the instantaneous temperature rise.

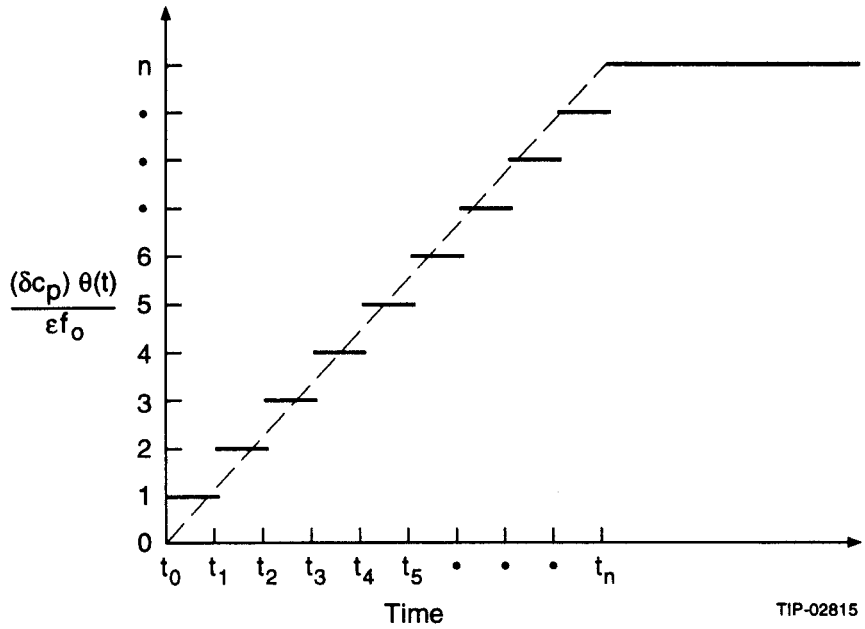

Figure 1. Normalized Temperature as a Function of Time for $k = 0$.

Although restricted herein to considering discrete time energy deposition, the method can be extended to an arbitrary time-depended situation as noted in Appendix B.

3.0 NON-ZERO THERMAL CONDUCTIVITY

Assume that the energy density distribution is the same in every event, and the events are equally temporally spaced, Eq. (1) is rewritten as

$$(\rho c_p)\frac{\partial\theta}{\partial t} = k\nabla^2\theta + \epsilon(\vec{r})f_o\sum_{i=0}^{n}\delta(t-t_i)\ . \tag{9}$$

Applying the spatial Fourier transformation, defined by Eq. (A4) in Appendix A, and after making some rearrangements, it follows

$$\frac{d\theta_\lambda}{dt} + a\ \theta_\lambda = \frac{\epsilon_\lambda f_o}{(\rho c_p)}\sum_{i=0}^{n}\delta(t-t_i)\ , \tag{10}$$

where θ_λ and ϵ_λ are the Fourier-transformed functions of the temperature and energy density distribution, and a is defined as

$$a = \frac{k\lambda^2}{(\rho c_p)}\ , \tag{11}$$

$$\lambda^2 = \sum_{j=1}^{3} \lambda_i^2 \ . \tag{12}$$

Using now the time Fourier transformation in Eq. (10) and making some rearrangements, the following expression is obtained

$$\theta_{\lambda\alpha} = \frac{\epsilon_\lambda f_o}{(\rho c_p)} \sum_{i=0}^{n} \frac{e^{-i\alpha t_i}}{-i\alpha + a} \ . \tag{13}$$

Applying the inverse time-Fourier transformation[1] and using Eq. (11), results in

$$\theta_\lambda = \frac{f_o}{(\rho c_p)} \sum_{i=0}^{n} H(t-t_i)\epsilon_\lambda \ e^{-\frac{k(t-t_i)}{(\rho c_p)} \lambda^2} \ . \tag{14}$$

Applying now the spatial inverse transformation, defined by Eq. (A5) in Appendix A, it follows

$$\theta = \frac{f_o}{(\rho c_p)} \sum_{i=0}^{n} H(t-t_i) F^{-1} \left[\epsilon_\lambda \ e^{-\frac{k(t-t_i)}{(\delta c_p)} \lambda^2} \right] \tag{15}$$

which can be expressed, using the convolution theorem,[2] as

$$\theta = \frac{f_o}{(\rho c_p)} \sum_{i=0}^{n} H(t-t_i) F^{-1}\left[\epsilon_\lambda\right] * F^{-1} \left[e^{-\frac{k(t-t_i)}{(\rho c_p)} \lambda^2} \right] \ , \tag{16}$$

where the convolution, $*$, between two functions is defined as

$$f(x) * g(x) = \int_{-\infty}^{+\infty} f(x-\xi) g(\xi) \ d\xi = \int_{-\infty}^{+\infty} f(\xi) g(x-\xi) \ d\xi \ . \tag{17}$$

It is not difficult to demonstrate, see Eq. (A6) in Appendix A, the following result

$$F^{-1} \left[e^{-\frac{k(t-t_i)}{(\rho c_p)} \lambda^2} \right] = \frac{1}{8 \left[\frac{\pi k (t-t_i)}{\rho c_p} \right]^{3/2}} \ e^{-\frac{\rho c_p}{4k(t-t_i)} r^2} \ . \tag{18}$$

Using this in Eq. (16), the general solution is given by

$$\theta(\vec{r}, t) = \frac{f_o \sqrt{\rho c_p}}{8[\pi k]^{3/2}} \sum_{i=0}^{n} \frac{H(t-t_i)}{[t-t_i]^{3/2}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \epsilon(\vec{r} - \vec{\xi}) \ e^{-\frac{\rho c_p}{4k(t-t_i)} \xi^2} \ d^3\vec{\xi} \ . \tag{19}$$

This is the general solution of the problem. However, one must show that this solution has the right limit wherever the conductivity is zero. This can be accomplished using the known relation

$$\lim_{\beta \to 0} \frac{1}{\sqrt{\pi}\beta} \, e^{-x^2/\beta^2} = \delta(x) \; . \tag{20}$$

Then, the following limit is obtained

$$\lim_{k \to 0} \theta(\vec{r}, t) = \frac{f_o}{(\rho c_p)} \sum_{i=0}^{n} H(t - t_i) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \epsilon(\vec{r} - \vec{\xi}) \delta(\vec{\xi}) \; d^3\vec{\xi} \tag{21}$$

which brings about the expression of Eq. (4). Defining the parameter q as

$$q = \frac{\rho c_p}{4k(t - t_i)} \; , \tag{22}$$

Eq. (9) can be written as

$$\theta(\vec{r}, t) = \frac{f_o}{(\rho c_p)} \sum_{i=0}^{n} \frac{H(t - t_i) \; q^{3/2}}{\pi^{3/2}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \epsilon(\vec{r} - \vec{\xi}) e^{-q\xi^2} \; d\xi \; . \tag{23}$$

Therefore, the limit of Eq. (21) can also be written as

$$\lim_{q \to \infty} \theta(\vec{r}, t) = \frac{f_o \epsilon(\vec{r})}{\rho c_p} \sum_{i=0}^{n} H(t - t_i) \; . \tag{24}$$

That is, the parameter q characterizes the type of energy deposition events such that the use of the instantaneous temperature rise can be justified. For the Superconducting Super Collider (SSC), the characteristic separation between events corresponds to the separation between bunches, 16.66 nSec. Then, q is of the order of

$$q = 1.5 \times 10^7 \; \frac{\rho c_p}{k} \tag{25}$$

which justifies completely the use of instantaneous temperature rise at room temperature since $\rho c_p / k$ is less or of the order of the unit for metals, and it can be four orders of magnitude higher for insulators. However, at low temperature the instantaneous temperature rise may be considered slightly overestimated since the specific heat for metals may be lower by four orders of magnitude, but its use automatically introduces a small safety margin in the calculations.

The following assumes that the instantaneous temperature rise is not justified. So, the thermal conductivity effect is considered.

4.0 GAUSSIAN DISTRIBUTION OF ENERGY DENSITY

The solution of Eq. (19) can be used once the distribution of energy is known. Assume that the distribution of energy is Gaussian in the three dimensions,

$$\epsilon(\vec{r}) = E_o \prod_{j=1}^{3} \frac{1}{\sqrt{\pi}\sigma_j} \, e^{-x_j^2/\sigma_j^2} \, , \tag{26}$$

where the normalization has been taken such that the integration all over the volume is the total energy, E_o, deposited in the material. The integration brings about the following expression, see Eq. (A7) in Appendix A,

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \epsilon(\vec{r} - \vec{\xi}) \, e^{-q\xi^2} \, d^3\vec{\xi} = E_o \prod_{j=1}^{3} \frac{e^{-\frac{qx_j^2}{\sqrt{1+q\sigma_j^2}}}}{\sqrt{1+q\sigma_j^2}} \, , \tag{27}$$

where q is given in Eq. (22). Substituting Eq. (26) in Eq. (19) and making some rearrangements, it follows

$$\theta(\vec{r}, t) = \frac{E_o f_o \sqrt{\rho c_p}}{8[\pi k]^{3/2}} \sum_{i=0}^{n} H(t - t_i) \prod_{j=1}^{3} \frac{e^{-qx_j^2/(1+q\sigma^2)}}{\sqrt{t - t_i + p\sigma_j^2}} \, , \tag{28a}$$

where p is defined by

$$p = \frac{\rho c_p}{4k} \, . \tag{28b}$$

From Eq. (28a), it is seen that the temperature depends inversely proportional on the $\sigma's$, and the hottest point is located at the origin. Defining $\theta_o(t) = \theta(\vec{0}, t)$, the following expression is obtained for the hottest point

$$\theta_o(t) = \frac{E_o f_o \sqrt{\rho c_p}}{8[\pi k]^{3/2}} \sum_{i=0}^{n} \frac{H(t - t_i)}{\prod_{j=1}^{3} \sqrt{t - t_i + p\sigma_j^2}} \, . \tag{29}$$

The particular case of a δ-like distribution of energy density can be obtained by making $\sigma_j = 0$ in the solutions of Eqs. (28a) and (29).

4.1 $\sigma_x = \sigma_y \neq \sigma_z$

In applications of energy deposition of hadronic or electromagnetic showers, the longitudinal energy distribution density is spread in larger extension than the transverse direction.

To estimate the hottest temperature rise in this case, assume $\sigma_x = \sigma_y = \sigma$ for the transverse direction. The expression obtained from Eq. (28) is

$$\theta_o(t) = \frac{E_o f_o \sqrt{\rho c_p}}{8[\pi k]^{3/2}} \sum_{i=0}^{n} \frac{H(t-t_i)}{(t-t_i+p\sigma^2)\sqrt{t-t_i+p\sigma^2}} , \tag{30}$$

and assume that the events happen so close together in time that it is possible to substitute the summation for an integration,

$$\theta_o(t) = \frac{E_o f_o \sqrt{\rho c_p}}{8[\pi k]^{3/2}} \int_0^{\tau_n} \frac{H(t-\tau)\, d\tau}{(t-\tau+p\sigma^2)\sqrt{t-\tau+p\sigma_z^2}} . \tag{31a}$$

Then, making this integration, the following solution results

$$\theta_o(t) = \frac{E_o f_o}{8\pi^{3/2} k \sqrt{\sigma_z^2 - \sigma^2}} \begin{cases} \log G(t,0) - \log G(\tau_n, \tau_n) & \text{if } t \le \tau_n \\ \log G(t,0) - \log G(t, \tau_n) & \text{if } t > \tau_n , \end{cases} \tag{31b}$$

where the function G is defined by

$$G(t, \tau_n) = \frac{\sqrt{t + p\sigma_z^2 - \tau_n} - \sqrt{p(\sigma_z^2 - \sigma^2)}}{\sqrt{t + p\sigma_z^2 - \tau_n} + \sqrt{p(\sigma_z^2 - \sigma^2)}} . \tag{31c}$$

Figure 2 shows $\theta_o(t)$ as a function of time. Eq. (31a) has a maximum value at $t = \tau_n$, and after this value, the temperature falls. The maximum temperature is given by

$$\theta_o^{max} = \frac{E_o f_o}{8\pi^{3/2} k \sqrt{\sigma_z^2 - \sigma^2}} \Big[\log G(\tau_n, 0) - \log G(\tau_n, \tau_n) \Big] . \tag{32}$$

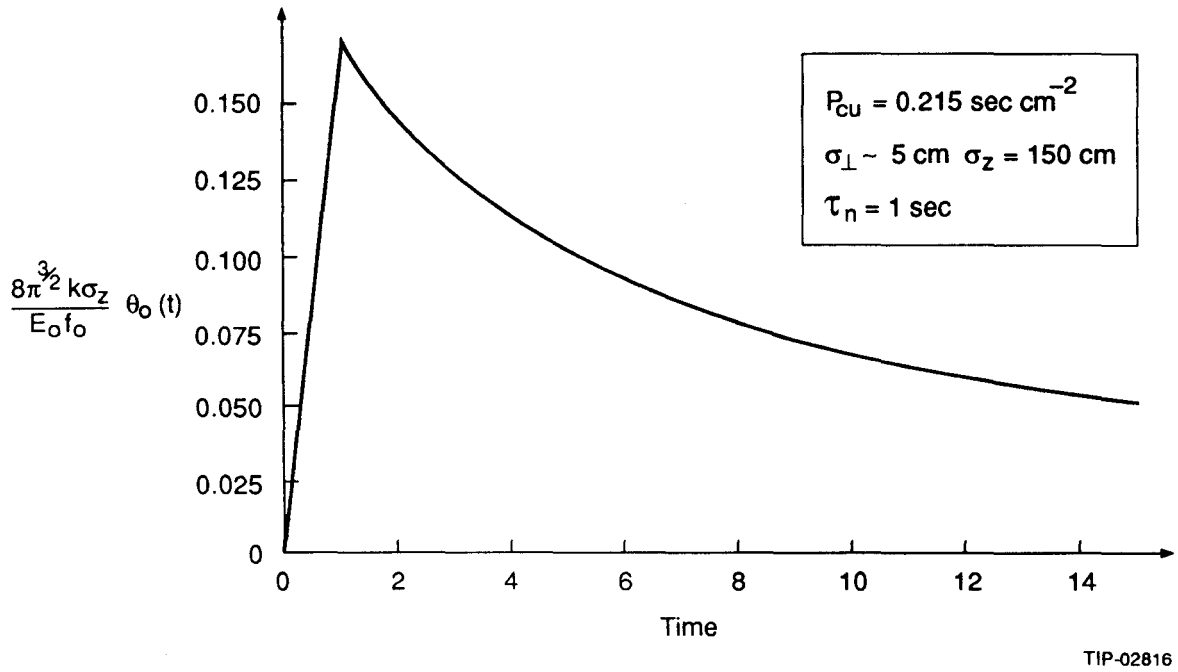

Figure 2a. Normalized Temperature for $k \neq 0$ and Large Time $\tau_n = 1$ sec.

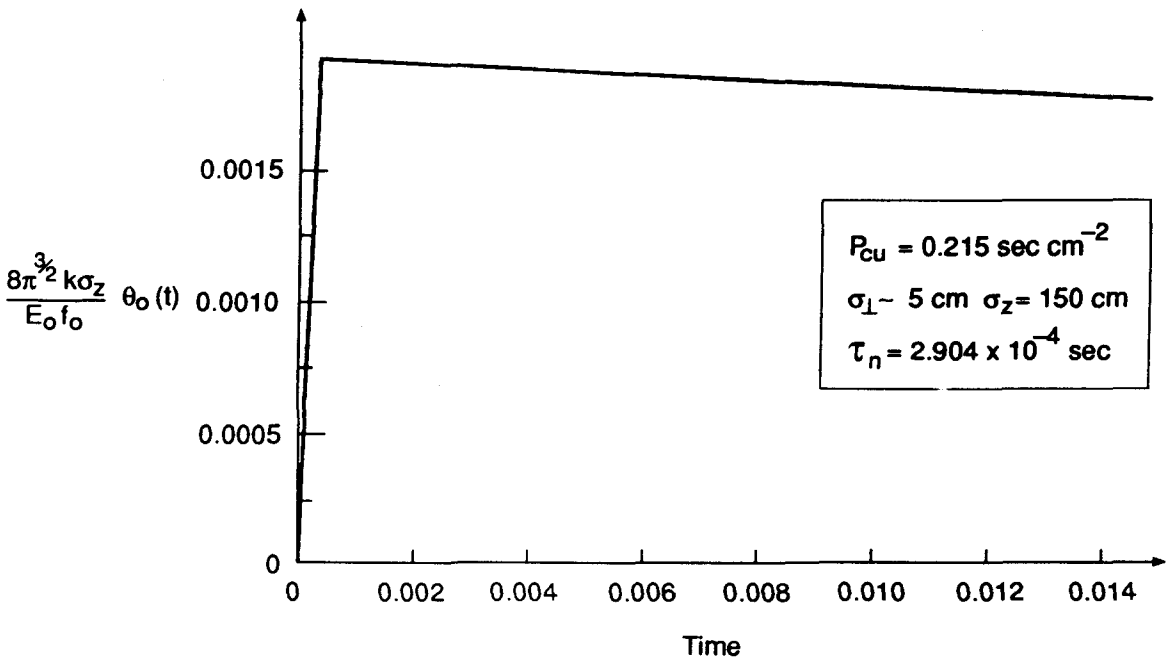

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Figure 2b. Normalized Temperature for $k \neq 0$ and Large Time $\tau_n = 2.9 \times 10^{-4}$ sec.

4.2 $\sigma_x = \sigma_y = \sigma_z$

For the case of a completely symmetrical distribution of energy density in the material, the solution can be obtained by taking the limit $\sigma_z \to \sigma$ in Eq. (31a). This limit is calculated with the help of Eq. (A8) of Appendix A. The resulting solution is given by

$$\theta_o(t) = \frac{E_o f_o \sqrt{p}}{4\pi^{3/2} k} \begin{cases} \frac{1}{\sqrt{p\sigma^2}} - \frac{1}{\sqrt{t+p\sigma^2}} & \text{if } t \leq \tau_n \\ \frac{1}{\sqrt{t+p\sigma^2-\tau_n}} - \frac{1}{\sqrt{t+p\sigma^2}} & \text{if } t > \tau_n \ , \end{cases} \tag{33}$$

where p is defined as in Eq. (26), and the maximum temperature can be written as

$$\theta_o^{max} = \frac{E_o f_o \sqrt{p}}{4\pi^{3/2} k} \left[\frac{1}{\sqrt{p\sigma^2}} - \frac{1}{\sqrt{\tau_n + p\sigma^2}} \right] \ , \tag{34}$$

which for the case $p\sigma^2 \ll \tau_n$, a simple expression results,

$$\theta_o^{max} = \frac{E_o f_o}{4\pi^{3/2} k\sigma} \ . \tag{35}$$

4.3 Uniform Distribution in the Z-Direction

For a thin layer of material, the energy density distribution may be uniform across the layer. Therefore, the temperature must not depend on the variable associated to this direction, says z. Eq. (9) is written as

$$(\rho c_p)\frac{\partial \theta}{\partial t} = k \left[\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right] + \epsilon(x,y) f_o \sum_{i=0}^{n} \delta(t - t_i) \ . \tag{36}$$

Using the same above procedure, the general solution can be given by

$$\theta(x,y,t) = \frac{f_o}{4\pi k}\sum_{i=0}^{n}\frac{H(t-t_i)}{t-t_i}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\epsilon(x-\xi,y-\mu)e^{-\frac{\rho c_p(\xi^2+\mu^2)}{4k(t-t_i)}}\,d\xi d\mu\ . \tag{37}$$

If a Gaussian distribution of energy density is gotten in the transverse direction,

$$\epsilon(x,y) = \frac{\epsilon_o}{\pi\sigma_x\sigma_y}\ e^{-x^2/\sigma_x^2 - y^2/\sigma_y^2}\ , \tag{38}$$

where ϵ_o is the total energy per unit length in the z-direction, the above convolution can easily be made, obtaining

$$\theta(x,y,t) = \frac{\epsilon_o f_o}{4\pi k}\sum_{i=0}^{n}\frac{H(t-t_i)\ e^{-qx^2/(1+q\sigma_x^2)-qy^2/(1+q\sigma_y^2)}}{\sqrt{t-t_i+p\sigma_x^2}\sqrt{t-t_i+p\sigma_z^2}}\ , \tag{39}$$

where p and q are given by Eqs. (28b) and (22). The temperature of the hottest point is

$$\theta_o(t) = \frac{\epsilon_o f_o}{4\pi k}\sum_{i=0}^{n}\frac{H(t-t_i)}{\sqrt{t-t_i+p\sigma_x^2}\sqrt{t-t_i+p\sigma_y^2}}\ . \tag{40}$$

Approximating the summation by an integration,

$$\theta_o(t) = \frac{\epsilon_o f_o}{4\pi k}\int_0^{\tau_n}\frac{H(t-\tau)\ d\tau}{\sqrt{(t+p\sigma_x^2-\tau)(t+p\sigma_y^2-\tau)}}\ , \tag{41}$$

the following solution is obtained

$$\theta_o(t) = \frac{\epsilon_o f_o}{2\pi k}\begin{cases}\log g(t,0) - \log g(\tau_n,\tau_n) & \text{if } t \le \tau_n \\ \log g(t,0) - \log g(t,\tau_n) & \text{if } t > \tau_n\end{cases}\ , \tag{42}$$

where the function g is defined as

$$g(t,\tau_n) = \sqrt{t+p\sigma_x^2-\tau_n} + \sqrt{t+p\sigma_y^2-\tau_n}\ . \tag{43}$$

Moreover, the maximum temperature is given by

$$\theta_o^{max} = \frac{\epsilon_o f_o}{2\pi k}[\log g(t,0) - \log g(\tau_n,\tau_n)]\ . \tag{44}$$

For the case where $\sigma_x = \sigma_y = \sigma$ and $\tau_n >> p\sigma^2$, Eq. (44) is expressed by a simpler expression,

$$\theta_o^{max} = \frac{\epsilon_o f_o}{2\pi k}\ \log\frac{1}{\sigma}\sqrt{4\tau_n k/\rho c_p}\ . \tag{45}$$

4.4 Event Dependence of the Gaussian Distribution

It could happen that every event may produce a Gaussian distribution which could change event-from-event even though they are equally time separated. In this case, Eq. (1) looks like

$$(\rho c_p)\frac{\partial\theta}{\partial t} = k\nabla^2\theta + f_o\sum_{i=0}^{n}\epsilon_i(\vec{r})\delta(t-t_i)\ , \tag{46}$$

and it is clear from Eqs. (19), (22) and (26) that the solution of this equation for a Gaussian distribution would be

$$\theta(\vec{r},t) = \frac{E_o f_o\sqrt{\rho c_p}}{8[\pi k]^{3/2}}\sum_{i=0}^{n}H(t-t_i)\prod_{j=1}^{3}\frac{e^{-qx_j^2/(1+q\sigma_{ji}^2)}}{\sqrt{t-t_i+p\sigma_{ji}^2}}\ , \tag{47}$$

where σ_{ji} is the standard deviation of the energy density deposited in the material in the j-direction at the i-event. The peak temperature evolution would be given by

$$\theta_o(t) = \frac{E_o f_o\sqrt{\rho c_p}}{8[\pi k]^{3/2}}\sum_{i=0}^{n}\frac{H(t-t_i)}{\prod_{j=1}^{3}\sqrt{t-t_i+p\sigma_{ji}^2}}\ . \tag{48}$$

5.0 MAXWELL-LIKE DISTRIBUTION ON THE LONGITUDINAL DIRECTION

As mentioned in Section 4.1, the energy density distribution deposited in the longitudinal direction may be different from the transverse direction. Generally, the energy deposited starts from zero, reaches a peak value and then falls very rapidly, see Figure 3. Keeping a Gaussian distribution in the transverse direction, assume a Maxwell-like energy density distribution in the longitudinal direction,

$$\epsilon(z) = \frac{2E_o^{1/3}}{\sigma_z^2}\ H(z)z\ e^{-z^2/\sigma_z^2}\ , \tag{49}$$

where $H(z)$ is the Heaviside function defined by Eq. (5). $\epsilon(z)$ has been normalized such that the total energy in the volume would be E_o. σ_z is related with the point where Eq. (49) reaches its maximum value, z_m, in the following way

$$z_m = \sigma_z/\sqrt{2}\ . \tag{50}$$

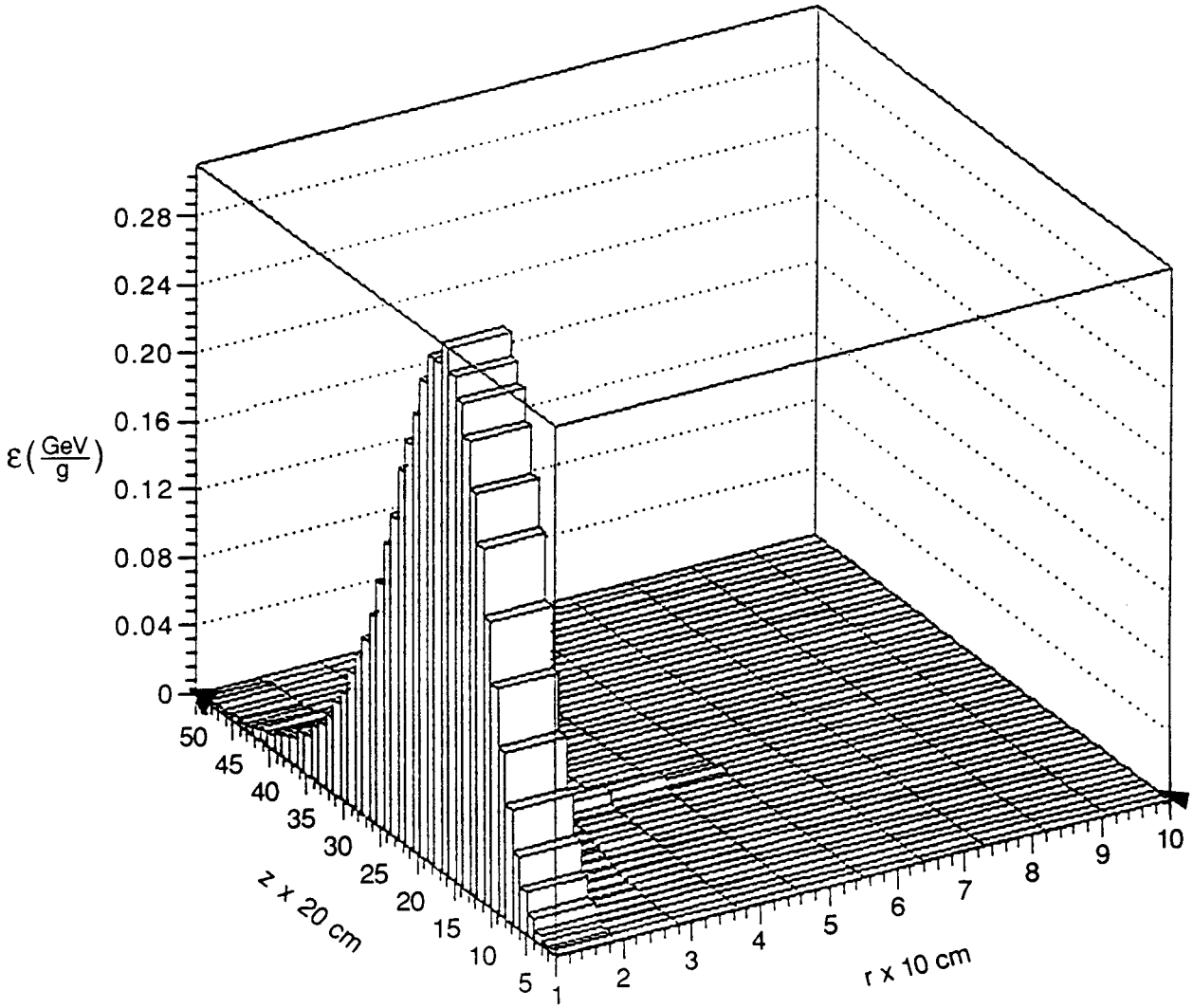

Figure 3. Energy Deposited in a Graphite Cylindrical Block for a 20 TeV Proton.

The convolution needed in Eq. (19) is calculated in Appendix A, Eq. (A9),

$$\int_{-\infty}^{+\infty} \epsilon(z-\xi)\, e^{-q\xi^2}\, d\xi = \frac{E_o^{1/3} e^{-\dfrac{qz^2}{1+q\sigma_z^2}}}{1+q\sigma_z^2}\left[1+\frac{\sqrt{\pi}q\sigma_z z}{\sqrt{1+q\sigma_z^2}}\right] . \tag{51}$$

Using this expression with Eqs. (39) and (19), the solution of the problem is given by

$$\theta(\vec{r},t) = \frac{E_o f_o \sqrt{\rho c_p}}{8[\pi k]^{3/2}} \sum_{i=0}^{n} \frac{H(t-t_i)\, e^{-\dfrac{q(x^2+y^2)}{1+q\sigma^2} - \dfrac{qz^2}{1+q\sigma_z^2}}}{(t-t_i+p\sigma^2)\sqrt{t-t_i}(1+q\sigma_z^2)}\left[1+\frac{\sqrt{\pi}q\sigma_z z}{\sqrt{1+q\sigma_z^2}}\right] , \tag{52}$$

where q and p are given by Eqs. (24) and (26). Note that the peak temperature in the transverse direction still occurs at the point $x = 0$, $y = 0$. However, in the longitudinal direction, for a fixed time, this peak occurs, (see Eq. (A10) in Appendix A, at the point

$$z_* = \frac{\sqrt{1+q\sigma_z^2}}{2\sqrt{\pi}\sigma_z q}[1+\sqrt{1+2\pi q\sigma_z^2}] \ . \tag{53}$$

Note that this value is always higher or equal to Eq. (50), $z_* \geq z_m$. In addition, the following limit is satisfied

$$\lim_{q\to\infty} z_* = z_m \ . \tag{54}$$

The peak value of the temperature is shifted away from the peak value of the longitudinal energy deposited by the amount $z_* - z_m$. Furthermore, for very bad thermal conductors (insulators), the peak temperature and the peak energy deposition are much closer for a longer time than the good thermal conductors (metals), as was expected. The peak temperature, $\theta_o(t) = \theta(0,0,z_*,t)$, is given by

$$\theta_o(t) = \frac{E_o f_o\sqrt{\rho c_p}}{16[\pi k]^{3/2}} \sum_{i=0}^{n} \frac{H(t-t_i)e^{-\frac{1+\sqrt{1+2\pi q\sigma_z^2}}{4\pi q\sigma_z^2}}}{(t-t_i+p\sigma^2)\sqrt{t-t_i}(1+q\sigma_z^2)} \left[3+\sqrt{1+2\pi q\sigma_z^2}\right] \ . \tag{55}$$

For an approximation, it is possible to substitute the summation for an integration although the integration may not be done analytically but numerically. However, in the limit of $q\sigma_z^2 \gg 1$, Eq. (55) can be written as

$$\lim_{q\sigma_z^2\gg 1} \theta_o(t) \simeq \frac{E_o f_o}{4\sqrt{2}\pi k\sigma_z} \sum_{i=0}^{n} \frac{H(t-t_i)}{t+p\sigma^2-t_i} \tag{55a}$$

and can be approximated by an integration which can bring about the solution

$$\theta_o(t) \approx \frac{E_o f_o}{4\sqrt{2}\pi k\sigma_z} \begin{cases} \log\dfrac{t+p\sigma^2}{p\sigma^2} & \text{if } t \leq \tau_n \\ \log\dfrac{t+p\sigma^2}{t+p\sigma^2-\tau_n} & \text{if } t > \tau_n \ . \end{cases} \tag{55b}$$

6.0 LONGITUDINAL EXPONENTIAL DECAY ENERGY DISTRIBUTION

Another type of longitudinal distribution that might be of some interest is that which has an exponential decay, for example

$$\epsilon(z) = \frac{E_o^{1/3}H(z)}{\sigma_z^2} z\ e^{-z/\sigma_z} \ , \tag{56}$$

where the same normalization has been taken as before, and the parameter σ_z is related with the peak value of Eq. (56) by

$$z_m = \sigma_z \ . \tag{57}$$

Using the result of Eq. (A11) of Appendix A with Eqs. (39) and (19), the following solution is obtained

$$\theta(\vec{r},t) = \frac{E_o f_o \sqrt{\rho c_p}}{2 \cdot 8[\pi k]^{3/2}} \sum_{i=0}^{n} \frac{H(t-t_i) e^{-\frac{q(x^2+y^2)}{1+q\sigma^2} - \frac{z}{\sigma_z} + \frac{1}{4q\sigma_z^2}}}{(t-t_i+p\sigma^2)\sqrt{t-t_i} 2q\sigma_z^2} \left[1 + 2\sqrt{\pi q}\,(z - \frac{1}{4q\sigma_z})\right] , \tag{58}$$

As before, most of the time the peak temperature does not coincide with the peak energy deposition. This peak occurs at

$$z_* = \sigma_z + \frac{1}{4q\sigma_z} - \frac{1}{2\sqrt{\pi q}} \tag{59}$$

which has the following limit

$$\lim_{q \to \infty} z_* = z_m \ . \tag{60}$$

The peak temperature, $\theta_o(t) = \theta(0,0,z_*)$, is given by

$$\theta_o(t) = \frac{E_o f_o}{8\pi e k \sigma_z} \sum_{i=0}^{n} \frac{H(t-t_i) e^{1/\sqrt{4\pi q \sigma_z^2}}}{(t-t_i+p\sigma^2)} \ . \tag{61}$$

If the same approximations as in Eq. (55a) and (55b) were made, the solution would be the same form of Eq. (55b) but multiplied by a factor of $1/e\sqrt{2}$.

7.0 LONGITUDINAL DEPENDENCE OF THE TRANSVERSAL DISTRIBUTION

Energy deposition from hadronic or electromagnetic showers has, in addition, other complications. The transverse size of the energy deposited grows with the penetration of the shower (see Figure 4), *i.e.*, if a Gaussian distribution in the transverse direction is assumed, the standard deviation of the energy deposited will depend on the longitudinal coordinate, z. Therefore, using a transverse Gaussian distribution and arbitrary distribution, $\epsilon(z)$, in the longitudinal direction, it follows from Eqs. (19) and (A7)

$$\theta(\vec{r},t) = \frac{E_o f_o \sqrt{\rho c_p}}{8[\pi k]^{3/2}} \sum_{i=0}^{n} \frac{H(t-t_i)}{\sqrt{t-t_i}} \int_{-\infty}^{+\infty} \frac{\epsilon(\xi) e^{-\frac{q(x^2+y^2)}{1+q\sigma(\xi)^2} - q(z-\xi)^2}}{t-t_i+p\sigma(\xi)^2} \, d\xi \ . \tag{62}$$

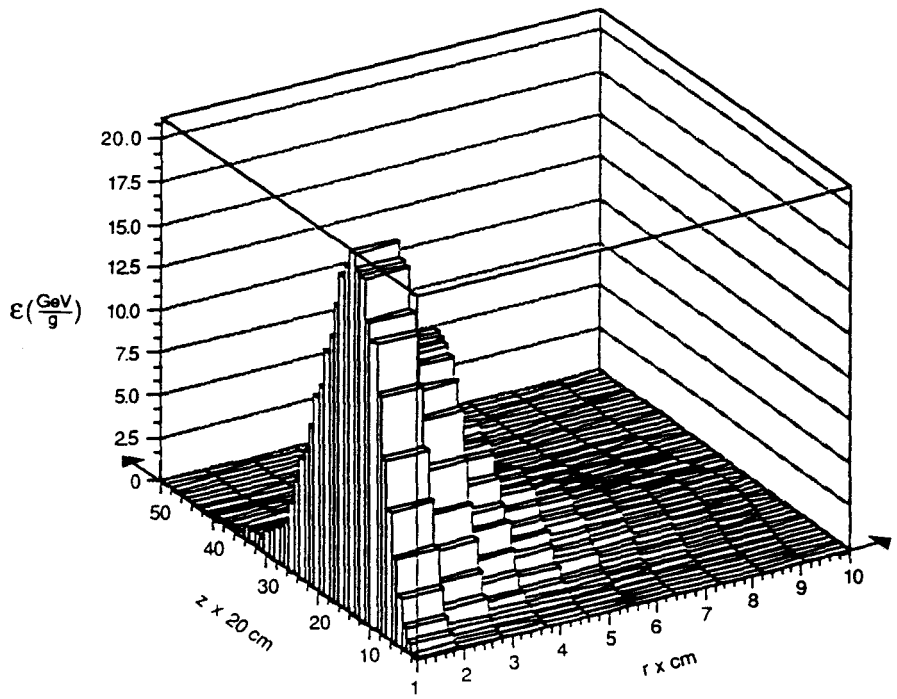

Figure 4a. Distribution of Energy Deposited for a 20 TeV Proton in Graphite.

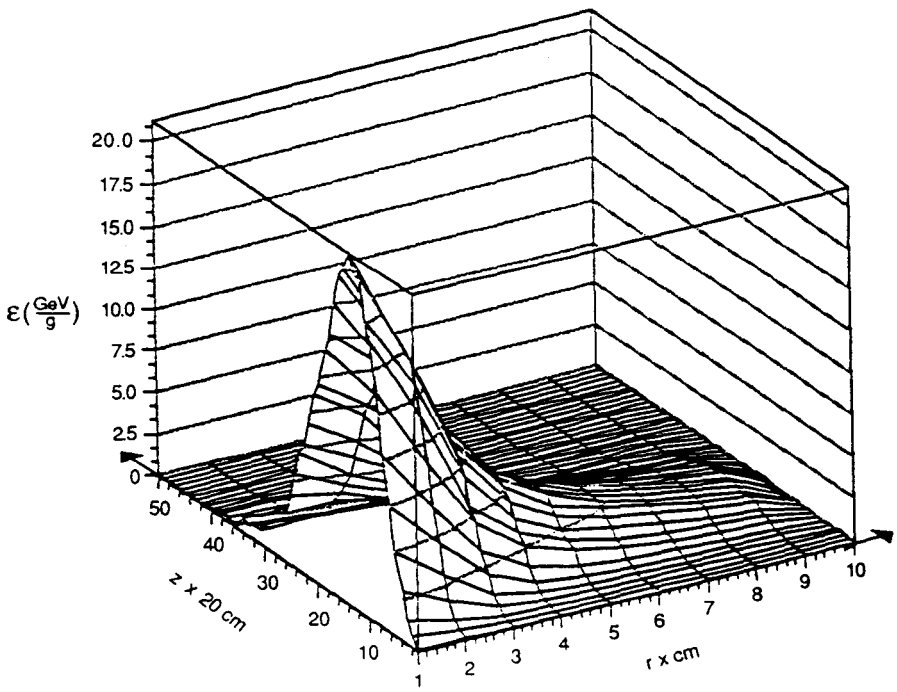

Figure 4b. Specific Energy Surface Distribution in Graphite for a 20 TeV Proton in Graphite.

This complicated expression can be solved numerically once $\sigma(z)$ is known. The peak temperature in the transverse direction still remains at the point $x = y = 0$, but the longitudinal peak depends on $\sigma(z)$. Defining $\theta_{oo}(z,t) = \theta(0,0,z,t)$, the peak temperature in the transverse direction is given by

$$\theta_{oo}(z,t) = \frac{E_o^{2/3} f_o \sqrt{\rho c_p}}{8[\pi k]^{3/2}} \sum_{i=0}^{n} \frac{H(t-t_i)}{\sqrt{t-t_i}} \int_{-\infty}^{+\infty} \frac{\epsilon(\xi) e^{-q(z-\xi)^2}}{[t-t_i+p\sigma(\xi)^2]} \, d\xi \ . \tag{63}$$

This expression is easier than Eq. (62). However, a numerical integration is required to calculate it. For example, using the distribution from Eq. (56), the following result can be demonstrated

$$\begin{aligned} \theta_{oo}(z,t) =& \frac{E_o f_o}{4\sigma_z^2 \pi^{3/2} k} \sum_{i=0}^{n} H(t-t_i) \ e^{-z/\sigma_z q + 1/(2\sigma_z q)^2} \\ & \times \int_0^{+\infty} \frac{\left[\beta/\sqrt{q} + z - 1/2\sigma_z q\right] e^{-\beta^2} \ d\beta}{t - t_i + p[\sigma(\beta/\sqrt{q} + z - 1/2\sigma_z q)]^2} \ . \end{aligned} \tag{64}$$

8.0 COOLING CONSIDERATION

Suppose that there is a uniform cooling such that it carries away a certain amount of energy deposited in the material. This cooling could be represented as a sink of power which is proportional to the difference in temperatures between the material and the batch, θ. Considering the material and the cooling substance as a homogeneous single material with average (ρc_p), total k, and constant of proportionality h for the cooling factor, the heat equation (9) becomes

$$(\rho c_p) \frac{\partial \theta}{\partial t} = k \ \nabla^2 \theta - h\theta + \epsilon(\vec{r}) f_o \sum_{i=0}^{n} \delta(t - t_i) \ . \tag{65}$$

Following the same procedure as in Section 4, the general solution of this equation is

$$\theta(\vec{r},t) = \frac{f_o}{(\rho c_p)} \sum_{i=0}^{n} \left[\frac{q}{\pi}\right]^{3/2} H(t-t_i) e^{-h(t-t_i)/(\rho c_p)} \int \epsilon(\vec{r} - \vec{\xi}) e^{-q\xi^2} d^3\vec{\xi} \ , \tag{66}$$

where q is given by Eq. (22). Therefore, if the cooling effect (of this type) was taken into consideration in the above examples, the solution would contain the factor $e^{-h(t-t_i)/(\rho c_p)}$. Before the next energy deposition events, the temperature drops by a factor of $e^{-h/f_o(\rho c_p)}$. Thus, for the characteristic bunch-to-bunch space frequency in the SSC ($f_o = 60 \ MHz$) and at room temperature, $f_o(\rho c_p) \sim 2 \times 10^8 \ W/cm^3 K$. Therefore, cooling effect is not

important for this event. Taken the limit $q \to \infty$ (or k equal to zero), the next expected limit is obtained

$$\lim_{q\to\infty} \theta(\vec{r},t) = \frac{f_o \epsilon(\vec{r})}{(\rho c_p)} \sum_{i=0}^{n} H(t-t_i) e^{-h(t-t_i)/(\rho c_p)} \ , \tag{67}$$

and wherever h is equal to zero, the solutions of Eqs. (19) or (24) are recovered. For small h (more precisely $h\tau_n/(\rho c_p) < 1$) and using the approximation of Eq. (6)

$$\theta(t) = \begin{cases} \frac{\epsilon f_o \ t}{(\rho c_p)}[1 - ht/2(\rho c_p)] & \text{if } t \leq \tau \\ \frac{\epsilon f_o \ \tau_n \ e^{-ht/(\rho c_p)}}{(\rho c_p)} & \text{if } t > \tau_n \ . \end{cases} \tag{68}$$

Thus, there is a decrease in the peak temperature and an exponential fall off of the temperature. The constant h must be determined experimentally.

9.0 CONCLUSION

The analytical expressions for temperature rise during energy deposition events were calculated through the solution of the time dependent heat equation, using the Fourier transformation approach. This calculation was made for several energy distributions. The solutions depend on the frequency of energy deposition. If this frequency is high, the solutions correspond to the instantaneous temperature rise. At the characteristic frequency of the bunch spacing (60 MHz) in the SSC, there is a clear justification to use the instantaneous temperature rise for energy deposition calculations at room temperature.

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APPENDIX A

A.1 THE FOURIER TRANSFORMATION

The time Fourier transformation of a function $f(t)$ is defined by

$$f_\alpha = F[f(t)] = \int_{-\infty}^{+\infty} e^{-i\alpha t} f(t)\, dt \ , \tag{A1}$$

and its inverse transformation is defined by

$$f(t) = F^{-1}[f_\alpha(\alpha)] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\alpha t} f_\alpha(\alpha)\, d\alpha \ . \tag{A2}$$

If $f(\alpha)$ is a even function, $f(-\alpha) = f(\alpha)$, then it follows

$$F^{-1}[f(\alpha)] = \frac{1}{2\pi} F[f(-\alpha)] \ . \tag{A3}$$

In three dimensions the definition may be given by

$$g_\lambda = F[g(\vec{r})] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-i\vec{\lambda}\cdot\vec{r}} g(\vec{r})\, d^3\vec{r} \ , \tag{A4}$$

where the vectors $\vec{\lambda}$ and $\vec{r}$ are $\vec{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ and $\vec{r} = (x_1, x_2, x_3) = (x, y, z)$. The inverse transformation is written as

$$g(\vec{r}) = F^{-1}[g_\lambda(\vec{\lambda})] = \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i\vec{\lambda}\cdot\vec{r}} g_\lambda(\vec{\lambda})\, d^3\vec{\lambda} \ . \tag{A5}$$

A.2 INVERSE FOURIER TRANSFORMATION OF A GAUSSIAN DISTRIBUTION

$$
\begin{aligned}
F^{-1}[e^{-\frac{k(t-t_i)}{\rho c_p}\lambda^2}] &= \frac{1}{(2\pi)^3}\prod_{j=1}^{3}\int_{-\infty}^{+\infty} e^{i\lambda_j x_j - \frac{k(t-t_i)}{\rho c_p}\lambda_j^2}\, d\lambda_j \\
&= \frac{1}{(2\pi)^3}\prod_{j=1}^{3} e^{-\frac{\rho c_p x_j^2}{4k(t-t_i)}} \int_{-\infty}^{+\infty} e^{-\frac{k(t-t_i)}{\lambda c_p}[\lambda_j - i\frac{x_j \rho c_p}{2k(t-t_i)}]^2}\, d\lambda_j \\
&= \frac{1}{(2\pi)^3}\prod_{j=1}^{3} \frac{e^{-\frac{\rho c_p x_j^2}{4k(t-t_i)}}}{\left[\frac{k(t-t_i)}{\delta cp}\right]^{1/2}} \int_{-\infty}^{+\infty} e^{-\xi^2}\, d\xi \\
&= \frac{1}{(2\pi)^3}\prod_{j=1}^{3} \frac{\sqrt{\pi}\, e^{-\frac{\rho c_p x_j^2}{4k(t-t_i)}}}{\left[\frac{k(t-t_i)}{\delta cp}\right]^{1/2}} \\
&= \frac{1}{8\pi^{3/2}\left[\frac{k(t-t_i)}{\rho c_p}\right]^{3/2}}\, e^{-\frac{\rho c_p}{4k(t-t_i)}\, r^2}.
\end{aligned}
\tag{A6}
$$

A.3 CONVOLUTION WITH A GAUSSIAN DISTRIBUTION

$$
\begin{aligned}
\int \epsilon(\vec{r}-\vec{\xi}) e^{-q\xi^2}\, d^3\vec{\xi} &= E_o \prod_{j=1}^{3} \frac{1}{\sqrt{\pi}\sigma_j} \int_{-\infty}^{+\infty} e^{-(x_j-\xi_j)^2/\sigma_j^2 - q\xi_j^2}\, d\xi_j \\
&= E_o \prod_{j=1}^{3} \frac{e^{-qx_j^2 + \frac{q^2 x_j^2}{q+1/\sigma_j^2}}}{\sqrt{\pi}\sigma_j} \int_{-\infty}^{+\infty} e^{-(q+\frac{1}{\sigma_j^2})[\xi_j - \frac{qx_j}{q+\frac{1}{\sigma_j^2}}]^2}\, d\xi_j \\
&= E_o \prod_{j=1}^{3} \frac{e^{\frac{-qx_j^2}{1+q\sigma_j^2}}}{\sqrt{\pi}\sigma_j\sqrt{q+\frac{1}{\sigma_j^2}}} \int_{-\infty}^{+\infty} e^{-\rho^2}\, d\rho \\
&= E_o \prod_{j=1}^{3} \frac{e^{-\frac{qx_j^2}{1+q\sigma_j^2}}}{\sqrt{1+q\sigma_j^2}}.
\end{aligned}
\tag{A7}
$$

A.4 LIMIT VALUE OF THE FUNCTION $\frac{1}{\beta}\log\frac{x-\beta}{x+\beta}$

$$\begin{aligned}
\lim_{\beta\to 0}\frac{1}{\beta}\log\frac{x-\beta}{x+\beta} &= \lim_{\beta\to 0}\frac{d}{d\beta}\log\frac{x-\beta}{x+\beta} \\
&= \lim_{\beta\to 0}\frac{\frac{d}{d\beta}\log\frac{x-\beta}{x+\beta}}{\frac{x-\beta}{x+\beta}} \\
&= \lim_{\beta\to 0}\frac{-2x}{x^2-\beta^2} \\
&= -\frac{2}{x}.
\end{aligned} \tag{A8}$$

A.5 CONVOLUTION WITH A MAXWELL-LIKE DISTRIBUTION

$$\begin{aligned}
\int_{-\infty}^{+\infty}\epsilon_3(\xi)e^{-q(z-\xi)^2}d\xi &= \frac{2E_o^{1/3}}{\sigma_z^2}\int_0^{+\infty}\xi\ e^{-\xi^2/\sigma_z^2-q(z^2-2z\xi+\xi^2)}\ d\xi \\
&= \frac{2E_o\ e^{-\frac{qz^2}{1+q\sigma_z^2}}}{\sigma_z^2}\int_0^{+\infty}\xi e^{-(q+\frac{1}{\sigma_z^2})[\xi-\frac{qz}{q+\frac{1}{\sigma_z^2}}]^2}\ d\xi \\
&= \frac{2E_o^{1/3}e^{-\frac{qz^2}{1+q\sigma_z^2}}}{\sigma_z^2\sqrt{q+\frac{1}{\sigma_z^2}}}\int_0^{+\infty}d\rho\Big[\frac{\rho}{\sqrt{q+\frac{1}{\sigma_z^2}}}+\frac{qz}{q+\frac{1}{\sigma_z^2}}\Big]e^{-\rho^2} \\
&= \frac{2E_o^{1/3}e^{-\frac{qz^2}{1+q\sigma_z^2}}}{\sigma_z^2\sqrt{q+\frac{1}{\sigma_z^2}}}\Big[\frac{1}{2\sqrt{q+\frac{1}{\sigma_z^2}}}+\frac{\sqrt{\pi}qz}{2(q+\frac{1}{\sigma_z^2})}\Big] \\
&= \frac{E_o^{1/3}\ e^{-\frac{qz^2}{1+q\sigma_z^2}}}{1+q\sigma_z^2}\Big[1+\frac{\sqrt{\pi}q\sigma_z z}{\sqrt{1+q\sigma_z^2}}\Big].
\end{aligned} \tag{A9}$$

A.6 LIMIT VALUE OF z_*

$$\lim_{q\sigma_z^2>>1} z_* = \lim_{q\sigma_z^2>>1}\frac{\sqrt{1+q\sigma_z^2}(1+\sqrt{1+2\pi q\sigma_z^2}}{2\sqrt{\pi}q\sigma_z} = \frac{\sigma_z}{\sqrt{2}}. \tag{A10}$$

A.7 CONVOLUTION WITH AN EXPONENTIAL DECAY DISTRIBUTION

If the distribution is given by Eq. (56), it follows

$$
\begin{aligned}
\int_{-\infty}^{+\infty} \epsilon_3(\xi) e^{-q(z-\xi)^2} d\xi &= \frac{E_o^{1/3}}{\sigma_z^2} \int_0^{+\infty} \xi e^{-\frac{z}{\xi} - q(z^2 - 2z\xi + \xi^2)} d\xi \\
&= \frac{E_o^{1/3}\, e^{-qz^2 + \frac{q}{4}(2z - \frac{1}{\sigma_z q})^2}}{\sigma_z^2} \int_0^{+\infty} \xi e^{-q[\xi - z + \frac{1}{2\sigma_z q}]^2} d\xi \\
&= \frac{E_o^{1/3} e^{-\frac{z}{\sigma_z} + \frac{1}{4q\sigma_z^2}}}{\sigma_z^2 \sqrt{q}} \int 0^{+\infty} \Big[\frac{\rho}{\sqrt{q}} + z - \frac{1}{2\sigma_z q}\Big] e^{-\rho^2} d\rho \\
&= \frac{E_o^{1/3} e^{-\frac{z}{\sigma_z} + \frac{1}{4q\sigma_z^2}}}{\sigma_z^2 \sqrt{q}} \Big[\frac{1}{2\sqrt{q}} + \frac{\sqrt{\pi}}{2}(z - \frac{1}{2\sigma_z q})\Big] \\
&= \frac{E_o^{1/3} e^{-\frac{z}{\sigma_z} + \frac{1}{4q\sigma_z^2}}}{2\sigma_z^2 q} \Big[1 + 2\sqrt{\pi q}(z - \frac{1}{4q\sigma_z})\Big] \,.
\end{aligned}
\tag{A11}
$$

APPENDIX B
ARBITRARY TIME DEPENDENT ENERGY DEPOSITION EVENTS

In this case, the heat equation is given by

$$(\rho c_p)\frac{\partial\theta}{\partial t} = k\nabla^2\theta + \sum_{i=0}^{n} f_i\epsilon_i(\vec{r})\Omega_i(t) \ , \tag{B1}$$

where $f_i = 1/(t_i - t_{i-1})$ is the frequency time separation per events that produces an energy deposition distribution $\epsilon_i(\vec{r})$ which evolves in time in the form $\Omega_i(t)$. The spatial Fourier transformation of Eq. (B1) brings about the equation

$$\frac{d\theta_\lambda}{dt} + a\theta_\lambda = \frac{1}{(\rho c_p)}\sum_{i=0}^{n} f_i\epsilon_i(\lambda)\Omega_i(t) \ , \tag{B2}$$

where a is defined by Eq. (11). The time Fourier transformation of Eq. (B2) gives

$$\theta_{\lambda\alpha} = \frac{1}{\rho c_p} = \sum_{i=0}^{n} f_i\epsilon_i(\lambda)\frac{\Omega_i(\alpha)}{-i\alpha + a} \ . \tag{B3}$$

Using the time-inverse Fourier transformation and the convolution theorem in Eq. (B3), it follows

$$\theta_\lambda = \frac{1}{\rho c_p}\sum_{i=0}^{n} f_i\epsilon_i(\lambda)\int_{-\infty}^{+\infty} H(t-\tau)e^{-\frac{k(t-\tau)}{\delta cp}\lambda^2}\Omega_i(\tau)\ d\tau \ . \tag{B4}$$

Finally, using the spatial inverse Fourier transformation and the convolution theorem in Eq. (B4), gives

$$\theta(\vec{r},t) = \frac{\sqrt{\rho c_p}}{8[\pi k]^{3/2}}\sum_{i=0}^{n} f_i\int_{-\infty}^{+\infty} d\tau\frac{H(t-\tau)\Omega_i(\tau)}{[t-\tau]^{3/2}}\int \epsilon_i(\vec{r}-\vec{\xi})e^{-\frac{\rho c_p\xi^2}{4k(t-\tau)}}d^3\vec{\xi} \ . \tag{B5}$$