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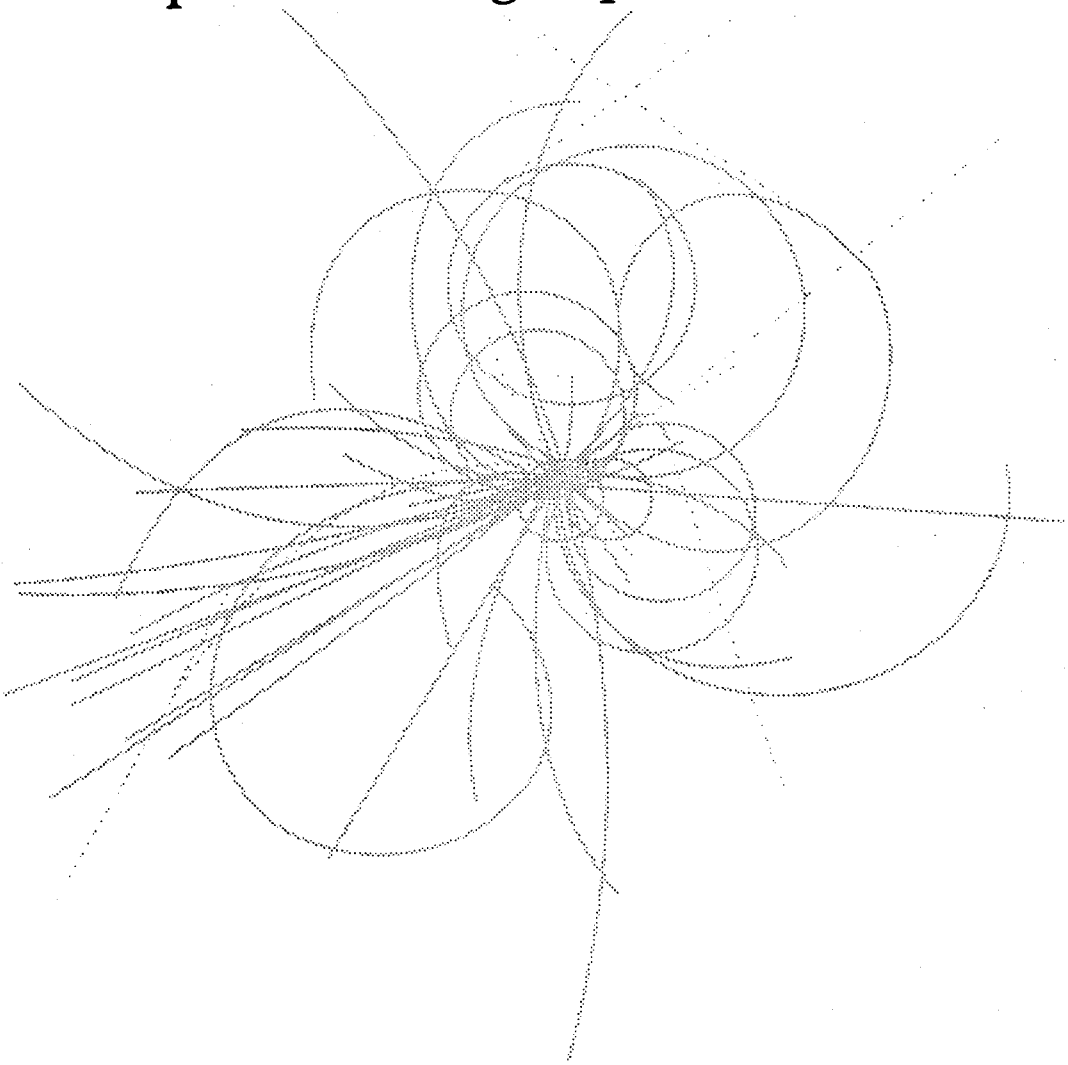

First Order Tune Shift Calculations for Transverse Betatron Dynamics

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First order tune shift calculations for transverse betatron dynamics†

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An effective Hamiltonian, with non-linear magnetic multipole terms and momentum dispersion contributions, is used to obtain the first order tune-shift results for transverse betatron motion for protons in the Superconducting Super Collider (SSC). This Hamiltonian is represented in terms of action angle variables, and analytical results are obtained using symbolic algebra methods. Mathematical derivations of the transverse multipole expansion and of the transverse betatron equations, using an invariant action and curvilinear coordinates, are given in the appendices. Numerical and graphical tune-space results are given that illustrate the dependence of tune-shifts on injection amplitude and momentum spread.

Tuneshift.tex

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I. INTRODUCTION

When protons are injected into the SSC they execute transverse betatron oscillation with betatron numbers (tunes) ν_x and ν_y. In a linear machine, if a proton of initial phase ϕ_0 is observed at the same fixed point along its path for j rotations around a ring of radius R, a transverse position coordinate follows the sinusoidal law $\cos(j\nu_0\omega_r T_0 + \phi_0)$. The phase advance for one rotation is $2\pi\nu_0$, and the betatron frequency is $\omega_r = c/R$, where c is the speed of light. The period for one rotation is $T_0 = 2\pi R/c$. Non-linear multipole contributions to the transverse magnetic field of the SSC dipole magnets can produce changes (tune-shifts) in the transverse betatron numbers. These changes can drive the tunes into regions of ν_x, ν_y tune-space where resonance lines are present. This can result in particle loss and degradation of the dynamic aperture of the beam. Such effects are seen clearly in computer simulation studies [1] done using the tracking code SSCTRK [2]. Using Fourier analysis methods, one can extract values for the betatron tunes from the data produced by SSCTRK. However, it is desirable to have an analytical method which will show the pattern in tune-space produced by particles with different injection positions (x, y) and with different values of momentum spread $\delta|\vec{p}|/\vec{p}|$, where $\delta|\vec{p}| = |\vec{p}| - |\vec{p}_0|$ is the difference between the proton scalar momentum $|\vec{p}|$ and the ideal orbit scalar momentum $|\vec{p}_0|$. The corrections to betatron tune-shifts resulting from higher order magnetic multipole contributions are calculated in first order perturbation theory using a non-linear effective Hamiltonian, represented in terms of action angle variables. The analytical results are obtained from this Hamiltonian using symbolic computational methods, and numerical results are obtained from these expressions. The analytical results are obtained from the REDUCE [3] program TSHIFT which is described in this paper. The results from this program are used in the numerical program TUNESHIFT.FOR to produce graphical results in the Top Drawer [4] file

TUNESHIFT.TOP. These methods have been successfully used to study the spread in tune-space associated with magnetic peaking [5]. Sample graphs resulting from these programs are given in the text, and the method for generating these results is described. For the numerical results, multipole contributions through B_{10} are included; however, the REDUCE code can be used to generate higher order contributions if these are desired. The numerical code TUNESHIFT.FOR is driven by and input file which contains data on the operating point (ν_{0x}, ν_{0y}) for the nominal transverse betatron tunes, values of the multipoles B_1 through B_{10}, and momentum offset $\delta|\vec{p}|/\vec{p}|$.

II. THE ACTION ANGLE EFFECTIVE HAMILTONIAN

The non-linear magnetic field multipole contributions are found from the multipole expansion for the transverse magnetic fields of the SSC dipole magnets

$$B_x(x,y,\bar{s}) + iB_y(x,y,\bar{s}) = \sum_{n=0}(B_n(\bar{s}) + A_n(\bar{s}))(x+iy)^n, \tag{2.1}$$

where $B_n(\bar{s})$ and $A_n(\bar{s})$ are, respectively, the normal multipole and skew-multipole coefficients. As described in Appendix A, these transverse fields are obtained from the real part of the scalar function $A(z,\bar{s})$ according to

$$B_y(x,y,\bar{s}) + iB_x(x,y,\bar{s}) = \bar{\mathcal{D}} Re A(z,\bar{s}), \tag{2.2}$$

where

$$\bar{\mathcal{D}} = \partial_x - i\partial_y,$$

and $z = x + iy$. The coordinates x, y, and $\bar{s}$ are defined in Appendix C.

In terms of the transverse coordinates x and y, the first order perturbed effective Hamiltonian, found from (C38), is written as

$$H = \frac{p_x^2}{2} + \frac{p_y^2}{2} + \frac{|\vec{p}_0|}{|\vec{p}|}\frac{K_x(\bar{s})}{2}x^2 - \frac{|\vec{p}_0|}{|\vec{p}|}\frac{K_y(\bar{s})}{2}y^2 + Re\mathcal{A}(z,\bar{s}) + \frac{|\vec{p}_0| - |\vec{p}|}{|\vec{p}|}\frac{x}{\rho(\bar{s})}, \tag{2.3}$$

with

$$Re\mathcal{A}(z,\bar{s}) = Re\sum_{n=0}\frac{\bar{B}_0(\bar{s})}{B(\bar{s})\rho(\bar{s})}\frac{(b_n+ia_n)}{(n+1)}(x+iy)^{n+1}. \tag{2.4}$$

In the above, use is made of the notations

$$K_x(\bar{s}) = \frac{1}{\rho^2(\bar{s})} + \frac{(e/c)\bar{B}_1(\bar{s})}{2|\vec{p}_0|}), \text{ and } K_y(\bar{s}) = \frac{(e/c)\bar{B}_1(\bar{s})}{2|\vec{p}_0|}), \tag{2.5}$$

and the relations $eB(\bar{s})\rho(\bar{s}) = c|\vec{p}|$, and $e\bar{B}_0(\bar{s})\rho(\bar{s}) = c|\vec{p}_0|$. The expression (2.4) represents the error coefficients associated with the multipole components within the dipole magnets of mean strength $\bar{B}_0$. The error coefficients associated with the mean values $\bar{B}_0$ and $\bar{B}_1$ are represented by the expressions

$$B_0(\bar{s}) = \bar{B}_0(\bar{s})(1+b_0)$$

$$B_1(\bar{s}) = \bar{B}_1(\bar{s}) + \bar{B}_0(\bar{s})b_1 \tag{2.6}$$

The other components of the error field are given by

$$B_n(\bar{s}) = \bar{B}_0(\bar{s})b_n \text{ and } A_n(\bar{s}) = \bar{B}_0(\bar{s})a_n, \tag{2.7}$$

where b_n and a_n are, respectively, a measure of the normal multipole and skew-multipole error fields. In deriving (2.3), terms of order B_0x, and B_1x^2 have been retained. However, terms of order B_0x^2/ρ, B_1x^3/ρ, $B_2x^4/\rho\ldots$, and A_0xy/ρ, $A_1x^2y/\rho\ldots$ have been neglected.

An additional non-linear contribution occurs as a result of having sextupole chromatic correctors [6]. This correction is of the form

$$\frac{|\vec{p}_0|}{|\vec{p}|}\frac{S(\bar{s})}{6}(x^3-3xy^2), \tag{2.8}$$

with

$$S(\bar{s}) = \frac{e}{|\vec{p}_0|c}B_2(\bar{s}). \tag{2.9}$$

Neglecting the error field contribution (2.4) and considering only the effect of the sextupole chromatic corrector, the effective Hamiltonian (2.3) becomes

$$\begin{aligned} H = \frac{p_x^2}{2} + \frac{p_y^2}{2} + (1 - \frac{\delta|\vec{p}|}{|\vec{p}|}) \left(\frac{K_x(\bar{s})}{2} x^2 - \frac{K_y(\bar{s})}{2} y^2 \right) \\ - \frac{\delta|\vec{p}|}{|\vec{p}|} \frac{x}{\rho(\bar{s})} + (1 - \frac{\delta|\vec{p}|}{|\vec{p}|}) \frac{S(\bar{s})}{6} (x^3 - 3xy^2). \end{aligned} \tag{2.10}$$

Following the method described in Appendix B, one finds the periodic closed orbit solution for an off-momentum particle to be of the form

$$x_c = \frac{\delta|\vec{p}|}{|\vec{p}|} D(\bar{s}, |\vec{p}|), \tag{2.11}$$

where the periodic dispersion function $D(\bar{s}, |\vec{p}|)$ satisfies

$$D''(\bar{s}, |\vec{p}|) + (1 - \frac{\delta|\vec{p}|}{|\vec{p}|}) K_x(\bar{s}) D(\bar{s}, |\vec{p}|) + (1 - \frac{\delta|\vec{p}|}{|\vec{p}|}) \frac{S(\bar{s})}{2} D^2(\bar{s}, |\vec{p}|) = \frac{1}{\rho(\bar{s})}. \tag{2.12}$$

At this point it is useful to perform a canonical transformation to eliminate certain terms which are linear in $\delta|\vec{p}|/\vec{p}|$. This is accomplished with the use of the generating function

$$F_2(x, p_{x_0}) = (x - D(\bar{s}, |\vec{p}|) \frac{\delta|\vec{p}|}{|\vec{p}|})(p_{x_0} + D'(\bar{s}, |\vec{p}|) \frac{\delta|\vec{p}|}{|\vec{p}|}), \tag{2.13}$$

which gives the transformed relations

$$\begin{aligned} x &= x_0 + D(\bar{s}, |\vec{p}|) \frac{\delta|\vec{p}|}{|\vec{p}|} \\ p_x &= p_{x_0} + D'(\bar{s}, |\vec{p}|) \frac{\delta|\vec{p}|}{|\vec{p}|} \\ H_0 &= H + \frac{\partial F_2}{\partial \bar{s}}. \end{aligned} \tag{2.14}$$

When these are used with (2.10), one finds

$$\begin{aligned} H_0 = \frac{p_x^2}{2} + \frac{p_y^2}{2} + \frac{K_x(\bar{s})}{2} x^2 - \frac{K_y(\bar{s}))}{2} y^2 + \frac{S(\bar{s})}{6} ({x_0}^3 - 3x_0 y^2) \\ - \frac{\delta|\vec{p}|}{|\vec{p}|} \left[(S(\bar{s}) D(\bar{s}, |\vec{p}|) + K_x(\bar{s})) \frac{{x_0}^2}{2} - (S(\bar{s}) D(\bar{s}, |\vec{p}|) + K_y(\bar{s})) \frac{y^2}{2} + \frac{S(\bar{s})}{6} ({x_0}^3 - 3x_0 y^2) \right] \\ + \left(\frac{\delta|\vec{p}|}{|\vec{p}|} \right)^2 \frac{S(\bar{s}) D(\bar{s}, |\vec{p}|)}{2} ({x_0}^2 - y^2). \end{aligned} \tag{2.15}$$

For the SSC $K_x(\bar{s}) \sim K_y(\bar{s})$, and one can choose the value of $S(\bar{s})$ to eliminate the first two $\delta|\vec{p}|/\vec{p}|$ dependent terms in (2.15). The remaining $\delta|\vec{p}|/\vec{p}|$ dependent term will not contribute to the first order tune-shifts as a result of the averaging process described later in the text. The remaining term coming from the sextupole correction is

$$-\left(\frac{\delta|\vec{p}|}{|\vec{p}|}\right)^2 \frac{K_x(\bar{s})}{2}({x_0}^2 - y^2), \tag{2.16}$$

and since $K_x(\bar{s}) \sim 1.6 \times 10^{-4}$ for the SSC this term will give a contribution which is small when compared to the $\delta|\vec{p}|/|\vec{p}|$ sextupole correction errors which occur in a real machine.

To obtain the first order tune-shifts, it is convenient to introduce the generating function

$$\begin{aligned} F_1(x_0, \phi_x, y, \phi_y) = &-\frac{{x_0}^2}{2\beta_{x_0}(\bar{s})}\left(\tan\phi_{x_0} - \frac{\beta'_{x_0}(\bar{s})}{2}\right) \\ &-\frac{y^2}{2\beta_y(\bar{s})}\left(\tan\phi_y - \frac{\beta'_y(\bar{s})}{2}\right) \end{aligned} \tag{2.17}$$

and to express the effective Hamiltonian obtained from (2.3), using

$$H_1 = H_0 + \frac{\partial F_1(x_0, \phi_{x_0}, y, \phi_y)}{\partial \bar{s}}, \tag{2.18}$$

in terms of action angle variables J_{x_0}, ϕ_{x_0} and J_y, ϕ_y, which are related to x, y, p_{x_0}, and p_y according to

$$\begin{aligned} x &= \sqrt{2\beta_{x_0}J_{x_0}}\cos\phi_{x_0} + \eta(\bar{s})\frac{\delta|\vec{p}|}{|\vec{p}|} \\ y &= \sqrt{2\beta_y J_y}\cos\phi_y \\ p_{x_0} &= -\sqrt{2J_{x_0}/\beta_{x_0}}(\sin\phi_{x_0} - \frac{\beta'_{x_0}(\bar{s})}{2}\cos\phi_{x_0}) \\ p_y &= -\sqrt{2J_y/\beta_y}(\sin\phi_y - \frac{\beta'_y(\bar{s})}{2}\cos\phi_y), \end{aligned} \tag{2.19}$$

where $D(\bar{s}, |\vec{p}|) \sim \eta(\bar{s})$ is the momentum independent part of the solution of the dispersion equation (2.12) with $S(\bar{s}) = 0$. The effective Hamiltonian (2.3) becomes

$$H_1(J_{x_0}, J_y, \phi_{x_0}, \phi_y) = \frac{J_{x_0}}{\beta_{x_0}(\bar{s})} + \frac{J_y}{\beta_y(\bar{s})} + Re\mathcal{A}(J_{x_0}, J_y, \phi_{x_0}, \phi_y). \tag{2.20}$$

The first order corrected tunes for transverse betatron motion are obtained from

$$\begin{aligned} \nu_x &= \nu_{0x} + \Delta\nu_x \\ \nu_y &= \nu_{0y} + \Delta\nu_y, \end{aligned} \tag{2.21}$$

where the nominal tunes are

$$\nu_{0x,y} = \frac{1}{2\pi}\int_{\bar{s}}^{\bar{s}+C_0} \frac{d\bar{s}'}{\beta_{x,y}(\bar{s}')}, \tag{2.22}$$

and the tune-shifts are

$$\begin{aligned} \Delta\nu_{x,y} &= \frac{C_0}{2\pi}\frac{\partial < Re\mathcal{A}(J_{x_0}, J_y, \phi_{x_0}, \phi_y)}{\partial J_{x_0,y}} > \\ &= \frac{\partial}{\partial J_{x_0,y}}\frac{1}{C_0}\int_0^{C_0} d\bar{s}\frac{1}{n+1}(\frac{b_n C_0}{2\pi\rho(\bar{s})})(\frac{1}{2\pi})^2\int_0^{2\pi}\int_0^{2\pi} d\phi_{x_0}d\phi_y \\ & Re(\sqrt{(2\beta_{x_0}J_{x_0})}\cos\phi_{x_0} + \eta(\bar{s})\frac{\delta|\vec{p}|}{|\vec{p}|} + i\sqrt{(2\beta_y J_y)}\cos\phi_y)^{n+1}, \end{aligned} \tag{2.23}$$

where $C_0 \sim 2\pi\rho(\bar{s}) \sim 2NL$ for a ring of N cells each of half-length L. The angular averages appearing in this expression are evaluated using

$$\frac{1}{2\pi}\int_0^{2\pi}(\cos\phi)^{2n}d\phi = \frac{1}{2^{2n}}\binom{2n}{n}. \tag{2.24}$$

The integrals involving odd powers of $\cos\phi$ are zero, and as a result of this no terms containing the skew-multipole coefficients $A_n(\bar{s})$ appear after performing the angular integration. The averaging around the circumference C_0 of the SSC involves integrals of the form

$$< \beta_{x_0}\beta_y^2\eta >= \frac{1}{C_0}\int_0^{C_0}\beta_{x_0}(\bar{s})\beta_y^2(\bar{s})\eta(\bar{s})d\bar{s} \sim \frac{1}{L}\int_0^L \beta_{x_0}(\bar{s})\beta_y^2(\bar{s})\eta(\bar{s})d\bar{s}, \tag{2.25}$$

which are evaluated numerically using the program TUNESHIFT.FOR. The symmetry properties and analytical expressions for the lattice functions $\beta_{x_0}(\bar{s})$, $\beta_y(\bar{s})$, and $\eta(\bar{s})$ are given in Appendix D. In a similar manner, an additional contribution to (2.23) is found when the chromatic non-linear term (2.16) is added to (2.20).

III. RESULTS AND CONCLUSIONS

The evaluation of (2.3) is performed using the REDUCE program TSHIFT (see Appendix E). The results agree with those found in [7]; however, there are some errors found in the expressions given in this reference. The REDUCE code has been used to generate FORTRAN code which has been substituted into the numerical code TUNESHIFT.FOR, where corrections up to order B_{10} have been included. The numerical results obtained from TUNESHIFT.FOR are generated from an input file INPUTS.DAT which has values for the multipole errors b_n, $\delta|\vec{p}|/\vec{p}|$ (DELTA), and injection amplitudes x_{max} mm (AMPX) and y_{max} mm (AMPY), which determine the size of the injection grid. In addition, this file contains the nominal tunes dnux and dnuy, the phase advance per cell D(degrees), the lattice tune q, and XIX and XIY, which represent chromaticity tune-shifts errors linear in $\delta|\vec{p}|/\vec{p}|$. These are errors which remain in (2.15) with imperfect cancellation resulting from chromatic correction. For the SSC, these terms are large compared to the quadratic correction (2.16). Particles are injected onto a grid from $(0,0)$ to (x_{max}, y_{max}). Typical result for the points in tune-space are shown in Fig. 1 through Fig. 4, which also show the input file. In these figures, resonance lines up to 7th order are shown, and the cross indicates the location of the nominal SSC tunes $\nu_{0x} = .425$ and $\nu_{0y} = .410$. Fig. 1 and Fig. 2 differ in the values of (x_{max}, y_{max}), which are, respectively, (0.5,0.5) and (0.6,0.6); however, they both result from the same value of $\delta|\vec{p}|/\vec{p}| = 5 \times 10^{-4}$. Comparing these figures, one can see clearly the effect of increased injection amplitude on the location of points in tune-space. Fig. 3 and Fig. 4 have the same injection amplitudes, respectively, as Fig. 1 and Fig. 2; however, they both result from $\delta|\vec{p}|/\vec{p}| = 10 \times 10^{-4}$, and one can see the dependence of the betatron tunes on momentum spread. The result of having a linear momentum error remaining after chromatic correction is seen in

Fig. 5 and Fig. 6, with XIX=XIY=5.0. These are to be compared with Fig. 1 and Fig. 2, respectively, where XIX=XIY=0.0. Here, one can see clearly that a linear momentum error has a large effect on the tune-spread.

In conclusion, one can see that reasonable results are found from the first order calculation. A comparison of the values of ν_x and ν_y obtained from the first order calculation with the values obtained from the tracking code SSCTRK shows good agreement. Results for the injection point $(0.8, 0.8)$, with all $b_n{}'s = 0$ except $b_{10} = 0.05$, and $\delta|\vec{p}|/\vec{p}| = 5. \times 10^{-4}$, are $\nu_x = .426$ and $\nu_y = .409$ from TUNESHIF.FOR, and $\nu_x = .425$ and $\nu_y = .409$ from SSCTRK. Furthermore, one can see the effect of multipole errors on the location of the betatron tunes in tune-space. In the case when the tune-spread crosses lower order resonance lines, one would expect to see a decrease of the dynamic aperture of the SSC during the injection phase. This, indeed, is in agreement with the results obtained from tracking codes.

APPENDIX A: MULTIPOLE EXPANSION FOR TRANSVERSE FIELDS

The transverse magnetic fields are determined easily in the form of complex functions of the variable $z = x + iy$. Defining the operators

$$\mathcal{D} = 2\partial_{\bar{z}}, \qquad \bar{\mathcal{D}} = 2\partial_z, \tag{A1}$$

one finds from

$$x = \frac{z + \bar{z}}{2}, \qquad y = \frac{z - \bar{z}}{2} \tag{A2}$$

that

$$\mathcal{D} = \partial_x + i\partial_y, \qquad \bar{\mathcal{D}} = \partial_x - i\partial_y. \tag{A3}$$

The Laplacian operator is found from (A1) and (A3) as

$$\nabla^2 = \bar{\mathcal{D}}\mathcal{D} = \partial_{xx} + \partial_{yy} = 4\partial_z\partial_{\bar{z}}. \tag{A4}$$

Considering the function

$$f(z) = B_y(x,y) + iB_x(x,y) = \sum_{n=0}^{\infty}(B_n + iA_n)z^n, \tag{A5}$$

one finds

$$\mathcal{D}f(z) = (\nabla \times \vec{B})_3 + i\vec{\nabla} \cdot \vec{B} = 0, \tag{A6}$$

and

$$\bar{\mathcal{D}}\mathcal{D}f(z) = \nabla^2(B_y(x,y) + iB_x(x,y)) = 0. \tag{A7}$$

These equations imply

$$(\vec{\nabla} \times \vec{B})_3 = \vec{\nabla} \cdot \vec{B} = \nabla^2\vec{B} = 0, \tag{A8}$$

and

$$\vec{B} = \vec{\nabla} \times \vec{A}.$$

These give the result

$$\vec{\nabla} \cdot \vec{B} = \vec{\nabla}\vec{\nabla} \cdot \vec{A} - \nabla^2 \vec{A} = 0, \tag{A9}$$

which is satisfied by

$$\vec{A} = (0, 0, A^3(x, y)).$$

Since

$$f(z) = \frac{1}{2}\bar{\mathcal{D}}A(z) = \frac{1}{2}\bar{\mathcal{D}}(A(z) + \bar{A}(z)), \tag{A10}$$

with

$$A(z) = \sum_{n=0}^{\infty} \frac{(B_n + iA_n)z^{n+1}}{n+1}, \tag{A11}$$

one finds

$$B_y(x, y) + iB_x(x, y) = \bar{\mathcal{D}}ReA(z). \tag{A12}$$

This allows the transverse components of the magnetic field to be written as

$$\begin{aligned} B^x(x, y) =& \partial_y A^3(x, y), \\ B^y(x, y) =& -\partial_x A^3(x, y), \end{aligned} \tag{A13}$$

with

$$A^3(x, y) = -ReA(z).$$

APPENDIX B: THE INVARIANT ACTION

The equations of motion for transverse betatron oscillations [8] can be found from a Lorentz invariant Hamiltonian [9]. This Hamiltonian, which is equal to the rest energy of the proton, is also invariant under general curvilinear coordinate transformations [10], and this suggests the introduction of an invariant action. This allows one to represent the special relativistic dynamical equations for betatron motion in the coordinate system related to the curved ideal orbit of a proton in the SSC. In addition, this formulation allows one to explicitly see the various approximations which are made to obtain the usual Courant-Snyder equations. For this invariant Hamiltonian, the Minkowski space contravariant coordinates are defined as the four-vector $q^i = (q^0, q^1, q^2, q^3) = (ct, x, y, z)$, where c is the speed of light. The contravariant components of the four-velocity are defined to be

$$u^i = dq^i/ds, \tag{B1}$$

and those of the four-momentum for a particle of rest mass m are

$$p^i = mcu^i, \tag{B2}$$

where ds is the invariant measure

$$ds = \sqrt{dq_i dq^i}, \tag{B3}$$

where use is made of the Einstein summation convention. The covariant differential components of a four-vector are found from

$$dq_i = g_{ij} dq^j. \tag{B4}$$

The non-zero Minkowski space components of the covariant metric tensor g_{ij} are

$$(g_{oo}, g_{11}, g_{22}, g_{33}) = (1, -1, -1, -1). \tag{B5}$$

The contravariant components of the metric tensor are found from the relation

$$g^{ij}g_{jk} = \delta^i_k \tag{B6}$$

to be

$$(g^{oo}, g^{11}, g^{22}, g^{33}) = (1, -1. - 1, -1). \tag{B7}$$

The invariant Hamiltonian is defined as

$$H(p^i, q^i)/c = p_i u^i - L(u^i, q^i) + mc, \tag{B8}$$

where the invariant Lagrangian for the motion of a particle of mass m and electric charge e in an electromagnetic field with four-potential $A(q)^i$ is defined as

$$L(u^i, q^i) = mc\sqrt{u_i u^i} + (e/c)A(q)_i u^i. \tag{B9}$$

The conjugate four-momentum is

$$p_i = \frac{\partial L}{\partial u^i} = \frac{mcu_i}{\sqrt{u_i u^i}} + (e/c)A(q)_i, \tag{B10}$$

and the Lagrangian (B9) becomes

$$L(u^i, q^i) = p_i u^i. \tag{B11}$$

For this Lagrangian, the action which is invariant under curvilinear coordinate transformations is defined as

$$S = \int L(u^i, q^i) ds. \tag{B12}$$

Using the method of stationary action and the approximations described in Appendix C, one finds equations (C41) and (C42), which describe transverse betatron motion.

As shown in Appendix C, the approximate Hamiltonian for this system may be written in terms of the transverse coordinates x and y and the local radius of curvature $\rho(\bar{s})$ of the ideal orbit as

$$\bar{p}_3 \approx \left(-(e/c)\frac{A^3(x,y,\bar{s})}{|\vec{p}|} - 1 + \frac{(p^x)^2 + (p^y)^2}{2}\right) \times (1 + x/\rho(\bar{s})), \qquad (B13)$$

with

$$A^3(x,y,\bar{s}) = -\frac{c|\vec{p}|}{e} Re\mathcal{A}(x,y,\bar{s}), \qquad (B14)$$

and

$$Re\mathcal{A}(x,y,\bar{s}) = Re\sum_{n=0} \frac{e}{c|\vec{p}|} \frac{(B_n(\bar{s}) + iA_n(\bar{s}))}{(n+1)} (x+iy)^{n+1}, \qquad (B15)$$

where $A_n(\bar{s})$, and $B_n(\bar{s})$ are, respectively, the normal and skew-multipole coefficients, and $|\vec{p}|$ is found in (C33).

APPENDIX C: THE BETATRON EQUATIONS OF MOTION

In this appendix it is shown how an invariant action is used to obtain the equation for betatron oscillations when these equations of motion are expressed in terms of curvilinear coordinates. This approach is motivated by the consideration of an invariant Hamiltonian. Considering (B8) and the identity

$$\frac{d}{ds}H(p^i,q^i)/c = \frac{1}{c}\frac{\partial H(p^i,q^i)}{\partial p_i}\frac{dp_i}{ds} + \frac{1}{c}\frac{\partial H(p^i,q^i)}{\partial q^i}\frac{dq^i}{ds}$$

$$= \frac{d}{ds}(p_i u^i) - \frac{\partial L(u^i,q^i)}{\partial q^i}u^i - \frac{\partial L(u^i,q^i)}{\partial u^i}\frac{du^i}{ds} = 0, \qquad (C1)$$

one can infer for Minkowski space the equations of motion

$$\begin{aligned} c\frac{dq^i}{ds} &= \frac{\partial H(p^i,q^i)}{\partial p_i}, \\ c\frac{dp_i}{ds} &= -\frac{\partial H(p^i,q^i)}{\partial q^i}, \\ p_i &= \frac{\partial L(u^i,q^i)}{\partial u^i}, \text{ and} \\ \frac{dp_i}{ds} &= \frac{\partial L(u^i,q^i)}{\partial q^i}, \end{aligned} \qquad (C2)$$

where the invariant Hamiltonian becomes

$$H(p^i,q^i) = mc^2 = c\sqrt{(p-(e/c)A(q))_i(p-(e/c)A(q))^i}. \qquad (C3)$$

Using the above, along with

$$\frac{d}{ds}A(q)_i = u^j\partial_j A(q)_i, \qquad (C4)$$

with $\partial_j = \partial/\partial x^j$, one finds the equation of motion

$$\frac{d}{ds}mcu_i = (e/c)u^j F_{ij}, \qquad (C5)$$

where the electromagnetic field tensor is

$$F_{ij} = \partial_i A(q)_j - \partial_j A(q)_i. \qquad (C6)$$

Since

$$F^{oj} = (0, -\vec{E}), \tag{C7}$$

with the electric field defined as

$$\vec{E} = \frac{\partial}{\partial ct}\vec{A}(q) + \vec{\nabla}A(q)^0, \tag{C8}$$

and since for the magnetic field $\vec{B}$ satisfies

$$F_{\alpha\beta} = -\epsilon_{\alpha\beta\gamma}B^{\gamma}, \tag{C9}$$

with $\alpha, \beta = 1$, 2, or 3 and $\epsilon_{123} = 1$ for the completely antisymmetric tensor, it follows that

$$\frac{d}{dt}\gamma mc^2 = \frac{d\mathcal{E}}{dt} = \vec{E}\cdot\vec{v}, \tag{C10}$$

$$\frac{d}{dt}\gamma m\vec{v} = e\vec{E} + (e/c)\vec{v}\times\vec{B}. \tag{C11}$$

The first equation gives the energy $\mathcal{E}$ gain or loss of the particle of rest mass m and velocity $\vec{v}$, and the second is the Lorentz equation. In obtaining these results, use has been made of $u^0 = \gamma, \vec{u} = \gamma\vec{\beta}$, $ds = cdt/\gamma$,$\vec{\beta} = \vec{v}/c$, and $\gamma = (1 - \vec{\beta}\cdot\vec{\beta})^{-1/2}$.

It is usual to express the equations of motion for betatron dynamics in curvilinear coordinates. The curvilinear coordinates used here are x, y, and $\bar{s}$, where x is the horizontal position of the particle relative to the ideal orbit, y is the vertical position coordinate relative to the orbital plane, and $\bar{s}$ is the arc length along the ideal orbit (see Fig. 7). The position vector of the particle, expressed in terms of the right-handed orthonormal basis $\hat{e}_1, \hat{e}_2$, and $\hat{e}_3$, is

$$\vec{r} = (\rho(\bar{s}) + x)\hat{e}_1 + y\hat{e}_2. \tag{C12}$$

The origin of this vector is at the instantaneous center of curvature of the ideal orbit. The coordinates relative to the system with origin at $\vec{r}_0 = \rho(\bar{s})\hat{e}_1$ are x, y,

and z, and they are measured, respectively, along the right-handed orthonormal basis. For motion relative to the torsion-free ideal orbit, the differential of the position vector is

$$d\vec{r} \approx dx\hat{e}_1 + dy\hat{e}_2 + (1 + x/\rho(\bar{s}))d\bar{s}\hat{e}_3 + O(d\theta^2), \qquad (C13)$$

where

$$d\bar{s} = \rho(\bar{s})d\theta, \qquad (C14)$$

with $d\theta$ the angular increment of $\vec{r}_0$ along the ideal orbit. It follows from (C13) that

$$(dr)^2 \approx (dx)^2 + (dy)^2 + (1 + x/\rho(\bar{s}))^2(d\bar{s})^2. \qquad (C15)$$

The contravariant coordinates in the curvilinear system are

$$(\bar{q}^0, \bar{q}^1, \bar{q}^2, \bar{q}^3) = (q^0, q^1, q^2, \bar{s}), \qquad (C16)$$

and it follows from (C15) that

$$d\bar{q}^3 = dq^3/(1 + x/\rho(\bar{s})). \qquad (C17)$$

The invariant space-time measure is

$$ds = \sqrt{\bar{g}_{ij}d\bar{q}^i d\bar{q}^j}, \qquad (C18)$$

and the components of the covariant metric tensor $\bar{g}_{ij}$ are

$$(\bar{g}_{oo}, \bar{g}_{11}, \bar{g}_{22}, \bar{g}_{33}) = (1, -1, -1, -(1 + x/\rho(\bar{s}))^2). \qquad (C19)$$

The curvilinear contravariant components of the four-momentum are

$$(\bar{p}^0, \bar{p}^1, \bar{p}^2, \bar{p}^3) = (p^0, p^1, p^2, p^3/(1 + x/\rho(\bar{s}))), \qquad (C20)$$

and the covariant components are

$$(\bar{p}_0, \bar{p}_1, \bar{p}_2, \bar{p}_3) = (p_0, p_1, p_2, p_3(1 + x/\rho(\bar{s}))). \tag{C21}$$

Similar relations are obtained for the coordinate differential four-vector dq^i and the vector-potential $A(q)^i$.

The betatron equations of motion are found in the curvilinear coordinate system using the fact that the action S is invariant under Lorentz transformations and orthogonal spatial coordinate transformations. The action (B12) is an invariant under the coordinate transformation $q^i \to \bar{q}^i$ so that

$$S = \int p_i u^i ds = \int \bar{p}_i \bar{u}^i ds. \tag{C22}$$

A variation of this action gives

$$\delta S = \int (\delta \bar{p}_i \bar{u}^i + \bar{p}_i \delta \bar{u}^i) ds. \tag{C23}$$

Using

$$\delta \bar{u}^i = \frac{d\delta \bar{q}^i}{ds}, \tag{C24}$$

and

$$\delta \bar{p}_3 = \frac{\partial \bar{p}_3}{\partial \bar{p}_\alpha} \delta \bar{p}_\alpha + \frac{\partial \bar{p}_3}{\partial \bar{q}^i} \delta \bar{q}^i, \quad \alpha \neq 3, \tag{C25}$$

one finds for δS

$$\delta S = \int \left((\frac{\partial \bar{p}_3}{\partial \bar{p}_\alpha} \bar{u}^3 + \bar{u}^\alpha) \delta \bar{p}_\alpha + (\frac{\partial \bar{p}_3}{\partial \bar{q}^i} \bar{u}^3 - \frac{d\bar{p}_i}{ds}) \delta \bar{q}^i + \frac{d(\bar{p}_i \delta \bar{q}^i)}{ds}) \right) ds. \tag{C26}$$

In the usual manner, this yields the dynamical equations of motion

$$\begin{aligned} \frac{d\bar{q}^\alpha}{ds} &= -\bar{u}^3 \frac{\partial \bar{p}_3}{\partial \bar{p}_\alpha}, \\ \frac{d\bar{p}_i}{ds} &= \bar{u}^3 \frac{\partial \bar{p}_3}{\partial \bar{q}^i}. \end{aligned} \tag{C27}$$

Using the representation of the invariant measure

$$ds = \sqrt{d\bar{q}_i d\bar{q}^i} = \sqrt{\frac{d\bar{q}_i}{d\bar{q}^3}\frac{d\bar{q}^i}{d\bar{q}^3}}\, d\bar{q}^3, \qquad (C28)$$

and the coordinates $(\bar{q}^0, \bar{q}^1, \bar{q}^2, \bar{q}^3) = (q^0, x, y, \bar{s})$, along with

$$\bar{u}^3 = 1/\sqrt{\frac{d\bar{q}_i}{d\bar{q}^3}\frac{d\bar{q}^i}{d\bar{q}^3}}, \qquad (C29)$$

one finds, using the notation $x' = dx/d\bar{s}$ etc., the dynamical equations

$$q^{0\prime} = -\frac{\partial \bar{p}_3}{\partial p^0} \qquad p^{0\prime} = \frac{\partial \bar{p}_3}{\partial \bar{q}^0}$$

$$x' = \frac{\partial \bar{p}_3}{\partial p^x} \qquad p^{x\prime} = -\frac{\partial \bar{p}_3}{\partial x}$$

$$y' = \frac{\partial \bar{p}_3}{\partial p^y} \qquad p^{y\prime} = -\frac{\partial \bar{p}_3}{\partial y}$$

$$(\bar{p}_3)' = \frac{\partial \bar{p}_3}{\partial \bar{s}}. \qquad (C30)$$

It is interesting to note in passing that the conventional form of Hamilton's equations can be found in the Cartesian coordinate system from (C22), if the coordinate $q^0 = ct$ is used in place of $\bar{q}^3$, and if one considers the variation δp_0 in finding the variation of the action. This leads to a system of equations similar to (C30)

$$\frac{dp^0 c}{dt} = \frac{\partial p^0 c}{\partial t}, \quad \frac{d\vec{p}}{dt} = -\frac{\partial p^0 c}{\partial \vec{q}}, \quad \text{and} \quad \frac{d\vec{q}}{dt} = \frac{\partial p^0 c}{\partial \vec{p}}, \qquad (C30a)$$

where the Hamiltonian is $p^0 c$. These equations lead to (C10) and C(11).

The effective Hamiltonian $\bar{p}_3$ for this system is found from the invariant

$$(mc)^2 = (p-(e/c)A(q))_i(p-(e/c)A(q))^i = (\bar{p}-(e/c)\bar{A}(q))_i(\bar{p}-(e/c)\bar{A}(q))^i. \quad (C31)$$

Defining the energy $\mathcal{E}$ from

$$\mathcal{E}/c = p^0 - (e/c)A(q)^0 = mc\gamma, \qquad (C32)$$

and the scalar three-momentum as

$$|\vec{p}| = \sqrt{\mathcal{E}/c)^2 - (mc)^2}, \tag{C33}$$

one can use

$$\bar{g}^{33} = -(1 + x/\rho(\bar{s}))^{-2}, \tag{C34}$$

and

$$\bar{A}_3 = -(1 + x/\rho(\bar{s}))A^3, \tag{C35}$$

to obtain

$$\frac{\bar{p}_3}{|\vec{p}|} = -(e/c)(1 + x/\rho(\bar{s}))\frac{A^3(x,y,\bar{s})}{|\vec{p}|} - \sqrt{1 - \left(\frac{p^x - (e/c)A^x(x,y,\bar{s})}{|\vec{p}|}\right)^2 - \left(\frac{p^y - (e/c)A^y(x,y,\bar{s})}{|\vec{p}|}\right)^2} \times (1 + x/\rho(\bar{s})). \tag{C36}$$

For constant $|\vec{p}|$, one can introduce the scaled variables for the four-momentum $p^i/|\vec{p}| \to p^i$ in the dynamical equation (C30) and in (C36). These equations simplify for the case of a purely transverse magnetic field, where $A^x = A^y = 0$, and for the approximation

$$\sqrt{1 - (p^x)^2 - (p^y)^2} \approx 1 - \frac{(p^x)^2 + (p^y)^2}{2}, \tag{C37}$$

which is valid for large $|\vec{p}|$. For this case, one finds that (C36) becomes

$$\bar{p}_3 \approx \left(-(e/c)\frac{A^3(x,y,\bar{s})}{|\vec{p}|} - 1 + \frac{(p^x)^2 + (p^y)^2}{2}\right) \times (1 + x/\rho(\bar{s})), \tag{C38}$$

and, neglecting terms of order $p^{x2}/\rho(\bar{s})$, (C30) gives

$$x' = p^x(1 + x/\rho(\bar{s})), \qquad y' = p^y(1 + x/\rho(\bar{s})), \tag{C39}$$

$$\frac{d}{d\bar{s}}\left(\frac{x'}{1 + x/\rho(\bar{s})}\right) = \frac{\partial}{\partial x}\left[(e/c)(1 + x/\rho(\bar{s}))\frac{A^3(x,y,\bar{s})}{|\vec{p}|} + 1 + x/\rho(\bar{s})\right],$$

and

$$\frac{d}{d\bar{s}}\left(\frac{y'}{1+x/\rho(\bar{s})}\right)=\frac{\partial}{\partial y}\left[(e/c)(1+x/\rho(\bar{s}))\frac{A^3(x,y,\bar{s})}{|\vec{p}|}+1+x/\rho(\bar{s})\right]. \qquad (C40)$$

Neglecting terms of order $(x/\rho(\bar{s}))^2$, $x'^2/\rho(\bar{s})$, and skew-multipole errors, one finds using $e\bar{B}_0(\bar{s})\rho(\bar{s})=c|\vec{p}_0|$,

$$x''+\frac{|\vec{p}_0|}{|\vec{p}|}\left(\frac{1}{\rho(\bar{s})^2}+\frac{e\bar{B}_1(\bar{s})}{c|\vec{p}_0|}\right)x=\frac{|\vec{p}|-|\vec{p}_0|}{\rho(\bar{s})|\vec{p}|}\sim 0, \qquad (C41)$$

and

$$y''-\frac{|\vec{p}_0|}{|\vec{p}|}\frac{e\bar{B}_1(\bar{s})}{c|\vec{p}_0|}y=0, \qquad (C42)$$

where $|\vec{p}_0|$ is the three-momentum along the ideal orbit.

APPENDIX D: LATTICE FUNCTIONS

In this appendix, the lattice functions used in the numerical calculations are derived. They are found for a lattice made up of similar cells of the FODO (focusing, drift, defocusing, drift) form. Focusing and defocusing are achieved with thin lens quadrupole magnets, and the drifts occur through bending dipole magnets of length L and strength B_0. These functions have period $2L$, and the functions on the interval $L < \bar{s} < 2L$ are found from those on the interval $0 < \bar{s} < L$ using

$$f(\bar{s})_{L<\bar{s}<2L} = f(2L - \bar{s})_{0<\bar{s}<L}. \tag{D1}$$

The beta functions for a lattice with phase advance μ per cell are found from the $(1,2)$ component of the transfer matrix. The function $\beta_x(\bar{s})$ is found from

$$\beta_x(\bar{s}) \sin\mu = (\mathbf{O}(\bar{s})\mathbf{F}\mathbf{O}(L)\mathbf{D}\mathbf{O}(L - \bar{s}))_{12}, \tag{D2}$$

where the focusing and defocusing matrices for lenses of focal length f are, respectively,

$$\mathbf{F} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{D} = \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}. \tag{D3}$$

The matrix for a drift of distance $\bar{s}$ is

$$\mathbf{O}(\bar{s}) = \begin{pmatrix} 1 & \bar{s} \\ 0 & 1 \end{pmatrix}. \tag{D4}$$

The resulting beta function for $0 < \bar{s} < L$ is

$$\beta_x(\bar{s}) = \frac{2L}{\sin\mu}\left(1 + \frac{L}{2f} - (\frac{1}{f} + \frac{L}{2f^2})\bar{s} + \frac{\bar{s}^2}{2f^2}\right), \tag{D5}$$

where $L = 2f\sin(\mu/2)$. The expression for $\beta_y(\bar{s})$ on the same interval is found from $\beta_y(\bar{s}) = \beta_x(L - \bar{s})$.

The dispersion function $\eta(\bar{s})$ can be found when the differential equation (2.12) is approximated by

$$\eta''(\bar{s}) + K(\bar{s})\eta(\bar{s}) = \frac{eB_0(\bar{s})}{c|\vec{p}_0|} = \frac{1}{\rho(\bar{s})}. \tag{D6}$$

For the interval $0 < \bar{s} < L$ where the dipole bending angle is θ, it is found to be

$$\eta(\bar{s}) = \eta(0)(1 - \bar{s}/2f) + C_0\bar{s}^2/8f^2, \tag{D7}$$

where $\eta(0) = C_0(1+L/4f)$ and $C_0 = 4\theta f^2/L$. Within the dipole magnet, $K(\bar{s}) = 0$, and $B_0(\bar{s}) \sim B_0(L/2)$. If follows from (D6) that

$$\eta''(L/2) = \frac{1}{\rho} = \frac{\theta}{L}. \tag{D8}$$

Within a focusing quadrupole of width ϵ, $B(\bar{s}) = 0$, and (D6) gives

$$\epsilon\eta''(0) = -\epsilon K(0)\eta(0) = -\eta(0)/f, \tag{D9}$$

where $\epsilon K(0) = 1/f$. This equation represents the change in the slope of $\eta(\bar{s})$ through the focusing quadrupole, and it is equivalent to the condition

$$\eta'(0) = -\eta(0)/2f. \tag{D10}$$

A similar argument for the defocusing quadrupole gives the condition

$$\eta'(L) = -\eta(L)/2f. \tag{D11}$$

From these three conditions, one can determine the quadratic approximation (D7) on the interval $0 < \bar{s} < L$. On the interval $L < \bar{s} < 2L$, $\eta(\bar{s})$ is found from (D1).

APPENDIX E: THE REDUCE CODE TSHIFT

The REDUCE program TSHIFT is given in this appendix, and the FORTRAN results which it produces for the tune-shifts (2.21) are given also. In the results, the notations $epsx = 2J_x$ and $epxy = 2J_y$ have been used. These quantities are related to the injection amplitudes through

$$\begin{aligned} \epsilon_x &= x_{amp}^2/\beta_{xmax} \\ \epsilon_y &= y_{amp}^2/\beta_{ymax}. \end{aligned} \tag{E1}$$

In addition, the notation $DELTA = \delta|\vec{p}|/\vec{p}|$ appears in the results. The angular averaging appearing in (2.23) has been performed in obtaining the results; however, the averaging of the type (2.25) around the circumference is done numerically in the program TUNESHIFT.FOR.

```
comment TSHIFT$
comment This programme calculates the multipole tuneshifs$
comment deltanux and deltanuy for B1 through B19$
comment 20 June 1991$
comment Tshift provides input for Tuneshift.for$
comment The A's are set to zero because they do not contribute$
comment as a result of the averaging on phix and phiy.$
  operator a$
  a(0):=0$
  a(1):=0$
  a(2):=0$
  a(3):=0$
  a(4):=0$
  a(5):=0$
  a(6):=0$
  a(7):=0$
  a(8):=0$
  a(9):=0$
  a(10):=0$
  a(11):=0$
  a(12):=0$

  operator b$
  b(0):=b0$
  b(1):=b1$
  b(2):=b2$
  b(3):=b3$
  b(4):=b4$
  b(5):=b5$
  b(6):=b6$
  b(7):=b7$
  b(8):=b8$
  b(9):=b9$
  b(10):=b10$
  b(11):=b11$
  b(12):=b12$

  operator c$

  let J**2=-1$
  for all n let c(n)=(x+j*y)**n$
  factor j$
  operator potential$
  comment potential=repot+i*impot$
  operator impot$
  operator repot$
  for all n let potential(n)=(b(n)+j*a(n))*c(n+1)/(n+1)$

  for all n let impot(n)=df(potential(n),j)$

  for all n let repot(n)=potential(n)-j*impot(n)$

  factor BO$
  factor a1,a2,a3,a4,a5,a6,a7a,a8,a9,a10,a11,a12$
  factor b1,b2,b3,b4,b5,b6,b7,b8,b9,b10,b11,b12$
  comment This gives the multipole potential for B3.$
  repot(3);

  x:=sqrt(2*betax*jx)*cos(phix)+eta*delta$
  y:=sqrt(2*betay*jy)*cos(phiy)$
  repot(1)$
  operator ix$
  operator iy$
  for all n match cos(phix)**n=ix(n)$
  for all n match cos(phiy)**n=iy(n)$
  match cos(phix)=ix(1)$
```

```
   match cos(phiy)=iy(1)$
   for i:=1 step 2 until 21 do let ix(i)=0$
   for i:=1 step 2 until 21 do let iy(i)=0$
   comment The operator for n!.$
   operator nfac$
   for all n let nfac(n)=for i:=1:n product i$
   comment Evaluate the integral(1/2pi)int(0>2pi)of cos**2m(phix)$
for i:=0 step 2 until 20 do let ix(i)=nfac(i)/(nfac(i/2)*nfac(i/2)*2**i)$
for i:=0 step 2 until 20 do let iy(i)=nfac(i)/(nfac(i/2)*nfac(i/2)*2**i)$
   jx:=epsx/2$
   jy:=epsy/2$
   comment The tune-shift operators.$
   operator dnux$
   operator dnuy$
   comment Tuneshifts$
   for all n let dnux(n)=df(repot(n),epsx)*2$
   for all n let dnuy(n)=df(repot(n),epsy)*2$
   off allfac$
   on div$
   order delta,epsx,epsy,betax,betay,eta$
   on fort$
   out dnux$
   for i:=1:10 do write"dnux(",i,"):=",dnux(i)$
   shut dnux$
   out dnuy$
   for i:=1:10 do write"dnuy(",i,"):=",dnuy(i)$
   shut dnuy$
   end$
```

```
        for i:=1:10 do write"dnux(",i,"):=",dnux(i)$

dnux(1.):=1./2.*B1*BETAX
dnux(2.):=B2*DELTA*BETAX*ETA
dnux(3.):=B3*(3./2.*DELTA**2*BETAX*ETA**2+3./8.*EPSX*BETAX
      . **2-3./4.*EPSY*BETAX*BETAY)
dnux(4.):=B4*(2.*DELTA**3*BETAX*ETA**3+3./2.*DELTA*EPSX*
      . BETAX**2*ETA-3.*DELTA*EPSY*BETAX*BETAY*ETA)
dnux(5.):=B5*(5./2.*DELTA**4*BETAX*ETA**4+15./4.*DELTA**2*
      . EPSX*BETAX**2*ETA**2-15./2.*DELTA**2*EPSY*BETAX*
      . BETAY*ETA**2+5./16.*EPSX**2*BETAX**3-15./8.*EPSX*
      . EPSY*BETAX**2*BETAY+15./16.*EPSY**2*BETAX*BETAY**2)
dnux(6.):=B6*(3.*DELTA**5*BETAX*ETA**5+15./2.*DELTA**3*
      . EPSX*BETAX**2*ETA**3-15.*DELTA**3*EPSY*BETAX*BETAY*
      . ETA**3+15./8.*DELTA*EPSX**2*BETAX**3*ETA-45./4.*
      . DELTA*EPSX*EPSY*BETAX**2*BETAY*ETA+45./8.*DELTA*
      . EPSY**2*BETAX*BETAY**2*ETA)
dnux(7.):=B7*(7./2.*DELTA**6*BETAX*ETA**6+105./8.*DELTA**4
      . *EPSX*BETAX**2*ETA**4-105./4.*DELTA**4*EPSY*BETAX*
      . BETAY*ETA**4+105./16.*DELTA**2*EPSX**2*BETAX**3*ETA
      . **2-315./8.*DELTA**2*EPSX*EPSY*BETAX**2*BETAY*ETA**
      . 2+315./16.*DELTA**2*EPSY**2*BETAX*BETAY**2*ETA**2+
      . 35./128.*EPSX**3*BETAX**4-105./32.*EPSX**2*EPSY*
      . BETAX**3*BETAY+315./64.*EPSX*EPSY**2*BETAX**2*BETAY
      . **2-35./32.*EPSY**3*BETAX*BETAY**3)
dnux(8.):=B8*(4.*DELTA**7*BETAX*ETA**7+21.*DELTA**5*EPSX*
      . BETAX**2*ETA**5-42.*DELTA**5*EPSY*BETAX*BETAY*ETA**
      . 5+35./2.*DELTA**3*EPSX**2*BETAX**3*ETA**3-105.*
      . DELTA**3*EPSX*EPSY*BETAX**2*BETAY*ETA**3+105./2.*
      . DELTA**3*EPSY**2*BETAX*BETAY**2*ETA**3+35./16.*
      . DELTA*EPSX**3*BETAX**4*ETA-105./4.*DELTA*EPSX**2*
      . EPSY*BETAX**3*BETAY*ETA+315./8.*DELTA*EPSX*EPSY**2*
      . BETAX**2*BETAY**2*ETA-35./4.*DELTA*EPSY**3*BETAX*
      . BETAY**3*ETA)
dnux(9.):=B9*(9./2.*DELTA**8*BETAX*ETA**8+63./2.*DELTA**6*
      . EPSX*BETAX**2*ETA**6-63.*DELTA**6*EPSY*BETAX*BETAY*
      . ETA**6+315./8.*DELTA**4*EPSX**2*BETAX**3*ETA**4-
      . 945./4.*DELTA**4*EPSX*EPSY*BETAX**2*BETAY*ETA**4+
      . 945./8.*DELTA**4*EPSY**2*BETAX*BETAY**2*ETA**4+315./
      . 32.*DELTA**2*EPSX**3*BETAX**4*ETA**2-945./8.*DELTA**
      . 2*EPSX**2*EPSY*BETAX**3*BETAY*ETA**2+2835./16.*
      . DELTA**2*EPSX*EPSY**2*BETAX**2*BETAY**2*ETA**2-315.
      . /8.*DELTA**2*EPSY**3*BETAX*BETAY**3*ETA**2+63./256.
      . *EPSX**4*BETAX**5-315./64.*EPSX**3*EPSY*BETAX**4*
      . BETAY+945./64.*EPSX**2*EPSY**2*BETAX**3*BETAY**2-
      . 315./32.*EPSX*EPSY**3*BETAX**2*BETAY**3+315./256.*
      . EPSY**4*BETAX*BETAY**4)
dnux(10.):=B10*(5.*DELTA**9*BETAX*ETA**9+45.*DELTA**7*EPSX
      . *BETAX**2*ETA**7-90.*DELTA**7*EPSY*BETAX*BETAY*ETA
      . **7+315./4.*DELTA**5*EPSX**2*BETAX**3*ETA**5-945./
      . 2.*DELTA**5*EPSX*EPSY*BETAX**2*BETAY*ETA**5+945./4.*
      . DELTA**5*EPSY**2*BETAX*BETAY**2*ETA**5+525./16.*
      . DELTA**3*EPSX**3*BETAX**4*ETA**3-1575./4.*DELTA**3*
      . EPSX**2*EPSY*BETAX**3*BETAY*ETA**3+4725./8.*DELTA**
      . 3*EPSX*EPSY**2*BETAX**2*BETAY**2*ETA**3-525./4.*
      . DELTA**3*EPSY**3*BETAX*BETAY**3*ETA**3+315./128.*
      . DELTA*EPSX**4*BETAX**5*ETA-1575./32.*DELTA*EPSX**3*
      . EPSY*BETAX**4*BETAY*ETA+4725./32.*DELTA*EPSX**2*
      . EPSY**2*BETAX**3*BETAY**2*ETA-1575./16.*DELTA*EPSX*
      . EPSY**3*BETAX**2*BETAY**3*ETA+1575./128.*DELTA*EPSY
      . **4*BETAX*BETAY**4*ETA)

        shut dnux$
```

```
        for i:=1:10 do write"dnuy(",i,"):=",dnuy(i)$

dnuy(1.):=-1./2.*B1*BETAY
dnuy(2.):=-B2*DELTA*BETAY*ETA
dnuy(3.):=-B3*(3./2.*DELTA**2*BETAY*ETA**2+3./4.*EPSX*
      . BETAX*BETAY-3./8.*EPSY*BETAY**2)
dnuy(4.):=-B4*(2.*DELTA**3*BETAY*ETA**3+3.*DELTA*EPSX*
      . BETAX*BETAY*ETA-3./2.*DELTA*EPSY*BETAY**2*ETA)
dnuy(5.):=-B5*(5./2.*DELTA**4*BETAY*ETA**4+15./2.*DELTA**2
      . *EPSX*BETAX*BETAY*ETA**2-15./4.*DELTA**2*EPSY*BETAY
      . **2*ETA**2+15./16.*EPSX**2*BETAX**2*BETAY-15./8.*
      . EPSX*EPSY*BETAX*BETAY**2+5./16.*EPSY**2*BETAY**3)
dnuy(6.):=-B6*(3.*DELTA**5*BETAY*ETA**5+15.*DELTA**3*EPSX*
      . BETAX*BETAY*ETA**3-15./2.*DELTA**3*EPSY*BETAY**2*
      . ETA**3+45./8.*DELTA*EPSX**2*BETAX**2*BETAY*ETA-45./
      . 4.*DELTA*EPSX*EPSY*BETAX*BETAY**2*ETA+15./8.*DELTA*
      . EPSY**2*BETAY**3*ETA)
dnuy(7.):=-B7*(7./2.*DELTA**6*BETAY*ETA**6+105./4.*DELTA**
      . 4*EPSX*BETAX*BETAY*ETA**4-105./8.*DELTA**4*EPSY*
      . BETAY**2*ETA**4+315./16.*DELTA**2*EPSX**2*BETAX**2*
      . BETAY*ETA**2-315./8.*DELTA**2*EPSX*EPSY*BETAX*BETAY
      . **2*ETA**2+105./16.*DELTA**2*EPSY**2*BETAY**3*ETA**
      . 2+35./32.*EPSX**3*BETAX**3*BETAY-315./64.*EPSX**2*
      . EPSY*BETAX**2*BETAY**2+105./32.*EPSX*EPSY**2*BETAX*
      . BETAY**3-35./128.*EPSY**3*BETAY**4)
dnuy(8.):=-B8*(4.*DELTA**7*BETAY*ETA**7+42.*DELTA**5*EPSX*
      . BETAX*BETAY*ETA**5-21.*DELTA**5*EPSY*BETAY**2*ETA**
      . 5+105./2.*DELTA**3*EPSX**2*BETAX**2*BETAY*ETA**3-
      . 105.*DELTA**3*EPSX*EPSY*BETAX*BETAY**2*ETA**3+35./2.
      . *DELTA**3*EPSY**2*BETAY**3*ETA**3+35./4.*DELTA*EPSX
      . **3*BETAX**3*BETAY*ETA-315./8.*DELTA*EPSX**2*EPSY*
      . BETAX**2*BETAY**2*ETA+105./4.*DELTA*EPSX*EPSY**2*
      . BETAX*BETAY**3*ETA-35./16.*DELTA*EPSY**3*BETAY**4*
      . ETA)
dnuy(9.):=-B9*(9./2.*DELTA**8*BETAY*ETA**8+63.*DELTA**6*
      . EPSX*BETAX*BETAY*ETA**6-63./2.*DELTA**6*EPSY*BETAY
      . **2*ETA**6+945./8.*DELTA**4*EPSX**2*BETAX**2*BETAY*
      . ETA**4-945./4.*DELTA**4*EPSX*EPSY*BETAX*BETAY**2*
      . ETA**4+315./8.*DELTA**4*EPSY**2*BETAY**3*ETA**4+
      . 315./8.*DELTA**2*EPSX**3*BETAX**3*BETAY*ETA**2-2835.
      . /16.*DELTA**2*EPSX**2*EPSY*BETAX**2*BETAY**2*ETA**2
      . +945./8.*DELTA**2*EPSX*EPSY**2*BETAX*BETAY**3*ETA**
      . 2-315./32.*DELTA**2*EPSY**3*BETAY**4*ETA**2+315./
      . 256.*EPSX**4*BETAX**4*BETAY-315./32.*EPSX**3*EPSY*
      . BETAX**3*BETAY**2+945./64.*EPSX**2*EPSY**2*BETAX**2
      . *BETAY**3-315./64.*EPSX*EPSY**3*BETAX*BETAY**4+63./
      . 256.*EPSY**4*BETAY**5)
dnuy(10.):=-B10*(5.*DELTA**9*BETAY*ETA**9+90.*DELTA**7*
      . EPSX*BETAX*BETAY*ETA**7-45.*DELTA**7*EPSY*BETAY**2*
      . ETA**7+945./4.*DELTA**5*EPSX**2*BETAX**2*BETAY*ETA
      . **5-945./2.*DELTA**5*EPSX*EPSY*BETAX*BETAY**2*ETA**
      . 5+315./4.*DELTA**5*EPSY**2*BETAY**3*ETA**5+525./4.*
      . DELTA**3*EPSX**3*BETAX**3*BETAY*ETA**3-4725./8.*
      . DELTA**3*EPSX**2*EPSY*BETAX**2*BETAY**2*ETA**3+
      . 1575./4.*DELTA**3*EPSX*EPSY**2*BETAX*BETAY**3*ETA**3
      . -525./16.*DELTA**3*EPSY**3*BETAY**4*ETA**3+1575./
      . 128.*DELTA*EPSX**4*BETAX**4*BETAY*ETA-1575./16.*
      . DELTA*EPSX**3*EPSY*BETAX**3*BETAY**2*ETA+4725./32.*
      . DELTA*EPSX**2*EPSY**2*BETAX**2*BETAY**3*ETA-1575./
      . 32.*DELTA*EPSX*EPSY**3*BETAX*BETAY**4*ETA+315./128.*
      . DELTA*EPSY**4*BETAY**5*ETA)

        shut dnuy$
```

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* Operated by the Universities Research Association, Inc., for the U. S. Department of Energy under Contract No. DE-AC35-89ER40486.

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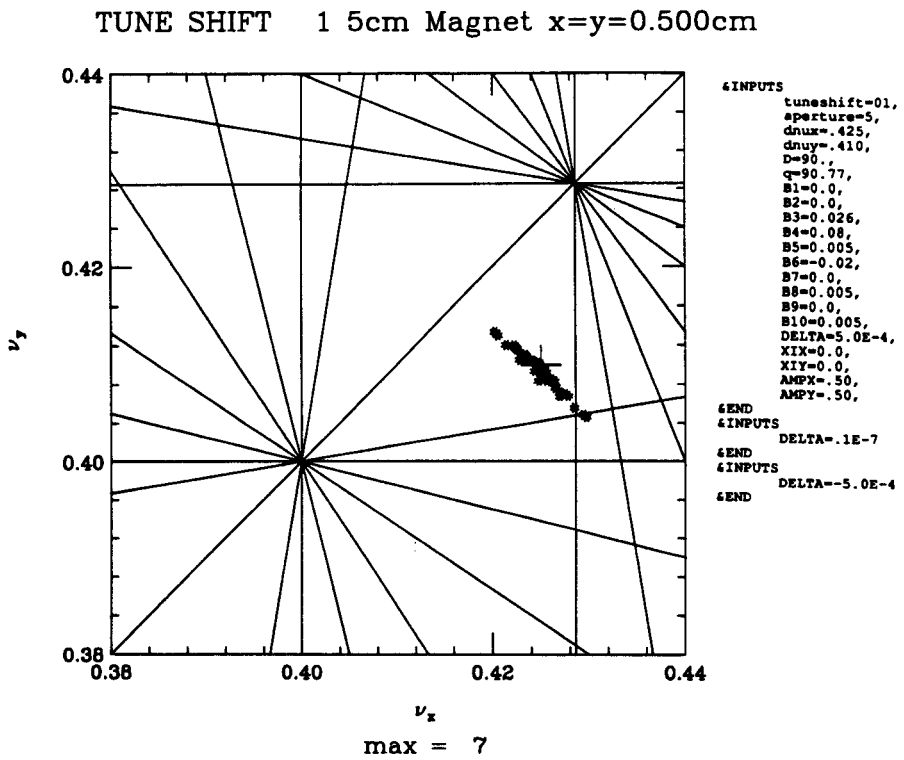


FIG. 1. The tune-spread for $\delta|\vec{p}|/\vec{p}| = 0$, and $\pm 5. \times 10^{-4}$, with injection amplitude grid from $(0, 0)$ to $(0.5\text{ mm}, 0.5\text{ mm})$.

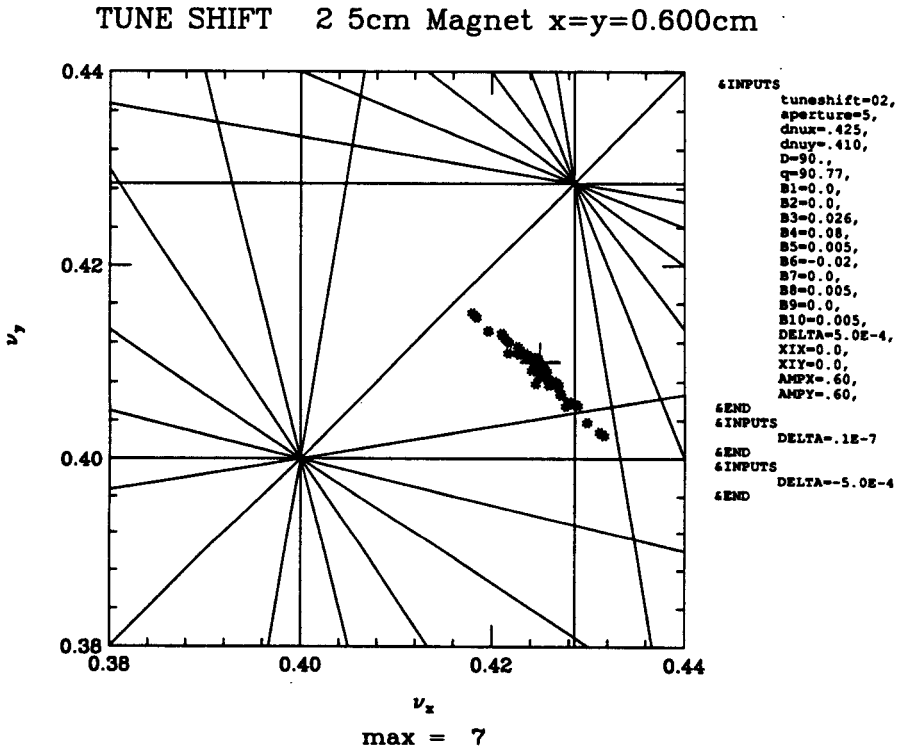
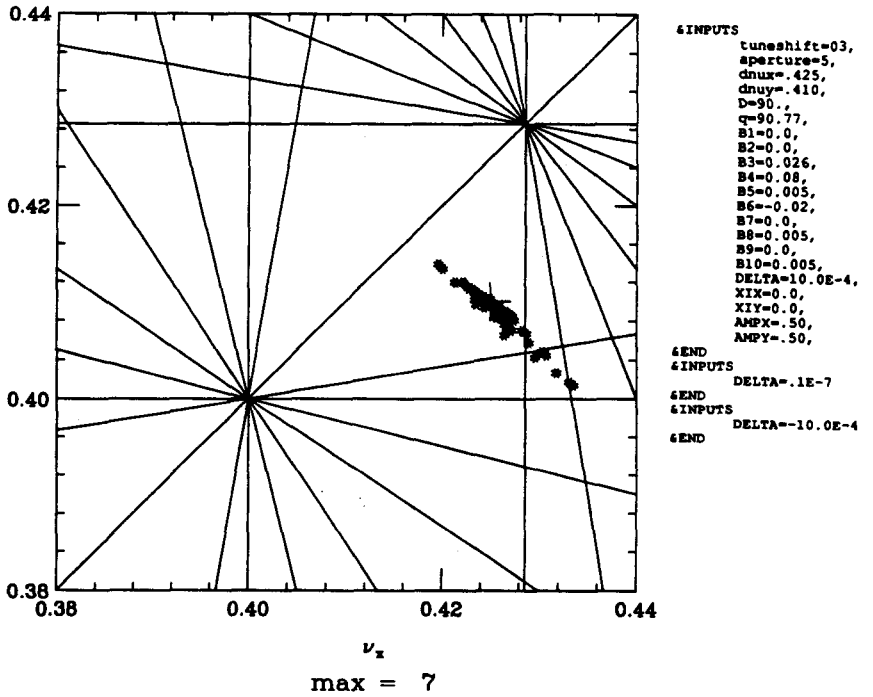


FIG. 2. The tune-spread for $\delta|\vec{p}|/\vec{p}| = 0$, and $\pm 5. \times 10^{-4}$, with injection amplitude grid from $(0, 0)$ to $(0.6\text{ mm}, 0.6\text{ mm})$.

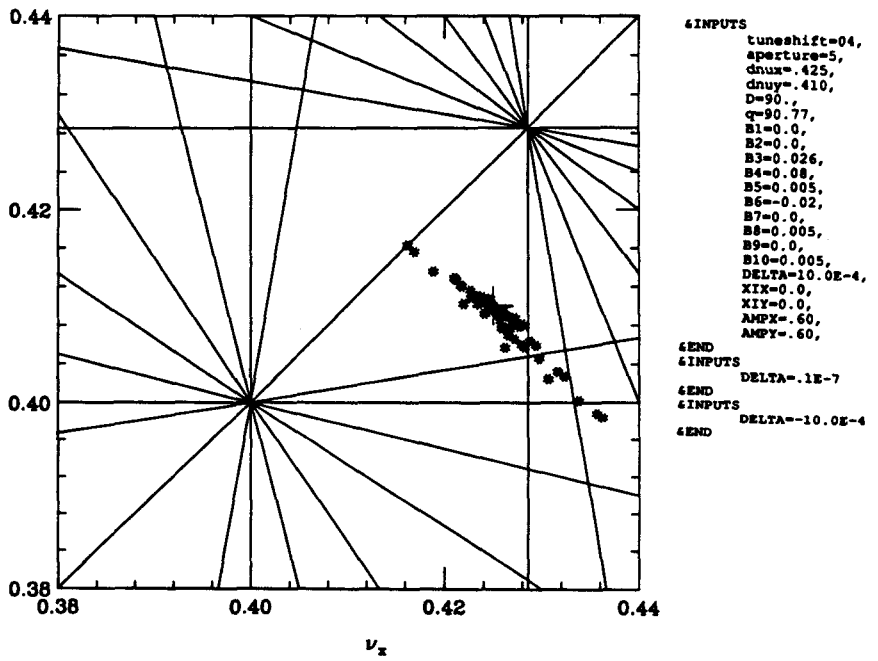
TUNE SHIFT 3 5cm Magnet x=y=0.500cm



max = 7

FIG. 3. The tune-spread for $\delta|\vec{p}|/\vec{p}| = 0$, and $\pm 10. \times 10^{-4}$, with injection amplitude grid from $(0,0)$ to $(0.5\text{ mm}, 0.5\text{ mm})$.

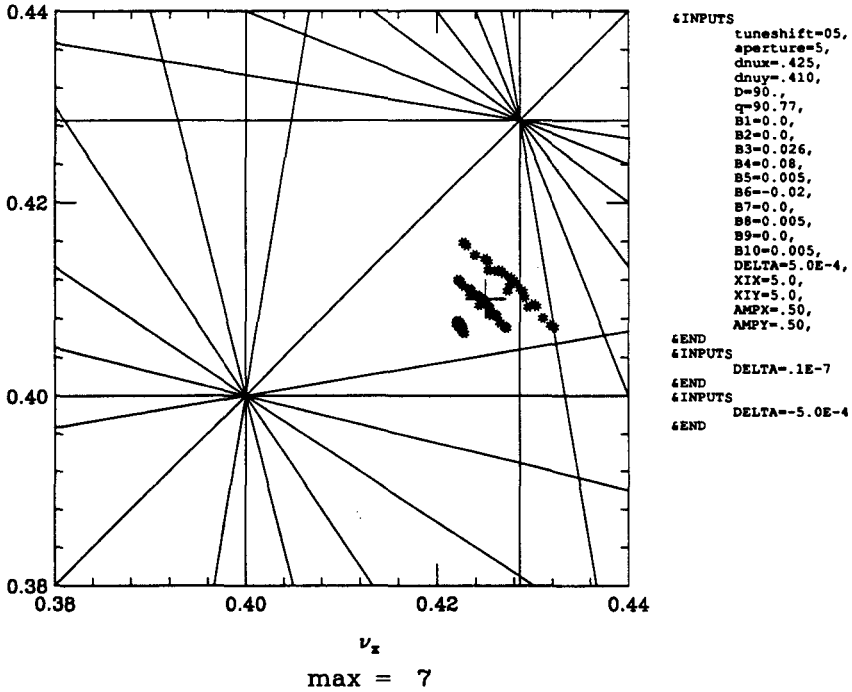
TUNE SHIFT 4 5cm Magnet x=y=0.600cm



max = 7

FIG. 4. The tune-spread for $\delta|\vec{p}|/\vec{p}| = 0$, and $\pm 10. \times 10^{-4}$, with injection amplitude grid from $(0,0)$ to $(0.6\text{ mm}, 0.6\text{ mm})$.

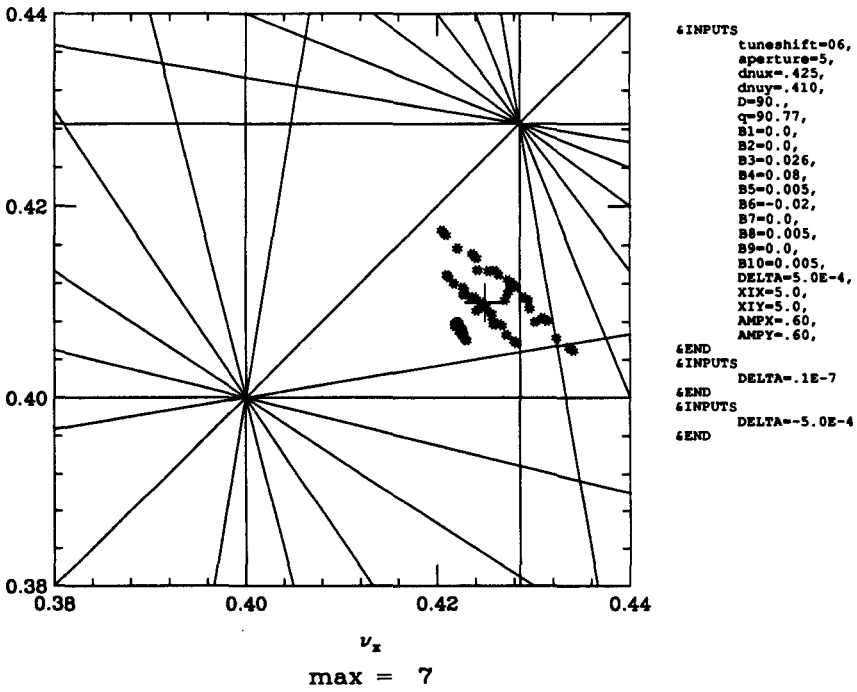
TUNE SHIFT 5 5cm Magnet x=y=0.500cm



max = 7

FIG. 5. The tune-spread with linear momentum error for $\delta|\vec{p}|/\vec{p}| = 0$, and $\pm 5. \times 10^{-4}$, with injection amplitude grid from (0,0) to (0.5 mm, 0.5 mm).

TUNE SHIFT 6 5cm Magnet x=y=0.600cm



max = 7

FIG. 6. The tune-spread with linear momentum error for $\delta|\vec{p}|/\vec{p}| = 0$, and $\pm 5. \times 10^{-4}$, with injection amplitude grid from (0,0) to (0.6 mm, 0.6 mm).

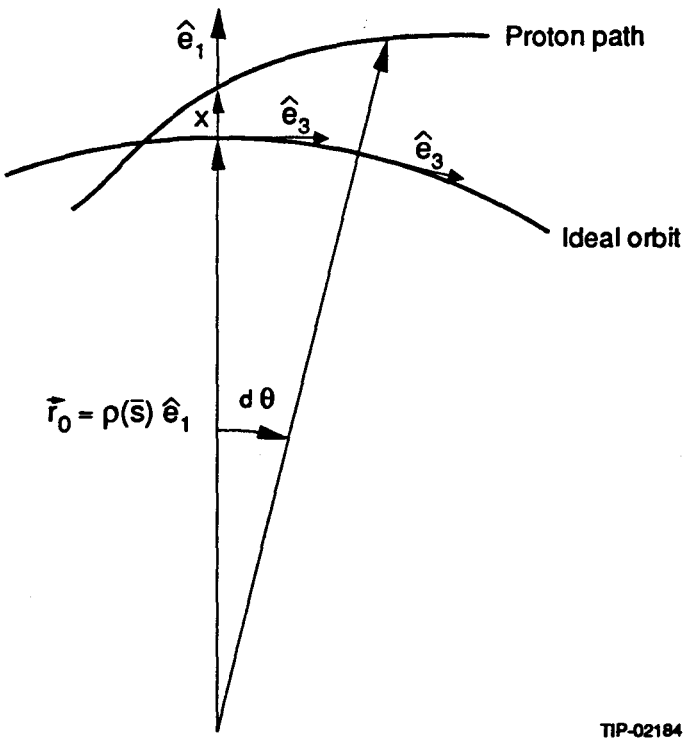


FIG. 7. The curvilinear geometry and coordinate systems for proton motion relative to the ideal orbit.